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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF REACTION-DIFFUSION SYSTEMS OF LOTKA-VOLTERRA TYPE

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Dedicated to the memory of Professor P. Hess

1. Introduction. As a mathematical model for the population dynamics of N-species in biology, Lotka [12] and Volterra [17] proposed the ordinary differential system of the form:

$$dv_j/dt = (-e_j + b_j^{-1} \sum_{k=1}^{N} a_{jk} v_k) v_j, \quad j = 1, \ldots, N, \tag{LV}$$

where e_j, $b_j (> 0)$, a_{jk} are given constants; and v_j denotes the biomass of the j-species; and investigated the asymptotic behavior of $v_1, \ldots, v_N$ for large time t.

For $N = 2$, there are extensive literatures on (LV) (or (RD) below), e.g., Copell [5], Henry [7], Rothe [16]. However, for $N \geq 3$, little seems to have been known; see Amann [2, 3], Krikorian [11], Fife-Mimura [6], Friedmann-Tzavars [8], Oshime [14] and others.

In the present paper we consider the reaction-diffusion's version of (LV) of the form:

$$\begin{aligned} &\frac{\partial}{\partial t} u_j = d_j \Delta u_j + u_j f_j(u) \quad (x \in \Omega, \ t > 0) \\ &\frac{\partial}{\partial \nu} u_j \Big|_{\partial\Omega} = 0, \quad (t > 0); \ \ u_j \Big|_{t=0} = \phi_j \quad (j = 1, \ldots, N), \end{aligned} \tag{RD}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, d_j is a positive constant, $\partial/\partial\nu$ denotes the outer normal derivative to $\partial\Omega$, and ϕ_j given smooth non-negative, and not identically zero function satisfying the compatibility condition: $\partial\phi_j/\partial\nu = 0$ on $\partial\Omega$. The purpose of the present paper is to study the asymptotic behavior of solutions of (RD) for large t under some assumptions on f_j.

We suppose that f_j, $j = 1, \ldots, N$, satisfies the following assumptions.

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Assumption 1.1. $f_j(\xi)$ is a smooth function of $\xi \in \mathbb{R}_+^N$ such that there is a positive constant b_j with

$$\sum_{j=1}^N b_j \xi_j f_j(\xi) \le 0, \quad \xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}_+^N, \tag{1}$$

$(\mathbb{R}_+^N = \{\xi \in \mathbb{R}^N;\ \xi_j \ge 0,\ (j = 1, \dots, N)\})$.

To state the theorem, we decompose the index set $\Lambda \equiv \{1, \dots, N\}$ into three disjoint sets $\Lambda_1, \Lambda_2, \Lambda_3 : \Lambda = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$ (disjoint). We define the set $\Lambda_{1,s}$ inductively. $\Lambda_{1,1}$ is the set of all j such that

$$\sum_{k=1}^N b_k \xi_k f_k(\xi) \le -\delta_1 \xi_j, \quad (\xi \in \mathbb{R}_+^N) \tag{2}$$

for some positive constant δ_1. If $\Lambda_{1,s}$ is defined, then by definition $j \in \Lambda_{1,s+1}$ if and only if either $j \in \Lambda_{1,s}$ or there is an i $(1 \le i \le N)$ such that

$$f_i(\xi) \ge -\delta_2 \xi_j - \rho(|\xi|) \sum_{k \in \Lambda_{1,s}} \xi_k, \ (\xi \in \mathbb{R}_+^N), \tag{3}$$

where δ_2 is some positive constant, and $\rho(s)$ some positive increasing function of s: if an index set Λ_0 is empty, we understand $\sum_{j \in \Lambda_0} a_j = 0$. We set

$$\Lambda_1 = \bigcup_s \Lambda_{1,s}$$

Λ_2 is the set of all $j \in \Lambda \setminus \Lambda_1$ such that

$$|f_j(\xi)| \le \rho(|\xi|) \sum_{k \in \Lambda_1} \xi_k, \quad (\xi \in \mathbb{R}_+^N). \tag{4}$$

Here and in what follows, $\rho(s)$ denotes a positive increasing function of s. Finally we set

$$\Lambda_3 = \Lambda - \Lambda_1 - \Lambda_2. \tag{5}$$

Let C be the set of all positive vectors $c = (c_i)_{i \in \Lambda_3}$ (Λ_3 is naturally ordered) such that

$$\Big| \sum_{i \in \Lambda_3} c_i f_i(\xi) \Big| \le \rho(|\xi|) \sum_{j \in \Lambda_1} \xi_j, \quad (\xi \in \mathbb{R}_+^N). \tag{6}$$

Assumption 1.2. If Λ_3 is non-empty, there is a vector in C.

We denote by r the number of the linearly independent vectors in C, and independent vectors by $c^{(j)} = (c_i^{(j)})_{\Lambda_3}$, $j = 1, \dots, r$, where a positive vector c means that all the component c_i is positive.

For vector $\gamma = (\gamma_k)_{k\in\Lambda_2} \geq 0$, non-negative real μ_0, and vector $\mu = (\mu_j)_{j=0}^{r}$, we define $\Theta(\gamma, \mu)$ by the set of all $\xi \in \mathbb{R}_+^N$ such that

$$\begin{aligned}
&\xi_j = 0 \quad (j \in \Lambda_1); \;\; \xi_j = \gamma_j \quad (\text{for } j \in \Lambda_2); \\
&\sum_{j=1}^{N} b_j \xi_j = \mu_0; \qquad \text{and} \qquad \sum_{k\in\Lambda_3} c_k^{(j)} \log \xi_k = \mu_j, \quad (j = 1, \dots, r),
\end{aligned} \tag{7}$$

where we understand that if Λ_3 is empty set, then $\mu = \mu_0$.

Remark 1.1. Suppose that $\Lambda_3 = \Lambda$, and $r = N - 2$. Suppose also that

$$\sum_{j=1}^{N} b_j \xi_j f_j(\xi) = 0, \tag{1'}$$

$$\sum_{j=1}^{N} c_j^{(k)} f_j(\xi) = 0, \quad (k = 1, \dots, N-2). \tag{2'}$$

Then the set

$$\Theta = \{\xi \in \mathbb{R}_+^N; \sum_{j=1}^{N} b_j \xi_j = \mu_0, \sum_{j\in\Lambda_3} c_j^{(k)} \log \xi_j = \mu_j, \ (1 \leq k \leq r)\}$$

is a bounded closed orbit, and so it is a periodic orbit for (RD).

Then our main theorem reads as follows:

Theorem 1.1. *Let the above assumptions* 1.1 *and* 1.2 *hold. Then any bounded solution u of* (RD) *converges to a $\Theta(\gamma, \mu)$ as $t \to \infty$, uniformly on $\bar{\Omega}$ for some $\gamma = (\gamma_j)_{j\in\Lambda_2}$ and $\mu = (\mu_j)_{j=0}^{r}$:*

$$\operatorname{dist}(u(x,t), \theta(\gamma,\mu)) \to 0 \quad \textit{as } t \to \infty, \quad \textit{uniformly on } \bar{\Omega},$$

where

$$\begin{aligned}
&\gamma_j > 0 \quad (\textit{if } \Lambda_2 \textit{ is not empty}); \\
&\mu_0 > 0 \quad (\textit{if } \Lambda_2 \cap \Lambda_3 \textit{ is not empty}); \\
&\mu_j = 0 \quad (\textit{if } \Lambda_3 \textit{ is empty}).
\end{aligned}$$

Corollary 1.1. *In addition to the assumptions* 1.1 *and* 1.2, *assume that $\Lambda_3 = \Lambda$, and $r = N - 2$, and that* (1′), *and* (2′) *hold. Then any bounded solution u converges to a periodic orbit for* (RD).

The proof is a direct consequence of Theorem 1.1 and Remark 1.1.

We now consider the reaction-diffusion system of the form considered originally by Lotka and Volterra, which is a special case of (RD):

$$
\begin{aligned}
&\frac{\partial}{\partial t}u_j = d_j\Delta u_j + (-e_j + b_j^{-1}\sum_{k=1}^{N} a_{jk}u_k)u_j, \quad (x \in \Omega,\ t > 0) \\
&\frac{\partial}{\partial \nu}u_j = 0, \quad (x \in \partial\Omega,\ t > 0); \quad u_j = \phi_j, \quad (x \in \Omega,\ t = 0).
\end{aligned} \tag{8}
$$

In this case it is easy to see that assumption 1.1 in Theorem 1.1 implies that $e_j \geq 0$, $(j = 1, \dots, N)$ and that the matrix $(a_{jk} + a_{kj})$ is non-positive definite. Let us see the construction of $\Lambda_1, \Lambda_2, \Lambda_3$ more concretely. $j \in \Lambda_{1,1}$ if

$$
\sum_{i=1}^{N} b_i(-e_i + b_i^{-1}\sum_{k=1}^{N} a_{ik}\xi_k)\xi_i \leq -\delta_1\xi_j, \quad (\xi \in \mathbb{R}_+^N) \tag{9}
$$

with some $\delta_1 > 0$. The above condition is equivalent to the one that $e_j > 0$ and the matrix $(a_{jk} + a_{kj})$ is non-positive definite.

If $\Lambda_{1,s}$ is constructed, then $\Lambda_{1,s+1} = \Lambda_{1,s} \cup \Lambda'_{1,s+1}$. Here $j \in \Lambda'_{1,s+1}$ if and only if there is an $i \in \Lambda$ with $a_{ik} \geq 0$ (for $k \in \Lambda - \Lambda_{1,s}$) and with $a_{ij} > 0$. Then we set $\Lambda_1 = \bigcup_s \Lambda_{1,s}$ and $\Lambda_3 = \Lambda - \Lambda_1 - \Lambda_2$. Suppose that there is a positive vector $c = (c_j)_{j\in\Lambda_3}$ with

$$
\sum_{i\in\Lambda_3} c_i b_i^{-1} a_{ik} = 0, \quad (k \in \Lambda_2 \cup \Lambda_3). \tag{10}
$$

Then the number of the linearly independent vectors in C is the dimension of the kernel of the matrix $(a_{jk})^*_{j\in\Lambda_3, k\in\Lambda_2\cup\Lambda_3}$, (* denotes adjoint matrix). We can now state Theorem 1.1 in a more concrete form.

Theorem 1.2. *Assume that $e_j \geq 0$, $(j = 1, 2, \dots, N)$ and that the matrix $(a_{jk}+a_{kj})$ is non-positive definite. Assume also that if Λ_3 is non-empty, then there is a c in C satisfying* (10). *Then any bounded solution of* (8) *converges to some $\theta(\gamma, \mu)$, uniformly on $\bar{\Omega}$ as $t \to \infty$, uniformly on $\bar{\Omega}$.*

Remark 1.2. Let the assumptions in Theorem 1.2 hold. If $n = 1$, (n: space dimension), then it can be shown that any solution is bounded, and so converges to some $\theta(\gamma, \mu)$ as $t \to \infty$, uniformly on $\bar{\Omega}$.

Corollary 1.2. *In addition to the assumptions in Theorem 1.2, assume that $a_{jk} = -a_{kj}$, $r = N - 2$, and $\Lambda_3 = \Lambda$ $(\equiv \{1, \dots, N\})$. Then any bounded solution of* (8) *converges to some periodic orbit as $t \to \infty$, uniformly in $\bar{\Omega}$.*

Example 1.1. Consider the reaction-diffusion system

$$
\begin{aligned}
&\frac{\partial}{\partial t}u_j = d_j\partial_x^2 u_j + (u_{j+1} - u_{j-1})u_j, \quad x \in \Omega, \quad t > 0 \\
&\frac{\partial}{\partial \nu}u_j = 0 \quad (x \in \partial\Omega,\ t > 0); \quad u_j(x, 0) = \phi_j(x), \quad (x \in \Omega) \\
&\qquad (j = 1, 2, 3; \quad u_4 = u_1, \ u_0 = u_3)
\end{aligned} \tag{11}
$$

$(\partial_x^2 = \partial^2/\partial x^2)$. It is easy to verify that Assumption 1.1 is satisfied, that $\Lambda_1 = \Lambda_2 =$ empty set and $\Lambda_3 = \{1, 2, 3\}$, and that the kernel of $(a_{jk})^*$ is spanned by the positive vector (1, 1, 1). Thus any bounded solution of (11) tends to the set

$$\Theta = \{\xi \in \mathbb{R}_+^3;\ \xi_1 + \xi_2 + \xi_3 = \gamma_0,\ \ \xi_1\xi_2\xi_3 = \gamma_1\}$$

for some positive γ_0, γ_1. Clearly Θ is a periodic orbit. Hence any bounded solution of (11) with positive initial data converges to some periodic orbit as $t \to \infty$, uniformly on $\bar{\Omega}$.

Example 1.2. Consider the reaction-diffusion system:

$$\begin{aligned} \frac{\partial}{\partial t}u_j &= d_j\partial_x^2 u_j - e_j u_j + (u_{j+1} - u_{j-1})u_j, \\ \frac{\partial}{\partial \nu}u_j &= 0\ (x \in \partial\Omega,\ t > 0),\ u_j(x,0) = \phi_j(x)\ (> 0) \\ &\qquad (j = 1, \ldots, N;\ \ u_0 = u_N,\ u_{N+1} = u_1), \end{aligned} \tag{12}$$

where $e_k = 1$ $(k = 1, \ldots, L)$; $e_k = (k = L+1, \ldots, N)$. Then we can see that

$$\Lambda_1 = \{1, \ldots, L, L+2, \ldots, 2[\frac{N-L}{2}]\};$$

$$\Lambda_2 = \{L+1, \ldots, 2[\frac{N-L+1}{2}] + L - 1\};\ \ \Lambda_3 = \text{empty set},$$

where [ ] denotes Causs symbol; thus any bounded solution u of (12) converges to a constant vector $(\gamma_1, \ldots, \gamma_N)$ where $\gamma_j = 0$, $(j \in \Lambda_1)$ and $\gamma_j > 0$, $(j \in \Lambda_2)$.

2. Some estimates. In this section we give some estimates to be used later. We begin by introducing some notations. $\| \ \|_p$ denotes the usual L^p-norm over Ω; we simply write $\| \ \|$ for $\| \ \|_2$. $H^{p,2}$ denotes the L^p-Sobolev space of order 2 with the norm $\| \ \|_{p,2}$. We define the operator P by

$$Pw = \frac{1}{|\Omega|}\int_\Omega w(x)\,dx.$$

In what follows, M denotes various constant independent of t; and set

$$\begin{aligned} &g_j(\xi) = \xi_j f_j(\xi) \\ &K = \sup |u_j(x,t)| \quad (x \in \bar{\Omega},\ t \geq 0,\ j = 1, \ldots, N) \\ &K_1 = \sup_{|\xi| \leq K, j} |f_j(\xi)|, \quad K_2 = \sup_{|\xi| \leq K, j} |\nabla_\xi g_j(\xi)|. \end{aligned} \tag{13}$$

Since the initial function is non-negative and not identically zero, it follows from the elementary property of parabolic equations that

$$u_j(x,t) > 0, \quad (x \in \bar{\Omega},\ t > 0), \quad j = 1, \ldots, N. \tag{14}$$

Since we are concerned with behavior of solution for large t, we may assume that (14) hold for $x \in \bar{\Omega}$ and $t \geq 0$.

Lemma 2.1. *Let u be a solution of* (RD). *Then*

$$\|u_j(t)\|_1 \le M, \quad (j \in \Lambda), \tag{15}$$

$$\int_0^t \|u_j(s)\|_1\, ds \le M, \quad (j \in \Lambda_{1,1}), \tag{16}$$

where M is a constant.

(Here and in what follows we shall simply write $u_j(t)$ for $u_j(x, t)$)

Proof. Integrating the j-th equation in (RD) in x and t over $\Omega \times (0, t)$, multiplying b_j, and taking the summation in j, we get by (2)

$$\int_\Omega \langle b, u(x, t)\rangle\, dx + \delta_1 \sum_{j\in\Lambda_{1,1}} \int_0^t \int_\Omega b_j u_j(x, s)\, dx\, ds \le \int_\Omega \langle b, \phi(x)\rangle\, dx, \tag{17}$$

($\langle , \rangle$: the inner product in $\mathbb{R}^N$) from which (15) and (16) follow immediately in view of (14).

Lemma 2.2. *Let u be a bounded solution of* (RD). *Then*

$$\int_0^t \|\nabla u_j(s)\|^2\, ds \le M, \quad j \in \Lambda; \tag{18}$$

$$\int_0^t \|u_j(s)\|_1\, ds \le M, \quad j \in \Lambda_1; \tag{19}$$

and

$$\int_0^t \|g_k(u)\|_1\, ds \le M, \quad k \in \Lambda_1 \cup \Lambda_2, \tag{20}$$

where M is independent of t.

Proof. By the assumption

$$\sup |u(x, t)|\, (\equiv K) < \infty, \quad (x \in \bar{\Omega},\ t \ge 0). \tag{21}$$

If $j \in \Lambda_{1,1}$, then (16) implies (19). Let $j \in \Lambda_{1,2} - \Lambda_{1,1}$. Then for some $i \in \Lambda$,

$$f_i(u) \ge \delta_2 u_j - \rho(K) \sum_{k\in\Lambda_{1,1}} u_k.$$

Integration of the i-th equation (divided by u_i) in (RD) over $\Omega \times (0, t)$ gives

$$\begin{aligned} d_i &\int_0^t \|(\nabla u_i(s))/u_i(s)\|^2\, ds + \delta_2 \int_0^t \|u_j(s)\|_1\, ds \\ &\le \int_\Omega \log u_i(x, t)\, dx - \int_\Omega \log \phi_i(x)\, dx + \rho(K) \sum_{k\in\Lambda_{1,1}} \int_0^t \|u_k(s)\|_1\, ds. \end{aligned} \tag{22}$$

The first term on the right hand side of (22) is, by (21), bounded, since $\log u_i(x,t) \leq \log_+ u_i(x,t) \leq |u_i(x,t)|$. The second term is, by (14), bounded. The third term is, by (16), bounded. Hence the right-side is bounded, and so is the left hand side. Thus (19) holds for $j \in \Lambda_{1,2}$. Inductively we can show (19) holds for $j \in \Lambda_1$. Clearly (20) holds for $j \in \Lambda_1$ in view of (21) and (19). For $j \in \Lambda_2$ we have, by (4) and (21),

$$\|g_j(u)\|_1 \leq \rho(K) \sum_{i\in\Lambda_1} \|u_i\|_1,$$

which together with (19) gives (20) with $j \in \Lambda_2$. Taking the inner product of the j-th equation in (RD) with u_j, and then integrating the result in t, we see

$$\begin{aligned} &\|u_j(t)\|^2 + 2d_j \int_0^t \|\nabla u_j(s)\|^2\,ds \\ =&\|\phi_j\|^2 + 2\int_0^t (g_j(u(s)),\ u_j(s))\,ds,\ \ (j \in \Lambda_1 \cup \Lambda_2) \end{aligned} \tag{23}$$

($(,) : L^2$-inner product). The right hand side is, by (20), bounded, in view of the boundedness of u. This shows that (18) holds for $j \in \Lambda_1 \cup \Lambda_2$. Finally let $j \in \Lambda_3$. Similarly to (22), integrating the i-th equation (multiplied by c_i/u_i), and taking the sum in i one finds that

$$\begin{aligned} \sum_i c_i \int_0^t \|(\nabla u_i)/u_i\|^2\,ds = &\sum c_i\left[\int_\Omega \log u_i(x,t)\,dx - \int_\Omega \log\phi_i(x)\,dx\right] \\ &- \sum_i c_i \int_0^t \int_\Omega f_i(u)\,dx\,ds. \end{aligned} \tag{24}$$

By Assumption 1.2 the right-hand side of (24) is bounded by

$$(\|u_i(t)\|_1 + \|\log\phi_i\|_1) + \rho(K) \sum_{k\in\Lambda_1} \int_0^t \|u_k\|_1\,ds,$$

which is, by (15) and (19), bounded. Hence the left hand side of (24) is also bounded. Consequently, by the positivity of c_i,

$$\int_0^t \|\nabla u_j\|^2\,ds \leq K^2 \int_0^t \|(\nabla u_j)/u_j\|^2\,ds \leq M,$$

showing that (18) holds for $j \in \Lambda_3$. This proves Lemma 2.2. □

To get the L^∞-bounds for solutions u of (RD), we introduce an operator $A_{j,p}$ in $L^p(\Omega)$:

$$\begin{aligned} &D(A_{j,p}) = \{v \in H^{p,2}(\Omega);\ (\partial/\partial\nu)v = 0\ \ (\text{on } \partial\Omega)\}; \\ &A_{j,p}v = -d_j\Delta v + \delta_1 v\ \ (j \in \Lambda_{1,1});\ \ = -d_j\Delta v\ \ (\text{otherwise}). \end{aligned}$$

We first note that $A_{j,2}$ is a non-negative self-adjoint operator in $L^2(\Omega)$. Let us fix p so that $p > n$, and write A_j for $A_{j,p}$ for simplicity. Then A_j has the following properties:

i) the spectral set of A_j consists only of isolated eigenvalues $\{\lambda_j\}$ with $0 \leq \lambda_1 < \lambda_2 \leq \dots$, and with finite multiplicities;

ii) the first eigenvalue λ_1 is positive if and only if $j \in \Lambda_{1,1}$;

iii) the estimate holds:

$$\|v\|_{p,2} \leq M\{\|A_j v\|_p + \|v\|_p\}, \quad v \in D(A_j); \tag{25}$$

iv) if we define the operator Q by $Q = I$, (if $\lambda_1 > 0$); $Q = I - P$, (if $\lambda_1 = 0$) (I = identity operator), then $Q\, e^{-tA_j} = Qe^{-tA_i} Q$;

v) A_j generates the holomorphic semigroups $\{e^{-tA_j}\}$ in $L^p(\Omega)$ so that

$$\|e^{-tA_j}\| \leq M; \;\; \|Qe^{-tA_j}\| \leq Me^{-\beta t}; \;\; \|A_j e^{-tA_j}\| \leq Me^{-\beta t}/t, \tag{26}$$

with some positive β;

vi) the solution u of (RD) can be written as

$$u_j(t) = e^{-tA_j}\phi_j + \int_0^t e^{-(t-s)A_j} g_j^*(u(s))\, ds, \tag{27}$$

where $g_j^*(u) = g_j(u) + \delta_1 u_j$, ($j \in \Lambda_{1,1}$); $= g_j(u)$ (otherwise) (see Agmon-Douglis-Nirenberg [1], Friedmann [7]).

Lemma 2.3. *We have*

$$\|v\|_\infty \leq M\|A_j v\|_p + M|Pv|, \quad (v \in D(A_j)) \tag{28}$$

(| | : *absolute value*)

Proof. Using the a priori estimate (25) for solutions of elliptic equations, $\|Qv\|_p \leq M\|A_j v\|_p$, and the Sobolev inequality, we get

$$\|v\|_\infty \leq M(\|A_j v\|_p + \|Pv\|_p),$$

showing (28); note $\|Pv\|_p \leq |Pv||\Omega|^{1/p}$.

Lemma 2.4. *Let u be a bounded solution of* (RD). *Then*

$$\int_0^\infty |Qg_j(u(s))|_p^p\, ds < \infty, \quad (j = 1, \dots, N). \tag{29}$$

Proof. If $j \in \Lambda_{1,1}$, then (29) is clear from (16). Suppose $j \in \Lambda - \Lambda_{1,1}$. By integration by parts, $(-\Delta u_j, u_j) = |\nabla u_j|^2$, which is integrable by (18). Since

$$\|Qu_j\|^2 \leq M\|A_{j,2}^{1/2} u_j\|^2 = M(A_{j,2}u_j, u_j) = Md_j(-\Delta u_j, u_j) = Md_j\|\nabla u_j\|^2,$$

which is integrable by (18) and since $\| Qu_j(t)\|_p^p \le 2^{p-2}K^{p-2}\| Qu_j(t)\|^2$, we see that $\| Qu_j(t)\|_p^p$ is integrable on $[0, \infty)$. By the mean-value theorem,

$$\begin{aligned}\| Qg_j(u(t))\|_p &= \| Q[g_j(u) - g_j(Pu)]\|_p \\ &\le \sum_{i=1}^{N} \int_0^1 \| Q[\partial_{\xi_i} g_j(u + s(Pu - u)) \cdot (Pu_i - u_i)]\|_p \, ds \\ &\le K_2 M \sum_{i=1}^{N} \| Pu_i - u_i\|_p \le K_2 M \sum_{1=1}^{N} \| Qu_i\|_p,\end{aligned} \tag{30}$$

from which (29) follows. This proves Lemma 2.4.

Lemma 2.5. *Let u be a bounded solution of* (RD). *Then*

$$\| A_j u_j(t)\|_p \to 0 \quad as \quad t \to \infty, \tag{31}$$

$$\| Qu_j(t)\|_p \to 0 \quad as \quad t \to \infty, \tag{32}$$

$$\| Qu_j(t)\|_\infty \to 0 \quad as \quad t \to \infty. \tag{33}$$

In particular,

$$\| u_j(t)\|_\infty \to 0 \quad as \quad t \to \infty \quad (j \in \Lambda_{1,1}). \tag{34}$$

Proof. We first show (32). Applying the Q to both sides of (27), and using (26) and iv), we find that

$$\| Qu_j(t)\|_p \le Me^{-t\beta}\|\phi_j\|_p + M\int_0^t e^{-(t-s)\beta}\| Qg_j(u(s))\|_p \, ds.$$

Letting $t \to \infty$ in the above inequality, we have, by (29), (32). We next show (31). To this end we express $A_j u_j$ in the form:

$$\begin{aligned}A_j u_j(t) =& A_j e^{-tA_j}\phi_j + \int_{t/2}^t A_j e^{-(t-s)A_j}(Qg_j(u(s)) - Qg_j(u(t))\, ds \\ &+ \int_0^{t/2} A_j e^{-(t-s)A_j} Qg_j(u(s))\, ds + (I - e^{-(t/2)A_j})Qg_j(u(t)) \\ &(\equiv J_1 + J_2 + J_3 + J_4).\end{aligned}$$

Clearly, $J_1 \to 0$ as $t \to \infty$ (in L^p). By(26) and (32), $J_4 \to 0$ as $t \to \infty$. From (25) it is easy to see that

$$\| J_3\|_p \le MK \int_0^{t/2} (t-s)^{-1} e^{-(t-s)\beta}\| Qg_j(u(s))\|_p \, ds$$

from which it follows that $J_3 \to 0$. It remains only to show $J_2 \to 0$. Similarly to (30),

$$\| Qg_j(u(t)) - Qg_j(u(s))\|_p \le MK_2\| u(t) - u(s)\|_p. \tag{35}$$

On the other hand, by standard arguments in the theory of (linear) evolution operators, we can show:

$$\|u_j(t) - u_j(s)\|_p \leq MK_1(|t-s| + |t-s|^{1/2}). \tag{36}$$

(For the proof see the appendix). Thus it follows from (26), (30), (23) and (36) that

$$\|J_2\|_p \leq \rho(K) \int_{t/2}^{t} (|t-s|^{-1/2} + |t-s|^{-3/4}) e^{-(t-s)\beta} ds\, \omega(t)^{1/2},$$

where $\omega(t) = \sup_{t/2 \leq s \leq t} \|Qu(s)\|_p$. Since $\omega(t) \to 0$, it follows that $J_2 \to 0$. This shows (31). (33) and (34) are immediate consequences of (28), (31) and the Sobolev inequality.

Lemma 2.6. *Let $j \in \Lambda_3$. Then there is a $\delta > 0$ such that*

$$|Pu_j(t)| \geq \delta > 0 \quad (t > 0). \tag{37}$$

Proof. By (24) and (2),

$$\begin{aligned}
&\sum_{j \in \Lambda_3} c_j (P(\log u_j(t)) - P(\log \phi_j)) \\
&\qquad \geq \int_0^t \int_\Omega \sum_{j \in \Lambda_3} c_j f_j(u)\, dx\, ds \\
&\qquad \geq -\rho(K) \sum_{j \in \Lambda_1} \int_0^t \|u_j(s)\|_1\, ds \qquad \text{[by (6)]} \\
&\qquad \geq -M, \quad (> -\infty) \qquad \text{[by (19)]}.
\end{aligned}$$

Consequently, by the positivity of c_j,

$$P \log u_j(t) \geq -M_0, \quad (t > 0) \tag{38}$$

with some constant M_0. By Jenssen's inequality, (37) follows from (38).

3. Proof of Theorem 1.1. We first show that $u_j(t)$ $(j \in \Lambda_1 \cup \Lambda_2)$ converges to some constant γ_j as $t \to \infty$, uniformly on Ω. We decompose u_j in the form:

$$u_j(t) = Pu_j(t) + Qu_j(t), \quad (= I_1(t) + I_2(t)). \tag{39}$$

Then by (33)

$$I_2(t) \to 0, \quad \text{uniformly on } \Omega. \tag{40}$$

Applying P to both side of (RD), integrating in x and t and noting $P\Delta u_j = 0$, we see

$$I_1(t) - I_1(s) = \frac{1}{|\Omega|} \int_s^t \left(\int_\Omega g(u)\, dx \right) ds. \tag{41}$$

Since $\|g_j(u)\|_1$, $(j \in \Lambda_1 \cup \Lambda_2)$ is, by (20), integrable on $[0, \infty)$, it follows that $\{I_1(t)\}$ is a Cauchy sequence. Hence there is a constant γ_j with

$$I_1(t) \to \gamma_j$$

which together with (40) shows that $u_j(t)$ converges to γ_j as $t \to \infty$, uniformly on $\bar{\Omega}$. Clearly $\gamma_j \geq 0$ since $u_j \geq 0$. Since $\|u_j(t)\|_1$, $(j \in \Lambda_1)$ is integrable on $(0, \infty)$ by (19), it follows that

$$\gamma_j = 0, \quad (j \in \Lambda_1).$$

Let $j \in \Lambda_2$. Similarly to (24) we get

$$\int_0^t \int_\Omega f_j(u)\, dx\, ds + \int_\Omega \log \phi_j(x)\, dx \leq \int_\Omega \log u_j(x, t)\, dx. \tag{42}$$

The first term on the left hand side is bounded from below, and so

$$-M_1 \leq \text{ the left hand side of (42),}$$

M_1 being some positive constant independent of t, since $\|f_j(u)\|_1$ is integrable on $[0, \infty)$ in view of (4) and (19). Hence letting $t \to \infty$ in (42) we see that the limit of the right hand side is bounded by $-M_1$ from below. Since the limit of the right side is $|\Omega| \log \gamma_j$, it follows that

$$\gamma_j > 0 \quad (j \in \Lambda_2). \tag{43}$$

Set

$$h(t) = \sum_{k=1}^N b_k P u_k(t)$$

for simplicity. Then by (1)

$$(d/dt)h(t) = P\left(\sum_{k=1}^n b_k g_k(u)\right) \leq 0$$

since $P\Delta u_j = 0$. Hence $h(t)$ is monotone decreasing in t. $h(t)$ is non-negative, since $u_j(t)$ is non-negative. Thus the limit of $h(t)$ exists, and we denote it by μ_0. If $\Lambda_1 \neq \Lambda$, then $\mu_0 > 0$ in view of (37), and (43). Finally we show

$$J \equiv \sum_{k \in \Lambda_3} c_k^{(j)} \log u_k(t) \to \mu_j \quad (t \to \infty), \text{ uniformly on } \Omega. \tag{44}$$

Put

$$J_1 = PJ, \quad J_2 = QJ, \quad c_k = c_k^{(j)}.$$

Then

$$\begin{aligned} J_2 &= \sum_k c_k Q[\log(Pu_k(t) + Qu_k(t)) - \log Pu_k(t)] \\ &= \sum_k c_k Q \log(1 + Qu_k(t)/Pu_k(t)) \end{aligned}$$

which tends to zero as $t \to \infty$, uniformly on $\bar{\Omega}$, in view of (33) and (37). Simple calculation gives

$$J_1(t) - J_2(s) = \sum_k c_k \int_s^t \left[\int_\Omega (d_k|\nabla u_k|/u_k|^2 + f_k(u))\,dx\right] ds. \tag{45}$$

By (24), $|(\nabla u_k)/u_k|$ is square integrable on $\Omega \times (0, \infty)$. Also the absolute value of the integral of $\sum_k c_k f_k(u)$ on the right hand side of (44) is, by (6), dominated by the integrable function

$$\rho(K) \sum_{k \in \Lambda_1} \int_s^t \|u_k(\tau)\|_{L^1}\,d\tau.$$

Hence the right hand side of (45) tends to zero as $s, t \to \infty$. Thus $\{J_1(t)\}$ is a Cauchy sequence, and so is $J(t)$. This shows (44). This completes the proof of Theorem 1.1.

Appendix. Proof of (36). Here we shall give the proof of (36). Set $A_j = A$, $g_j = g$, etc. for simplicity. To show (36), we estimate each term on the right-hand side of the equation

$$\begin{aligned} u(t) - u(s) &= (e^{-tA} - e^{-sA})\phi + \int_s^t e^{-(t-\tau)A} g(u(\tau))\,d\tau \\ &+ \int_0^s (e^{-(t-\tau)A} - e^{-(s-\tau)A}) f(u(\tau))\,d\tau \quad (\equiv J_1 + J_2 + J_3). \end{aligned}$$

From the elementary properties in semigroup theory it follows that

$$\begin{aligned} \|(e^{-tA} - e^{-sA})\| &\le (t-s)\|(e^{-(t-s)A} - I)/((t-s)A)\|\,\|Ae^{-sA}\| \\ &\le M(t-s)s^{-1}e^{-s\beta}; \quad \text{and} \\ \|e^{-tA} - e^{-sA}\| &\le Me^{-s\beta}. \end{aligned}$$

Hence by the interpolation theorem,

$$\|e^{-tA} - e^{-sA}\| \le M(t-s)^\theta s^{-\theta} e^{-s\beta} \quad (0 < \theta < 1;\ 0 < s < t). \tag{A1}$$

Similarly,

$$\|(e^{-tA} - e^{-sA})\phi\|_p \le M(t-s)e^{-s\beta}\|A\phi\|_p.$$

Thus by (26)

$$\|J_1\|_p \le M(t-s), \quad \|J_2\|_p \le MK_3(t-s),$$

where $K_3 = \sup|g(\xi)|$, $(|\xi| \le K)$. Also by (A1) with $\theta = 1/2$,

$$\|J_3\| \le M(t-s)^{1/2}.$$

Collecting all the estimates above we get (36).

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