ASYMPTOTIC BEHAVIOR OF SOME RANK TESTS FOR ANALYSIS OF VARIANCE[1]

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1. Summary. The H test and the median test have been proposed for testing the hypothesis of the equality of c probability distributions. Assuming translation-type alternatives, we find the limiting distributions of the H and median test statistics. These results are used to derive general formulas for the asymptotic relative efficiencies of these tests with respect to one another and to the classical F test. A short discussion of the computation of approximate power functions of these tests is also included.

2. Introduction. A few tests of a non-parametric nature have been proposed for the problem of testing the equality of c probability distributions, the so called c-sample problem. Tests for the two-sample problem have been proposed by Wilcoxon [22], Mann and Whitney [11], J. Westenberg, [21], and Mood and Brown [12], among others. Consistency and power properties of some of these tests have been discussed by van Dantzig [3], Lehmann [8], [9], and Mood [13], among others.

The H test proposed by Wallis and Kruskal [20] is a direct generalization of the two-sided Wilcoxon test discussed in detail by Mann and Whitney [11]. The H test is similar to a classical F test, with ranks replacing the original observations. The Mood-Brown median test [12] makes use of the construction of a 2-by-c table and the resulting large sample theory thereof. Pitman [16] derives the general formula for the asymptotic relative efficiency of the Wilcoxon test with respect to the ordinary t test, when quite general translation-type alternative hypotheses are assumed. Mood [13] assumes only normal alternative hypotheses and computes the asymptotic relative efficiencies, with respect to the t test, to be $3/\pi$ for the Wilcoxon test and $2/\pi$ for the median test; the former is a special case of the Pitnam result.

After setting up suitable alternative hypotheses and finding the limiting distributions of the relevant statistics, we find general formulas for the asymptotic relative efficiencies in the c-sample case for translation alternatives but almost arbitrary distributions. These formulas do not in general depend on c.

Mood [12] and Kruskal [7] derive the limiting distributions of their respective statistics in the case of the hypothesis of equal distributions. The methods used here to derive the distributions under the alternative hypothesis duplicate their results.

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A rather complete bibliography on nonparametric c-sample tests and a discussion of the rationale for applying them is given by Wallis and Kruskal [20].

The c-sample problem may be expressed formally as follows. Let $\{X_{ij}\}$ for $i = 1, 2, \cdots, c$, and $j = 1, 2, \cdots, n_1$ be a set of independent random variables. The probability distribution function of X_{ij} is denoted by F_i, so that $F_i(x)$ is the probability of the event $[X_{ij} \leqq x]$. The set of admissible hypotheses designates that each F_i belongs to some class of distribution functions Ω. The hypothesis to be tested, say K_0, specifies that F_i is an element of Ω for each i and that furthermore $F_1(x) = F_2(x) = \cdots = F_c(x)$ for all real x. Alternative to K_0 is the hypothesis that each F_i belongs to Ω but that K_0 does not hold. To avoid the problem of ties, it is assumed throughout that the class Ω is the class of continuous distribution functions.

So as to pay particular attention to translation-type alternatives, the class of admissible hypotheses will be limited to include only those hypotheses which state that $F_i(x) = F(x + \epsilon_i)$ for all $i = 1, 2, \cdots, c$, for some arbitrary choice of F in the class Ω and real numbers $\epsilon_1, \cdots, \epsilon_c$. It is seen immediately that specifying all $\epsilon_1 = 0$ yields hypothesis K_0, while if $\epsilon_i \neq \epsilon_j$ for some pair (i, j) then an alternative to K_0 is given.

The H test is based on the statistic

$$H = \frac{12}{N(N+1)} \sum_{i=1}^{c} n_i \left(\bar{R}_i - \frac{N+1}{2}\right)^2, \tag{1}$$

when $\bar{R}_i$ is the average rank of the members of the ith sample obtained after ranking all of the $N = \sum n_i$ observations. The test consists in rejecting K_0 at a significance level α if H exceeds some predetermined number h_α. Kruskal [7] proves that if K_0 is true, the statistic H has a limiting chi square distribution with $c - 1$ degrees of freedom as all $n_i \to \infty$ simultaneously. This provides the user of this H test with a large sample approximation of the value of h_α for any $0 < \alpha < 1$.

The Mood-Brown median test is based on the statistic

$$M = \frac{N(N-1)}{b(N-b)} \sum_{i=1}^{c} \frac{1}{n_i} \left(m_i - \frac{bm_i}{N}\right)^2 \tag{2}$$

where $N = \sum n_i$, and $b = \frac{1}{2}(N - 1)$ when N is odd, and $b = \frac{1}{2}N$ when N is even, while m_i denotes the number of observations in the ith sample which are less than the median of all of the observations. Mood proves that whenever K_0 is true, the statistic M has a limiting chi square distribution with $c - 1$ degrees of freedom as all $n_i \to \infty$ simultaneously. The median test is then to reject K_0 at the level of significance α whenever M exceeds a number m_α. Use of the limiting distribution allows one to determine an approximate value for m_α for large samples.

Because both the H test and the median test are consistent against translation alternatives, the distributions of H and M will be studied, in following sections, assuming a sequence of admissible alternative hypotheses K_n for $n = 1, 2, \cdots$.

The hypothesis K_n specifies, for each $i = 1, 2, \cdots, c$, that $F_i(x) = F(x + \theta_i/\sqrt{n})$, with $F \varepsilon \Omega$ but not specified further, and for some pair (i, j) that $\theta_i \neq \theta_j$. The letter n will be used to index a sequence of situations in which K_n is the true hypothesis. Limiting probability distributions will then be found as $n \to \infty$. The problem will be so formulated that N will be proportional to n. For large n the hypothesis K_n is "near" K_0, so that this type of limit process provides a way of studying the effect of small translations on these tests.

The notation $\chi_r^2(\lambda)$ will denote the possibly noncentral chi square distribution with degrees of freedom r and noncentral parameter λ. Thus $\chi_r^2(\lambda)$ is the probability distribution of the sum of r squares of independent normal random variables whose variances are all unity and whose sum of squared expectations is denoted by λ. For $\lambda = 0$ we see that $\chi_r^2(0)$ is the ordinary chi square distribution. The $\chi_r^2(\lambda)$ distribution has been studied and used by Fisher [4], Tang [19], and Patnaik [15], among others. A partial tabulation of some percentage points of $\chi_r^2(\lambda)$ is given by Fix [5].

3. The limiting distribution of H under hypothesis K_n. The purpose of this section is to prove

THEOREM 3.1. *For each index n assume that $n_\alpha = s_\alpha n$, with s_α a positive integer, and the truth of hypothesis K_n. If for any real number t,*

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} \sqrt{n}\, \{F(x + t/\sqrt{n}) - F(x)\}\, dF(x)$$

exists finite, then, for $n \to \infty$, the limiting distribution of the statistic H is $\chi_{c-1}^2(\lambda^H)$, where

$$\lambda^H = \left[12\left(\sum_{j=1}^{c} s_j\right)^{-2}\right] \sum_{\alpha=1}^{c} s_\alpha \cdot \left\{\sum_{i=1}^{c} s_i \lim_{n\to\infty} \int_{-\infty}^{+\infty} \sqrt{n}\left[F\left(x + \frac{\theta_i - \theta_\alpha}{\sqrt{n}}\right) - F(x)\right] dF(x)\right\}^2. \tag{3}$$

From (1) and definitions (5), (6), and (9) below one can write

$$H = \left[12\Big/\left(\sum_{i=1}^{c} s_i\right)\left(\sum_{i=1}^{c} s_i + \frac{1}{n}\right)\right] \sum_{\alpha=1}^{c} s_\alpha^3 \left[\sqrt{n}\left(U'^\alpha - \tfrac{1}{2} \sum_{i=1\neq\alpha}^{c} \frac{s_i}{s_\alpha}\right)\right]^2. \tag{4}$$

The proof of Theorem 3.1 then quite naturally depends upon showing that the random variables

$$\sqrt{n}\left(U'^\alpha - \tfrac{1}{2} \sum_{i=1\neq\alpha}^{c} \frac{s_i}{s_\alpha}\right), \qquad \alpha = 1, 2, \cdots, c,$$

have a certain joint limiting normal distribution as $n \to \infty$. The methods used in the proof are mainly adaptations of results of Hoeffding [6] and Lehmann [8].

We begin the proof by defining the functions h^α by

$$h^\alpha(y_1, \cdots, y_\alpha, \cdots, y_c) = \sum_{\beta=1}^{c} \frac{s_\beta}{s_\alpha} \delta(y_\beta, y_\alpha), \tag{5}$$

with the convention that $\delta(y_\beta, y_\alpha) = 1$ whenever $y_\beta < y_\alpha$ and is otherwise zero. Throughout this discussion α will range over the integers 1, 2, $\cdots$, c. Recalling that $n_i = s_i n$, we construct for $k = 1, 2, \cdots, n$ the random vectors

$$\begin{aligned} X_k = (&X_{1\ (k-1)s_1+1}, X_{1\ (k-1)s_1+2}, \cdots, X_{1\ ks_1};\\ &X_{2\ (k-1)s_2+1}, \cdots, X_{2\ ks_2}; \quad \cdots \quad ; X_{c\ (k-1)s_c+1}, \cdots, X_{c\ ks_c}) \end{aligned} \tag{6}$$

and the random variables φ^α, U^α, and U'^α, defined by

$$\varphi^\alpha(X_1, \cdots, X_c) = \frac{1}{(s_1 s_2 \cdots s_c)(c!)} \sum h^\alpha(X_{1j_1}, \cdots, X_{cj_c}), \tag{7}$$

with the summation extending over all indices $(j_1, \cdots, j_c)$ in such a manner the arguments of a single h^α are components of distinct vectors;

$$U^\alpha = \sum \varphi^\alpha(X_{\beta_1}, X_{\beta_2}, \cdots, X_{\beta_c}) \Big/ \binom{n}{c} \tag{8}$$

where the summation extends over all indices $1 \leqq \beta_1 < \beta_2 < \cdots < \beta_c \leqq n$; and

$$U'^\alpha = \frac{1}{n_1 n_2 \cdots n_c} \sum_{j_1=1}^{n_1} \cdots \sum_{j_c=1}^{n_c} h^\alpha(X_{1j_1}, X_{2j_2}, \cdots, X_{cj_c}). \tag{9}$$

Then U'^α is recognized as the average of all kinds of h^α terms while U^α is an average of only those h^α terms in which the arguments of a given h^α are each elements of a different vector. Setting j^α equal to the sum of all h^α terms appearing in U'^α but not in U^α, we have

$$U'^\alpha = \frac{1}{n_1 n_2 \cdots n_c} \left\{ \binom{n}{c} U^\alpha + J^\alpha \right\}.$$

Let

$$D^\alpha = U'^\alpha - U^\alpha = \frac{1}{n_1 n_2 \cdots n_c} \left\{ \left[\binom{n}{c} c! - n_1 n_2 \cdots n_c \right] U^\alpha + J^\alpha \right\}.$$

Adopting a method of proof given by Lehmann ([8], p. 168), we use the inequality $(\sum_1^k a_i)^2 \leqq k \sum_1^k a_i^2$ for real numbers a_i, and the fact that

$$E\{h^\alpha(X_{1j_1}, X_{2j_2}, \cdots, X_{cj_c})\}^2 \leqq \left(\sum_{\beta=1}^{c} \frac{s_\beta}{s_\alpha} \right)^2.$$

Thus we establish that

$$E(\sqrt{n}\, D^\alpha)^2 \leqq 4 \left(\sum_{\beta=1}^{c} \frac{s_\beta}{s_\alpha} \right)^2 \left[\sqrt{n} \left(1 - \frac{1}{n_1 n_2 \cdots n_c} \binom{n}{c} \right) \right]^2 \to 0 \quad \text{as } n \to \infty$$

With the notation $W^\alpha = \sqrt{n}\,(U'^\alpha - EU'^\alpha)$ and $Z^\alpha = \sqrt{n}\,(U^\alpha - EU^\alpha)$,

$$E(W^\alpha - Z^\alpha)^2 = E(\sqrt{n}\,D^\alpha)^2 \to 0 \qquad \text{as } n \to \infty.$$

Applying Lemma 7.1 of Hoeffding ([6], p. 305), we have

LEMMA 3.1. *If either of the random vectors* $W = (W^1, W^2, \cdots, W^c)$ *or* $Z = (Z^1, Z^2, \cdots, Z^c)$ *has a limiting probability distribution as* $n \to \infty$, *then the other random vector has this same limiting distribution as* $n \to \infty$.

The next step in the proof is to compare the random vector Z with the random vector $Y = (Y^1, Y^2, \cdots, Y^c)$, whose components are defined by $Y = (c/\sqrt{n})\sum_1^n \psi_1^\alpha(X_i)$, with

$$\psi_j^\alpha(x_1, x_2, \cdots, x_j) = E\varphi^\alpha(x_1, x_2, \cdots, x_j, X_{j+1}, X_{j+2}, \cdots, X_c) - E\varphi^\alpha(X_1, X_2, \cdots, X_c).$$

The functions $\psi_j^\alpha(x_1, x_2, \cdots, x_j)$ are the same as those defined by Hoeffding [6] except that they are applied to this special problem. Now Hoeffding ([6], p. 299, (5.13)) has shown that

$$E(Z^\alpha)^2 = n\sigma^2(U^\alpha) = cn\binom{n}{c}^{-1}\binom{n-c}{c-1}a_1^\alpha + R_{nc}^\alpha,$$

$$R_{nc}^\alpha = n\binom{n}{c}^{-1}\sum_{j=2}^{c}\binom{c}{j}\binom{n-c}{c-j}a_j^\alpha, \qquad a_j^\alpha = E\{\psi_j^\alpha(X_1, X_2, \cdots, X_j)\}^2.$$

By expanding binomial coefficients we calculate

$$R_{nc}^\alpha = \sum_{j=2}^{c}(j!)\binom{c}{j}^2\prod_{k=1}^{c-j}\left[1 - \frac{c-1}{n-k}\right]\sum_{l=1}^{j-1}[n - c + j - l]^{-1}a_j^\alpha;$$

however, $a_j^\alpha \leqq 4(\sum_{\beta=1}^{c} s_\beta/s_\alpha)^2$ for all α, j so that $R_{nc}^\alpha \to 0$ as $n \to \infty$. Referring to Hoeffding ([6], p. 308, (7.10) and (7.12)), we find that

$$E(Y^\alpha)^2 = E(Y^\alpha Z^\alpha) = c^2 a_1^\alpha.$$

Substitution yields

$$E(Y^\alpha - Z^\alpha)^2 = E(Y^\alpha)^2 + E(Z^\alpha)^2 - 2E(Z^\alpha Y^\alpha) = R_{nc}^\alpha + \left[cn\binom{n}{c}^{-1}\binom{n-c}{c-1} - c^2\right]a_1^\alpha \to 0 \qquad \text{as } n \to \infty.$$

Another application of Lemma 7.1 of Hoeffding ([6], p. 305), produces

LEMMA 3.2. *If either of the random vectors* Z *or* Y *has a limiting probability distribution as* $n \to \infty$, *then the other random vector has this same limiting distribution as* $n \to \infty$.

It now remains to find the limiting distribution of Y. Each Y^α is a sum of independent and identically distributed random variables,

$$Y^\alpha = \frac{c}{\sqrt{n}}\sum_{j=1}^{n}\psi_1^\alpha(X_j); \qquad E(Y^\alpha) = 0.$$

Also, $\psi_1^\alpha(X_j) \leqq 2(\sum_{\beta=1}^c s_\beta/s_\alpha)$ with probability one. Adopting the notation for the column vectors $s = (s_1, \cdots, s_c)'$ a vector of real numbers, and $\psi_1 = (\psi_1^1, \psi_1^2, \cdots, \psi_1^c)'$, the characteristic function of Y is expressed as

$$f_n(s) = E(e^{is'Y}) = E(\exp\{ics'\psi_1/\sqrt{n}\})^n,$$

because of independence. Taking logarithms, expanding the real and imaginary parts of the exponential in finite Taylor series, using the almost sure boundedness of $\psi_1(X_j)$, noting that $E[\psi_1(X_j)] = 0$, and finally expanding the logarithm in a finite Taylor series, produces the usual type of result that

$$\log f_n(s) = -\tfrac{1}{2}c^2 s'[E(\psi_1\psi_1')]s + O(n^{-1/2}) \qquad \text{as } n \to \infty$$

for any fixed real vector s. From the continuity theorem for characteristic functions ([2], p. 96), we conclude

LEMMA 3.3. *The random vector Y has a limiting normal distribution with $E(Y)$ the zero vector and variance-covariance matrix $\Sigma = \lim_{n\to\infty} c^2E(\psi_1\psi_1')$.*

Adopting the notation

$$A_{\beta\alpha} = \frac{1}{s_\alpha}\sum_{j=1}^{s_\alpha}\left[F_\beta(X_{\alpha j}) - \int F_\beta(x)\,dF_\alpha(x)\right] - \frac{1}{s_\beta}\sum_{j=1}^{s_\beta}\left[F_\alpha(X_{\beta j}) - \int F_\alpha(x)\,dF_\beta(x)\right],$$

we can recognize $\psi_1^\alpha(X_1) = (1/c)\sum_{\beta=1}^c (s_\beta/s_\alpha)A_{\beta\alpha}$. A lengthy computation and an application of the Lebesgue bounded convergence theorem, in view of the boundedness of each F_j and $\lim_{n\to\infty} F_j(x) = F(x)$, yields the result that

$$\Sigma = \lim_{n\to\infty} c^2E(\psi_1\psi_1') = 1/12\left[\frac{1}{s_\alpha}\left(\sum_{j=1}^c \frac{s_j}{s_\beta}\right)\left(\delta_{\alpha\beta}\sum_{j=1}^c \frac{s_j}{s_\beta} - 1\right)\right].$$

Combining the previous three lemmas produces

LEMMA 3.4. *If for each index n the hypothesis K_n is valid and W^α denotes the random variable $\sqrt{n}\,(U'^\alpha - EU'^\alpha)$, then the random vector $W = (W^1, W^2, \cdots, W^c)$ has a limiting normal distribution with zero mean vector and variance-covariance matrix Σ.*

Recalling $W^\alpha = \sqrt{n}\,(U'^\alpha - E(U'^\alpha))$ and (4), and letting

$$m^\alpha = \sqrt{n}\left[E(U'^\alpha) - \tfrac{1}{2}\sum_{i=1}^c \frac{s_i}{s_\alpha}\right],$$

we write H as

$$H = \left[12\Big/\left(\sum_{i=1}^c s_i\right)\left(\sum_{i=1}^c s_i + \frac{1}{n}\right)\right]\sum_{\alpha=1}^c s_\alpha^3(W^\alpha + m^\alpha)^2.$$

Now H will have the same limiting distribution as

$$H^* = \left[12\Big/\left(\sum_{i=1}^c s_i\right)^2\right]\sum_{\alpha=1}^c s_\alpha^3(W^\alpha + m^\alpha)^2,$$

but because $\sum n_\alpha \bar{R}_\alpha = \frac{1}{2}(\sum n_\alpha)(\sum n_\alpha + 1)$, we have $\sum s_\alpha^2 W^\alpha = O(n^{-1/2})$ as $n \to \infty$. So, except for terms of higher order,

$$H^* = \left[12 \Big/ \left(\sum_{i=1}^{c} s_i\right)^2\right] \sum_{\alpha=1}^{c} \sum_{\beta=1}^{c-1} s_\alpha^2 \left(s_\alpha \delta_{\alpha\beta} + \frac{s_\beta^2}{s_c}\right)(W^\alpha + m^\alpha)(W^\beta + m^\beta).$$

We recognize the matrix of the quadratic form H^* as the inverse of the limiting variance-covariance matrix of the random variables $W^1, W^2, \cdots, W^{c-1}$.

Lemma 3.5. *If the vector x has a normal distribution, with mean vector μ and non-singular variance-covariance matrix Λ, then the quadratic form $x'\Lambda^{-1}x$ has a $\chi_r^2(\lambda)$ distribution, with $\lambda = \mu'\Lambda^{-1}\mu$ and r the rank of Λ.*

A proof of this lemma is given by Rao ([17], p. 57). We now calculate

$$\lim_{n\to\infty} m^\alpha = \sum_{\beta=1}^{c} \frac{s_\beta}{s_\alpha} \lim_{n\to\infty} \sqrt{n} \int_{-\infty}^{+\infty} \left[F\left(x + \frac{\theta_\beta - \theta_\alpha}{\sqrt{n}}\right) - F(x)\right] dF(x),$$

and combine Lemmas 3.4 and 3.5 with a theorem of Mann and Wald ([10], p. 223) to complete the proof of Theorem 3.1.

In many instances λ^H can easily be computed with the aid of

Lemma 3.6. *If the distribution function F possesses a continuous derivative F' except at most on a set S where $\int_S dF(x) = 0$, and if there exists a function g which bounds the difference quotient $|[F(x + \theta) - F(x)]/\theta| \leqq g(x)$ for which $\int_{-\infty}^{+\infty} g(x)\, dF(x) < \infty$, then*

$$\lim_{n\to\infty} \sqrt{n} \int_{-\infty}^{+\infty} [F(x + \theta/\sqrt{n}) - F(x)]\, dF(x) = \theta \int_{-\infty}^{+\infty} F'(x)\, dF(x).$$

This lemma is proved by a direct application of the Lebesgue bounded convergence theorem and the definition of the derivative. In the event that the conditions of Lemma 3.6 are satisfied, then

$$\lambda^H = 12\left\{\int_{-\infty}^{+\infty} F'(x)\, dF(x)\right\}^2 \sum_{\alpha=1}^{c} s_\alpha(\theta_\alpha - \bar{\theta})^2,$$

$$\bar{\theta} = \sum_{\alpha=1}^{c} s_\alpha \theta_\alpha \Big/ \sum_{\alpha=1}^{c} s_\alpha .$$

4. The limiting distribution of M under hypothesis K_n. The purpose of this section is to derive the limiting distribution of the statistic M as $n \to \infty$. The result is stated in

Theorem 4.1. *Assume for each index $n = 1, 2, \cdots$ the validity of hypothesis K_n, that F has a continuous derivative F' at its median a, and that $n_i = s_i n$, for each $i = 1, 2, \cdots, c$, with s_i a positive integer. With these assumptions the limiting distribution of M is $\chi_{c-1}^2(\lambda^M)$ with*

(10) $$\lambda^M = 4[F'(a)]^2 \sum_{i=1}^{c} s_1(\theta_i - \bar{\theta})^2, \qquad \theta = \sum_{i=1}^{c} s_i \theta_i \Big/ \sum_{i=1}^{c} s_i .$$

The proof of the theorem is a generalization of a type of proof sketched by Mood [13] in his discussion of the two-sample problems. Because the two cases N odd and N even require slight differences in exposition, only the proof for N odd will be given here. A similar proof for N even could readily be constructed. In this case N odd,

$$M = \frac{4}{1 + 1/N} \sum_{i=1}^{c} n_i \left(\frac{m_i}{n_i} - \tfrac{1}{2} + \frac{1}{2N}\right)^2.$$

Defining the random variables $v_j = \sqrt{n_j}\,[(m_j/n_j) - \tfrac{1}{2}]$, permits M to be written

$$M = \frac{4}{1 + 1/N} \sum_{i=1}^{c} \left\{v_j^2 + \frac{\sqrt{n_j}}{N} v_j + \frac{n_j}{4N^2}\right\}. \tag{9}$$

Provided that we can demonstrate that v_j has a limiting distribution, since $\sqrt{n_j}/N$, n_j/N^2, and $1/N$ all converge to zero as $n \to \infty$, M will have the same limiting distribution as the statistic $4\sum v_j^2$. The first part of proof consists in proving

LEMMA 4.1. *Assuming the hypothesis of Theorem* 4.1, the limiting distribution of the vector $(v_1, \cdots, v_{c-1})$ *is normal with* $E(v_j) = F'(a)\sqrt{s_j}(\theta_j - \bar{\theta})$ *and covariance matrix* A_c *given by*

$$A_c^{-1} = \left(\frac{4}{s_c}(s_c\delta_{ij} + \sqrt{s_i s_j})\right), \qquad i, j = 1, 2, \cdots, c-1,$$

where δ_{ij} *is the usual Kronecker delta.*

Let $r_1, \cdots, r_c$ be a set of independent random variables each with a uniform probability distribution on the unit interval and let

$$v_j' = \sqrt{n_j}\left[\frac{m_i + r_j}{n_j} - \tfrac{1}{2}\right] = v_j + \frac{r_j}{\sqrt{n_j}}, \qquad j = 1, 2, \cdots, c.$$

The difference $v_j' - v_j$ tends to zero in probability and so, by a well known theorem ([2], p. 299) the vectors $(v_1', \cdots, v_c')$ and $(v_1, \cdots, v_c)$ possess the same limiting distribution if they have one at all. Because the v_j are discrete while the v_j' are continuous random variables, it is easier to examine the limiting distribution of $(v_1', \cdots, v_{c-1}')$.

Denoting by Z the median of all the samples combined, the probability of the joint event $m_1 = a_1$ and $m_2 = a_2$ and $\cdots$ and $m_{c-1} = a_{c-1}$ and $z_1 \leqq Z \leqq z_2$ is

$$P[m_1 = a_1, \cdots, m_{c-1} = a_{c-1}, z_1 \leqq Z \leqq z_2]$$
$$= \sum_{i=1}^{c} \int_{z_1}^{z_2} \frac{(n_i - a_i)}{1 - F_i(z)} F_i'(z) \prod_{j=1}^{c} \binom{n_j}{a_j} \{F_j(z)\}^{a_j}\{1 - F_j(z)\}^{n_j - a_j}\, dz$$

for $\sum a_i = \tfrac{1}{2}(N - 1)$ with a_i a nonnegative integer, and is zero otherwise. Writing $m_i' = m_i + r_i$ and square brackets to indicate the "largest integer contained in," we see that the joint probability density function of the random variables $m'_i, \cdots, m'_{c-1}, Z$ is

$$g(m'_1, \cdots, m'_{c-1}, z) = \sum_{i=1}^{c} \frac{n_i - [m'_i]}{1 - F_i(z)} F'(z) \sum_{j=1}^{c} \binom{n_j}{[m'_j]} \{F_j(z)\}^{[m'_j]} \{1 - F_j(z)\}^{n_j - [m'_j]}$$

for $\sum [m'_i] = \frac{1}{2}(N - 1)$, and otherwise zero. With the transformation

$$w = \sqrt{n}\,(Z - a); \quad v'_j = \sqrt{n_j}\{(m'_j/n_j) - \tfrac{1}{2}\}, \quad j = 1, 2, \cdots, c,$$

the probability density of $(v'_1, \cdots, v'_{c-1}, w)$ becomes

$$h(v'_1, \cdots, v'_{c-1}, w) = \sum_{i=1}^{c} \frac{[d_i](n_i n_c)^{-1/2}}{1 - F_i(a + w/\sqrt{n})} F'_i\left(a + \frac{w}{\sqrt{n}}\right)$$
$$\times \prod_{j=1}^{c} \sqrt{n_j} \binom{n_j}{[d_j]} \left\{F_j\left(a + \frac{w}{\sqrt{n}}\right)\right\}^{[d_j]} \left\{1 - F_j\left(a + \frac{w}{\sqrt{n}}\right)\right\}^{n_j - [d_j]}$$

where $d_j = \frac{1}{2}n_j + v'_j\sqrt{n_j}$ and square brackets indicate the "largest integer contained in."

Noting that $\sum_1^c \sqrt{s_i}v'_i = o(1)$ as $n \to \infty$, employing Stirling's formula for $\log n!$, and using series expansions and the continuity of F' at $x = a$, we compute

$$\lim_{n\to\infty} h(v'_1, \cdots, v'_{c-1}, w) = \left(\frac{1}{\sqrt{2\pi}}\right)^{c-1} \frac{2^{c-1}\sqrt{s}}{\sqrt{s_c}}$$
$$\times \exp\left\{-\tfrac{1}{2} \sum_{j=1}^{c} 4[v_j - F'(a)\sqrt{s_j}(\theta_j - \bar{\theta})]^2\right\}$$
$$\times \frac{2F'(a)\sqrt{s}}{\sqrt{2\pi}} \exp\{-2(F'(a))^2 s(\bar{\theta} - w)^2\},$$

where $s = s_1 + s_2 + \cdots + s_c$. Letting A_c denote the variance-covariance matrix of $(v'_1, \cdots, v'_{c-1})$, we find

$$A_c^{-1} = \{(4/s_c)(s_c\delta_{ij} + \sqrt{s_i s_j})\}, \qquad i, j = 1, 2, \cdots, c - 1.$$

Applying a theorem of Scheffé [18] yields the result that the limiting distribution of $(v'_1, \cdots, v'_{c-1}, w)$ is the foregoing normal distribution. Integrating out the variable w, we obtain the desired limiting probability distribution of $(v'_1, \cdots, v'_{c-1})$ and hence of $(v_1, \cdots, v_{c-1})$, which proves Lemma 4.1.

Earlier in this section it was remarked that if $(v_1, \cdots, v_{c-1})$ has a limiting distribution, then H has the same limiting distribution as

$$4 \sum_{i=1}^{c} v_i^2 = \left\{\sum_{j=1}^{c-1} (s_c + s_j)v_j^2 + \sum_{j=1}^{c-1} \sum_{k=1\neq j}^{c-1} \sqrt{s_j s_k}\, v_j v_k\right\} + \eta.$$

However, η tends to zero in probability, since $\sum \sqrt{s_i}v_i = o(1)$ is satisfied with probability one. We recognize then that, except for the term η, $4\sum v_i^2$ is equal to the quadratic form in the limiting distribution of $(v_1, \cdots, v_{c-1})$, provided that the means are shifted to zero. As in Section 3, we employ Lemma 3.5 to obtain the main Theorem 4.1 of this section.

5. Asymptotic relative efficiency. The concept of asymptotic relative efficiency of one consistent test with respect to another is due to Pitman [16]. An application and account of this method of comparing consistent tests is presented by Noether ([14], p. 241). Briefly, the idea of asymptotic relative efficiency is to choose a sequence of alternative hypotheses which vary with the sample sizes in such a manner that the powers of the two tests for this sequence of alternatives have a common limit less than one. The comparison of the two tests is then made on a sample size basis.

To be more definite, suppose that two consistent tests T and T' require N and N' observations, respectively, to attain the power β at level of significance α for testing the hypothesis K_0 against hypothesis K_n. The difference in the sample sizes N and N' results from the fact that we demand that the tests yield a common power for a given alternative $K_{n'}$. The asymptotic relative efficiency of T' with respect to T is defined to be

$$\lim_{N\to\infty} N/N' = \lim_{n\to\infty} n/n' = \epsilon_{T',T}(\alpha, \beta, K_0, \{K_n\}).$$

The asymptotic relative efficiency of the median test with respect to the H test is stated in

THEOREM 5.1. *If $n_i = s_i n$ and if the distribution function F has the two properties,*

(i) *F is continuous at its median, and*

(ii) $\lim_{n\to\infty} \sqrt{n} \int_{-\infty}^{+\infty} [F(x + \theta/\sqrt{n}) - F(x)]\, dF(x)$ *exists,*

then the asymptotic relative efficiency of the median test with respect to the H test for testing the hypothesis K_0 against K_n is

$$\epsilon_{M,H} = \frac{\left(\sum_{j=1}^{c} s_j\right)^2 \cdot [F'(a)]^2 \sum_{i=1}^{c} s_i(\theta_i - \bar{\theta})^2}{3 \sum_{\alpha=1}^{c} s_\alpha \left\{\sum_{i=1}^{c} s_i \lim_{n\to\infty} \int_{-\infty}^{+\infty} \sqrt{n}\left[F\left(x + \frac{\theta_i - \theta_\alpha}{\sqrt{n}}\right) - F(x)\right] dF(x)\right\}^2}.$$

To prove Theorem 5.1, let n' and n index the sample sizes for the H test and the median test, respectively. The alternative hypothesis K_n states that $F_i(x) = F(x + \theta_i/\sqrt{n})$ and so is characterized by the numbers $\theta_i/\sqrt{n}$. If the level of significance is fixed at α and the limiting power fixed at β, then, since from Theorems 3.1 and 4.1 H has a limiting $\chi^2_{c-1}(\lambda^H)$ distribution and M has a limiting $\chi^2_{c-1}(\lambda^M)$ distribution under K_n, we must have $\lambda^H = \lambda^M$ to achieve the same limiting power for the two tests. To have the same alternatives for each test we must have $\theta_i/\sqrt{n} = \theta'_i/\sqrt{n'}$. The substitution $\theta'_i = \theta_i \sqrt{n'/n}$ in (10) along with the requirement $\lambda^H = \lambda^M$ (to guarantee equal power) yields formula (11), which proves Theorem 5.1.

COROLLARY 5.1. *If in addition to the hypothesis of Theorem 6.1, the hypothesis of Lemma 3.6 is assumed, then*

$$\epsilon_{M,H} = \tfrac{1}{3}\left[F'(a) \Big/ \int_{-\infty}^{+\infty} F'(x)\, dF(x)\right]^2.$$

Here $\epsilon_{M,H}$ does not depend upon α, β, $(\theta_1, \cdots, \theta_c)$, or c, but is a function of F only.

The comparison of the H test with respect to the ordinary analysis of variance $\mathfrak{F}$ test is contained in

THEOREM 5.2. *If the distribution function F satisfies the conditions of Lemma 3.6 and if* $\int_{-\infty}^{+\infty} x^2\, dF(x) - \left[\int_{-\infty}^{+\infty} x\, dF(x)\right]^2 = \sigma_F^2$ exists, then

$$\epsilon_{H,F} = 12\sigma_F^2 \left[\int_{-\infty}^{+\infty} F'(x)\, dF(x)\right]^2.$$

The classical $\mathfrak{F}$ statistic in this instance is defined by

$$\mathfrak{F} = \frac{\dfrac{1}{c-1} \sum_{i=1}^{c} n_i(x_{i\cdot} - \bar{x})^2}{\left[1 \Big/ \sum_{k=1}^{c} (n_k - 1)\right] \sum_{i=1}^{c} \sum_{j=1}^{n_i} (x_{ij} - x_{i\cdot})^2}.$$

Now Fisher [4] and Tang [19] have shown that if $F(x)$ is the normal distribution function, then under hypothesis K_n the statistic $\mathfrak{F}$ has a limiting $\chi^2_{c-1}(\chi^{\mathfrak{F}})$ distribution with $\lambda^{\mathfrak{F}} = \sum_1^c s_i[(\theta_i - \bar{\theta})/\sigma_F]^2$. However, it is a well known result of the weak Law of Large Numbers that $[1/(n-1)] \sum_{j=1}^{n} (x_{ij} - x_{1\cdot})^2 \to \sigma_F^2$ in probability as $n \to \infty$. Also the Lindeberg-Levy central limit theorem shows that $\sqrt{n_i}[x_{i\cdot} - E(x_{i\cdot})]/\sigma_F$ has a limiting $N(0, 1)$ distribution. Application of the Mann-Wald theorem used previously gives the result that under hypothesis K_n the statistic $\mathfrak{F}$ has a limiting $\chi^2_{c-1}(\lambda^{\mathfrak{F}})$ distribution whenever F satisfies the hypothesis of Theorem 5.2. A calculation similar to that for the proof of Theorem 5.1 completes the proof of Theorem 5.2.

Theorems 5.1 and 5.2 show that, depending upon F, $\epsilon_{M,H}$ can be $\gtreqless 1$, similarly for $\epsilon_{H,\mathfrak{F}}$ and $\epsilon_{M,\mathfrak{F}} = \epsilon_{M,H}\epsilon_{H,\mathfrak{F}}$. In the event that F is some normal distribution function, then $\epsilon_{M,H} = 2/3$, $\epsilon_{H,\mathfrak{F}} = 3/\pi$, and $\epsilon_{M,\mathfrak{F}} = 2/\pi$. When F is the uniform distribution function on the unit interval, $F(x) = x$ if $0 \leqq x \leqq 1$, then $\epsilon_{M,H} = 1/3$, $\epsilon_{H,\mathfrak{F}} = 1$, and $\epsilon_{M,\mathfrak{F}} = 1/3$.

6. Power functions. The power of a test for a given simple alternative hypothesis is the probability that the test will reject the hypothesis tested when the given alternative is true. In terms of this power definition, the power function is defined on the class of alternative hypotheses.

As we have seen in Sections 3 and 4, both the H and M statistics have limiting noncentral chi square distributions when the alternatives K_n are true for each n. In the event that Lemma 3.6 is satisfied, the noncentral parameter in each of these limiting distributions is a function of $\theta_1, \cdots, \theta_c$ only through the variable $\sum s_i(\theta_i - \bar{\theta})^2$. In fact

$$\lambda^H = 12\left[\int_{-\infty}^{+\infty} F'(x)\, dF(x)\right]^2 \sum_{i=1}^{c} s_i(\theta_i - \bar{\theta})^2, \qquad \lambda^M = 4[F'(a)]^2 \sum_{i=1}^{c} s_i(\theta_i - \bar{\theta})^2.$$

For particular choices of F the power function of each of these tests could be considered as a function of $\sum s_i(\theta_i - \bar{\theta})^2$. This type of power function approximation is discussed in Cochran's paper on the chi square test for goodness of fit [1].

The tables of Fix [5] may be employed to find approximate values for these power functions. The procedure would be as follows. Suppose that F is the uniform distribution function on the unit interval, then $\lambda^H = 12\sum s_i(\theta_i - \bar{\theta})^2$ and $\lambda^M = 4\sum s_i(\theta_i - \bar{\theta})^2$. If the approximate power is desired for the test using $n_i^0 = s_i n^0$ observations in the ith sample, when alternative $F_i(x) = F(x + \epsilon_i)$ for $i = 1, 2, \cdots, c$, is true, set $\epsilon_i = \theta_i/\sqrt{n^0}$ and compute

$$\lambda^H = 12\sum_{i=1}^{c} n_i^0(\epsilon_i - \bar{\epsilon})^2, \qquad \lambda^M = 4\sum_{i=1}^{c} n_i^0(\epsilon_i - \bar{\epsilon})^2.$$

For the given level of significance and $c - 1$ degrees of freedom, enter the Fix tables and find the approximate powers for these two tests at the given alternative. Because of the limited extent of the Fix tables, the power can be found only to the first decimal place without some sort of interpolation. In most instances, however, this accuracy should be sufficient, as it is not known how close these approximations are to the true power.

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