Asymptotic Behavior of Solutions of Pseudo-parabolic Partial Differential Equations (*).

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Summary. - *Consider a solution to a second-order pseudo-parabolic equation with sufficiently smooth time-independent coefficients in a cylindrical domain. If it vanishes on the cylindrical surface for all times and if its restriction to a fixed instant belongs to* $C_{2+\alpha}$, *then its pointwise values decay exponentially as* $t \to \infty$ *while its Dirichlet norm grows expontially as* $t \to -\infty$. *Similar conclusion still hold for solutions to non-homogeneous equations under non-homogeneous boundary conditions provided the free term and the boundary data posses these asymptotic behaviors.*

1. Introduction. - Let G be a bounded domain in R^n with ∂G, boundary of G, being of class $C^{2+\alpha}$, [4]. Denote by $C_{m+\alpha}(\bar{G})$ the BANACH space of functions with partial derivatives up to and including order $m \geqq 0$ uniformly HÖLDER continuous in $\bar{G}$ with exponent α. The well known HÖLDER norm [3-5] on the space $C_{m+\alpha}(\bar{G})$ will be denoted by $\| \;\|_{m+\alpha}$.

We shall be concerned with strict solutions $u(t, x)$ of the pseudo-parabolic partial differential equations,

$$Mu_t - Lu = f(t, x), \tag{1.1}$$

in a cylindrical domain $G \times (-T, T)$. Here $u_t = \partial u/\partial t$ and L and M stand for second-order differential operator in the divergence form:

$$Lu \equiv \frac{\partial}{\partial x_i}\left(l_{ij}(x)\frac{\partial u}{\partial x_j}\right) - l(x)u, \tag{1.2}$$

$$Mu \equiv m(x)u - \frac{\partial}{\partial x_i}(m_{ij}(x)\frac{\partial u}{\partial x_j}$$

In writing (1.2) as well as what follows, we have adopted the summation convention over the repeated indices. Also, we shall assume that for some α, $0 < \alpha < 1$, the real coefficients $l_{ij}(x)$ and $m_{ij}(x)$, belong to $C_{1+\alpha}(\bar{G})$ and that $l(x)$ and $m(x)$ belong to $C_\alpha(\bar{G})$. Further, L abd M will be restricted to be symmetric and uniformly elliptic in G in the sense that there are constants

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k_L, K_L, k_M and K_M such that for all real vectors, $\xi = (\xi_1, \xi_2, \dots, \xi_n)$, the inequalities

$$k_L |\xi|^2 \leqq l_{ij}(x)\,\xi_i\xi_j \leqq K_L\,|\xi|^2,$$

$$k_M |\xi|^2 \leqq m_{ij}(x)\xi_i\xi_j \leqq K_M |\xi|^2, \tag{1.3}$$

$$|\xi|^2 \equiv \sum_{i=1}^{n} (\xi_i)^2$$

hold for all x in $\bar{G}$. Finally, the function $l(x)$ will be assumed to be non-negative and $m(x)$ to be strictly positive throughout $\bar{G}$.

Equations of the form (1.1) appears in the flows of second-order fluids [13, 18], the consolidation of clay and the seepage of fluid through fissured rocks [14, 15]. It has been shown [7] that mixed initial and boundary value problems for (1.1) are well-posed ones. Also, the solutions of the initial value problems of parabolic equatinos are limiting cases of that of the pseudo-parabolic equations [17]. The asymptotic behaviors of weak solutions of (1.1) have been studied in [7] and the present objective is to study these behaviors for strict solutions at $t \to \infty$. We are indebted to Professor TRUESDELL for pointing out the references [14] and [15].

2. Solutions of homogeneous problems. - Denote by $C^0_{2+\alpha}(\bar{G})$ the closed subspace of $C_{2+\alpha}(\bar{G})$ consisting of functions which vanish on ∂G. Consider the problem of finding a strict solution $u(t, x)$ such that

$$Mu_t = Lu \text{ for } x \text{ in } G, \; -\infty < t < \infty,$$

$$u = 0 \text{ on } \partial G \text{ for all time } t, \tag{2.1}$$

$$u(0, x) = u_0(x) \text{ in } \bar{G}, \; u_0 \in C^0_{2+\alpha}(\bar{G}).$$

It is known [7] that a unique solution of (2.1) exists and that it belongs to $C^0_{2+\alpha}(\bar{G})$ for all time t. We wish to show that

THEOREM 1. - *If $u(t, x)$ is the solution of the problem (2.1) in more than one space variable, then there are positive constants C and λ depending only on α, L, M and the domain G such that for all $t \geqq 0$*

$$\| u(t, \cdot) \|_{2+\alpha} \leqq C \| u_0 \|_{2+\alpha}\, e^{-\lambda t}.$$

To this end, it is convenient to change the time scale. Upon setting $\tau = t/k$, the function $u(t, x) = u(t(\tau), x)$ becomes the solution of the following problem:

$$
\begin{aligned}
& k^{-1} M u_\tau = L u \text{ in } G \times (-\infty, \infty), \\
(2.2) \qquad & u = 0 \text{ on } \partial G \text{ for all time } \tau, \\
& u(0, x) = u_0(x) \text{ in } \overline{G}.
\end{aligned}
$$

Now we choose the constant k so large that

$$
k_L - K_M / k \equiv A > 0 \tag{2.3}
$$

where k_L and K_M are the constants appearing in (1.3). In what follows we shall consider the solution $u(\tau, x)$ of (2.2) instead of $u(t, x)$. For the solution $u(\tau, x)$ we have

LEMMA 1. - *If $u(\tau, x)$ is the strict solution of* (2.2), *then there is a positive constant λ_0 depending only on α, L, M and the domain G such that:*

$$
\| u(\tau, \cdot) \|_{2+\alpha} e^{-\lambda_0 |\tau|}, \; \| u_\tau(\tau, \cdot) \|_{2+\alpha} e^{-\lambda_0 |\tau|} \tag{2.4}
$$

remains uniformly bounded for all values of τ.

PROOF. - Denote by $kM^{-1}Lu_0$ the unique solution in $\overset{0}{C}_{2+\alpha}(\overline{G})$ of the DIRICHLET problem,

$$
k^{-1} M u = L u_0 \text{ in } G, \; u = 0 \text{ on } \partial G, \tag{2.5}
$$

where u_0 is a given function in $\overset{0}{C}_{2+\alpha}(\overline{G})$. Then, we have the SCHAUDER boundary estimate [1-5],

$$
\| u \|_{2+\alpha} = \| k M^{-1} L u_0 \|_{2+\alpha} \leqq \text{const.} \, \| L u_0 \|_\alpha \leqq \lambda_0 \| u_0 \|_{2+\alpha}, \tag{2.6}
$$

where the constant λ_0 depends only on α, L, M and G and is independent of u_0. Since for all u_0 in $\overset{0}{C}_{2+\alpha}(\overline{G})$, problem (2.5) always has a unique solution in $\overset{0}{C}_{2+\alpha}(\overline{G})$, the operator $M^{-1}L$ is defined on the entire space $\overset{0}{C}_{2+\alpha}(\overline{G})$. Also the estimate in (2.6) shows that as a mapping of $\overset{0}{C}_{2+\alpha}(\overline{G})$ onto $\overset{0}{C}_{2+\alpha}(G)$ the operator $M^{-1}L$ is bounded with respect to the HÖLDER norm $\| \; \|_{2+\alpha}$.

Based on this fact it is shown in [7] that (2.2) alwasy has a unique solution in $C^0_{2+\alpha}(\bar{G})$ and it is given by the formula:

$$u(\tau, x) = \{ \exp (\tau k M^{-1} L \} u_0(x). \tag{2.7}$$

$$\exp (\tau k M^{-1} L) \equiv \sum_{n=0}^{\infty} (\tau k M^{-1} L)^n / n!, \tag{2.8}$$

where the equality sign holds in the sense of $\| \|_{2+\alpha}$ — morm.

For later applications we also note that as a consequence of (2.7) the solution $u(\tau, x)$ is analytic in τ for all time τ.

From the estimate (2.6) and the formula (2.7) it follows that

$$\|u(\tau, \cdot)\|_{2+\alpha} \leqq e^{\lambda_0 |\tau|} \|u_0\|_{2+\alpha}$$

which is the first part of the lemma.

For the second part of the lemma we note that u_τ also belongs to $C^0_{2+\alpha}(\bar{G})$ for all values of τ and that it is given by [7]

$$u_\tau(\tau, x) = kM^{-1}L \{ \exp (\tau k M^{-1} L) \} u_0(x).$$

It follows from this formula and the estimate in (2.6) that

$$\|u_\tau(\tau, \cdot)\|_{2+\alpha} \leqq \lambda_0 e^{\lambda_0 |\tau|} \|u_0\|_{2+\alpha},$$

which is the second part of the lemma.

Lemma 1 ensures that the Laplace transform of the solution of (2.2)

$$v(\gamma, x) \equiv \int_0^\infty u(\tau, x) e^{-\gamma\tau} d\tau$$

and its partial derivatives $\partial v / \partial x_j$ and $\partial^2 v / \partial x_i \partial x_j$ will exist for all values of γ with $Re\ \gamma > \lambda_0$. Further, upon an integration by parts and upon applying the initial condiiion in (2.2) and the lemma 1, we find

$$\int_0^\infty u_\tau(\tau, x) e^{-\lambda\tau} d\tau = -u_0(x) + \gamma v(\gamma, x).$$

Hence by multiplying the equations in (2.2) by $e^{-\gamma\tau}$ and then integrating the resulting equations from $\tau = 0$ to $\tau = \infty$ we get

$$(L - \gamma k^{-1} M) v = - M u_0 \text{ in } G, \; v(\gamma, x) = 0 \text{ on } \partial G. \tag{2.9}$$

This shows that the Laplace transform $v(\gamma, x)$ of the solution of (2.2) is necessarily a solution of (2.9) provided $Re\,\gamma > \lambda_0$.

We proceed to consider the solutions of (2.2) when γ takes on various complex values. To this end we check that

LEMMA 2. - *For all values of γ with $Re\,\gamma \geqq -1$, the differential operator $L - \gamma k^{-1}M$ is uniformly and properly elliptic in $\bar{G}$*, [3, 5, 6].

PROOF. - Denote by L' and M' the principal part of L and M respectively. If $Re\,\gamma \geqq -1$, then for all real vector $\xi \equiv (\xi_1, \ldots, \xi_n)$ the inequality

$$\begin{aligned} |L'(x, \xi) - \gamma k^{-1}M'(x, \xi)| &\equiv |[l_{ij}(x) + \gamma k^{-1}m_{ij}(x)]\xi_i\xi_j| \\ &\geqq (k_L - |Re\,\gamma|k^{-1}K_M)|\xi|^2 \\ &\geqq A|\xi|^2 > 0 \end{aligned}$$

holds for all x in $\bar{G}$ in view of the choice of the contants k in (2.3). This verifies the uniform ellipticity condition.

Since the *root* condition (or the *supplementary* condition or the *proper ellipticity* condition) is automatically satisfied if the number of space variables are greater than two, we need only to check it for the two dimensional case. To do this we choose a rectangular coordinate system with origin at the point under consideration such that relative to this particular chosen coordinate system both the matrix (l_{ij}) and the matrix (m_{ij}) become diagonal ones. Indeed, such a choice of the coordinate system is always possible, because both (l_{ij}) and (m_{ij}) are real symmetric positive definite matrices [4, vol. I, pp. 37-39]. Let Ξ and Ξ' be two unit vectors along the coordinate axes, i.e., $\Xi = (1,0)$, and $\Xi' = (0,1)$. Let $\gamma = \lambda + i\beta$ and let w be the solution of the equation

$$(l_{ij} + \lambda m_{ij} + i\beta m_{ij})(\Xi_i + w\Xi_i')(\Xi_j' + w\Xi_j') = 0.$$

Since (l_{ij}) and (m_{ij}) are now diagonal matrices and since Ξ and Ξ' are unit vector along two distinct coordinate axes, we find, upon expanding the left-hand side of the equation, that

$$w^2 + (c_1 + ik_1)/(c_3 + ik_3) = 0 \tag{2.10}$$

$$c_1 \equiv l_{11} + \lambda m_{11}, \ c_3 = \beta m_{11}, \ k_1 = l_{22} + \lambda m_{22},$$

$$k_3 \equiv \beta m_{22}.$$

Note that the positive definiteness of (m_{ij}) ensures that the denominator $c_3 + ik_3$ never vanishes. Thus if $w_1 = u_1 + iv_1$ and $w_2 = u_2 + iv_2$ are the two solutions of (2.10), then it is necessary that

$$u_1 + u_2 + i(v_1 + v_2) = 0.$$

Hence $v_1 + v_2 = 0$ and we conclude from the ellipticity condition that v_1 and v_2 are non-zero constants equal in magnitude and opposite in sign. This proves that the supplement condition on $L - \gamma k^{-1}M$ is satisfied for the two particular orthogonal unit vectors Ξ and Ξ'.

However our justification of the supplement condition is completed by the observation [3, p. 625] that the root condition holds generally provided it is satisfied at a particular point for some pair of linearly independent vectors. This observation was based on the ellipticity condition and the continuous dependence of w upon the coefficients l_{ij} and m_{ij} and the fact that if w is a solution of (2.10) for given Ξ and Ξ' then $-w$ is also a solution of (2.10) for $-\Xi$ and Ξ'.

LEMMA 3. - *For all values of* γ *with* $Re\,\gamma \geqq -1$, *problem* (2.9) *always has a unique solution* $v(\gamma, x)$ *which is analytic in* γ.

PROOF. - First, we check that if $Re\,\gamma \geqq -1$ then the uniqueness of solution holds. Indeed, if w is a solution in $C_{2+\alpha}(\overline{G})$ of the homogeneous differential equation in (2.9) and it vanishes on ∂G then

$$\int_G [(L - \gamma k^{-1}M)w]\overline{w}dx = 0.$$

By setting $\gamma = \lambda + i\beta$, $w(\gamma, x) = \varphi(\gamma, x) + i\psi(\gamma, x)$ we find, in virtue of the symmetry of L and M, that

$$\int_G [\varphi(L - \lambda k^{-1}M)\varphi + \psi(L - k^{-1}M)\psi]dx = 0,$$

$$\int_G \beta k^{-1}[\varphi M\varphi + \psi M\psi]dx = 0.$$

Since the operators L and M are assumed to be elliptic and the operator $L - \lambda k^{-1}M$ is also elliptic because of the choice of k and the restriction on λ, the above two relations hold for all values of β only when $\varphi(\gamma, x)$ and $\psi(\gamma, x)$ vanish identically in $\overline{G}$.

Because of Lemma 2, the existence of a solution to problem (2.9) follows from its uniqueness, [3, Th. 12.7]. Thus, we are assured of the existence of the solution $v(\gamma, x)$ in $C^0_{2+\alpha}(\overline{G})$ to (2.9) for all values of γ with $Re\,\gamma \geqq -1$. Consequently, the half-plane $Re\,\gamma \geqq -1$ is contained in the resolvent set of the operator $kM^{-1}L - \gamma$.

The analyticy of $v(\gamma, x)$ in γ for $Re\,\gamma \geqq -1$ now follows the fact that the resolvent $R(x) \equiv (kM^{-1}L - \gamma)^{-1}$ is a holomorphic function of γ in each component of the resolvent set of the operator $kM^{-1}L - \gamma$, [16, Th. 1, p. 211]. So $v(\gamma, x)$, as an element in $C^0_{2+\alpha}(\bar{G})$, is analytic in γ and so is its value at any point x in $\bar{G}$.

PROOF OF THEOREM 1. - First, we recall that problem (2.1) always has a unique solution $u(\tau, x)$ in $C^0_{2+\alpha}(\bar{G})$ and its Laplace transform $v(\gamma, x)$ is the unique solution $v(\gamma, x)$ of the problem (2.9). Since $u(\tau, x)$ must satisfy the estimate in Lemma 1, the inversion theorem for Laplace transform [4] ensures that

$$u(\tau, x) = \frac{1}{2\pi i}\int_{L_1} v(\gamma, x)e^{\gamma\tau}d\gamma, \tag{2.17}$$

where the path L_1 is the line $Re\,\gamma = \lambda_1$ and λ_1 is a positive constant greater than λ_0 which appears in Lemma 1. We wish to show that $u(\tau, x)$ in (2.17) is also given by

$$u(\tau, x) = \frac{1}{2\pi i}\int_{L_2} v(\gamma, x)^{\gamma\tau}d\gamma \tag{2.17$'$}$$

where the path L_2 is defined by $Re\,\gamma = -1$.

For $\gamma \neq 0$, we can write (2.9) in the form:

$$(\gamma^{-1}L - k^{-1}M)v = -\gamma^{-1}Mu_0 \text{ in } G,\ v(\gamma, x) = 0 \text{ on } \partial G. \tag{2.9$'$}$$

Clearly, for an appropriately given constant k', the operator $\gamma^{-1}L - k^{-1}M$ will be in the (ε, k')-neighborhood of the operator $-k^{-1}M$ for all sufficiently large $|\gamma|$, [3, p. 687]. Also, the DIRICHLET problem,

$$k^{-1}Mv(\gamma, x) = 0 \text{ in } G,\ v(\gamma, x) = 0 \text{ on } \partial G,$$

has only the solution zero. Hence for all sufficiently large $|\gamma|$, we have the estimate, [3, Th. 12.3],

$$\|v(\gamma, \cdot)\|_{2+\alpha} \leqq \text{const. } \|Mu_0\|_\alpha/|\gamma|. \tag{2.18}$$

where the constant depends only on α, M and G and is independent of γ. In particular, it implies that

$$|v(\gamma, x)| = 0(1/|\gamma|) \text{ as } |\gamma| \to \infty. \tag{2.19}$$

Now it follows from (2.17) and (2.19) and the CAUCHY integral theorem that

$$\int_{L_2} v(\gamma, x)e^{\gamma\tau}d\tau = \int_{L_1} v(\gamma, x)e^{\gamma\tau}d\tau .$$

This estanlishes the representation formula (2.17)' for $u(\tau, x)$.

We proceed to derive the desired conclusion in the theorem by combining the formulas in (2.17) and (2.17'). Denote by γ_1, γ_2 the complex variables related by

$$Re\,\gamma_1 = \lambda_1, \; Re\,\gamma_2 = -1, \; Im\,\gamma_1 = Im\,\gamma_2 \equiv \beta . \tag{2.20}$$

Further, let $v_1(\gamma_1, x)$, $v_2(\gamma_2, x)$ be the solutions of (2.9) with $\gamma = \gamma_1$ and $\gamma = \gamma_2$ respectively. Then we can write (2.17)' as

$$\begin{aligned} u(\tau, x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \{ v_1(\lambda_1 + i\beta, x) + v_2(-1 + i\beta, x) \\ &\qquad - v_1(\lambda_1 + i\beta, x)\} e^{(-1+i\beta)} d\beta \\ &= u(\tau, x)e^{-(1+\lambda_1)\tau} + \frac{e^{-\tau}}{2\pi}\int_{-\infty}^{\infty}(v_2 - v_1)e^{-i\beta\tau} d\beta , \end{aligned}$$

where (2.17) has been used in deriving the second equality. Accordingly,

$$\| u(\tau, \cdot)\|_{2+\alpha} \leqq \| u(\tau, \cdot)\|_{2+\alpha} e^{-(1+\lambda_1)\tau} + \frac{e^{-\tau}}{2\pi}\int_{-\infty}^{\infty} \| v_2 - v_1 \|_{2+\alpha}\, d\beta . \tag{2.21}$$

On the other hand, in the proof of Lemma 1 we have shown that

$$\| u(\tau, \cdot)\|_{2+\alpha} \leqq \| u_0 \|_{2+\alpha}\, e^{\lambda_0\tau} . \tag{2.22}$$

By combining (2.21) and (2.22) we find

$$\| u(\tau, \cdot)\|_{2+\alpha} \leqq \| u_0 \|_{2+\alpha}\, e^{-(1+\lambda_1-\lambda_0)\tau} + \frac{e^{-1}}{2\pi}\int_{-\infty}^{\infty} \| v_2 - v_1 \|_{2+\alpha}\, d\beta .$$

By letting $\lambda_1 \to \lambda_0$ in the above inequality it yields

$$\| u(\tau, \cdot)\|_{2+\alpha} \leqq \| u_0 \|_{2+\alpha}\, e^{-\tau} + e^{-\tau}\frac{1}{2\pi}\int_{-\infty}^{\infty} \| v_2 - v_1 \|_{2+\alpha}\, d\beta . \tag{2.23}$$

To complete a proof of the theorem it remains to show that the integral on the right-hand side of (2.23) exists and is finite. To do this we note that for fixed γ_1 and γ_2 the function $v_2 - v_1$ satisfies the equations

(2.24) $(L - \gamma_1 k^{-1} M)(v_2 - v_1) = (1 + \lambda_1) k^{-1} M v_2$ in G, $v_2 - v_1 = 0$ and ∂G,

in virtue of the fact that $Im\,\gamma_1 = Im\,\gamma_2 = \beta$. By precisely the same reasoning as that for deriving the estimate in (2.18) we conclude that for all sufficiently large $|\gamma_1|$

$$\|v_2 - v_1\|_{2+\alpha} \leqq C'(1 + \lambda_1) k^{-1} \|M v_2\|_\alpha / |\gamma_1| . \tag{2.25}$$

where the constant C' depends only on α, G and M and is independent of γ_1 and v_2. Thus, (2.25) and (2.18) combined imply that for all sufficiently large $|\gamma_1 \gamma_2|$

$$\|v_2 - v_1\|_{2+\alpha} \leqq C'' \|M u_0\|_\alpha / |\gamma_1 \gamma_2|$$

with C'' being dependent of u_0, γ_1 and γ_1. For convenience, we state this estimate in the form: for all large $|\beta|$ greater than some positive constant β_0

$$\|v_2 - v_1\|_{2+\alpha} \leqq C''' \|u_0\|_{2+\alpha} / |\beta|^2 \tag{2.26}$$

with constant C''' depending only on α, β_0, G, L and M and independent of u_0 and β. On the other hand, for all $|\beta| \leqq \beta_0$ the solution $v_2(-1 + i\beta, x)$ of (2.9) and the solution $v_2 - v_1$ of (2.24) are subject to the estimates

$$\|v_2 - v_1\|_{2+\alpha} \leqq b' \|M u_0\|_\alpha , \quad \|v_2 - v_1\|_{2+\alpha} \leqq b'' \|v_2\|_{2+\alpha}$$

and so for all $|\beta| \leqq \beta_0$

$$\|v_2 - v_1\|_{2+\alpha} \leqq b''' \|M u_0\|_\alpha \leqq b \|u_0\|_{2+\alpha} \tag{2.27}$$

with the constant b depending only on α, β_0, G, L and M. Now from (2.26) and (2.27) we see that for all values of β

$$\|v_2 - v_1\|_{2+\alpha} \leqq C_1 \|u_0\|_{2+\alpha} / (1 + \beta^2)$$

where the constant C_1 depends only on α, G, L and M. Hence

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \|v_2 - v_1\|_{2+\alpha} d\beta \leqq C_1 \|u_0\|_{2+\alpha} \int_{-\infty}^{\infty} \frac{d\beta}{1 + \beta^2} \leqq C_2 \|u_0\|_{2+\alpha} . \tag{2.28}$$

where the constant C_2 depends only on α, G, L and M. By combining the estimates in (2.23) and (2.28), we find

$$\|u(\tau, \cdot)\|_{2+\alpha} \leqq \|u_0\|_{2+\alpha} e^{-\tau} + C_3 \|u_0\|_{2+\alpha} e^{-\tau} \leqq C \|u_0\|_{2+\alpha} e^{-\tau} \tag{2.29}$$

with the constant C being independent of u_0. Replacing τ by t/k in (2.29) it gives

$$\|u(t, \cdot)\|_{2+\alpha} \leqq C \|u_0\|_{2+\alpha} e^{-t/k}$$

which is what to be proved.

As we already mentioned, for $u_0(x)$ in $C^0_{2+\alpha}(\bar{G})$ the solution $u(t, x)$ of (2.1) is, as an element in $C^0{}_{c}(\bar{G})$, analytic in t for all values of t. Thus $u_t(t, x)$ is the unique solution of the problem

$$M(u_t)_t = Lu_t \text{ in } G \times (-\infty, \infty),$$

$$u_t(t, x) = 0 \text{ on } \partial G \text{ for all time } t, \; u_t(0, x) = u_t(0, x) \text{ in } \bar{G}.$$

By repeating the same reasoning as was shown in the proof of Theorem 1, we conclude that in the case of more than one space variable $u_t(t, x)$ decays expontially as $t \to \infty$. Since we can repeat the same argument as many times as we like it follows that

COROLLARY 1. - *If $u(t, x)$ is the solution of (2.1) in the case of more than one space variable, then $u(t, x)$ tegether with its time derivatives of all orders decays expontially as $t \to \infty$.*

To simplify notation, we write $E(t) = \exp(tM^{-1}L)$. Since the bounded operator $M^{-1}L$ maps the complete space $C^0_{2+\alpha}(\bar{G})$ onto itself. Theorem 1 says that for a given pair of elliptic operators L and M and for a given smooth domain G the inequality

$$\|u(t, \cdot)\|_{2+\alpha} \equiv \|E(t)u_0(\cdot)\|_{2+\alpha} \leqq C \|u_0\|_{2+\alpha} e^{-\lambda t}$$

holds for all function $u_0(x)$ in $C^0_{2+\alpha}(\bar{G})$ with the constants C and λ being independent of u_0. Thus we have

COROLLARY 2, - *The group of bounded operators $E(t)$ decays exponentially as $t \to \infty$. That is*

$$\|E(t)\|_{2+\alpha} \leqq Ce^{-\lambda t}$$

with the constant C and λ depending only on α, G, L and M.

REMARK 1. - Since the operator $L-\gamma k^{-1}M$ is *not properly elliptic* if the domain G is a set on the real line, we have excluded this special case in Theorem 1. However, we can multiply the first equation in (2.1) by u and integrate the resulting equation over G to obtain

$$\frac{1}{2}\frac{d}{dt}\int_G (m_{ij}u_iu_j + mu^2)dx = -\int_G (l_{ij}u_iu_j + lu^2)dx. \tag{2.31}$$

The essential feature of this equation is that the value of the right-hand side is non-negative for all functions u. By applying the ellipticity condition L and M and by applying POINCARÉ's inequality [9] we derive from (2.31) that

$$\frac{d}{dt}\|u(t,\ \cdot)\|_M \leqq -C\|u(t,\ \cdot)_M, \tag{2.32}$$

where the constant D depends only on L and M and the domain G and where

$$\|u(t,\ \cdot)\|_M^2 \equiv \int_G (m_{ij}u_iu_j + mu^2)dx. \tag{2.33}$$

upon integrating (2.32) from 0 to t with $t>0$, it gives

$$\|u(t,\ \cdot)\|_M \leqq \|u(0,\ \cdot)\|_M e^{-ct},\ t>0.$$

This inequality together with SOBOLEV's lemma [8] implies that *if* $u(t,\ \cdot)$ *is a strict solution of* (2.1) *in one space variable, then for all* $t>0$ *the* L_2*-norm of* u, $\|u\|_0$, *is subject to the estimate*

$$\|u(t,\ \cdot)\|_0 \leqq \text{const. } \|u(0,\ \cdot)\|_M e^{-ct}, \tag{2.34}$$

where the positive constant depends only on α, L, M *and* G.

REMARK 2. - Theorem 1 and Lemma 1 furnish upper bounds for $\|u(t,\ \cdot)\|_{2+\alpha}$ for $-\infty<t<\infty$. For the lower bound for $\|u(t,\ \cdot)\|_M$ as $t\to-\infty$, we integrate (2.32) from 0 to t with $t<0$ te obtain

$$\|u(t,\ \cdot)\|_M \geqq \|u(0,\ \cdot)\|_M e^{C|t|},\ t<0,\ C>0.$$

This means that *if* $u(t,\ x)$ *is the solution of* (2.1) *then for all* $t<0$,

$$\|u(t,\ \cdot)\|_1 \geqq \text{const. } \|u(t,\ \cdot)\|_M \geqq \text{const. } \|u(0,\ \cdot)\|_M e^{C|t|},$$

where the constants depend only on α, L, M *and* G *and where* $\|u\|_1$ *stands for the Dirichlet norm of* u.

3. **Solution of non-homogeneous problem.** - By establishing the asymptotic behavior of the solutions of homogeneous problems, it is easy now to extend these results to the general case. In what follows we assume for all time t, $f(t, x)$ as a map: $t \to C_\alpha(\bar{G})$ is strongly continuous in t and $h(t, x)$ as a map: $t \to C_{2+\alpha}(\bar{G})$ is strongly continuously differentiable in t. Let $u(t, x)$ be the solution in $C_{2+\alpha}(\bar{G})$ of the problem:

$$(3.1)\qquad \begin{aligned} &Mu_t - Lu = f(t, x) \text{ in } Gx(-\infty, \infty) \\ &u(t, x) = h(t, x) \text{ in } \partial G \text{ for all time } t, \\ &u(0, x) = u_0(x) \text{ in } \bar{G}, \end{aligned}$$

where $u_0(x)$ is a given function in $C_{2+\alpha}(\bar{G})$. We wish to show that.

THEOREM 2. - *Let $u(t, x)$ be the solution of (3.1) in more than one space variable. If for some positive constant μ*

$$(3.2)\qquad \|f(t, \cdot)\|_\alpha e^{+\mu t}, \ \|h(t, \cdot)\|_{2+\alpha} e^{+\mu t}, \ \|h_t(t, \cdot)\|_{2+\alpha} e^{+\mu t},$$

remain uniformly bounded for all $t \geqq 0$, then for all $t \geqq 0$,

$$(3.3)\qquad \begin{aligned} &\|u(t, x)\|_{2+\alpha} \leqq \text{const. } e^{-t \min. (\lambda, \mu)}, \ \lambda \neq \mu \\ &\|u(t, x)\|_{2+\alpha} \leqq \text{const. } te^{-\lambda t}, \ \lambda = \mu, \end{aligned}$$

where the constants depend on α, λ, μ, f, h, L, M and G and λ is the same as in Theorem 1; if for some constant $\mu > 0$

$$(3.4)\qquad \|f(t, \cdot)\|_\alpha, \ \|h(t, \cdot)\|_{2+\alpha}, \ \|h_t(t, \cdot)\|_{2+\alpha} = 0(t^{-\mu})$$

as $t \to \infty$, then

$$(3.5)\qquad \|u(t, \cdot)\|_{2+\alpha} = 0(t^{-\mu}) \text{ as } t \to \infty.$$

In (3.4) and (3.5) and throughout this section we use the notation that for any $\mu \geqq 0$, $t^\mu \cdot 0(t^{-\mu}) \to 0$ as $t \to \infty$.

PROOF. - First, we consider the case $h(t, x) = 0$ identically. For each fixed t, let $v(t, x)$ be the solution of the DIRICHLET problem:

$$(3.6)\qquad Mv(t, x) = f(t, x) \text{ in } G, \ v(t, x) = 0 \text{ on } \partial G.$$

Then $v(t, x)$ belongs to $C^0_{2+\alpha}(\bar{G})$ for all time t. Further we have the SCHAUDER estimate,

$$\|v(t, \cdot)\|_{2+\alpha} \leqq \text{const.}\ \|f(t, \cdot)\|_\alpha$$

with the constant depending only on α, M and the domain G. This estimate shows that $v(t, x)$ as a map: $t \to C^0_{2+\alpha}(\bar{G})$ is strongly continuous in t. Furthermore, if the conditions in (3.2) hold, then for all time t

$$\|v(t, \cdot)\|_{2+\alpha} \leqq \text{const.}\ e^{-\mu t} \tag{3.7}$$

with the constant depending only on α, M, f and G; if the conditions in (3.4) holds then

$$\|v(t, \cdot)\|_{2+\alpha} = 0(t^{-\mu}) \text{ as } t \to \infty. \tag{3.8}$$

Since $h(t, x)$ is assumed to vanish identically, $u_0(x)$ must belong to $C^0_{2+\alpha}(\bar{G})$ so as to be compatible. By principle of superposition or DUHAMEL'S principle, one sees that the solution (3.1) is given by the formula,

$$u(t, x) = E(t)u_1 + \int_0^t E(t-\tau)v(\tau, x)d\tau, \qquad -\infty < t < \infty, \tag{3.9}$$

where the equality sign holds in the sense of $\|\ \|_{2+\alpha}$-norm and $E(t) \equiv \exp(tM^{-1}L)$. It follows immediately from (3.9) that

$$\|u(t, \cdot)\|_{2+\alpha} \leqq \|E(t)u_0\|_{2+\alpha} + \int_0^t \|E(t-\tau)\|_{2+\alpha}\|v(t, \cdot)\|_{2+\alpha}\,d\tau \tag{3.10}$$

for all time $t \geqq 0$. Indeed, the dependence of the operaȷor $E(t)$ on t is continuous in the uniform operator topology [7] and $v(\tau, x)$ has been shown to be strongly continuous in τ. Hence the integral on the right-hand side of (3.10) exists in the RIEMANNIAN sense for all $t \geqq 0$. Also from Corollary 2 we see that for all $\tau \leqq t$.

$$\|E(t-\tau)\|_{2+\alpha} \leqq \text{const.}\ e^{-\lambda(t-\tau)} \tag{3.11}$$

with the constant depending only on α, L, M and the domain G.

If the inequality in (3.7) holds, then it follows from (3.11) that

$$\int_0^t \|E(t-\tau)\|_{2+\alpha}\|v(\tau, \cdot)\|_{2+\alpha}\,d\tau \leqq \text{const.}\ |e^{-\mu t} - e^{-\lambda t}| \tag{3.12}$$

with the constant depending only on α, λ, μ, f, L, M and G. Thus the assertion in (3.3) follows from (3.10), Theorem 1 and (3.12).

In case $\lambda = \mu$, then instead of (3.12) we have

$$\int_0^t \|E(t-\tau)\|_{2+\alpha} \|v(\tau, \cdot)\|_{2+\alpha} d\tau \leqq \text{const. } te^{-\lambda t} \tag{3.12$'$}$$

with the constant depending only on α, f, L, M and G. Thus we have proved the first part of Theorem 2 under the homogeneous boundary conditions.

Suppose that the hypotheses in (3.8) hold. Then for $t \geqq 0$

$$\|v(t, \cdot)\|_{2+\alpha} \leqq \text{const. } (1+t)^{-\mu}$$

with the constant being independent of t. Hence

$$\int_0^t \|E(t-\tau)\|_{2+\alpha} \|v(\tau, \cdot)\|_{2+\alpha} d\tau \leqq \text{const. } e^{-\lambda t} \int_0^t (1+\tau)^{-\mu} e^{+\lambda \tau} d\tau, \tag{3.13}$$

with the constant depending only on α, f, L, M and G. We now assert that there are positive constants t_0 and k such that

$$\varphi(t) = \int_0^t (1+\tau)^{-\mu} e^{\lambda\tau} d\tau \leqq k(1+t)^{-\mu} e^{+\lambda t} = \psi(t) \tag{3.14}$$

for all $t \geqq t_0$. Indeed, for the functions $\varphi(t)$ and $\psi(t)$ so defined we have

$$\frac{d}{dt}(\psi - \varphi) = (1+t)^{-\mu} e^{\lambda t} \left[k\left(\lambda - \frac{\mu}{1+t}\right) - 1 \right].$$

If we choose t_0 so that $\mu/(1+t_0) \leqq \lambda/2$ and choose k so large that $k\lambda \geqq 2$ then

$$\frac{d}{dt}(\psi - \varphi) \geqq 0 \text{ for all } t \geqq t_0. \tag{3.15}$$

Moreover, it is immediately seen that we can choose the positive constant k so large that the inequality in (3.14) holds for $t = t_0$. But then (3.14) holds for all $t \geqq t_0$ in virtue of the differential inequality in (3.15). By combining the estimates in (3.10), (3.13) and (3.14) we find

$$\begin{aligned} \|u(t, \cdot)\|_{2+\alpha} &\leqq \|E(t)u_0\|_{2+\alpha} + \text{const. } (1+t)^{-\mu} \\ &\leqq \text{Const. } e^{-\lambda t} + \text{const. } (1+t)^{-\mu}, \end{aligned}$$

with the constants depending only on α, f, L, M and G. This proves the assertion in (3.5) under the homogeneous boundary conditions.

If $h(t, x)$ does not vanish identically, then $u_0 - h(0, x)$ belongs to $\overset{0}{C}_{2+\alpha}(\bar{G})$. Consider the function $w(t, x) = u(t, x) - h(t, x)$. It is in $\overset{0}{C}_{2+\alpha}(\bar{G})$ for all time t and satisfies the equation,

$$MW_t - Lw = f(t, x) - (Mh_t - Lh) \text{ in } G \times (-\infty, \infty).$$

Accordingly, the above proof applies to the function $w(t, x)$ in virtue of the conditions in (3.2) and (3.4) on the given function $h(t, x)$. Consequently, $u(t, x) = w(t, x) = h(t, x)$ has the properties as stated in Theorem **2**. The proof is now complete.

We proceed to consider the case of one space variable. Let $f(t, x)$, $h(t, x)$ ‖be given functions as specified at the beginning of this section. Denote by $f\|_0$ the L_2-norm of f and by $\|f\|_M$ the DIRICHLET norm of f as defined in (2.33). We wish to show that

THEOREM 2′. - *Let $u(t, x)$ be the solution in $C_{2+\alpha}(\bar{G})$ of* (3.1). *If for some constant $b > 0$*

$$\|f(t, \cdot)\|_0 e^{+bt}, \ \|h(t, \cdot)\|_M e^{+bt}, \ \|h_t(t, \cdot)\|_M e^{+bt} \tag{3.16}$$

remains uniformly bounded for all $t \geqq 0$, then, with C given in Remark 1,

$$\|u(t, \cdot)\|_0 \leqq \text{const. } e^{-t \min(b, c)} \text{ for all } t \geqq 0 \tag{3.17}$$

with the constant being independent of t; if for some constant $b > 0$.

$$\|f(t, \cdot)\|_0, \ \|h(t, \cdot)\|_M, \ \|h_t(t, \cdot)\|_M = 0(t^{-b}) \tag{3.18}$$

as $t \to \infty$, then

$$\|u(t, \cdot)\|_0 = 0(t^{-b}) \text{ as } t \to \infty. \tag{3.19}$$

PROOF. - It is now clear that for all time t the solntion of (3.1) has the representation

$$u(t, x) = h(t, x) + E(t)(u_0 - h(0, x)) + \int_0 E(t - \tau)v(\tau, x)d\tau,$$

where $v(t, x)$ belongs to $\overset{0}{C}_{2+\alpha}(\bar{G})$ for all time t and

$$Mv = f(t, x) - (Mh_t - Lh) \text{ in } G. \tag{3.20}$$

Hence for all $t \geqq 0$

$$\|u(t,\ \cdot)\|_0 \leqq \|h(t,\ \cdot)\|_0 + \|E(t)(u_0 - h(0,\ \cdot)\|_0 + \int_0^t \|E(t-\tau)v(\tau,\ \cdot)\|\,d\tau. \tag{3.21}$$

we prooceed to estimate each term on the right of this inequality.

If the hypotheses in (3.16) hold, then SOBOLEV'S lemma ensures that there are constants independent of t such that

$$\|h(t,\ \cdot)\|_0 \leqq \text{const.}\ \|h(t,\ \cdot)\|_M \leqq \text{const}\ e^{-bt}. \tag{3.22}$$

Since the function $u_0 - h(0,\ x)$, belongs to $\overset{0}{C}_{2+\alpha}(\bar{G})$, an application of (2.34) gives

$$\|E(t)(u_0 - h(0,\ \cdot)\|_0 \leqq \text{const},\ \|u_0 - h(0,\ \cdot)\|_M e^{-ct}. \tag{3.23}$$

Also $v(t,\ x)$ belongs to $\overset{0}{C}_{2+\alpha}(\bar{G})$ for all time t, further application of (2.34) leads to

$$\int_0^t \|E(t-\tau)v(\tau,\ \cdot)\|_0\,d\tau = \int_0^t \|E(t)v(t,\ \tau,\ \cdot)\|_0\,d\tau \leqq \text{const.}\ e^{-ct}\int_0^t \|v(t-\tau,\ \cdot)\|_M\,d\tau, \tag{3.24}$$

with the constant being independent of t. By multiplying (3.20) by $v(t,\ x)$ and then integrating over G for fixed t, we obtain, after integration by parts and application of ellipticity condition and POINCARE'S inequality,

$$\|v(t,\ \cdot)\|_M \leqq \text{const.}\ \{\|f(t,\ \cdot)\|_0 + \|h(t,\ \cdot)\|_M + \|h_t(t,\ \cdot)\|_M\} \leqq \text{const.}\ e^{-bt},$$

with the constants being independent of t. By combining this estimate with (3.24) we find

$$\int_0^t \|E(t-\tau)v(\tau,\ \cdot)\|_0 d\tau \leqq \text{const.}\ e^{-ct}\int_0^t e^{-b(t-\tau)}\,d\tau \leqq \text{const.}\ |e^{-ct} - e^{-bt}|. \tag{3.25}$$

Thus, the assertion in (3.17) follows from (3.21)-(3.23) and (3.25).

If the hypotheses in (3.18) hold, then by the same reasoning as that for deriving (3.14) we have for all t greater than $t_0 > 0$.

$$\int_0^t \|E(t-\tau)v(\tau,\ \cdot)\|_0\, d\tau \leqq \text{const.} \int_0^t e^{-c(t-\tau)}(1+\tau)^{-b} d\tau \tag{3.26}$$

$$\leqq \text{const.}\ (1+t)^{-b}$$

with the constant independent of t. Thus, the assertion in (3 19) follows from (3.21), (3.18) and (3.26).

REMARK 3. - Let $v(x)$ be the solution of the DIRICHLET problem:

$$Lv = -g(x) \text{ in } G,\ v(x) = k(x) \text{ on } \partial G, \tag{3.27}$$

with $g(x)$, $h(x)$ being given functions in $C_\alpha(\bar{G})$ and $C_{2+\alpha}(\bar{G})$ respectively. If the functions $f(t, x) - g(x)$ and $h(t, x) - k(x)$ satisfy the hypotheses in (3.2) or (3.4), then the corresponding solutions $u(t, x) - v(x)$ have the asymptoic behaviours as stated in (3.3) or that in (3.5). Similar statements hold for the case of one space variable. This shows how the solution $u(t, x)$ of (3.1) converges to the steady solution $v(x)$ of (3.27).

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