ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO QUASILINEAR HYPERBOLIC EQUATIONS WITH NONLINEAR DAMPING

By

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1. Introduction. In this paper we consider the following quasilinear hyperbolic system with damping:

$$v_t - (h(v)p)_x = 0, \tag{1.1}$$

$$p_t + \sigma(v)_x = f(v)p, \tag{1.2}$$

where $\sigma'(v) < 0, h(v) > 0, f(v) < 0$ and $v > 0$.

The above is a system derived in [23], [24], and describes the propagation of heat wave for rigid solids at very low temperatures, below about $20°$ K. The first equation (1.1) comes from the balance of energy, which in the one-dimensional case takes the form

$$\varepsilon(\vartheta)_t + q_x = 0, \tag{1.3}$$

where $\vartheta > 0$ is the absolute temperature, ε is the internal energy, and q is the one-dimensional heat flux. Equation (1.2) is the evolution equation for an internal parameter p, which is introduced to account for memory effects of the heat flux. The effect of memory may be considered, for example, as a functional of a history of a temperature gradient,

$$q = -\alpha(\vartheta) \int_{-\infty}^{t} e^{-b(t-s)} \vartheta_x(x, s) ds, \quad \alpha(\vartheta) > 0, \quad b > 0 \tag{1.4}$$

by defining

$$p = \int_{-\infty}^{t} e^{-b(t-s)} \vartheta_x(x, s) ds. \tag{1.5}$$

Equation (1.4) can then be equivalently replaced with

$$q = -\alpha(\vartheta)p, \tag{1.6}$$

$$p_t = -bp + \vartheta_x. \tag{1.7}$$

Equation (1.7), related to (1.4) via (1.5), is, however, linear and does not fully describe the properties of heat propagation in solids (cf. [23], [24], and references therein). To improve the model one may generalize the history dependence of q by modifying Eq. (1.4) or, as was done in [24], by introducing a suitable nonlinear dependence in (1.7),

$$p_t = g_1(\vartheta)\vartheta_x + g_2(\vartheta)p. \tag{1.8}$$

The functions α, g_1, and g_2 present in (1.6) and (1.8) are material functions. The second law of thermodynamics imposes the restrictions that $\alpha(\vartheta) = \psi_{20}\vartheta^2 g_1(\vartheta)$ and $g_2(\vartheta) < 0$, where the constant $\psi_{20} > 0$ comes from the Helmholtz free energy ψ which has the form $\psi = \psi_1(\vartheta) + \frac{1}{2}\psi_{20}\vartheta p^2$. We additionally make an assumption that $g_1(\vartheta) > 0$ (cf. [24]). Combining (1.3) with (1.8) gives the following system:

$$\varepsilon(\vartheta)_t - (\alpha(\vartheta)p)_x = 0, \tag{1.9}$$

$$p_t + G_1(\vartheta)_x = g_2(\vartheta)p, \quad G_1'(\vartheta) = -g_1(\vartheta). \tag{1.10}$$

In the steady-state case, $p_t = 0$, equations (1.9), (1.10) lead to a nonlinear diffusion equation,

$$c_v(\vartheta)\vartheta_t - (\mathcal{K}(\vartheta)\vartheta_x)_x = 0, \tag{1.11}$$

where $\mathcal{K}(\vartheta) = -\psi_{20}\frac{\vartheta^2 g_1^2(\vartheta)}{g_2(\vartheta)} > 0$ is the steady-state conductivity measured experimentally, $q = -\mathcal{K}(\vartheta)\vartheta_x$, and $\varepsilon'(\vartheta) = c_v(\vartheta)$ is the specific heat.

System (1.1), (1.2) can be obtained from (1.9), (1.10) by employing the substitution $\varepsilon(\vartheta) = v > 0$ with $\sigma = G_1 \circ \varepsilon^{-1}, f = g_2 \circ \varepsilon^{-1}, h = \alpha \circ \varepsilon^{-1}, \sigma'(v) < 0, h(v) > 0$, and $f(v) < 0$.

In terms of the new variable v, the steady-state diffusion equation (1.11) takes the form

$$v_t + P(v)_{xx} = 0, \tag{1.12}$$

$$h(v)p = -P(v)_x, \tag{1.13}$$

or

$$v_t - (h(v)p)_x = 0, \tag{1.14}$$

$$h(v)p = -P(v)_x, \tag{1.15}$$

where

$$P'(v) = -\frac{F'(v)}{f(v)} < 0, \quad F'(v) = h(v)\sigma'(v) < 0. \tag{1.16}$$

When $f(v) = -b$ with $b > 0$ a constant, and $h(v) = 1$, the system (1.1), (1.2) reduces to the isentropic Euler equations in Lagrangian coordinates with an external function $-bp$ in the momentum equation, modeling compressible flow through porous media, and the corresponding simplified parabolic equation (1.12) represents Darcy's law. (As mentioned above, $f(v) = -b$ is not physical for our application.)

It is known that the damping mechanism is dissipative and can guarantee the regularity of solutions. The investigation of the case of a p-system ($h(v) = 1$) with linear damping ($f(v) = -b$) has been extensively studied. Global existence of smooth solutions was established by Nishida [18] for sufficiently small initial data. Hsiao and Liu [4, 5] and Hsiao [3] first investigated asymptotic behavior, where they showed that the solution (v, p) to the hyperbolic p-system with linear damping and data (1.20), (1.21),

has parabolic structure, *i.e.*, it behaves as $(\bar{v}, \bar{p})$ where $\bar{v}$ is a self-similar solution to equation (1.12) and $\bar{p}$ satisfies equation (1.13) for $h(v) = 1$. This is called a nonlinear diffusive phenomenon. Under the restriction of weak diffusive waves and small initial perturbation, the convergence rates were obtained by Nishihara [19, 20] in L_2 and L_∞ norms, and by Nishihara, Wang and Yang [22] in optimal L_p norms in terms of an approximate Green's function and pointwise estimates. Zhao [30] obtained the optimal L_p convergence rates for strong diffusive waves and large initial data, where only the initial oscillation is required to be small. Based on the resolution of the Riemann problem by Hsiao and Tang [11, 12], Hsiao and Luo [7] discussed nonlinear diffusive phenomena for piecewise smooth solution. There are other related works, such as on the boundary effects [17, 21, 8, 13], on large initial data [29, 30], on weak solutions and large time behavior [2, 16, 25], on wave interactions [15], and on the regularity and formation of singularities [26, 28], and references therein.

However, no result has been obtained on the qualitative behavior (particularly, large time behavior and nonlinear diffusive phenomena) of solutions to a non p-system with nonlinear damping. The particular properties of f, the function which physically should provide a damping mechanism, are responsible for the asymptotic behavior of the system (1.1), (1.2). (In the case of $f(v) = -b$, the only condition required for decay is $b > 0$.) In the paper we will show that it is sufficient for the nonlinear damping $f(v)$ to satisfy the following conditions:

$$f(v) < 0, \tag{1.17}$$

$$f''(v) < 0, \tag{1.18}$$

$$3(f'(v))^2 - f(v)f''(v) < 0, \tag{1.19}$$

in order to establish that the solution (v, p) to (1.1), (1.2) ((1.14), (1.15)) with initial data

$$v(x, 0) = v_0(x), \quad p(x, 0) = p_0(x), \tag{1.20}$$

$$(v_0, p_0)(\pm\infty) = (v_\pm, 0), \tag{1.21}$$

has the property that a pair $(v, h(v)p)$ approaches time-asymptotically $(\bar{v}, h(\bar{v})\bar{p})$, where $(\bar{v}, \bar{p})$ is the solution of (1.12), (1.13) with the same constant states at $x = \pm\infty$. $\bar{v}$ is a self-similar solution to equation (1.12) and $\bar{p}$ satisfies equation (1.13). We should point out that $(v, h(v)p) = (v, -q)$, where q is the heat flux (see (1.6)) *i.e.*, (v, q) asymptotically approaches $(\bar{v}, \bar{q})$. Thus, as $t \to \infty$, the difference between the hyperbolic (thermal) wave and the solution to the corresponding diffusion equation with steady-state conductivity approaches zero. This means that the solution (v, p) to the hyperbolic system (1.1), (1.2), with data (1.20), (1.21), eventually has the same parabolic structure as $(\bar{v}, \bar{p})$.

Let us set

$$V_0(x) = \int_{-\infty}^{x} (v_0(s) - \bar{v}(s + x_0, 0))ds, \quad V_1(x) = p_0(x) - \bar{p}(x + x_0, 0), \tag{1.22}$$

where x_0 is uniquely selected such that

$$\int_{-\infty}^{\infty} (v_0(s) - \bar{v}(s + x_0, 0))ds = 0 \tag{1.23}$$

and

$$V(x,t) = \int_{-\infty}^{x} (v(s,t) - \bar{v}(s + x_0, t))ds. \tag{1.24}$$

The main result in the present paper is the following

THEOREM 1.1. Assume that $F \in C^3$, $F' < 0$, $f \in C^2$ with (1.17)–(1.19) hold, and $|v_+ - v_-| \ll 1$. Suppose $V_0 \in H^3$ and $V_1 \in H^2$. Then, there exists a $\delta_0 > 0$ such that if $\|V_0\|_3 + \|V_1\|_2 \leq \delta_0$, a unique global solution (v, p) to (1.1)–(1.2) and (1.20)–(1.21) exists and satisfies

$$\sum_{k=0}^{3} (1+t)^k \|\partial_x^k V(.,t)\|^2 + \sum_{k=0}^{2} (1+t)^{(k+2)} \|\partial_x^k V_t(.,t)\|^2$$

$$+ (1+t)^4 \|V_{tt}(.,t)\|^2 + (1+t)^5 (\|V_{ttt}(.,t)\|^2 + \|V_{xtt}(.,t)\|^2)$$

$$+ \int_0^t \left\{ \sum_{k=1}^{3} (1+s)^{(k-1)} \|\partial_x^k V(.,s)\|^2 + \sum_{k=0}^{2} (1+s)^{(k+1)} \|\partial_x^k V_t(.,s)\|^2 \right\} ds$$

$$+ \int_0^t \|\bar{v}_x V(.,s)\|^2 ds$$

$$+ \int_0^t (1+s)^5 \|V_{ttt}(.,s)\|^2 ds$$

$$\leq C(\|V_0\|_3^2 + \|V_1\|_2^2 + |v_+ - v_-|). \tag{1.25}$$

In addition for $h(v) = 1$, if it also holds that $V_0 \in L_1$ and $V_1 \in L_1$, the following optimal L_p ($2 \leq p \leq \infty$) decay rates are satisfied:

$$\|\partial_x^k (v - \bar{v})(.,t)\|_{L_p} \leq C\delta_0 (1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{k+1}{2}}, \tag{1.26}$$

$$\|\partial_x^k (p - \bar{p})(.,t)\|_{L_p} \leq C\delta_0 (1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{k+2}{2}}, \tag{1.27}$$

for any $k \leq 2$ if $p = 2$ and $k \leq 1$ if $p \in (2, \infty]$.

This theorem can be proved by energy methods, but with a nonstandard multiplier due to nonlinear damping and pointwise estimates. However, though it is dissipative, the nonlinear damping causes the strong interaction of hyperbolic thermal waves and nonlinear diffusive waves, similar to those in the stability analysis of elementary hyperbolic waves (shock profiles and rarefaction waves). It is the hyperbolic character and the structure of nonlinear damping that control the interaction instead of convexity. However, because of the strong interaction, one can only control the interaction if the two wave strengths are both weak enough, prove the global existence of smooth nonlinear hyperbolic waves starting from the neighborhood of the nonlinear diffusive wave, and obtain the L_2 convergence rate to the diffusive wave. It means that the nonlinear diffusive phenomena still occur, no matter how the damping mechanism performs. Using global existence of smooth solutions and the associated L_2 convergence rates to nonlinear diffusive waves, we may, employing the methods of [22], obtain optimal L_p convergence rates in the case of $h(v) = 1$, and $f(v)$ nonconstant.

The next section of the paper gives some useful properties of self-similar solutions to the diffusion equation (1.12). In Sec. 3, we present apriori estimates for the local smooth

solution to (1.1)–(1.2) in order to prove global existence and obtain time decay rates. In Sec. 4, we establish optimal L_p convergence rates of the solution (v, p) to the nonlinear diffusive waves $(\bar{v}, \bar{p})$, $h(v) = 1$. An example of the function $f(v)$ is finally given.

Notation. L_p $(1 \leq p \leq +\infty)$ denotes the space of measurable functions g on $\mathbb{R}$ with norm defined by

$$\|g\|_{L_p} = \left(\int_{-\infty}^{\infty} |g(x)|^p dx \right)^{\frac{1}{p}}.$$

$H^m(\mathbb{R})$ is the standard Sobolev space normed by

$$\|f\|_m = \sum_{k=0}^{m} \|\frac{\partial^k f}{\partial x^k}\|_{L_2}.$$

2. Diffusive equation. In this section, for the reader's convenience, we will give a summary of the results concerning the self-similar solution of the nonlinear parabolic equation (1.12) (cf. [3, 4, 22]). These are necessary in obtaining the asymptotic properties of the solution to (1.1), (1.2) with initial data (1.20), (1.21).

The nonlinear parabolic equation,

$$v_t + P(v)_{xx} = 0, \quad P'(v) < 0 \tag{2.1}$$

possesses a unique and strictly monotone self-similar solution $\bar{v}$ (see [27])

$$\bar{v}(x, t) \overset{\Delta}{=} \Phi(\zeta), \quad \zeta = \frac{x}{\sqrt{t+1}}, \tag{2.2}$$

satisfying

$$\Phi''(\zeta) + \frac{P''(\Phi(\zeta))\Phi'(\zeta) - \frac{1}{2}\zeta}{P'(\Phi(\zeta))} \Phi'(\zeta) = 0,$$
$$\Phi(\pm\infty) = v_{\pm}, \quad (v_+ \neq v_-). \tag{2.3}$$

The following estimate is needed for our analysis (see [3, 4, 22] for details):

$$\sum_{k=1}^{6} |\frac{d^k}{d\zeta^k}\Phi(\zeta)| + |\Phi(\zeta) - v_+|_{\zeta>0} + |\Phi(\zeta) - v_-|_{\zeta<0} \leq C|v_+ - v_-|e^{-c\zeta^2}, \tag{2.4}$$

$$|\bar{v}_t(x, t)| \leq C|v_+ - v_-|(1+t)^{-1}, \quad |\bar{v}_x(x, t)| \leq C|v_+ - v_-|(1+t)^{-\frac{1}{2}}, \tag{2.5}$$

$$\int_{-\infty}^{\infty} |\bar{v}_x(x, t)|^2 dx \leq C|v_+ - v_-|^2(1+t)^{-\frac{1}{2}}, \tag{2.6}$$

$$\int_{-\infty}^{\infty} (|\bar{v}_t(x, t)|^2 + |\bar{v}_{xx}(x, t)|^2) dx \leq C|v_+ - v_-|^2(1+t)^{-\frac{3}{2}}, \tag{2.7}$$

$$\int_{-\infty}^{\infty} (|\bar{v}_{xt}(x, t)|^2 + |\bar{v}_{xxx}(x, t)|^2) dx \leq C|v_+ - v_-|^2(1+t)^{-\frac{5}{2}}, \tag{2.8}$$

$$\int_{-\infty}^{\infty} |\bar{v}_{tt}(x, t)|^2 dx \leq C|v_+ - v_-|^2(1+t)^{-\frac{7}{2}}, \tag{2.9}$$

$$\int_{-\infty}^{\infty} |\bar{v}_{xtt}(x, t)|^2 dx \leq C|v_+ - v_-|^2(1+t)^{-\frac{9}{2}}, \tag{2.10}$$

$$\int_{-\infty}^{\infty} |\bar{v}_{xttt}(x,t)|^2 dx \le C|v_+ - v_-|^2 (1+t)^{-\frac{13}{2}}, \tag{2.11}$$

for some constant $C, c > 0$ independent of ζ, and

$$\|\partial_t^l \partial_x^k \bar{v}(.,t)\|_{L_p} \le C|v_+ - v_-|(1+t)^{-l-\frac{k}{2}+\frac{1}{2p}} \tag{2.12}$$

for $1 \le l + k \le 6$ and $p \in [2, \infty]$.

3. The apriori estimates. In this section, we prove the global existence of the smooth solution (v, p) to the hyperbolic system (1.1)–(1.2) with initial data (1.20)–(1.21) and discuss its parabolic structure. Then, we estimate the decay rates (1.25).

The local existence of smooth solutions to (1.1) and (1.2) can be obtained by the standard methods (see [14]), and what we do is to obtain uniform apriori estimates in order to extend the local solution for any $T > 0$.

The differences $v - \bar{v}$ and $h(v)p - h(\bar{v})\bar{p}$ satisfy the following system of equations:

$$(v - \bar{v})_t - (h(v)p - h(\bar{v})\bar{p})_x = 0, \tag{3.1}$$

$$(h(v)p - h(\bar{v})\bar{p})_t + (F(v) - F(\bar{v}))_x - f(\bar{v})(h(v)p - h(\bar{v})\bar{p})$$
$$- (f(v) - f(\bar{v}))h(v)p - P(\bar{v})_{xt} + H(\bar{v})P(\bar{v})_x(v - \bar{v})_t$$
$$- [H(\bar{v})(h(v)p - h(\bar{v})\bar{p}) + (H(v) - H(\bar{v}))h(v)p]v_t - H(\bar{v})P(\bar{v})_x P(\bar{v})_{xx} = 0, \tag{3.2}$$

where we used the relation (1.13) for $\bar{v}$ and $h(\bar{v})\bar{p}$, and introduce the notation

$$H(v) = \frac{h'(v)}{h(v)}. \tag{3.3}$$

We notice that integration of (3.1) over $\mathbb{R}$ gives

$$\frac{d}{dt} \int_{-\infty}^{\infty} (v - \bar{v})(x,t) dx = 0 \tag{3.4}$$

which implies, in terms of (1.23), that

$$\int_{-\infty}^{\infty} (v(x,t) - \bar{v}(x + x_0, t)) dx = 0. \tag{3.5}$$

Let us recall that V as in (1.24) is

$$V(x,t) = \int_{-\infty}^{x} (v(s,t) - \bar{v}(s + x_0, t)) ds.$$

By the above, (3.5) and (1.24), we have the property for V that

$$V(+\infty, t) = V(-\infty, t) = 0, \tag{3.6}$$

and also that

$$V_t = h(v)p - h(\bar{v})\bar{p}, \quad V_x = v - \bar{v}. \tag{3.7}$$

Then, the system (3.1), (3.2) can be written as a single second order "wave" equation for V,

$$V_{tt} + (F(V_x + \bar{v}) - F(\bar{v}))_x - f(\bar{v})V_t - P(\bar{v})_{xt}$$
$$- (f(V_x + \bar{v}) - f(\bar{v}))V_t + (f(V_x + \bar{v}) - f(\bar{v}))P(\bar{v})_x$$

$$+ H(\bar{v})[P(\bar{v})_x V_t]_x - H(\bar{v})V_t V_{xt} - (H(V_x + \bar{v}) - H(\bar{v}))V_t V_{xt}$$

$$+ (H(V_x + \bar{v}) - H(\bar{v}))[P(\bar{v})_x V_t]_x - (H(V_x + \bar{v}) - H(\bar{v}))P(\bar{v})_x P(\bar{v})_{xx}$$

$$- H(\bar{v})P(\bar{v})_x P(\bar{v})_{xx} = 0, \tag{3.8}$$

$$V(x,0) = V_0(x), \quad V_t(x,0) = V_1(x), x \in \mathbb{R}. \tag{3.9}$$

Assume on the existence domain it holds apriori that

$$N(T) = \max_{0 \le t \le T} \left\{ \sum_{k=0}^{3}(1+t)^k \|\partial_x^k V(.,t)\|^2 + \sum_{k=0}^{2}(1+t)^{k+2}\|\partial_x^k V_t(.,t)\|^2 \right\} \ll 1. \tag{3.10}$$

One can verify that under the apriori assumption (3.10), it holds for two constants v_1, v_2,

$$0 < v_1 \le \bar{v} + V_x \le v_2. \tag{3.11}$$

What we do next is to obtain the following lemma.

LEMMA 3.1. Under the assumptions of Theorem 1.1, it holds, for $0 \le t \le T$, that

$$\int_{-\infty}^{\infty}(V^2 + V_x^2 + V_t^2)(x,t)dx + \int_0^t\int_{-\infty}^{\infty}(\bar{v}_x^2 V^2 + V_x^2 + V_t^2)dxds$$

$$\le C(\|V_0\|_3^2 + \|V_1\|_2^2 + |v_+ - v_-|), \tag{3.12}$$

$$\int_{-\infty}^{\infty}(V_{xt}^2 + V_{tt}^2 + V_{xx}^2)(x,t)dx + \int_0^t\int_{-\infty}^{\infty}(V_{xt}^2 + V_{tt}^2 + V_{xx}^2)dxds$$

$$\le C(\|V_0\|_3^2 + \|V_1\|_2^2 + |v_+ - v_-|), \tag{3.13}$$

$$\int_{-\infty}^{\infty}(V_{xxt}^2 + V_{xtt}^2 + V_{xxx}^2)(x,t)dx + \int_0^t\int_{-\infty}^{\infty}(V_{xxt}^2 + V_{xtt}^2 + V_{xxx}^2)dxds$$

$$\le C(\|V_0\|_3^2 + \|V_1\|_2^2 + |v_+ - v_-|), \tag{3.14}$$

and that

$$\sum_{k=0}^{3}(1+t)^k\|\partial_x^k V(.,t)\|^2 + \sum_{k=0}^{2}(1+t)^{(k+2)}\|\partial_x^k V_t(.,t)\|^2$$

$$+ (1+t)^4\|V_{tt}(.,t)\|^2 + (1+t)^5(\|V_{ttt}(.,t)\|^2 + \|V_{xtt}(.,t)\|^2)$$

$$+ \int_0^t\left\{\sum_{k=1}^{3}(1+s)^{(k-1)}\|\partial_x^k V(.,s)\|^2 + \sum_{k=0}^{2}(1+s)^{(k+1)}\|\partial_x^k V_t(.,s)\|^2\right\}ds$$

$$+ \int_0^t(1+s)^5\|V_{ttt}(.,s)\|^2ds$$

$$\le C(\|V_0\|_3^2 + \|V_1\|_2^2 + |v_+ - v_-|), \tag{3.15}$$

provided that (3.10) holds.

Proof. Multiplying (3.8) with $-f(\bar{v})V$ and integrating it over $\mathbb{R}$ leads to

$$\frac{d}{dt}\int_{-\infty}^{\infty}(\frac{1}{2}f(\bar{v})^2V^2 - f(\bar{v})V_t V)dx$$

$$+ \int_{-\infty}^{\infty}[f(\bar{v})V_t^2 + \bar{v}_t f'(\bar{v})VV_t - f(\bar{v})f'(\bar{v})\bar{v}_t V^2]dx$$

$$+ \int_{-\infty}^{\infty} (F(\bar{v} + V_x) - F(\bar{v}))(f(\bar{v})V_x + f'(\bar{v})\bar{v}_x V)dx$$

$$+ \int_{-\infty}^{\infty} [f(\bar{v})P(\bar{v})_{xt}V + f(\bar{v})(f(\bar{v} + V_x) - f(\bar{v}))V_t V]dx$$

$$- \int_{-\infty}^{\infty} f(\bar{v})P(\bar{v})_x(f(\bar{v} + V_x) - f(\bar{v}))V dx$$

$$- \int_{-\infty}^{\infty} f(\bar{v})H(\bar{v})[P(\bar{v})_x V_t]_x V dx + \int_{-\infty}^{\infty} f(\bar{v})H(\bar{v})V_t V_{xt} dx$$

$$+ \int_{-\infty}^{\infty} f(\bar{v})(H(V_x + \bar{v}) - H(\bar{v}))V_t V_{xt} V dx$$

$$- \int_{-\infty}^{\infty} f(\bar{v})(H(V_x + \bar{v}) - H(\bar{v}))[P(\bar{v})_x V_t]_x V dx$$

$$+ \int_{-\infty}^{\infty} f(\bar{v})(H(V_x + \bar{v}) - H(\bar{v}))P(\bar{v})_x P(\bar{v})_{xx} V dx + \int_{-\infty}^{\infty} f(\bar{v})H(\bar{v})P(\bar{v})_x P(\bar{v})_{xx} V dx = 0,$$

$$(3.16)$$

which implies

$$\frac{d}{dt} \int_{-\infty}^{\infty} (\frac{1}{2}f(\bar{v})^2 V^2 - f(\bar{v})V_t V)dx + \int_{-\infty}^{\infty} [f(\bar{v})V_t^2 + \bar{v}_t f'(\bar{v})VV_t]dx$$

$$+ \int_{-\infty}^{\infty} [F(\bar{v} + V_x) - F(\bar{v}) - F'(\bar{v})V_x][f(\bar{v})V_x + f'(\bar{v})\bar{v}_x V]dx$$

$$- \int_{-\infty}^{\infty} [f(\bar{v} + V_x) - f(\bar{v}) - f'(\bar{v})V_x]f(\bar{v})P(\bar{v})_x V dx$$

$$- \int_{-\infty}^{\infty} [f'(\bar{v})\bar{v}_x V + f(\bar{v})V_x]P(\bar{v})_t dx + \int_{-\infty}^{\infty} [f(\bar{v} + V_x) - f(\bar{v})]f(\bar{v})V_t V dx$$

$$+ \int_{-\infty}^{\infty} [f(\bar{v})H(\bar{v})]'P(\bar{v})_x \bar{v}_x VV_t dx + \int_{-\infty}^{\infty} f(\bar{v})H(\bar{v})P(\bar{v})_x V_x V_t dx$$

$$- \frac{1}{2} \int_{-\infty}^{\infty} [f(\bar{v})H(\bar{v})]'\bar{v}_x VV_t^2 dx - \frac{1}{2} \int_{-\infty}^{\infty} f(\bar{v})H(\bar{v})V_t^2 V_x dx$$

$$+ \int_{-\infty}^{\infty} (H(V_x + \bar{v}) - H(\bar{v}))f(\bar{v})V_t V_{xt} V dx - \int_{-\infty}^{\infty} (H(V_x + \bar{v}) - H(\bar{v}))f(\bar{v})[P(\bar{v})_x V_t]_x V dx$$

$$+ \int_{-\infty}^{\infty} (H(V_x + \bar{v}) - H(\bar{v}))f(\bar{v})P(\bar{v})_x P(\bar{v})_{xx} V dx - \frac{1}{2} \int_{-\infty}^{\infty} [f(\bar{v})H(\bar{v})]'[P(\bar{v})_x]^2 \bar{v}_x V dx$$

$$- \frac{1}{2} \int_{-\infty}^{\infty} f(\bar{v})H(\bar{v})[P(\bar{v})_x]^2 V_x dx$$

$$= \int_{-\infty}^{\infty} [f(\bar{v})f'(\bar{v})\bar{v}_t V^2 - F'(\bar{v})f(\bar{v})V_x^2 - (f'(\bar{v})F'(\bar{v}) - f(\bar{v})f'(\bar{v})P'(\bar{v}))\bar{v}_x VV_x]dx$$

$$=: I_0.$$

$$(3.17)$$

By (2.1), (1.16), and (1.17)–(1.19), the I_0 can be estimated as

$$I_0 = - \int_{-\infty}^{\infty} [4F'(\bar{v})f'(\bar{v})\bar{v}_x VV_x + f(\bar{v})F'(\bar{v})V_x^2 + F'(\bar{v})(f''(\bar{v}) + \frac{(f'(\bar{v}))^2}{f(\bar{v})})\bar{v}_x^2 V^2)]dx$$

$$\leq -c_0 \int_{-\infty}^{\infty} (V_x^2 + \bar{v}_x^2 V^2) dx. \tag{3.18}$$

Substituting (3.18) into (3.17) yields

$$\frac{d}{dt} \int_{-\infty}^{\infty} (\frac{1}{2} f(\bar{v})^2 V^2 - f(\bar{v}) V_t V) dx + \int_{-\infty}^{\infty} [f(\bar{v}) V_t^2 + c_0 (V_x^2 + \bar{v}_x^2 V^2)] dx$$

$$\leq C \int_{-\infty}^{\infty} \bar{v}_t^2 dx + CN(T) \int_{-\infty}^{\infty} [V_x^2 + V_t^2] dx + \frac{c_0}{4} \int_{-\infty}^{\infty} \bar{v}_x^2 V^2 dx. \tag{3.19}$$

Multiplying (3.8) with V_t and integrating it by parts over $\mathbb{R}$ leads to

$$\frac{1}{2} \frac{d}{dt} \int_{-\infty}^{\infty} [V_t^2 - F'(\bar{v}) V_x^2] dx - \int_{-\infty}^{\infty} f(\bar{v}) V_t^2 dx$$

$$+ \int_{-\infty}^{\infty} [f(\bar{v} + V_x) - f(\bar{v})][P'(\bar{v}) \bar{v}_x V_t - V_t^2] dx$$

$$- \int_{-\infty}^{\infty} (F(\bar{v} + V_x) - F(\bar{v}) - F'(\bar{v}) V_x) V_{xt} + \frac{1}{2} \int_{-\infty}^{\infty} F''(\bar{v}) \bar{v}_t V_x^2 dx$$

$$- \int_{-\infty}^{\infty} P(\bar{v})_{xt} V_t dx - \frac{1}{2} \int_{-\infty}^{\infty} H(\bar{v})_x P(\bar{v})_x V_t^2 dx + \frac{1}{2} \int_{-\infty}^{\infty} H(\bar{v}) P(\bar{v})_{xx} V_t^2 dx$$

$$- \int_{-\infty}^{\infty} H(\bar{v}) V_t^2 V_{xt} dx + \int_{-\infty}^{\infty} [H(V_x + \bar{v}) - H(\bar{v})][(P(\bar{v})_x V_t)_x - V_{xt} V_t] V_t dx$$

$$- \int_{-\infty}^{\infty} [H(V_x + \bar{v}) - H(\bar{v})] P(\bar{v})_x P(\bar{v})_{xx} V_t dx - \int_{-\infty}^{\infty} H(\bar{v}) P(\bar{v})_x P(\bar{v})_{xx} V_t dx = 0, \tag{3.20}$$

which means

$$\frac{1}{2} \frac{d}{dt} \int_{-\infty}^{\infty} [V_t^2 - F'(\bar{v}) V_x^2] dx - \int_{-\infty}^{\infty} f(\bar{v}) V_t^2 dx$$

$$\leq C \int_{-\infty}^{\infty} (\bar{v}_{xt}^2 + \bar{v}_t^2) dx + C(N(T) + |v_+ - v_-|) \int_{-\infty}^{\infty} (V_x^2 + V_t^2) dx. \tag{3.21}$$

Integrating $[(3.19) + 2 \times (3.21)]$ over $[0, t]$ leads, in terms of (3.10) and (2.4)-(2.11), to (3.12).

Similarly, integrating $((3.8))_x \times [-f(\bar{v}) V_t + 2 V_{tt}]$ and $((3.8))_{xx} \times [-f(\bar{v}) V_{tt} + 2 V_{ttt}]$ over $(-\infty, +\infty) \times [0, t]$, we can prove high order estimates (3.13) and (3.14) in terms of (3.12) and (3.8). $\qquad\square$

Next, we prove the decay rates (3.15) (cf. [19, 20]). Multiplying (3.8) with V_t and integrating it by parts over $(-\infty, +\infty)$ leads to

$$\frac{d}{dt} \int_{-\infty}^{\infty} (\frac{1}{2} V_t^2 + Q)(x, t) dx - \int_{-\infty}^{\infty} f(\bar{v} + V_x) V_t^2 dx$$

$$= - \int_{-\infty}^{\infty} \int_0^{V_x} (F'(\bar{v} + \theta) - F'(\bar{v})) \bar{v}_t d\theta dx$$

$$- \int_{-\infty}^{\infty} [f(\bar{v} + V_x) - f(\bar{v})] P(\bar{v})_x V_t dx + \int_{-\infty}^{\infty} P(\bar{v})_{xt} V_t dx$$

$$+ \frac{1}{2} \int_{-\infty}^{\infty} H(\bar{v})_x P(\bar{v})_x V_t^2 dx - \frac{1}{2} \int_{-\infty}^{\infty} H(\bar{v})(P(\bar{v})_{xx} V_t^2 dx + \int_{-\infty}^{\infty} H(V_x + \bar{v}) V_{xt} V_t^2 dx$$

$$- \int_{-\infty}^{\infty} (H(V_x + \bar{v}) - H(\bar{v}))[P(\bar{v})_x V_t]_x V_t dx + \int_{-\infty}^{\infty} H(V_x + \bar{v}) P(\bar{v})_x P(\bar{v})_{xx} V_t dx,$$

$$(3.22)$$

where

$$Q = - \int_0^{V_x} [F(\bar{v} + \theta) - F(\bar{v})] d\theta,$$

which satisfies

$$Q_- V_x^2 \leq Q \leq Q_+ V_x^2, \tag{3.23}$$

for two constants $0 < Q_- < Q_+$.

Multiplying (3.22) by $[1 + t]$ and integrating it by parts over $[0, t]$ yields, in terms of (2.4)–(2.9), and (3.23), that

$$(1 + t) \int_{-\infty}^{\infty} (V_t^2 + V_x^2)(x, t) dx + \int_0^t \int_{-\infty}^{\infty} (1 + s) V_t^2 dx ds$$

$$\leq C \int_{-\infty}^{\infty} (V_t^2 + V_x^2)(x, 0) dx + C \int_0^t \int_{-\infty}^{\infty} (V_t^2 + V_x^2)(x, s) dx ds$$

$$+ C \int_0^t \int_{-\infty}^{\infty} (1 + s)[V_x^2(|\bar{v}_t| + \bar{v}_x^2) + P(\bar{v})_{xt}^2 + \bar{v}_x^2 \bar{v}_{xx}^2] dx ds$$

$$\leq C(\|V_0\|_3^2 + \|V_1\|_2^2 + |v_+ - v_-|). \tag{3.24}$$

Similarly, in terms of (3.12)–(3.14), we can prove the decay rates (3.15) by considering $((3.8))_x \times V_{xt}$, $((3.8)) \times V_{xx}$, $((3.8))_t \times V_{tt}$, $((3.8))_t \times V_t$, $((3.8))_x \times V_{xxx}$, $((3.8))_{xx} \times V_{xxt}$, $((3.8))_{tt} \times V_{ttt}$, respectively, integrating over $(-\infty, +\infty)$, multiplying with $(1 + t)^k$ for some integer $k > 0$ and integrating over $[0, t]$. $\qquad \square$

Applying the local existence result and the continuity argument, we can prove, by Lemma (3.1), that for $\|V_0\|_3^2 + \|V_1\|_2^2 + |v_+ - v_-| \leq \delta_0 \ll 1$, there is the global smooth solution to (3.8) and (3.9), and then to (1.1) and (1.2) with decay rates (1.25).

4. L_p convergence rates. In this section, we will obtain optimal L_p decay rates (1.26) and (1.27) for the smooth solution V, for the case $h(v) = 1$, which gives $F'(v) = \sigma'(v)$, cf. (1.16). By (1.16), one can rewrite (3.8), for $(y, s) \in \mathbb{R} \times (0, t)$, as

$$V_s(y, s) - (a(y, s) V_y(y, s))_y = \frac{1}{f(\bar{v})} V_{ss}(y, s) + r_1(y, s) + r_2(y, s) + r_3(y, s), \tag{4.1}$$

where $a(y, s) = -P'(\bar{v}(y, s)) > A_0 > 0$, and

$$r_1(y, s) = \frac{1}{f(\bar{v})} [\sigma(\bar{v} + V_y) - \sigma(\bar{v}) - \sigma'(\bar{v}) V_y + P(\bar{v})_s]_y(y, s), \tag{4.2}$$

$$r_2(y, s) = - \frac{1}{f(\bar{v})} [f(\bar{v} + V_y) - f(\bar{v})] V_s(y, s) \tag{4.3}$$

$$r_3(y, s) = \frac{1}{f(\bar{v})} [f(\bar{v} + V_y) - f(\bar{v}) - f'(\bar{v}) V_y] P(\bar{v})_y(y, s). \tag{4.4}$$

Denote the approximate Green's function $G(x, t; y, s)$ by (cf. [22])

$$G(x, t; y, s) = \left(\frac{1}{4\pi a(x, t)(t - s)} \right)^{1/2} \exp \left\{ -\frac{(x - y)^2}{4A(t, y, s)(t - s)} \right\},$$

$$(4.5)$$

where $A(t, y, s) = -P'(\Phi(\eta))$, and

$$\eta = \begin{cases} \frac{y}{\sqrt{1+s}}, & s > \frac{t}{2}, \\ \frac{y}{\sqrt{1+t/2}}, & s \leq \frac{t}{2}. \end{cases} \tag{4.6}$$

Multiplying (4.1) with $G(x, t; y, s)$ and integrating with respect to (y, s) by parts over $\mathbb{R} \times (0, t)$, one obtains that the solution to (4.1) can be expressed as

$$V(x, t) = \int_{-\infty}^{\infty} G(x, t; y, 0)V_0(y)dy$$

$$+ \int_0^t \int_{-\infty}^{\infty} G_R(x, t; y, s)V(y, s)dyds$$

$$+ \int_0^t \int_{-\infty}^{\infty} G(x, t; y, s)\frac{1}{f(\bar{v})}V_{ss}(y, s)dyds$$

$$+ \int_0^t \int_{-\infty}^{\infty} G(x, t; y, s)r_1(y, s)dyds$$

$$+ \int_0^t \int_{-\infty}^{\infty} G(x, t; y, s)r_2(y, s)dyds$$

$$+ \int_0^t \int_{-\infty}^{\infty} G(x, t; y, s)r_3(y, s)dyds, \tag{4.7}$$

where

$$G_R(x, t; y, s) = G_s(x, t; y, s) + (a(y, s)G_y(x, t; y, s))_y. \tag{4.8}$$

Then, we have for each $k \leq 3$ that

$$\partial_x^k V(x, t) = \int_{-\infty}^{\infty} \partial_x^k G(x, t; y, 0)V_0(y)dy$$

$$+ \int_0^t \int_{-\infty}^{\infty} \partial_x^k G_R(x, t; y, s)V(y, s)dyds$$

$$+ \int_0^t \int_{-\infty}^{\infty} \partial_x^k G(x, t; y, s)\frac{1}{f(\bar{v})}V_{ss}(y, s)dyds$$

$$+ \int_0^t \int_{-\infty}^{\infty} \partial_x^k G(x, t; y, s)r_1(y, s)dyds$$

$$+ \int_0^t \int_{-\infty}^{\infty} \partial_x^k G(x, t; y, s)r_2(y, s)dyds$$

$$+ \int_0^t \int_{-\infty}^{\infty} \partial_x^k G(x, t; y, s)r_3(y, s)dyds$$

$$=: \sum_{i=1}^{6} J_i^k, \tag{4.9}$$

and for each $k \leq 2$ that

$$
\partial_x^k V_t(x, t) = \int_{-\infty}^{\infty} \partial_t \partial_x^k G(x, t; y, 0) V_0(y) dy
$$

$$
+ \partial_t \int_0^t \int_{-\infty}^{\infty} \partial_x^k G_R(x, t; y, s) V(y, s) dy ds
$$

$$
+ \partial_t \int_0^t \int_{-\infty}^{\infty} \partial_x^k G(x, t; y, s) \frac{1}{f(\bar{v})} V_{ss}(y, s) dy ds
$$

$$
+ \partial_t \int_0^t \int_{-\infty}^{\infty} \partial_x^k G(x, t; y, s) r_1(y, s) dy ds
$$

$$
+ \partial_t \int_0^t \int_{-\infty}^{\infty} \partial_x^k G(x, t; y, s) r_2(y, s) dy ds
$$

$$
+ \partial_t \int_0^t \int_{-\infty}^{\infty} \partial_x^k G(x, t; y, s) r_3(y, s) dy ds
$$

$$
=: \sum_{i=1}^{6} I_i^k. \tag{4.10}
$$

First, we introduce the properties of the approximate Green's function (cf. [22]).

$$
|\partial_t^l \partial_s^h \partial_x^k \partial_y^m G(x, t; y, s)|
$$
$$
= O(1) \sum_{m_1 + m_2 = m} (t - s)^{-(k + m_1)/2} \xi^{m_2}
$$
$$
\cdot (\xi_2^2 + (t - s)^{-1})^l (\xi_1^2 + (t - s)^{-1})^h G_H(x - y, t - s), \tag{4.11}
$$

where $l \leq 1$, $h \leq 1$, and $\xi = \xi_1 + \xi_2$ with

$$
\xi_1 = \begin{cases} (1 + s)^{-1/2}, & s > t/2, \\ 0, & s \leq t/2, \end{cases} \tag{4.12}
$$

$$
\xi_2 = \begin{cases} 0, & s > t/2, \\ (1 + t)^{-1/2}, & s \leq t/2. \end{cases} \tag{4.13}
$$

The $G_H(x, t)$ is a heat kernel which satisfies

$$
G_H(x, t) = \left(\frac{1}{4\pi A_0 t} \right)^{\frac{1}{2}} \exp \left\{ -\frac{x^2}{A_1 t} \right\}. \tag{4.14}
$$

where $A_1 > 4 \max A(t, y, s) + O(1)\delta_0$, and

$$
\|G_H(., t)\|_{L_p} \leq C t^{-\frac{1}{2}(1 - \frac{1}{p})}. \tag{4.15}
$$

One can verify that

$$
\partial_x G = -\partial_y G - \left(\frac{(x - y)^2}{4A^2(t - s)} \partial_y A(t, y, s) + \frac{\partial_x a(x, t)}{2a(x, t)} \right) G, \tag{4.16}
$$

$$
\partial_t G = -\partial_s G - \left(\frac{(x - y)^2}{4A^2(t - s)} (\partial_s A + \partial_t A)(t, y, s) - \frac{\partial_t a(x, t)}{2a(x, t)} \right) G, \tag{4.17}
$$

and

$$|\partial_y A| + |\partial_x a| = O(1)\delta_0 \xi, \quad |\partial_s A| + |\partial_t A| + |\partial_t a| = O(1)\delta_0 \xi^2. \tag{4.18}$$

Moreover, it holds that for $t/2 < s \leq t$

$$G_R(x,t;y,s) \leq C\delta_0((1+s)^{-1} + (t-s)^{-1/2}(1+s)^{-1/2})$$

$$\cdot \exp\left\{-\frac{Cy^2}{1+s}\right\} G_H(x-y,t-s), \quad t/2 < s \leq t, \tag{4.19}$$

$$|\partial_t^l \partial_x^k G_R(x,t;y,s)| \leq C\delta_0(1+s)^{-1/2}(1+t)^{-(l-\frac{k+1}{2})}$$

$$\cdot \exp\left\{-\frac{Cy^2}{1+t}\right\} G_H(x-y,t-s), \quad s < t/2 \tag{4.20}$$

$$\lim_{s\to t/2\pm} |\partial_x^k G_R(x,t;y,s)| \leq C\delta_0(1+t)^{-(l-\frac{k+1}{2})} \exp\left\{-\frac{Cy^2}{1+t/2}\right\} G_H(x-y,t/2). \tag{4.21}$$

Denote

$$E_p^{l,k}(t) = (1+t)^{\frac{1}{2}(1-\frac{1}{p})+l+\frac{k}{2}}, \tag{4.22}$$

and set

$$Q(t) = \sup_{\substack{2<p.0\leq s\leq t, \\ l+k\leq 2, l\leq 1}} E_p^{l,k}(s)\|\partial_t^l \partial_x^k V(.,s)\|_{L_p} + \sup_{\substack{0\leq s\leq t, \\ l+k=3, l\leq 1}} E_p^{l,k}(s)\|\partial_t^l \partial_x^k V(.,s)\|_{L_2},$$

$$Q_1(t) = \delta_0 + |v_+ - v_-|Q(t) + Q(t)^2. \tag{4.23}$$

The above terms J_i^k, I_i^k can be estimated as follows. First of all, we deal with J_6^k and I_6^k. In terms of Lemma 3.1, (2.12) and (4.22), one has

$$\|\partial_s^l \partial_y^k r_3(.,s)\|_{L_p} \leq C(\delta_0 + |v_+ - v_-|Q(t) + Q(t)^2)(1+s)^{-\frac{1}{2}(1-\frac{1}{p})-l-\frac{k+3}{2}-\frac{1}{2}}, \tag{4.24}$$

for $l \leq 1$, $k \leq 2$, and

$$\|\partial_s^2 r_3(.,s)\|_{L_p} \leq C(\delta_0 + |v_+ - v_-|Q(t) + Q(t)^2)(1+s)^{-\frac{1}{2}(1-\frac{1}{p})+\frac{1}{4}-3-\frac{1}{2}}. \tag{4.25}$$

A direct calculation together with (4.11) and (4.22) leads to

$$\|J_6^k\|_{L_p} \leq C \int_0^{t/2} \|\int_{-\infty}^{\infty} \partial_x^k G(.,t;y,s)r_3(y,s)dy\|_{L_p}ds$$

$$+ C \int_{t/2}^t \|\int_{-\infty}^{\infty} \partial_x^k G(.,t;y,s)r_3(y,s)dy\|_{L_p}ds$$

$$\leq CQ_1(t) \int_0^{t/2} (t-s)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{k}{2}}(1+s)^{-2}ds$$

$$+ CQ_1(t) \int_{t/2}^t (t-s)^{-k/2}(1+s)^{-\frac{1}{2}(1-\frac{1}{p})-2}ds$$

$$\leq C(\delta_0 + |v_+ - v_-|Q(t) + Q(t)^2)(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{k}{2}}, \tag{4.26}$$

for $k \leq 1$, and

$$\|I_6^0\|_{L_p} \leq C\|\int_{-\infty}^{\infty} G(.,t;y,t)r_3(y,t)dy\|_{L_p}ds$$

$$+ C \int_0^{t/2} \| \int_{-\infty}^{\infty} \partial_t G(.,t;y,s) r_3(y,s) dy \|_{L_p} ds$$

$$+ C \int_{t/2}^{t} \| \int_{-\infty}^{\infty} \partial_t G(.,t;y,s) r_3(y,s) dy \|_{L_p} ds$$

$$\leq C Q_1(t)(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-2}$$

$$+ C Q_1(t) \int_0^{t/2} [(t-s)^{-1} + (1+t)^{-1}](t-s)^{-\frac{1}{2}(1-\frac{1}{p})}(1+s)^{-2} ds$$

$$+ C Q_1(t) \int_{t/2}^{t} [(t-s)^{-k/2} + (1+s)^{-1/2}](1+s)^{-\frac{1}{2}(1-\frac{1}{p})-2} ds$$

$$\leq C(\delta_0 + |v_+ - v_-|Q(t) + Q(t)^2)(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-1}. \qquad (4.27)$$

Via (4.16)–(4.18) and (2.4), one can get

$$\partial_x^2 G = - \partial_x \partial_y G - \left(\frac{(x-y)^2}{4A^2(t-s)} \partial_y A(t,y,s) + \frac{\partial_x a(x,t)}{2a(x,t)} \right) \partial_x G$$

$$+ O(1)\delta_0 [(t-s)^{-\frac{1}{2}} + (1+s)^{-\frac{1}{2}}](1+s)^{-\frac{1}{2}} G_H, \qquad (4.28)$$

$$\partial_t \partial_x G = - \partial_s \partial_x G - \left(\frac{(x-y)^2}{4A^2(t-s)}(\partial_s A + \partial_t A)(t,y,s) - \frac{\partial_t a(x,t)}{2a(x,t)} \right) \partial_x G$$

$$+ O(1)\delta_0 [(t-s)^{-\frac{1}{2}} + (1+s)^{-\frac{1}{2}}](1+s)^{-1} G_H. \qquad (4.29)$$

Therefore, the combination of integration by parts, Hausdorff-Young inequality, (4.11)–(4.21), and (4.28)–(4.29) yields

$$\|J_6^2\|_{L_p} \leq C \int_0^{t/2} \| \int_{-\infty}^{\infty} \partial_x^2 G(.,t;y,s) r_3(y,s) dy \|_{L_p} ds$$

$$+ C \int_{t/2}^{t} \| \int_{-\infty}^{\infty} \partial_x G(..t;y,s) \partial_y r_3(y,s) dy \|_{L_p} ds$$

$$+ C\delta_0 \int_{t/2}^{t} [(t-s)^{-1/2} + (1+s)^{-1/2}](1+s)^{-\frac{1}{2}}$$

$$\cdot \| \int_{-\infty}^{\infty} G_H(.,t;y,s) r_3(y,s) dy \|_{L_p} ds$$

$$\leq C \int_0^{t/2} (t-s)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{2}{2}}(1+s)^{-2} ds$$

$$+ C Q_1(t) \int_{t/2}^{t} [(t-s)^{-1/2} + (1+s)^{-1/2}](1+s)^{-\frac{1}{2}(1-\frac{1}{p})-2-\frac{1}{2}} ds$$

$$\leq C(\delta_0 + |v_+ - v_-|Q(t) + Q(t)^2)(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{2}{2}}, \qquad (4.30)$$

and

$$\|I_6^1\|_{L_p} \leq C \int_0^{t/2} \| \int_{-\infty}^{\infty} \partial_t \partial_x G(.,t;y,s) r_3(y,s) dy \|_{L_p} ds$$

$$+ C \int_{t/2}^t \| \int_{-\infty}^\infty \partial_x G(.,t;y,s) \partial_s r_3(y,s) dy \|_{L_p} ds$$

$$+ C \delta_0 \int_{t/2}^t [(t-s)^{-1/2} + (1+s)^{-1/2}](1+s)^{-1}$$

$$\cdot \| \int_{-\infty}^\infty G_H(.,t;y,s) r_3(y,s) dy \|_{L_p} ds$$

$$+ C \| \int_{-\infty}^\infty \partial_x G(.,t;y,t/2) r_3(y,t/2) dy \|_{L_p} ds$$

$$\leq C \delta_0 \int_0^{t/2} [(t-s)^{-1} + (1+t)^{-1}](t-s)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{k}{2}}(1+s)^{-2} ds$$

$$+ C \delta_0 \int_{t/2}^t [(t-s)^{-1/2} + (1+s)^{-1/2}](1+s)^{-\frac{1}{2}(1-\frac{1}{p})-3} ds$$

$$\leq C(\delta_0 + |v_+ - v_-|Q(t) + Q(t)^2)(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-1-\frac{1}{2}}. \tag{4.31}$$

Noticing (4.8), (4.16)–(4.17) and (4.28)–(4.29), one can get, after integration by parts, that

$$\int_{t/2}^t \int_{-\infty}^\infty \partial_x^3 G(x,t;y,s) r_3(y,s) dy ds$$

$$= O(1) \int_{t/2}^t \int_{-\infty}^\infty [G_R(x,t;y,s) + (1+s)^{-\frac{1}{2}} \partial_y G(x,t;y,s)] \partial_y r_3(y,s) dy ds$$

$$+ O(1) \int_{t/2}^t \int_{-\infty}^\infty \partial_y G(x,t;y,s) \partial_s r_3(y,s) dy ds$$

$$+ O(1) \int_{-\infty}^\infty G(x,t;y,s) \partial_y r_3(y,s) dy |_{s=t/2}^t$$

$$+ O(1) \delta_0 \sum_{h \leq 1} \int_{t/2}^t \int_{-\infty}^\infty [(t-s)^{-\frac{1}{2}} + (1+s)^{-\frac{1}{2}}]$$

$$\cdot (1+s)^{-(2-h)/2} G_H \partial_y^h r_3(y,s) dy ds$$

$$+ O(1) \delta_0 \int_{t/2}^t \int_{-\infty}^\infty (1+s)^{-\frac{1}{2}} G_H \partial_s r_3(y,s) dy ds, \tag{4.32}$$

$$\int_{t/2}^t \int_{-\infty}^\infty \partial_t \partial_x^2 G(x,t;y,s) r_3(y,s) dy ds$$

$$= O(1) \int_{t/2}^t \int_{-\infty}^\infty G_R(x,t;y,s) \partial_s r_3(y,s) dy ds$$

$$+ O(1) \int_{t/2}^t \int_{-\infty}^\infty G(x,t;y,s) \partial_s^2 r_3(y,s) dy ds$$

$$+ O(1) \int_{-\infty}^\infty G(x,t;y,s) \partial_s r_3(y,s) dy |_{s=t/2}^t$$

$$- \int_{-\infty}^\infty \partial_x^2 G(x,t;y,s) \partial_s r_3(y,s) dy |_{s=t/2}^t$$

$$+ O(1)\delta_0 \int_{t/2}^t \int_{-\infty}^\infty [\partial_y G + (1+s)^{-\frac{1}{2}} G](x,t;y,s)(1+s)^{-\frac{1}{2}}\partial_s r_3(y,s)$$

$$+ O(1)\delta_0 \sum_{h\leq 1} \int_{t/2}^t \int_{-\infty}^\infty [(t-s)^{-\frac{1}{2}} + (1+s)^{-\frac{1}{2}}]$$

$$\cdot (1+s)^{-\frac{3}{2}+\frac{h}{2}} G_H \partial_y^h r_3(y,s)dyds. \tag{4.33}$$

Thus, it follows from (4.32)–(4.33) that

$$\|J_6^3\|_{L_p} \leq C \int_0^{t/2} \| \int_{-\infty}^\infty \partial_x^3 G(.,t;y,s)r_3(y,s)dy\|_{L_p}ds$$

$$+ C \int_{t/2}^t \| \int_{-\infty}^\infty G_R(.,t;y,s)\partial_y r_3(y,s)dy\|_{L_p}ds$$

$$+ C \int_{t/2}^t \| \int_{-\infty}^\infty \partial_y G(.,t;y,s)\partial_s r_3(y,s)dy\|_{L_p}ds$$

$$+ C \int_{t/2}^t (1+s)^{-1/2}\| \int_{-\infty}^\infty \partial_y G(.,t;y,s)\partial_y r_3(y,s)dy\|_{L_p}ds$$

$$+ C\| \int_{-\infty}^\infty G(.,t;y,s)\partial_y r_3(y,s)dy|_{s=t/2}^t\|_{L_p}$$

$$+ C\delta_0 \sum_{h\leq 1} \int_{t/2}^t [(t-s)^{-1/2} + (1+s)^{-1/2}](1+s)^{-1+h/2}$$

$$\cdot \| \int_{-\infty}^\infty G_H(.,t;y,s)\partial_y^h r_3(y,s)dy\|_{L_p}ds$$

$$+ C\delta_0 \int_{t/2}^t (1+s)^{-1/2}\| \int_{-\infty}^\infty G_H(.,t;y,s)\partial_s r_3(y,s)dy\|_{L_p}ds$$

$$\leq CQ_1(t) \int_0^{t/2} (t-s)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{3}{2}}(1+s)^{-2}ds$$

$$+ CQ_1(t) \int_{t/2}^t [(t-s)^{-1/2} + (1+s)^{-1/2}](1+s)^{-\frac{1}{2}(1-\frac{1}{p})-2}ds$$

$$\leq C(\delta_0 + |v_+ - v_-|Q(t) + Q(t)^2)(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{3}{2}}, \tag{4.34}$$

$$\|I_6^2\|_{L_p} \leq C \int_0^{t/2} \| \int_{-\infty}^\infty \partial_t \partial_x^2 G(.,t;y,s)r_3(y,s)dy\|_{L_p}ds$$

$$+ C\| \int_{t/2}^t \int_{-\infty}^\infty G_R(.,t;y,s)\partial_s r_3(y,s)dy\|_{L_p}ds$$

$$+ C \int_{t/2}^t \| \int_{-\infty}^\infty \partial_x G(.,t;y,s)\partial_s r_3(y,s)dy\|_{L_p}ds$$

$$+ C\| \int_{-\infty}^\infty G(.,t;y,s)\partial_s r_3(y,s)dy|_{s=t/2}^t\|_{L_p}ds$$

$$+ C\| \int_{-\infty}^\infty \partial_x^2 G(.,t;y,t/2)r_3(y,t/2)dy\|_{L_p}ds$$

$$+ C \int_{t/2}^{t} \| \int_{-\infty}^{\infty} G(.,t;y,s) \partial_s^2 r_3(y,s) dy \|_{L_p} ds$$

$$+ C\delta_0 \int_{t/2}^{t} (1+s)^{-1/2} \| \int_{-\infty}^{\infty} \partial_y G(.,t;y,s) \partial_s r_3(y,s) dy \|_{L_p} ds$$

$$+ C\delta_0 \int_{t/2}^{t} (1+s)^{-1} \| \int_{-\infty}^{\infty} G(.,t;y,s) \partial_s r_3(y,s) dy \|_{L_p} ds$$

$$+ C\delta_0 \sum_{h \leq 1} \int_{t/2}^{t} [(t-s)^{-1/2} + (1+s)^{-1/2}](1+s)^{-3+h/2}$$

$$\cdot \| \int_{-\infty}^{\infty} G_H(.,t;y,s) \partial_y^h r_3(y,s) dy \|_{L_p} ds$$

$$\leq C Q_1(t) \int_0^{t/2} [(t-s)^{-1} + (1+t)^{-1}](t-s)^{-\frac{1}{2}(1-\frac{1}{p})-1}(1+s)^{-2} ds$$

$$+ C Q_1(t) \int_{t/2}^{t} [(t-s)^{-1/2} + (1+s)^{-1/2}](1+s)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{5}{2}} ds$$

$$+ C Q_1(t)(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-2}$$

$$\leq C(\delta_0 + |v_+ - v_-|Q(t) + Q(t)^2)(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-2}. \tag{4.35}$$

Next, by repeating the same argument above, and in terms of Lemma 3.1, (4.11)–(4.21), and (2.4)–(2.12), one can estimate J_5^k, I_5^k, and, similarly to [22], the other terms as

$$\|J_i^k\|_{L_p} \leq O(1)(\delta_0 + |v_+ - v_-|Q(t) + Q(t)^2)(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{k}{2}}, \tag{4.36}$$

$$\|I_i^k\|_{L_p} \leq O(1)(\delta_0 + |v_+ - v_-|Q(t) + Q(t)^2)(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-1-\frac{k}{2}}, \tag{4.37}$$

for $i = 1,2,3,4,5$.

Combining the above estimates, we have finally

$$Q(t) \leq C(\delta_0 + |v_+ - v_-|Q(t) + Q(t)^2), \tag{4.38}$$

which together with the assumption of $\delta_0 \ll 1$ and a continuity argument leads to

LEMMA 4.1. Under the assumptions of Theorem 1.1, it holds that

$$\|\partial_x^k V_x(.,t)\|_{L_p} \leq C\delta_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{k+1}{2}}, \tag{4.39}$$

$$\|\partial_x^k V_t(.,t)\|_{L_p} \leq C\delta_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{k+2}{2}}, \tag{4.40}$$

for any $k \leq 2$ if $p = 2$ and $k \leq 1$ if $p \in (2,\infty]$.

Then, the optimal decay rates (1.26)–(1.27) follow from Lemma 3.1. □

5. A remark on the function $f(v)$. A simple example of a function f which satisfies conditions (1.17)–(1.19) can be given by

$$f(v) = -f_0 v^{-c}, \quad 0 < c < \frac{1}{2}, \quad f_0 > 0.$$

This function, going back to the notation of Sec. 1 where $\varepsilon(\vartheta) = v > 0$, determines $g_2(\vartheta)$ (recall that $f = g_2 \circ \varepsilon^{-1}$). Crucial here is then the choice of the internal energy ε. If we use the widely accepted Debye's law for the heat capacity c_v at very low temperatures, $c_v(\vartheta) = \varepsilon'(\vartheta) = c_0 \vartheta^3$, then $\varepsilon(\vartheta) = \frac{c_0}{4} \vartheta^4$. Taking this into account, we obtain the following function for g_2,

$$g_2(\vartheta) = -g_0 \vartheta^{-d}, \quad 0 < d < 2,$$

where $f_0 = g_0 (\frac{4}{c_0})^c$.

The function g_2, together with g_1, can be chosen to give agreement with experimental data obtained for heat conductivity $\mathcal{K}$, and thermal wave speed U_E through the relations (cf. [24])

$$\mathcal{K}(\vartheta) = -\psi_{20} \frac{\vartheta^2 g_1^2(\vartheta)}{g_2(\vartheta)}, \qquad U_E{}^2(\vartheta) = \psi_{20} \frac{\vartheta^2 g_1^2(\vartheta)}{c_v(\vartheta)}.$$

The second relation allows us to choose the function g_1 (hence σ) based on data for U_E. It is worth noticing that this would not be possible if the system (1.1) and (1.2) was a p-system. Using an empirical relation, $U_E(\vartheta) = (A + B\vartheta^n)^{-\frac{1}{2}}$ (A, B and n are constitutive constants, $1K \le \vartheta \le 20K$, cf. [1]), with $g_2(\vartheta) = -g_0 \vartheta^{-d}$ leads to the heat conductivity being in qualitative agreement with the data for the ranges of temperature where U_E and $\mathcal{K}$ have been measured.

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