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ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO THE STEFAN PROBLEM WITH A KINETIC CONDITION AT THE FREE BOUNDARY

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Abstract

We study the large time behaviour of the free boundary for a one-phase Stefan problem with supercooling and a kinetic condition $u = -\varepsilon|\dot{s}|$ at the free boundary $x = s(t)$. The problem is posed on the semi-infinite strip $[0, \infty)$ with unit Stefan number and bounded initial temperature $\varphi(x) \leq 0$, such that $\varphi \to -1 - \delta$ as $x \to \infty$, where δ is constant. Special solutions and the asymptotic behaviour of the free boundary are considered for the cases $\varepsilon \geq 0$ with δ negative, positive and zero, respectively. We show that, for $\varepsilon > 0$, the free boundary is asymptotic to $k\sqrt{t}$, $\delta t/\varepsilon$ if $\delta < 0$, $\delta > 0$ respectively, and that when $\delta = 0$ the large time behaviour of the free boundary depends more sensitively on the initial temperature. We also give a brief summary of the corresponding results for a radially symmetric spherical crystal with kinetic undercooling and Gibbs-Thomson conditions at the free boundary.

1. Introduction

We study the qualitative behaviour and special solutions of the Stefan problem with a kinetic condition at the free boundary [5], [6]. Several authors have considered this problem and the existence, uniqueness and regularity of the solution have been obtained (e.g. [14], [15]).

We consider here the one-phase Stefan problem on a semi-infinite strip $[0, \infty)$, with a kinetic condition at the free boundary, unit Stefan number and bounded initial temperature $\varphi(x) \leq 0$, so that the liquid is supercooled.

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That is, we study the following dimensionless problem which corresponds to the limit of a two phase Stefan problem when the thermal diffusivity in the solid is vanishingly small:

$$u_t - u_{xx} = 0, \qquad s(t) < x < \infty,\ t > 0 \tag{1.1}$$

$$u(x,0) = \varphi(x) \leq 0, \qquad 0 \leq x < \infty \tag{1.2}$$

$$u(s(t),t) = -\varepsilon\dot{s}(t), \qquad t > 0 \tag{1.3}$$

$$u_x(s(t),t) = -\dot{s}(t), \qquad t > 0 \tag{1.4}$$

where $\varepsilon \geq 0$ is a constant, and where $\varphi(x) \in C^1[0,\infty)$ is a given function, which is bounded together with its first derivative. The negativity of φ ensures that $\dot{s} > 0$.

The problem (1.1)–(1.4) reduces to the standard supercooled Stefan problem when $\varepsilon = 0$. It is known that when $\varepsilon = 0$ the solution of (1.1)–(1.4) can blow up with $\dot{s} \to \infty$ in finite time for certain initial data $\varphi(x)$, in particular when (but not only when) $\varphi(\infty) < -1$ [7, 9, 11], and also that the kinetic term with $\varepsilon > 0$ prevents blow-up for any initial data $\varphi(x)$, at least for the analogous problem posed on a finite spatial domain (see [15]).

In this paper we give some special solutions and discuss the asymptotic behaviour corresponding to initial data $\varphi(x)$ with

$$\varphi(x) \to -1 - \delta \quad \text{as } x \to \infty, \tag{1.5}$$

where δ is a constant. The plan of the paper is as follows. In Section 2 we review known results on similarity solutions and asymptotic behaviour when $\varepsilon = 0$. In Section 3 we present analogous exact solutions for $\varepsilon > 0$ and in Section 4, using an integral equation derived from the Laplace transform of (1.1)–(1.4), we analyse the asymptotic behaviour of $s(t)$ as $t \to \infty$ in the case that the initial data is not compatible with one of the similarity solutions previously noted. In Section 5 we summarise the results of Sections 2–4. Lastly, in Section 6 we give a brief review of the corresponding results for a spherical crystal growing in three dimensions, with surface energy effects incorporated via a Gibbs-Thomson condition on the free boundary. These are qualitatively the same as the one-dimensional results of Sections 2–5. Our analysis here complements the numerical work of Schaefer and Glicksman

[13]; they pointed out that values of δ as high as 0.8 are obtainable using certain materials.

2. Known results and exact solutions when $\varepsilon = 0$

We first review some results for the standard supercooled Stefan problem. Firstly, as a consequence of the finite time blow-up for the initial-value problem with initial data $\varphi(x)$ having $\varphi(x) < -1$ (see [7, 9, 11]), we know that there is no solution of (1.1)–(1.4) for large time if δ is a positive constant. Secondly, when $\delta < 0$, there is a similarity solution of the form[2]

$$u(x,t) = f(x/\sqrt{t}), \qquad s(t) = \beta\sqrt{t} \tag{2.1}$$

where

$$f(\xi) = \frac{\beta}{2}\exp(\beta^2/4)\int_\xi^\beta \exp(-y^2/4)\,dy \tag{2.2}$$

and β is to be determined from

$$\beta\exp(\beta^2/4)\int_{\beta/2}^\infty \exp(-y^2)\,dy = 1+\delta, \tag{2.3}$$

which has real positive solutions only if $-1 < \delta < 0$ [1]. The initial data for this solution is the step function $\varphi(x) = -1-\delta$, $x \geq 0$.

The asymptotic behaviour

$$s(t) \sim \beta\sqrt{t}, \quad \text{as } t \to \infty \tag{2.4}$$

was obtained by [3] for any initial data $\varphi(x)$ with $\varphi'' > 0$ and $\varphi(\infty) > -1/4$. It is a reasonable conjecture that (2.4) is true for any $\varphi(x)$ with $\varphi(\infty) > -1$ and for which finite-time blow-up does not occur; we shall support this conjecture with asymptotic results in Section 4.

Lastly when $\delta = 0$, we can find a travelling wave solution in the form

$$u(x,t) = \exp(-V(x-Vt)) - 1, \qquad s(t) = Vt \tag{2.5}$$

where V is any positive constant. We remark here that there is no similarity solution of the form (2.1) for $\delta = 0$ and no travelling wave solution of the

[2] In using β here, we are following the notation of Lamé & Clapeyron (1831) [12] who first considered the one-phase Stefan (*sic*) problem.

form (2.5) if $\delta < 0$. We also note that V in (2.5) is arbitrary, whereas β in (2.1) is determined by $\varphi(\infty)$.

3. Exact solutions for $\varepsilon > 0$

We begin our analysis of the case $\varepsilon > 0$ by noting two exact solutions analogous to the solutions given in Section 2.

(a) $\varepsilon > 0$, $\delta < 0$: similarity solution with $s(t) = \beta\sqrt{t}$

We begin with the case $\delta < 0$. We know that there are similarity solutions of the form (2.1) if $\varepsilon = 0$. When $\varepsilon > 0$, we incorporate the kinetic condition (1.3) by seeking similarity solutions of the form

$$u(x,t) = f(\xi) + g(\xi)/\sqrt{t}, \qquad \xi = x/\sqrt{t}, \tag{3.1}$$

$$s(t) = \beta\sqrt{t}, \qquad \beta > 0. \tag{3.2}$$

We find that $f(\xi)$ and $g(\xi)$ satisfy the ordinary differential equations $f'' + (\xi/2)f' = 0$, $g'' + (1/2)(\xi g)' = 0$ where primes denote differentiation with respect to ξ. Further, from the kinetic and Stefan conditions, (1.3)–(1.4) we find that

$$f(\beta) = 0, \qquad f'(\beta) = -\beta/2, \qquad g(\beta) = -\varepsilon\beta/2, \qquad g'(\beta) = 0,$$

and so

$$f(\xi) = \frac{\beta}{2} e^{\beta^2/4} \int_\xi^\beta e^{-y^2/4}\, dy,$$

$$g(\xi) = \frac{\beta}{2}\left\{\frac{\beta}{2} e^{-\xi^2/4} \int_\xi^\beta e^{y^2/4}\, dy - e^{(\beta^2-\xi^2)/4}\right\}.$$

Since $g(\xi) \to 0$ as $\xi \to \infty$, β is determined by (2.3) which, as already noted, has real positive solutions only when $-1 < \delta < 0$.

We observe that β is independent of ε. This surprising result is reminiscent of the fact that, without kinetic undercooling, the corresponding similarity solution for the growth of a spherical solid region expanding into supercooled liquid has a rate of growth unaffected by the inclusion of a Gibbs-Thomson condition at the free boundary [10] (see also Section 6). It is to be contrasted with the results of the next part of this section, where we find a travelling wave solution whose speed does depend on ε.

Finally, we note that $u(x, 0+) \sim O(1/x)$ as $x \to 0$, but that finite initial data can be obtained by shifting the origin of t.

(b) $\varepsilon > 0$, $\delta > 0$: travelling-wave solutions

When $\delta > 0$, a travelling-wave solution analogous to (2.5) can be found. We seek a solution

$$u(x,t) = f(z), \qquad z = x - Vt \tag{3.3}$$

$$s(t) = Vt, \qquad V > 0. \tag{3.4}$$

By direct calculation we establish that

$$u(x,t) = \exp[-V(x - Vt)] - (1 + \delta) \tag{3.5}$$

$$s(t) = Vt = \delta t/\varepsilon. \tag{3.6}$$

Here the wave speed V is uniquely determined.

We now investigate how the possible travelling-wave solutions of (1.1)–(1.4) behave as the parameters ε and δ approach zero. Suppose first that $\varepsilon = o(\delta)$ as $\varepsilon \to 0$, $\delta \to 0$. In this case $\varphi(x)$ tends to a step function and the velocity V becomes infinite as δ and $\varepsilon \to 0+$, suggesting that there is no solution for the problem with $\varepsilon = 0$, $\delta = 0$ when the initial data is a step function. In the case where $\delta = o(\varepsilon)$, we observe that $V \to 0+$ as ε, $\delta \downarrow 0$, the corresponding initial function goes to zero and we retrieve the trivial solution, although the limit is not uniform as $x \to \infty$.

Finally, in the case $\delta = O(\varepsilon)$, we notice that $V = \delta/\varepsilon$ is bounded as ε, δ go to zero. This gives the solution (2.5), and underlines the fact that V is indeterminate in the limit δ, $\varepsilon \downarrow 0$. Note that there is no bounded travelling wave solution of the form (3.5), (3.6) if δ is negative.

4. Asymptotic behaviour of $s(t)$ as $t \to \infty$

In this section, we discuss the large-time behaviour of the free boundary $s(t)$ by considering an integral equation formulation of problem (1.1)–(1.4). We begin with the assumption that there is indeed a unique classical solution for all $t > 0$. This is the case for $\varepsilon > 0$, provided the initial data $\varphi(x)$ satisfies some mild conditions, for example $\varphi \in C^1[0,\infty)$ and φ, φ' are bounded (see [14], [15]).

We now investigate the large-time behaviour of solutions with arbitrary bounded smooth initial data satisfying (1.5).

The first step is to reduce (1.1)–(1.4) to an integral equation by applying a Laplace transform in x [8]. We define the transform $\hat{u}(p,t)$ of $u(x,t)$ by

$$\hat{u}(p,t) = \int_{s(t)}^{\infty} e^{-px} u(x,t)\, dx; \tag{4.1}$$

by a direct calculation using (1.1)–(1.4) we find that

$$\partial\hat{u}/\partial t - p^2\hat{u} = [1 + \varepsilon p + \varepsilon\dot{s}]\dot{s}e^{-ps}, \qquad \hat{u}(p,0) = \hat{\varphi}(p) \tag{4.2}$$

where $\hat{\varphi}(p) = \int_0^\infty e^{-px}\varphi(x)\,dx$. Thus we have, from (4.2),

$$\hat{u}(p,t) = e^{p^2t}[\hat{\varphi}(p) + \int_0^t \dot{s}(\tau)(1 + \varepsilon p + \varepsilon\dot{s})e^{-ps(\tau)-p^2\tau}\,d\tau]. \tag{4.3}$$

Since $u(x,t)$ exists and is bounded for all t, it follows that $\hat{u}(p,t)$ exists and is bounded for all t and $\operatorname{Re} p > 0$. Thus taking $|\arg p| < \pi/4$ and letting $t \to \infty$, the quantity in square brackets in (4.3) must vanish identically, yielding

$$\hat{\varphi}(p) = -\int_0^\infty \dot{s}(\tau)[1 + \varepsilon p + \varepsilon\dot{s}(\tau)]e^{-ps(\tau)-p^2\tau}\,d\tau. \tag{4.4}$$

Integrating by parts, we obtain another more convenient form of (4.4):

$$\begin{aligned}\hat{\varphi}(p) = &-\frac{1}{p} - \varepsilon + p(1+\varepsilon p)\int_0^\infty e^{-ps-p^2t}\,dt \\ &- \varepsilon\int_0^\infty \dot{s}^2e^{-ps-p^2t}\,dt.\end{aligned} \tag{4.5}$$

The behaviour of $s(t)$ depends on the balance between the terms on the right-hand side of (4.5).

In order to obtain the asymptotic behaviour of $s(t)$ as $t \to \infty$, we must investigate (4.5) as $p \to 0$, in particular the behaviour of the integrals $\int_0^\infty e^{-ps-p^2t}\,dt$ and $\int_0^\infty \dot{s}^2e^{-ps-p^2t}\,dt$. We first note that, by a direct calculation, if we take $s(t) = \beta\sqrt{t}$ with β a positive constant, then

$$p\int_0^\infty e^{-ps-p^2t}\,dt = \frac{1}{p}\left[1 - \beta e^{\beta^2/4}\int_{\beta/2}^\infty e^{-x^2}\,dx\right],$$

$$\int_\eta^\infty \dot{s}^2e^{-ps-p^2t}\,dt \sim \log p\sqrt{\eta}, \quad \text{as } p \to 0,$$

and, if we take $s(t) = Vt$ with V a positive constant, then

$$p\int_0^\infty e^{-ps-p^2t}\,dt = \frac{1}{p+V} \tag{4.6}$$

$$\int_0^\infty \dot{s}^2e^{-ps-p^2t}\,dt = \frac{V^2}{(p+V)p}. \tag{4.7}$$

Comparing these forms for $s(t)$, when the initial data is such that $\hat{\varphi} = -(1+\delta)/p + \hat{\varphi}_1$, where $\hat{\varphi}_1 = o(1/p)$ as $p \to 0$ (for example, if $\varphi_1(x)$ is bounded and vanishes at ∞, or if $\varphi_1(x) \sim \sin\omega x$), we conclude that the asymptotic behaviour of $s(t)$ is $\beta\sqrt{t}$ if $\delta < 0$, $\varepsilon \geq 0$, and Vt if $\delta > 0$, $\varepsilon > 0$; β and V are determined as in Sections 2 and 3 respectively. This analysis further suggests that in the marginal case $\delta = 0$, $\varepsilon > 0$, the free boundary is in general asymptotic neither to Vt nor to $\beta\sqrt{t}$ as $t \to \infty$. We therefore investigate in more detail the remaining case $\delta = 0$, $\varepsilon \geq 0$, where the asymptotic form of $s(t)$ depends more sensitively on $\hat{\varphi}(p)$.

(1a) We start with $\delta = \varepsilon = 0$ and choose initial data $\varphi(x)$ with the form $\varphi(x) = -1 + \varphi_1(x)$ where $\hat{\varphi}_1(p)$ is finite (that is, $\varphi_1(x)$ is integrable over $[0, \infty)$) and nonzero at $p = 0$. Then we find the asymptotic behaviour of the free boundary to be $s(t) \sim Vt$ as $t \to \infty$, with $V = 1/\hat{\varphi}_1(0)$. We see from (4.5)–(4.7) that no similar result is valid if $\varepsilon > 0$.

(1b) We next discuss the case (still with $\delta = \varepsilon = 0$) where the initial data satisfies $\varphi(x) \sim -1 + cx^{-(1+\gamma)} + o(x^{-(1+\gamma)})$ as $x \to \infty$, so that

$$\hat{\varphi}(p) \sim -p^{-1} + ap^{\gamma} + o(p^{\gamma}) \quad \text{as } p \to 0 \tag{4.8}$$

where $-1 < \gamma < 0$ and $a = c\Gamma(-\gamma)$. If, in the first integral of (4.5), we take $s(t) \sim kt^{\alpha}$, $\frac{1}{2} < \alpha < 1$, then, by rescaling time so that $s(t) \sim pt$ (i.e. putting $t = (k/p)^{1/(1-\alpha)}\tau$) and applying Laplace's method [4] to estimate the behaviour of the integral as $p \to 0$, we find that

$$p\int_0^{\infty} e^{-ps-p^2t}\,dt \sim \frac{1}{\alpha}\Gamma\left(\frac{1}{\alpha}\right)k^{-1/\alpha}p^{(1-1/\alpha)} + o(p^{(1-1/\alpha)}) \tag{4.9}$$

as $p \to 0$. This gives an estimate of the order of singularity (with respect to p) of this integral as $p \to 0$. The relation between the order γ of this singularity and α is depicted in Figure 1.

Comparing (4.9) to (4.8), we see from (4.5) with $\varepsilon = 0$ that $s(t) \sim kt^{\alpha}$ where

$$\alpha = \frac{1}{1-\gamma}, \qquad k = \left(\frac{\Gamma(1/\alpha)}{a\alpha}\right)^{\alpha}. \tag{4.10}$$

The inequality $-1 < \gamma < 0$ implies that $\frac{1}{2} < \alpha < 1$. Note that in general k will be real and positive only if $a > 0$; that is, there will be a solution only if $\varphi(x) \geq -1$ in the far field. Indeed, it is likely that finite-time blow-up will occur if $a < 0$.

This method can clearly be extended to more complicated behaviour of $\hat{\varphi}(p)$.

Note that if we take $\varphi(x) = -1$, $0 \leq x < \infty$ (i.e. the initial data is a unit step function) then $\hat{\varphi}(p) = -\frac{1}{p}$. Then in (4.5) with $\varepsilon = 0$, all the terms cancel except for $p\int_0^{\infty} e^{-ps(t)-p^2t}\,dt$, which is strictly positive. Thus the supercooled Stefan problem with unit step function initial data $\varphi(x)$ has no solution that has a Laplace transform (4.1).

(1c) If we consider initial data of the form $\varphi(x) = -1 + \varphi_1(x)$ where $\hat{\varphi}_1(p)$ vanishes at $p = 0$, then from parts (1a, b) of this section, it is apparent that

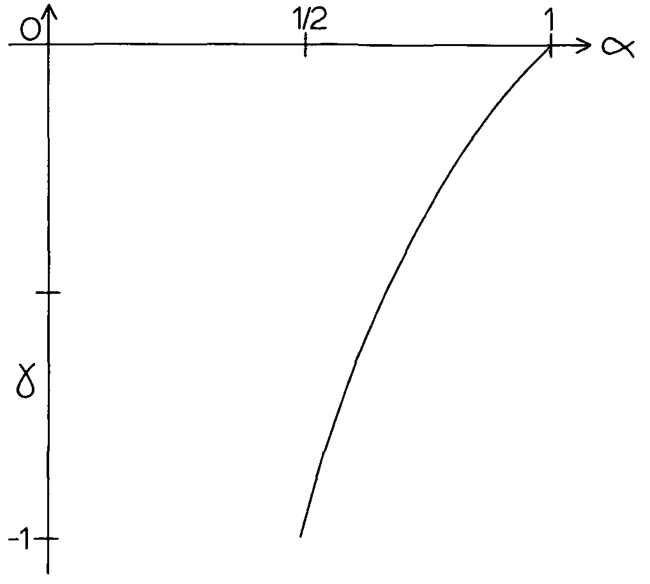

FIGURE 1. Relation between γ and α for $\delta = 0$, $\varepsilon = 0$.

it is impossible to have $s(t) \sim kt^\alpha$ for any $\frac{1}{2} \le \alpha \le 1$. We thus look for solutions which have $s(t) \sim kt^\alpha$ for $\alpha > 1$, as $t \to \infty$. The estimate (4.9) remains valid for $\alpha > 1$ (although a different scaling $\tau = p^2 t$ is necessary to obtain it). Thus if we assume that $\hat{\varphi}_1(p) \sim ap^\gamma$ as $p \to 0$ for $\gamma > 0$, we recover (4.10). Clearly, however, this is valid only for $0 < \gamma < 1$.

The condition that $\hat{\varphi}_1(0) = 0$ is simply the condition that $\int_0^\infty \varphi_1(x)\,dx = 0$, and since the assumption that $\hat{\varphi}_1(p) \sim ap^\gamma$ excludes the possibility that $\varphi_1 \equiv 0$, this implies that $\varphi_1(x)$ must change sign. In particular, it implies that there must be regions where $\varphi(x) < -1$.

By analogy with the finite-time blow-up case (where $\dot{s}(t)$ becomes infinite in a finite time) we can regard these cases as infinite-time blow up (since $\hat{s}(t)$ is unbounded at $t \to \infty$). Evidently such infinite time blow up cannot occur if $\varepsilon > 0$, for the maximum principle implies that $|\dot{s}| \le \sup|\varphi(x)|/\varepsilon$ (see [15]).

(1d) Now we consider the case $\varepsilon > 0$ and $\hat{\varphi}(p)$ of the form (4.8). Suppose we take $s(t) = kt^\alpha$, $\alpha \in (\frac{1}{2}, 1)$; then we can obtain an estimate for the second integral in (4.5) in the same manner as for the first integral in (4.5), namely

$$\varepsilon \int_0^\infty \dot{s}^2 e^{-ps-p^2 t}\,dt \sim \varepsilon\alpha k^{1/\alpha}\Gamma\left(2 - \frac{1}{\alpha}\right)p^{(1/\alpha-2)} + o(p^{(1/\alpha-2)}) \text{ as } p \to 0. \tag{4.11}$$

The orders of magnitude of these two integrals as $p \to 0$ (as determined by (4.9) and (4.11)) are shown as functions of α in Figure 2. According to

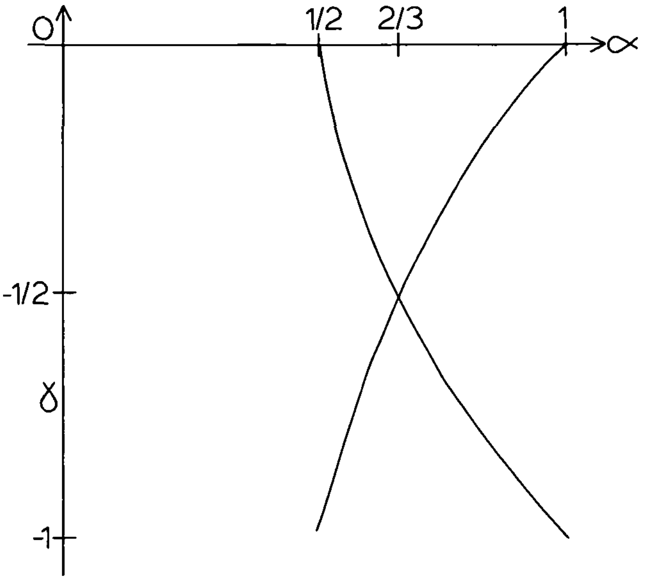

FIGURE 2. Relations between γ and α for $\delta = 0$, $\varepsilon > 0$.

Figure 2 we see, from (4.5), (4.8)–(4.11), that the following cases must be considered:

(i) $-1 < \gamma < -\frac{1}{2}$: There are two possible choices for γ (see Figure 2). The choice we make depends on the sign of a in (4.8). If $a > 0$, the first integral in (4.5) must balance ap^{γ}, so we choose

$$\gamma = 1 - \frac{1}{\alpha}, \qquad k = \left(\frac{\Gamma(1/\alpha)}{\alpha a}\right)^{\alpha}.$$

If on the other hand $a < 0$, the second integral balances ap^{γ} and thus

$$\gamma = \frac{1}{\alpha} - 2, \qquad k = \left(\frac{-a}{\varepsilon\alpha\Gamma(2 - 1/\alpha)}\right)^{\alpha}.$$

(ii) $\gamma = -\frac{1}{2}$: Now $\alpha = \frac{2}{3}$ and k is determined as the unique positive root of

$$\frac{2\varepsilon}{3}k^2 + \frac{a}{\sqrt{\pi}}k^{3/2} - \frac{3}{4} = 0;$$

that is

$$k = \frac{3a}{4\varepsilon\sqrt{\pi}}\left(\sqrt{\left(1 + \frac{2\pi\varepsilon}{a^2}\right)} - 1\right).$$

Note that as $\varepsilon \to 0$, $k \to 3\sqrt{\pi}/(4a)$.

(iii) $-\frac{1}{2} < \gamma < 0$: No matter which integral we choose in (4.5) to balance ap^{γ}, the other integral will be more singular than p^{γ} (see Figure 2). The only way to produce a term to balance ap^{γ} is to choose $\alpha = \frac{2}{3}$ and $k = \frac{1}{2}(\frac{9}{\varepsilon})^{1/3}$ (thereby causing the terms of $O(p^{-1/2})$ to cancel) and to then consider higher order terms in the expansion of $s(t)$ as $t \to \infty$.

We must therefore look at asymptotic behaviours of $s(t)$ of the form

$$s(t) \sim kt^{2/3} + k_1 t^{\alpha_1} + o(t^{\alpha_1}) \quad \text{as } t \to \infty$$

where

$$k = \frac{1}{2}\left(\frac{9}{\varepsilon}\right)^{1/3} \quad \text{and} \quad 0 < \alpha_1 < \frac{2}{3}.$$

The parameters k_1 and α_1 are to be found in terms of a and γ.

To investigate the behaviour of the integrals in (4.5), we first set $t = k^3\tau/p^3$, where $k = \frac{1}{2}(\frac{9}{\varepsilon})^{1/3}$; the first integral, for example, becomes

$$p\int_0^\infty \exp(-pkt^{2/3} - p^2 t - pk_1 t^{\alpha_1})\,dt$$
$$= \frac{k^3}{p^2}\int_0^\infty e^{-\eta(\tau^{2/3}+\tau)} \exp(-k_1 k^{3\alpha_1} p^{1-3\alpha_1}\tau^{\alpha_1})\,d\tau$$

where $\eta = k^3 p^{-1}$. Provided $0 < \alpha_1 < \frac{2}{3}$, (which is just the condition that $t^{\alpha_1} = o(t^{2/3})$ as $t \to \infty$), the term $e^{-\eta\tau^{2/3}}$ controls the asymptotic behaviour as $p \to 0$. In this case a straightforward application of Laplace's method [4] gives the estimate

$$p\int_0^\infty \exp(-pkt^{2/3} - p^2 t - pk_1 t^{\alpha_1})\,dt$$
$$\sim \frac{3\sqrt{\pi}}{2k^{3/2}}p^{-1/2} - \frac{3k_1}{2k^{(3+3\alpha_1)/2}}\Gamma\left(\frac{3}{2}(1+\alpha_1)\right)p^{(1/2-3\alpha_1/2)}\frac{-6}{k^2} + o(1).$$

A similar calculation can be made for the second integral in (4.5); in this case, however, some care must be taken in dealing with the lower limit of integration, as the integrand will not be integrable at $t = 0$ if $\alpha_1 \in (0, 1/3]$. As we are concerned with the asymptotic behaviour of $s(t)$ as $t \to \infty$, however, the lower limit can be replaced by any finite nonzero constant if necessary.

After a lengthy calculation, we find the following estimate:

$$p\int_0^\infty e^{-ps-p^2 t}\,dt - \varepsilon\int_0^\infty \dot{s}^2 e^{-ps-p^2 t}\,dt$$
$$\sim -\varepsilon k_1 k^{-3(\alpha_1-1)/2}\Gamma((3\alpha_1-1)/2)\alpha_1(3\alpha_1+1)p^{-(3\alpha_1-1)/2}$$
$$+ 2\varepsilon\alpha_1 k_1^2 k^{(6\alpha_1-3)/2}H(p) - 3\varepsilon\alpha_1^2 k_1^2 k^{-(6\alpha_1-3)/2}F(p)/2$$
$$+ o(p^{-(6\alpha_1-3)/2}) \quad \text{as } p \to 0$$

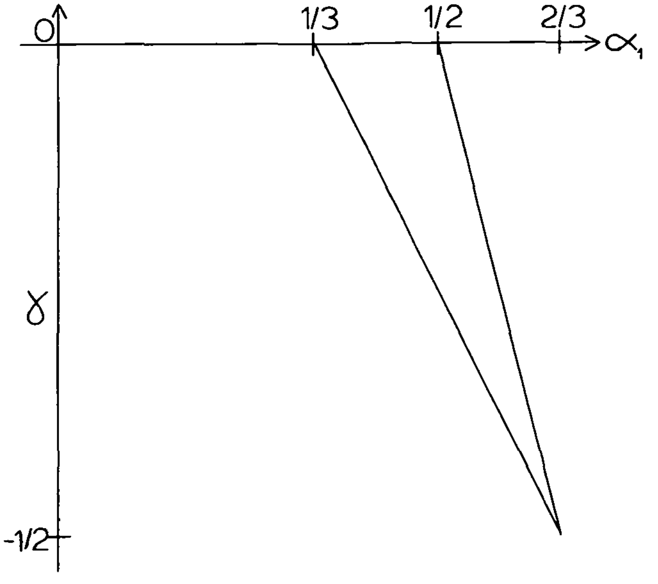

FIGURE 3. Relation between γ and α_1 for $\delta = 0$, $\varepsilon > 0$.

where

$$H(p) = \begin{cases} \Gamma\left(\frac{6\alpha_1 - 1}{2}\right) p^{(6\alpha_1-3)/2}, & \frac{1}{6} < \alpha_1 < \frac{2}{3} \\ O(p), & 0 < \alpha_1 \le \frac{1}{6} \end{cases}$$

$$F(p) = \begin{cases} \Gamma\left(\frac{6\alpha_1 - 3}{2}\right) p^{-(6\alpha_1-3)/2}, & \frac{1}{2} < \alpha_1 < \frac{2}{3} \\ O(\log p), & 0 < \alpha_1 \le \frac{1}{2}. \end{cases}$$

As previously, this allows us to choose α_1 in terms of γ (Figure 3). We therefore have $s(t) \sim kt^{2/3} + k_1 t^{\alpha_1}$ as $t \to \infty$ if $-\frac{1}{2} < \gamma < 0$, where $k = \frac{1}{2}(\frac{9}{\varepsilon})^{1/3}$, $\alpha_1 = (1 - 2\gamma)/3$, and where $\frac{1}{3} < \alpha_1 < \frac{2}{3}$ and k_1 is determined by $-\varepsilon k_1 k^{-3(\alpha_1-1)/2}\alpha_1(3\alpha_1 + 1)\Gamma((3\alpha_1 - 1)/2) = a$.

5. Summary for the planar problem

We have discussed the asymptotic behaviour of a one-dimensional Stefan problem with the kinetic condition $u = -\varepsilon\dot{s}(t)$ at the free boundary, and initial data $\varphi(x) \to -1 - \delta$ as $x \to \infty$. We have investigated the cases ε nonnegative, δ negative, zero and positive respectively. To summarise, we display our results in Figure 4.

(I) $\varepsilon = 0$, $\delta > 0$: finite time blow-up.

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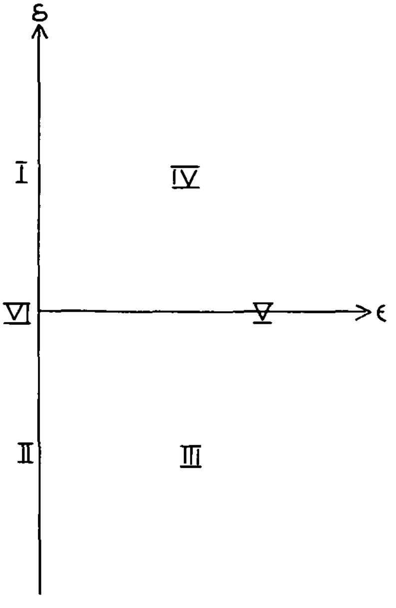

FIGURE 4. Regions of existence of classes of solution in the $\varepsilon - \delta$ plane.

(II) $\varepsilon = 0$, $\delta < 0$: similarity solutions exist with $s(t) = \beta\sqrt{t}$ and $\varphi(x) \equiv -1-\delta$. For other $\varphi(x)$ with $\varphi(\infty) = -1-\delta$, $s(t) \sim \beta\sqrt{t}$ as $t \to \infty$ provided that no blow-up occurs.

(III) $\varepsilon > 0$, $\delta < 0$: similarity solutions (but no travelling wave solution) exist with $s(t) = \beta\sqrt{t}$; for other initial data the free boundary is asymptotic to $\beta\sqrt{t}$ as $t \to \infty$.

(IV) $\varepsilon > 0$, $\delta > 0$: travelling wave solutions (but no similarity solution) exist with $s(t) = \delta t/\varepsilon$; for other initial data the free boundary is asymptotic to $\delta t/\varepsilon$ as $t \to \infty$.

(V) $\varepsilon > 0$, $\delta = 0$: $s(t) \sim kt^\alpha$ with $\frac{1}{2} < \alpha < 1$, as $t \to \infty$. Furthermore, if $\hat{\varphi}(p)$, the Laplace transform of $\varphi(x)$, has the form

$$\hat{\varphi}(p) = -p^{-1} + ap^\gamma + o(p^\gamma) \quad \text{as } p \to 0 \tag{5.1}$$

(i) If $-1 < \gamma < -\frac{1}{2}$, $a > 0$, then $\alpha = 1/(1-\gamma)$, $k = (\Gamma(1/\alpha)/\alpha a)^\alpha$.
(ii) If $-1 < \gamma < -\frac{1}{2}$, $a < 0$, then $\alpha = 1/(2+\gamma)$, $k = (-a/\varepsilon\alpha\Gamma(2-1/\alpha))^\alpha$.
(iii) If $\gamma = -\frac{1}{2}$, then $\alpha = \frac{2}{3}$, $k = (3/4\varepsilon\sqrt{\pi})[\sqrt{a^2 + 2\pi\varepsilon} - a]$.
(iv) If $-\frac{1}{2} < \gamma < 0$, $s(t) \sim kt^{2/3} + k_1 t^{\alpha_1}$, as $t \to \infty$, where

$$k = \frac{1}{2}\left(\frac{9}{\varepsilon}\right)^{1/3}, \; \alpha_1 = \frac{1-2\gamma}{3}, \; k_1 = -ak^{(3\alpha_1 - 3)/2}/\varepsilon\alpha_1(1+3\alpha_1)\Gamma(3\alpha_1 - 1)/2).$$

(VI) $\varepsilon = \delta = 0$:

(i) If $\varphi(x) = -1 + \varphi_0(x)$ with $\hat{\varphi}_0(p)$ finite and nonzero at $p = 0$ then $s(t) \sim Vt$ as $t \to \infty$, where $V = 1/\hat{\varphi}_1(0)$. This includes the travelling wave solution (2.5) as a special case.

(ii) If $\hat{\varphi}(p)$ has the same form as (5.1), $-1 < \gamma < 0$, then $s(t) \sim kt^\alpha$ with $\frac{1}{2} < \alpha < 1$, as $t \to \infty$, where $\alpha = 1/(1-\gamma)$, $k = (\Gamma(1/\alpha)/\alpha a)^\alpha$.

(iii) No solution exists for unit step function initial data.

(iv) If $\hat{\varphi}(p)$ has the form (5.1) for $0 < \gamma < 1$ then $s(t) \sim kt^\alpha$ with k and $\alpha > 1$ determined as in (ii). In this case $\dot{s}(t)$ is unbounded as $t \to \infty$ and we have "infinite-time" blow-up.

6. Three-dimensional solutions with radial symmetry

We briefly describe the extension of our previous results to a radially symmetric three-dimensional problem with an extra term incorporating surface tension effects at the free surface via a Gibbs-Thomson condition. The spherical version of the problem (1.1)–(1.4) is

$$\begin{aligned} u_t &= r^{-2}(r^2 u_r)_r, && s(t) < r < \infty \\ u &= -\varepsilon\dot{s} - 2\sigma/s, && r = s(t) \\ u_r &= -\dot{s}, && r = s(t) \\ u(r,0) &= \varphi(r), && s(0) \le r < \infty, \end{aligned}$$

where $\sigma \ge 0$ is the dimensionless surface tension. If we introduce a new variable

$$v(r,t) = ru(r,t)$$

then $v(r,t)$ satisfies

$$v_t = v_{rr}, \qquad s(t) < r < \infty \tag{6.1}$$

$$v = -\varepsilon s\dot{s} - 2\sigma, \qquad r = s(t) \tag{6.2}$$

$$v_r = -(\varepsilon + s)\dot{s} - 2\sigma/s, \qquad r = s(t) \tag{6.3}$$

with initial data

$$v(r,0) = r\varphi(r) = \psi(r), \tag{6.4}$$

say, where $\varphi(r)$ has the same form as (1.5).

We first mention that the problem (6.1)–(6.4) can blow up in finite time if $\varepsilon = 0$ and $\delta > 0$ (even with surface tension), and that when $\delta < 0$ there is a similarity solution with $s(t) = \beta\sqrt{t}$, where β is to be determined from

$$\frac{\beta^2}{2}\left(1 - \beta e^{\beta^2/4}\int_{\beta/2}^{\infty} e^{-x^2}\,dx\right) = 1 + \delta \tag{6.5}$$

(for details, see [2] and references therein). This similarity solution includes both surface tension and kinetic undercooling.

When $\delta > 0$ there is a pseudo-travelling-wave solution[3]

$$\begin{aligned} v(r,t) = &-(1+\varepsilon V)r + 2(1/V - \sigma) + [2(Vt - 1/V) - r]e^{-V(r-Vt)} \\ &- \frac{2\sigma}{V\sqrt{t}}e^{-r^2/4t}\int_{-r/2\sqrt{t}}^{(r-2Vt)/2\sqrt{t}} e^{y^2}\,dy, \\ s(t) = &\; Vt, \end{aligned}$$

where $V = \delta/\varepsilon$. For $\varepsilon = \delta = 0$, this is also a solution, for arbitrary $V > 0$. It is singular with $v = O(1/r^2)$ at the origin as $t \to 0+$, but this can be overcome by changing the time origin.

We now investigate the large-time behaviour of the free boundary $s(t)$ for problem (6.1)–(6.4). As previously, we define the Laplace transform by (4.1). This reduces (6.1)–(6.4) to an integral equation formulation. By a straightforward calculation, we get the integral equation

$$\begin{aligned} \psi(p) = &-d\hat{\varphi}/dp = -e^{-ps(0)}[p^{-2} + (2\sigma + 2\varepsilon + (1+\varepsilon p)s(0))p^{-1}] \\ &+ \int_0^{\infty}\left[1 + 2\varepsilon p + (1+\varepsilon p)ps(t) - \frac{2\sigma}{s(t)} - \varepsilon s(t)\dot{s}(t)^2\right]e^{-ps-p^2 t}\,dt. \end{aligned} \tag{6.6}$$

Repeating the method used in Section 4 we can find similar asymptotic results. Our results here confirm the numerical solutions of Schaefer and Glicksman [13].

For brevity, we state the main results only.

(1) $\varepsilon > 0$, $\delta > 0$: the asymptotic behaviour of $s(t)$ is Vt and $V = \delta/\varepsilon$.

(2) $\varepsilon > 0$, $\delta < 0$: the asymptotic behaviour of $s(t)$ is $\beta\sqrt{t}$ and β is determined by (6.5). Note that this is independent of both ε and σ.

(3) $\varepsilon > 0$, $\delta = 0$: the asymptotic behaviour of $s(t)$ is kt^{α} with $\frac{1}{2} < \alpha < 1$. In particular, if

$$\hat{\psi}(p) = -e^{-ps(0)}p^{-2} + ap^{\gamma-1} + o(p^{\gamma-1}), \quad \text{as } p \to 0 \tag{6.7}$$

where $-1 < \gamma < 0$, then

(a) if $-1 < \gamma < -\frac{1}{2}$, $a > 0$, then $\alpha = 1/(1-\gamma)$, $k = ((1+\alpha)\Gamma(1/\alpha)/\alpha^2 a)^{\alpha}$;

(b) if $-1 < \gamma < -\frac{1}{2}$, $a < 0$, then $\alpha = 1/(2+\gamma)$, $k = (-a/\varepsilon\alpha\Gamma(3 - 1/\alpha))^{\alpha}$;

[3] This solution does not appear to have been noted previously.

(c) if $\gamma = -\frac{1}{2}$ then $\alpha = \frac{2}{3}$, k is to be determined from

$$\frac{\varepsilon}{3}k^3 + \frac{a}{\sqrt{\pi}}k^{3/2} - \frac{15}{8} = 0;$$

(d) if $-\frac{1}{2} < \gamma < 0$, then $\alpha = \frac{2}{3}$, $k = \frac{3}{2}(\frac{5}{3\varepsilon})^{1/3}$, and we proceed to higher order terms as above.

(4) $\varepsilon = \delta = 0$.

(a) If $\hat{\psi}(p)$ has the form (6.7) and $-1 < \gamma < 0$ then

$$\alpha = \frac{1}{1-\gamma}, \quad k = \left[\frac{1+\alpha}{a\alpha^2}\Gamma\left(\frac{1}{\alpha}\right)\right]^{\alpha},$$

provided k is real and positive.

(b) If $\hat{\psi}(p)$ has the form (6.7) with $\gamma = 0$ then $s(t) \sim 2t/(a+2\sigma)$ as $t \to \infty$, provided $a + 2\sigma > 0$.

(c) If $\hat{\psi}(p)$ has the form (6.7) for $\gamma > 0$ and $\sigma > 0$ then $s(t) \sim t/\sigma + o(t)$ as $t \to \infty$. The precise form of the $o(t)$ term is determined by the higher order terms in $\hat{\psi}(p)$.

(d) If $\hat{\psi}(p)$ has the form (6.7) for $0 < \gamma < 1$ and $\sigma = 0$ then $s(t) \sim kt^{\alpha}$, where

$$\alpha = \frac{1}{1-\gamma}, \quad k = \left[\frac{1+\alpha}{a\alpha^2}\Gamma\left(\frac{1}{\alpha}\right)\right]^{\alpha}$$

and we have "infinite-time" blow-up.

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