Asymptotic Behavior of the Solutions of the Third Order Nonlinear Differential Equations $u''' \pm t^{\sigma} u^{n} = 0$ (*) (**).

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Summary. – *The asymptotic behavior of the proper nonoscillator solutions of the nonlinear, third order, ordinary differential equation* (*) $u''' \pm t^{\sigma} u^{n} = 0$, *where* $n > 1$ *and* σ *is an arbitrary real number, is considered. Cases for* σ *and* n *are studied and the possible asymptotic behavior* ($t \to \infty$) *of the solutions of* (*) *are found and conditions for their existence are demonstrated.*

1. – Introduction.

Little is known about the asymptotic properties of the solutions of third order nonlinear differential equations although there have been several investigations of third order linear equations. Among these are the papers of AHMAD and LAZER [1], M. HANNAN [11], LAZER [19], SCHURR [23], and SINGH [24].

Autonomous third order nonlinear differential equations have been considered by J. O. C. EZEILO [8], V. HAAS [12], R. R. D. KEMP [14], and B. S. LALLI [18], while nonautonomous third order nonlinear equations have been studied by D. BOBROWSKI [3], EZELIO [5]-[7], J. W. HEIDEL [13], KIGURADZE [15]-[16], LEGATOS and STARKOS [20], LICKO and M. SVEC [21], J. L. NELSON [22], K. E. SWICK [25]-[26], W. R. UTZ [27], and P. WALTMAN [28].

The above mentioned papers on nonlinear differential equations have primarily discussed oscillation or other asymptotic behavior of equations of a particular form. This paper follows the same pattern in that it discusses the asymptotic behavior of nonoscillatory solutions of two particular third order, nonlinear, nonautomous equations. The equations considered were chosen because of wide interest in the second order equations of the same form whose asymptotic behavior has been discussed by R. BELLMAN in Chapter 7 of his book [2].

A proper solution of a third order differential equation is one which is real and has a continuous second derivative for $t \geq t_0$. This paper introduces a technique (Theorem 3.1) whereby the proper solutions of the equation

$$u''' - t^{\sigma} u^{n} = 0\,, \quad n > 1\,, \quad \sigma \in (-\infty, \infty) \tag{1.1}$$

(*) This paper is part of the author's dissertation which was prepared under the dirrection of Professor THOMAS G. HALLAM at Florida State University. This research was supported by NSF grant GP 11534.

(**) Entrata in Redazione il 9 maggio 1973.

can be shown to be bounded by certain powers of t. If $q(t) \geqslant 0$ is bounded below by a power (possibly negative) of t, then the above mentioned technique can be applied to the equation

$$u''' - q(t)\, u^n = 0$$

with similar results being obtained.

The asymptotic expansions in this paper are similar to the expansions found in Hallam and Heidel [10]. However, when their results are restricted to our situation, we are able to obtain an additional term in each expansion.

Our discussion is restricted to positive solutions since either u^n is not real for negative values of u or $(-u)^n = \pm u^n$. Thus, in the case of negative solutions of

$$u''' \mp t^\sigma u^n = 0$$

the discussion can be reduced to that of positive solutions of the same equation or of the equation

$$u''' \pm t^\sigma u^n = 0\,.$$

2. – Fundamental concepts.

To see if either of the equations

$$u''' \pm t^\sigma u^n = 0\,, \qquad n > 1 \tag{2.1}$$

have solutions of the form ct^w we substitute $u = ct^w$ into (2.1). After a simple calculation we have that if

$$\begin{aligned} c &= [\pm (\sigma + 3)(\sigma + n + 2)(\sigma + 2n + 1)/(n-1)^3]^{1/(n-1)} \\ &= [\pm w(w-1)(w-2)]^{1/(n-1)} \end{aligned} \tag{2.2}$$

where

$$w = -(\sigma + 3)/(n-1) \tag{2.3}$$

then ct^w is an exact solution of (2.1). Since we are concerned only with real solutions, we need to take note of the values of σ and n which yield a real value for c. Before examining the various cases of σ and n, we will state lemmas that will be used often in our study.

Lemma 2.1. – If $\lim_{t\to\infty} u(t) = \infty$, and if $u'(t) \geqslant 0$ eventually, then $u' \leqslant u^{1+\varepsilon}$ for $t \geqslant t_0$ for any $\varepsilon > 0$, except perhaps in a set of intervals of finite total length which depends upon ε.

The proof of this lemma may be found on page 97 of [2].

LEMMA 2.2. – Let $f(t)\geqslant 0$ for $t\geqslant t_0$ If $\int_{t_0}^{\infty} f(t)\,dt=\infty$ then

$$\int_{t_0}^{t} f(\tau)[1+o(1)]\,d\tau=\big(1+o(1)\big)\int_{t_0}^{t} f(\tau)\,d\tau$$

LEMMA 2.3. – Let $f(t)\geqslant 0$ for $t\geqslant t_0$. If $\int_{t_0}^{\infty} f(t)\,dt<\infty$ then

$$\int_{t}^{\infty} f(\tau)[1+o(1)]\,d\tau=[1+o(1)]\int_{t}^{\infty} f(\tau)\,d\tau\,.$$

The proofs of Lemmas 2.2 and 2.3 are trivial.

LEMMA 2.4. – Let $X=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ be a vector in R^3. Define the matrix

$$A=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -\gamma & -\beta & -\alpha\end{bmatrix},$$

where $\alpha=3(w-1)$, $\beta=3w^2-6w+2$, and $\gamma=w(w-1)(w-2)$ for some real constant w. The characteristic roots of A are $-w$, $1-w$, and $2-w$. Then, for $t\in[t_0,\infty)$ and for any negative characteristic root λ of A there exists a solution $\varphi(t)$ of

$$X'(t)=AX(t)+F(X)\,, \tag{2.4}$$

where

$$F(X)=\begin{bmatrix}0\\0\\x_1^n\end{bmatrix},\quad n>1\,,$$

such that

$$\varphi(t)=k\exp[\lambda t]+o\big(\exp[\lambda t]\big)\,, \tag{2.5}$$

where k is some nonzero constant. Conversely, if $w\neq 0, 1$, or 2, then any solution $\varphi(t)$ of (2.4) which tends to zero as t goes to infinity must satisfy (2.5) for some negative characteristic root of A.

The proof is a direct consequence of Theorems 4.1, 4.3, and 4.4 in Chapter 13 of [4].

LEMMA 2.5. – If the continuously differentiable function f is oscillatory, then for any two real numbers α, β the function $\alpha f' + \beta f$ is oscillatory.

PROOF. – If $\alpha = 0$, the lemma is true. If $\alpha \neq 0$, then it is sufficient to prove the lemma for $f' + \gamma f$, $\gamma = \beta/\alpha$. We will assume that $\gamma > 0$ as the proof for $\gamma = 0$ or $\gamma < 0$ is analogous to that for $\gamma > 0$. Let x, y, z be any three consecutive zeros of f. Then by Rolle's Theorem there exist $c \in (x, y)$ and $d \in (y, z)$ such that $f'(c) = f'(d) = 0$. It can be assumed without loss of generality that $f(t) > 0$, $t \in (x, y)$ and $f(t) < 0$, $t \in (y, z)$. Then there exist points near c such that $f' + \gamma f$ is positive and points near d such that $f' + \gamma f$ is negative. Since this procedure can be repeated for each triple of consecutive zeros, the function $f' + \gamma f$ is oscillatory.

LEMMA 2.6. – Let $b(t) \geq 0$ for $t \geq t_0$, $\int_{t_0}^{\infty} b(t)\,dt < \infty$, and p be a positive number. Then,

$$t^{-p}\int_{t_0}^{t} s^p b(s)\,ds$$

approaches zero as t approaches infinity.

LEMMA 2.7. – Let $b(t) \geq 0$ for $t \geq t_0$ and $\int_{t_0}^{\infty} s^p b(s)\,ds < \infty$ where $p > 0$; then

$$t^p\int_{t}^{\infty} b(s)\,ds$$

approaches zero as t approaches infinity.

The proofs of Lemmas 2.7 and 2.8 are contained in [9].

LEMMA 2.8. – If $|a| \leq K$, $|b| \leq K$, and $r \geq 1$, then

$$|a^r - b^r| \leq Kr|a - b|.$$

The proof of the lemma may be found in [10].

LEMMA 2.9. – If $\lim_{t\to\infty} f(t) = k$, a finite number, and $f'(t)$ exists and is eventually nonpositive or nonnegative, then for any $\varepsilon > 0$ and for t sufficiently large $|f'(t)| < \varepsilon$ except possibly on a set of finite measure.

PROOF. – Suppose $f'(t) \geq 0$ for $t \geq t_0$. Then, if there exists an $\varepsilon > 0$ such that $f'(t) \geq \varepsilon$ on a set A of infinite measure then

$$k - f(t_0) = \int_{t_0}^{\infty} f'(t)\,dt \geq \varepsilon m(A) = \infty.$$

where $m(A)$ denotes the measure of A. This contradicts the finiteness of k. The proof for $f'(t)$ eventually nonpositive is analogous.

3. – The equation $u''' - t^\sigma u^n = 0$, $w > 2$.

THEOREM 3.1. – If $u(t)$ is a solution of Eq.

$$u''' - t^\sigma u^n = 0\,, \qquad n > 1 \tag{3.1}$$

such that $\lim_{t\to\infty} u(t) = \infty$ then there exist numbers $\varkappa$ and λ such that for any $\varepsilon > 0$

$$\lim_{t\to\infty} u\,(t)/t^{\varkappa+\varepsilon} = 0$$

and

$$\lim_{t\to\infty} u(t)/t^{\lambda-\varepsilon} = \infty\,.$$

Moreover, $\lambda \geqslant 2$ and $\varkappa \leqslant w + 3/(n-1)$.

PROOF. – From $\lim_{t\to\infty} u''(t) = \infty$ we have $\lim_{t\to\infty} u'(t) = \infty$ and $\lim_{t\to\infty} u(t) = \infty$. From (3.1) we get that $u'''(t) > 0$. Choosing $\varepsilon_i > 0$, $i = 1, 2, 3$ small enough so that for any $0 < \varepsilon < n-1$, $(1+\varepsilon_3)(1+\varepsilon_2)(1+\varepsilon_1) < n - \varepsilon$, we have by applying Lemma 2.1 three successive times

$$u''' \leqslant u''^{1+\varepsilon_1} \leqslant (u'^{1+\varepsilon_2})^{1+\varepsilon_1} \leqslant (u^{1+\varepsilon_3})^{(1+\varepsilon_2)(1+\varepsilon_1)}$$

except possibly on a set A_ε of finite measure. It follows that for arbitrary $\delta > 0$

$$u'''/u^{n-\varepsilon} = t^\sigma u^\varepsilon < \delta$$

that except possibly on a finite set $A_\delta \supset A_\varepsilon$. Defining $k = -\sigma/\varepsilon$ implies that

$$u/t^k < \delta^{1/\varepsilon} = \delta_1\,, \qquad t \notin A_\delta\,.$$

If the set A_δ is bounded for all positive δ sufficiently small then clearly $\lim_{t\to\infty} (u/t^k) = 0$. If the set A_δ is not bounded for all positive δ sufficiently small then for a given δ the set A_δ is a sequence of intervals, $\{I_i\}_{i=1}^\infty$ whose length tends to zero. Also, there exists a subsequence $\{I_{i_j}\}_{i_j=1}^\infty$ of $\{I_i\}_{i=1}^\infty$, henceforth noted as $\{I_j\}_{j=1}^\infty$, such that there exists t_j, $t_j + h_j \in I_j$, $h_j > 0$ with $u(t_j) = 2\delta_1 t_j^k$ and $u(t_j + h_j) = \delta_1 (t_j + h_j)^k$. From the monotonicity of $u(t)$ we have

$$\delta_1 (t_j + h_j)^k = u(t_j + h_j) > u(t_j) = 2\delta_1 t_j^k\,.$$

If follows that

$$((t_j + h_j)/t_j)^k > 2\,. \tag{3.2}$$

Since the length of the intervals I_j tends to zero as j goes to infinity, $\lim_{j\to\infty} h_j = 0$. Taking the limit in expression (3.2) as j goes to infinity yields $1 \geqslant 2$, an obvious contradiction. Hence, $\lim_{t\to\infty} u(t)/t^k = 0$. Defining $\varkappa = \inf\{k\colon \lim_{t\to\infty} u(t)/t^k = 0\}$ we see that $\varkappa$ does exist and $\varkappa \leqslant -\sigma/\varepsilon$. By choosing ε arbitraryly close to $n-1$ we see that

$$\varkappa \leqslant -\sigma/(n-1) = w + 3/(n-1)\,.$$

From l'Hospital's Rule $\lim_{t\to\infty} u(t)/t^2 = \lim_{t\to\infty} u''(t)/2) = \infty$.
Therefore, $\lambda = \sup\{k\colon \lim_{t\to\infty} u(t)/t^k = \infty\}$ is well-defined and $\lambda \geqslant 2$.

COROLLARY 3.1. Under the hypothesis of Theorem 3.1 $\lambda \leqslant w$. Therefore, if $w < 2$ there are no solutions $u(t)$ of (3.1) such that $\lim_{t\to\infty} u''(t) = \infty$.

PROOF. – Suppose that the λ given by Theorem 3.1 is greater than two and greater than w. Then there exists a $k < \lambda$ such that $k > 2$, $k > w$ and $\lim_{t\to\infty} u(t)/t^k = \infty$. For t sufficiently large $u(t) > (nk + \sigma + 1)^{1/n} t^k$ which implies from (3.1)

$$u'''(t) > (nk + \sigma + 1)\, t^{nk+\sigma}\,, \qquad t \geqslant t_0\,.$$

Since $k > w$ and $k > 2$, it follows that

$$nk + \sigma > k - 3 > -1\,.$$

By successive integrations we get

$$u(t) > t^{nk+\sigma+3}$$

for t sufficiently large.

Since $k > 2$, we have $k(n-1) > -(\sigma + 3)$, or that there exists a $b_1 > 0$ such that

$$u(t) > (k + b_1 + 1)^{1/n} t^{k+b_1}\,. \tag{3.3}$$

Now substituting (3.3) into (3.1) we get again by successive integrations

$$u(t) > t^{(k+b_1)n+\sigma+3}$$

for t sufficiently large. We can define $b_{i+1} = b_1 + nb_i$, $i = 1, 2, 3, \ldots$, and repeat our process substituting $u(t) > (k + b_i + 1)^{1/n} t^{k+b_i}$ into (3.1) until we obtain after a finite number of steps

$$u(t) > t^{\varkappa+1}$$

for t sufficiently large, which contradicts the definition of $\varkappa$ in Theorem 3.1 and shows that $\lambda \leqslant w$.

If $\lambda = 2$ then clearly $\lambda \leqslant w$ whenever $w \geqslant 2$. Suppose now that $w < 2$ and set $k = \lambda = 2$. The inequality $w < 2$ implies that $2n + \sigma + 1 > 0$. We can now repeat the above argument to get $\lambda \leqslant w$. For $w < 2$ we have $2 \leqslant \lambda \leqslant w < 2$, which is absurd. In particular when $w < 2$ then there are no solutions $u(t)$ of (3.1) such that $\lim_{t\to\infty} u''(t) = \infty$.

THEOREM 3.2. – If $2n + \sigma + 1 < 0$, $(w > 2)$ then any positive, proper solution $u(t)$ of Eq. (3.1) has one of the asymptotic behaviors given by (3.4)-(3.11) below.
In the following statements a, b, d, and t_0 are constants.

(3.4) $$u(t) = ct^{w}(1 + o(1))$$

(3.5) $$u(t) = a + b(t - t_0) + d(t - t_0)^2 + \\ + \frac{d^n (t - t_0)^{2n+\sigma+3}}{(2n + \sigma + 3)(2n + \sigma + 2)(2n + \sigma + 1)} (1 + o(1))$$

(3.6) $$u(t) = a + b(t - t_0) + d(t - t_0)^2 + d^n[t \ln (t/t_0) - (t - t_0)](1 + o(1))$$

(3.7) $$u(t) = a + (t - t_0) + \tfrac{1}{2}d(t - t_0)^2 + \tfrac{1}{2}d^n \ln (t/t_0)(1 + o(1))$$

(3.8) $$u(t) = a + b(t - t_0) + \frac{b^n (t - t_0)^{n+\sigma+3}}{(n + \sigma + 1)(n + \sigma + 2)(n + \sigma + 3)} (1 + o(1))$$

(3.9) $$u(t) = a + b(t - t_0) + \tfrac{1}{2}b^n \ln (t/t_0)(1 + o(1))$$

(3.10) $$u(t) = a + \frac{a^n t^{\sigma+3}}{(\sigma + 1)(\sigma + 2)(\sigma + 3)} (1 + o(1))$$

There exists λ, $\varkappa$ such that

(3.11) There exists $\lambda, \varkappa$ such that $2 \leqslant \lambda,\ \varkappa \leqslant w + 3/(n - 1)$

such that for any $\varepsilon > 0$, $t^{\lambda-\varepsilon} \leqslant u(t) \leqslant t^{\varkappa+\varepsilon}$ for t sufficiently large and

$$\lim_{t\leftarrow\infty} u''(t) = \lim_{t\to\infty} u'(t) = \lim_{t\to\infty} u(t) = \infty; \qquad u(t) \neq ct^{w}.$$

PROOF. – Since $u(t)$ is a positive, proper solution of (3.1), $u''' = t^{\sigma} u^n > 0$ for positive t. It follows then that we have the following three cases:

(a) $$\lim_{t\to\infty} u''(t) = 0;$$

(b) $$\lim_{t\to\infty} u''(t) = d_1 > 0;$$

(c) $$\lim_{t\to\infty} u''(t) = \infty.$$

First, we will consider the case (a). From $u''(t)$ increasing and tending to zero as t tends to infinity, we have $u''(t) < 0$; therefore, $u'(t)$ is a decreasing function.

If $\lim_{t\to\infty} u'(t) = b > 0$, then by integrating Eq. (3.1) twice from t to infinity and then from t_0 to t we obtain either the asymptotic expansion (3.8) or (3.9).

If $\lim_{t\to\infty} u'(t) = 0$, then since $u'(t) > 0$, we must have $\lim_{t\to\infty} u(t) = a > 0$ or $\lim_{t\to\infty} u(t) = \infty$.

In the case where $\lim u(t) = a > 0$, by integrating (3.1) from t_0 to infinity three successive times, we obtain the asymptotic expansion (3.10).

To rule out the case where $\lim_{t\to\infty} u'(t) = 0$ and $\lim_{t\to\infty} u(t) = \infty$, the transformations $u = ct^w v$ and $t = e^s$ are applied in (3.1). This leads to the equation

$$v''' + \alpha v'' + \beta v' + \gamma v = \gamma v^n \tag{3.12}$$

where $\alpha = 3(w-1)$, $\beta = 3w^2 - 6w + 2$ and $\gamma = w(w-1)(w-2)$. A simple calculation yields $-w$, $1-w$, and $2-w$ as the characteristic roots associated with the linear part of (3.12),

$$v''' + \alpha v'' + \beta v' + \gamma v = 0 \,. \tag{3.13}$$

Our hypothesis $w > 2$ implies that the three characteristic roots are negative. Placing (3.12) into system form we get

$$X' = AX + F(X) \tag{3.14}$$

where

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad x_1 = v,\ x_2 = v',\ x_3 = v'',\ A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\gamma & -\beta & -\alpha \end{bmatrix}, \quad F(X) = \begin{bmatrix} 0 \\ 0 \\ x_1^n \end{bmatrix}.$$

By Lemma 2.4, it follows that

$$v(s) = k \exp[\lambda s] + o(\exp[\lambda s])$$

where $k \neq 0$ and λ is one of the three characteristic roots. Transforming these three forms to there corresponding forms for Eq. (3.1), we see that

$$u(t) = kct^{\lambda+w} + o(t^{\lambda+w}) \,.$$

But for none of the three possible values of λ, $-w$, $1-w$, $2-w$, can $u(t)$ satisfy $\lim_{t\to\infty} u(t) = \infty$ and $\lim_{t\to\infty} u'(t) = 0$.

In case (b) we have $u(t) = t^2(d_1/2 + o(1))$. Letting $d = d_1/2$ and integrating (3.1) first from t to infinity and then twice from t_0 to t we obtain one of the asymptotic expansions (3.5), (3.6), or (3.7).

Case (c) yields the asymptotic expansion (3.4) or by Theorem 3.1 the asymptotic behavior described in statement (3.11).

Now we will show the existence of solutions which possess the asymptotic expansions (3.5)-(3.10). The method of successive approximations will be used in a manner similar to that used by HALLAM and HEIDEL [10]. Let the constants c_i, $i = 0, 1$ and d_j, $j = 1, 2, 3$ be given as follows:

$$c_0 = -\tfrac{1}{2}, \quad c_1 = 1$$
$$d_1 = \tfrac{1}{2}, \quad d_2 = -1, \quad d_3 = \tfrac{1}{2}.$$

LEMMA 3.1. – Let α be one of the integers 0, 1 or 2; suppose that a_p, $p = 0, \ldots, \alpha$ are given real constants. If $y(t)$ is a solution of the integral equations

$$y(t) = \sum_{p=0}^{\alpha} \frac{a_p}{p!}(t-t_0)^p + \sum_{p=0}^{1-\alpha} c_p (t-t_0)^{2-p} \int_t^{\infty} (s-t_0)^p s^\sigma y^n(s)\, ds + \sum_{p=1}^{\alpha+1} d_{2+\alpha-p} (t-t_0)^{\alpha-p+1} \int_{t_0}^{t} (s-t_0)^{1-\alpha+p} s^\sigma y^n(s)\, ds\,,$$

(3.15)
$$y^{(k)}(t) = \sum_{p=k}^{\alpha} \frac{a_p}{(p-k)!}(t-t_0)^{p-k} + \sum_{p=0}^{1-\alpha} (-1)^{p+1} (t-t_0)^{2-p-k} \int_t^{\infty} (s-t_0)^p s^\sigma y^n(s)\, ds + \sum_{p=1}^{\alpha+1-k} (-1)^{\alpha+p-1} (t-t_0)^{\alpha-p-k+1} \int_{t_0} (s-t_0)^{1-\alpha+p} s^\sigma y^n(s)\, ds \qquad 0 < k \leqslant \alpha\,,$$

and for $0 < k \leqslant 2-\alpha$

$$y^{(\alpha+k)}(t) = \sum_{p=0}^{2-\alpha-k} (-1)^{p+1} (t-t_0)^{2-\alpha-p-k} \int_t^{\infty} (s-t_0)^p s^\sigma y^n(s)\, ds\,,$$

then $y(t)$ is a solution of (3.1) such that $y^{(i)}(t_0) = a_i$, $i = 0, 1, \ldots, \alpha$.

The proof follows by direct verification.

Now define for $\alpha = 0, 1, 2$ and for given constants a_i, $i = 0, \ldots, \alpha$

$$B_k = \sum_{j=k}^{\alpha} \frac{|a_j|}{(j-k)!} + 1\,, \qquad k = 0, 1, 2\,,$$

$$B = \max_{k=0,1,2} \{B_k^n, nB_k\}\,,$$

$$R_\alpha = \min_{i=0\ldots,\alpha} \{|in+\sigma+3|, |in+\sigma+2|, |in+\sigma+1|\}\,,$$

and

$$T_\alpha = [R_\alpha/(4B)]^{1/[\alpha n+\sigma+2+sgn(1-\alpha)]}\,.$$

LEMMA 3.2. – For $t \geqslant t_0 \geqslant T_\alpha$ the following statements are true:

(*a*) for $0 < k \leqslant 2 - \alpha$

$$B \sum_{p=0}^{2-\alpha-k} t^{2-2\alpha-p} \int_t^\infty s^{p+\sigma+n(\alpha-k)}\, ds \leqslant \tfrac{1}{2};$$

(*b*) for $0 < k \leqslant \alpha$

$$B \sum_{p=0}^{1-\alpha} t^{2-p-\alpha} \int_t^\infty s^{p+\sigma+n\alpha-kn}\, ds$$

$$+ B \sum_{p=1}^{\alpha+1-k} t^{1-p} \int_{t_0}^t s^{1-\alpha+p+\sigma+n\alpha-kn}\, ds \leqslant \tfrac{1}{2};$$

and

(*c*) for $k = 0$

$$B \sum_{p=0}^{1-\alpha} |c p| t^{2-p-\alpha} \int_t^\infty s^{p+\sigma+\alpha n}\, ds$$

$$+ B \sum_{p=1}^{\alpha+1} |d_{2+\alpha-p}| t^{1-p} \int_{t_0}^t s^{1-\alpha+p+\sigma+n\alpha}\, ds \leqslant \tfrac{1}{2}.$$

The proof of Lemma 3.2 is straightforward and is, therefore, omitted.

THEOREM 3.3. – If $\alpha n + \sigma + 3 - \alpha < 0$, $\alpha = 0$, 1 or 2 and a_p, $p = 0, \dots, \alpha$ are given constants, then for $t_0 \geqslant T_\alpha$ there exists a solution $y(t)$ of (3.1) which possesses the asymptotic behavior (3.10) whenever $\alpha = 0$; (3.8) and (3.9) whenever $\alpha = 1$; and (3.5), (3.6) and (3.7) whenever $\alpha = 2$. Furthermore, the derivatives of $y(t)$ have the asymptotic expansions given by

$$y^{(i)}(t) = \sum_{p=i}^{\alpha} \frac{a_p}{(p-i)!} (t - t_0)^{p-i} + \frac{a_\alpha^n (t - t_0)^{\alpha n + \sigma + 3 - i}}{(\alpha n + \sigma + 3 - i) \dots (\alpha n + \sigma + 1)} \bigl(1 + o(1)\bigr), \qquad i = 1, 2 \tag{3.16}$$

provided that when $i = 1$, $2n + \sigma + 1 \neq -1$. In the case where $i = 1$ and $2n + \sigma + 1 = -1$

$$y^{(1)}(t) = a_1 + a_2(t - t_0) + a_2^n \ln (t/t_0)\bigl(1 + o(1)\bigr). \tag{3.17}$$

Because this proof is similar to the existence proof in [10], we will leave out many of the details.

PROOF. – Define

$$\Phi_0^0(t;\,t_0) \quad = \sum_{p=0}^{\alpha} \frac{a_p}{p!}(t-t_0)^p$$

$$\Phi_0^k(t;\,t_0) \quad = \sum_{p=k}^{\alpha} \frac{a_p}{(p-k)!}(t-t_0)^{p-k}\,, \qquad k=1,\ldots,\alpha$$

$$\Phi_0^{\alpha+k}(t;\,t_0) = t^{\alpha-k} \qquad k=1,\ldots,2-\alpha\,.$$

Inductively, we define for $\eta = 0, 1, 2,$

$$\Phi_{\eta+1}^0(t;\,t_0) = \Phi_0^0(t;\,t_0) + \sum_{p=0}^{1-\alpha} c_p(t-t_0)^{2-p}\int_t^{\infty}(s-t_0)^p s^\sigma \Phi_\eta^0(s;\,t_0)^n\,ds$$

$$+\sum_{p=1}^{\alpha+1} d_{2+\alpha-p}(t-t_0)^{\alpha-p+1}\int_{t_0}^{t}(s-t_0)^{1-\alpha+p}s^\sigma \Phi_\eta^0(s;\,t_0)^n\,ds$$

$$\Phi_{\eta+1}^k(t;\,t_0) = \Phi_0^k(t;\,t_0) + \sum_{p=0}^{1-\alpha}(-1)^{p+1}(t-t_0)^{2-p-k}\int_t^{\infty}(s-t_0)^p s^\sigma \Phi_\eta^k(s;\,t_0)^n\,ds$$

$$+\sum_{p=1}^{\alpha+1-k}(-1)^{\alpha+p-1}(t-t_0)^{\alpha-p-k+1}\int_{t_0}^{t}(s-t_0)^{1-\alpha+p}s^\sigma \Phi_\eta^k(s;\,t_0)^n\,ds \qquad 0<k\leqslant\alpha$$

and for $0<k\leqslant 2-\alpha$

$$\Phi_{\eta+1}^{(\alpha+k)}(t;\,t_0) = \sum_{p=0}^{2-\alpha-k}(-1)^{p+1}(t-t_0)^{2-\alpha-p-k}\int_t^{\infty}(s-t_0)^p s^\sigma \Phi_\eta^{\alpha+k}(s;\,t_0)^n\,ds\,.$$

For $t\geqslant T_\alpha$ it is easy to see that

$$|\Phi_0^k(t;\,t_0)|\leqslant B_k t^{\alpha-k}\,, \qquad k=0,1,2\,. \tag{3.18}$$

Now, it can be shown by induction that for $t\geqslant T_\alpha$

$$|\Phi_{\eta+1}^k(t;\,t_0)|\leqslant B_k t^{\alpha-k}\,, \qquad k = 0,1,2\,,\ \eta = 0,1,2,\ldots\,.$$

In order to establish the existence of a solution $(t;\,t_0)$ of the integrals (3.15), it can be shown that

$$|\Phi_{\eta+1}^k(t;\,t_0) - \Phi_\eta^k(t;\,t_0)|\leqslant t^{\alpha-k}/2^{\eta+1} \qquad \text{for } t\geqslant T_\alpha,\ k=0,1,\ldots\,. \tag{3.19}$$

Therefore, for each $k = 0, 1, 2$ $\lim_{\eta\to\infty} \Phi_\eta^k(t; t_0) = \Phi^k(t; t_0)$ exists uniformly on compact subintervals of $[t_0, \infty)$. Also,

$$|\Phi^k(t; t_0)| \leq B_k t^{\alpha-k}, \qquad k = 0, 1, 2 .$$

We will now show that $\Phi^0(t; t_0)$ has the proper asymptotic behavior. Let us first note that

$$\lim_{t\to\infty} [\Phi^0(t; t_0)/t - t_0)^\alpha] = a_\alpha/a! + d_{2+\alpha}\int_{t_0}^{\infty}(s - t_0)^{2-\alpha} s^\sigma \Phi^0(s; t_0)^n s\, ds = \lambda_\alpha .$$

If $\alpha n + \sigma + 3$ and $\alpha n + \sigma + 2$ are negative for $\alpha = 2$ or if $\alpha n + \sigma + 3$ is negative for $\alpha = 1$, then Lemma 2.6 and 2.7 assure of having the indeterminate form $0/0$ in the expression

$$\Big[\sum_{p=0}^{1-\alpha} c_p (t - t_0)^{2-p}\int_t^\infty (s - t_0)^p s^\sigma \Phi^0(s; t_0)^n ds + \sum_{p=1}^{\alpha+1} d_{2+\alpha-p}(t - t_0)^{\alpha-p+1}\int_{t_0}^{t}(s - t_0)^p s^\sigma \Phi^0(s; t_0)^n ds + d_{1+\alpha}\int_t^\infty (s - t_0) s^\sigma \Phi^0(s; t_0)^n ds\Big]\Big/(t - t_0)^{\alpha n+\sigma+3} .$$

If these numbers are positive then one may also apply L'Hospital's Rule. Thus, in either case we have

$$\begin{aligned}(3.20)\quad &\lim_{t\to\infty}\Big[\sum_{p=0}^{1-\alpha} c_p(t - t_0)^{2-p}\int_t^\infty (s - t_0)^p s^\sigma \Phi^0(s; t_0)^n ds \\ &+ \sum_{p=2}^{\alpha+1} d_{2+\alpha-p}(t - t_0)^{\alpha-p+1}\int_{t_0}^{t}(s - t_0)^p s^\sigma \Phi^0(;s; t_0)^n ds \\ &+ d_{1+\alpha}\int_t^\infty (s - t_0) s^\sigma \Phi^0(s; t_0)^n ds\Big]\Big/(t - t_0)^{\alpha n+\sigma+3} \\ &= \lim_{t\to\infty}\Big[\sum_{p=0}^{1-\alpha}(-1)^{p+1}(t - t_0)^{1-p}\int_t^\infty (s - t_0)^p s^\sigma \Phi^0(s; t_0)^n ds \\ &+ \sum_{p=2}^{\alpha+1}(-1)^{p-1}(t - t_0)^{p-2}\int_{t_0}^{t}(s - t_0)^{3-p} s^\sigma \Phi^0(s; t_0)^n ds\Big]\Big/(\alpha n + \sigma + 3)(t - t_0)^{\alpha n+\sigma+2}\end{aligned}$$

$$=\lim_{t\to\infty}\begin{cases} -\int_t^\infty s^\sigma \Phi^0(s;\,t_0)^n\,ds/[(\alpha n+\sigma+3)(\alpha n+\sigma+2)(t-t_0)^{\alpha n+\sigma+1}] & \text{if } \alpha=0 \text{ or } 1\\ \int_{t_0}^t s^\sigma \Phi^0(s;\,t_0)^n\,ds/[(\alpha n+\sigma+3)(\alpha n+\sigma+2)(t-t_0)^{\alpha n+\sigma+1} & \text{if } \alpha=2\end{cases}$$

$$=\lim_{t\to\infty} t^\sigma\Phi^0(t;\,t_0)^n/[(\alpha n+\sigma+3)(\alpha n+\sigma+2)(\alpha n+\sigma+1)(t-t_0)^{\alpha n+\sigma}$$

$$=\lambda_\alpha^n/[(\alpha n+\sigma+3)(\alpha n+\sigma+2)(\alpha n+\sigma+1)]\,.$$

This shows the existence of the asymptotic expansions (3.3), (3.6) and (3.8). If $\alpha n+\sigma+3=0$ for $\alpha=1$, then

$$\lim_{t\to\infty}\Big[-\tfrac{1}{2}(t-t_0)^2\int_t^\infty s^\sigma\Phi^0(s;\,t_0)^n\,ds+(t-t_0)\int_t^\infty(s-t_0)s^\sigma\Phi^0(s;\,t_0)^n\,ds$$

$$+\tfrac{1}{2}\int_{t_0}^t(s-t_0)^2s^\sigma\Phi^0(s;\,t_0)^n\,ds\Big]/\ln(t/t_0)$$

$$=\lim_{t\to\infty}\Big[-(t-t_0)\int_t^\infty s^\sigma\Phi^0(s;\,t_0)^n\,ds+\int_t^\infty(s-t_0)s^\sigma\Phi^0(s;\,t_0)^n\,ds\Big]/t^{-1}$$

$$=\lim_{t\to\infty}\int_t^\infty s^\sigma\Phi^0(s;\,t_0)^n\,ds/t^{-2}=\tfrac{1}{2}\lim_{t\to\infty}\Phi^0(t;\,t_0)^n=\tfrac{1}{2}\lambda_1^n\,.$$

From the above we have the existence of the asymptotic behavior (3.7). Similarly, one can establish the desired asymptotic behavior for $\Phi^0(t;\,t_0)$ when $2n+\sigma+3=0$ or $2n+\sigma+2=0$ for $\Phi^k(t;\,t_0)$, $k=1,2$.

THEOREM 3.4. – If $k>\varkappa$, where $\varkappa$ is the number given by Theorem 3.1, then any solution $u(t)$ of Eq. (3.1) such that $u(t_0)>at_0^k$, $u'(t_0)>akt_0^{k-1}$, and $u''(t_0)>ak(k-1)t_0^{k-2}$, where $a=[k(k-1)(k-2)]^{1/(n-1)}$, has a finite escape time, that is, there exists a T, $t_0<T<\infty$ such that $\lim_{t\to T^-}u(t)=\infty$.

PROOF. – Suppose that $u(t)$ does not have a finite escape time and hence exists on $[t_0,\infty)$. Let $\psi(t)=at^k$ and $J=[t_0,x)$ be the maximum interval such that $u(t)>\psi(t)$. From the definition of $wk>\varkappa\geqslant w$ implies that $\sigma+nk>k-3$. For $t\in J$ we have from Eq. (3.1)

$$u'''(t)>a^nt^{\sigma+nk}>a^nt^{k-3}=\psi(t)'''\,. \tag{3.21}$$

From $u''(t_0)>\psi''(t_0)$ and (3.12) it follows that $u''(t)>\psi''(t)$, $t\in J$. Similarly, we can obtain that $u'(t)>\psi'(t)$, $t\in J$. In order for $u(t)$ and $\psi(t)$ to intersect at time t_1, there must exist a time t_2 prior to t_1 such that $u'(t_2) < \psi'(t_2)$. Therefore, we must have $J = [t_0, \infty)$. An application of Theorem 3.1 yields that $u(t) < t^{\varkappa}$ for t sufficiently large which contradicts $u(t) > t^k > t^{\varkappa}$ for $t \in J$.

COROLLARY 3.2. – Given $u(t_0)$ and $u'(t_0)$ to be any two arbitrary positive numbers, it is possible to choose $u''(t_0)$ in such a way that the corresponding solution $u(t)$ has a finite escape time.

PROOF. – By integrating Eq. (3.1) and using integration by parts we see that

$$u(t) = u(t_0) + u'(t_0)(t - t_0) + \tfrac{1}{2} u''(t_0)(t - t_0)^2 + \tfrac{1}{2}\int_{t_0}^{t} (s - t)^2 s^{\sigma} u^n(s)\, ds\,,$$

$$u'(t) = u'(t_0) + u''(t_0)(t - t_0) + \int_{t_0}^{t} (t - s) s^{\sigma} u^n(s) ds\,,$$

and

$$u''(t) = u''(t_0) + \int_{t_0}^{t} s^{\sigma} u^n(s)\, ds$$

which implies respectively that for $t_* > t_0$

$$2u(t_*)/(t_* - t_0)^2 > u''(t_0)\,,$$

$$u'(t_*)/(t_* - t_0) > u''(t_0)\,,$$

and

$$u''(t_*) > u''(t_0)\,.$$

Given $t_* > t_0$, we choose $u''(t_0)$ so that $u''(t_0)$ so that $u''(t_0) > ak(k-1)t_*^{k-2}$, where a and k are as in Theorem 3.4, then by Theorem 3.4 $u(t)$ has a finite escape time.

COROLLARY 3.3. – If $u(t)$ is a solution of Eq. (3.1) such that $u(t_0) \geqslant ct_0^w$, $u'(t_0) \geqslant cwt_0^{w-1}$ and $u''(t_0) \geqslant cw(w-1)t_0^{w-2}$ which strict inequality holding in at least one of the three inequalities, then $u(t) > ct^w$ for $t > t_0$.

PROOF. – This follows from the proof of Theorem 3.4 by using the properties of c and w from Eq. (2.2) and (2.3).

4. – The equation $u''' - t^{\sigma} u^n = 0$, $w \leqslant 2$.

Now we will consider the remaining cases of σ and n for Eq. (3.1).

THEOREM 4.1. – Any positive, proper solution $u(t)$ of Eq. (3.1) with $w \leqslant 2$ has the following asymptotic behavior:

σ, n	w	asymptotic behavior
$2n+\sigma+1=0$	$w=2$	(3.8) (3.9) (3.10) (3.11) (4.2)
$2n+\sigma+1>0,\ \sigma+3<0,\ n+\sigma+2<0$	$1<w<2$	(3.8) (3.9) (3.10)
$n+\sigma+2=0$	$w=1$	(3.10) (4.2)
$2n+\sigma+1>0,\ \sigma+3<0,\ n+\sigma+2>0$	$0<w<1$	(3.4) (3.10) (4.1)
$\sigma+3=0$	$w=0$	no positive proper
$\sigma+3>0$	$w<0$	solutions exist

where

(4.1) the solution $u(t)$ intersects with the function $v(t)=ct^w$ infinitely often,

and

$$\lim_{t\to\infty} u''(t)=\lim_{t\to\infty} u'(t)=0;\ \lim_{t\to\infty} u(t)=\infty;\ u(t)\neq ct^w . \tag{4.2}$$

In this section we have, as in Section 3, $u'''(t)>0$ and the three cases

(a) $$\lim_{t\to\infty} u''(t)=0;$$

(b) $$\lim_{t\to\infty} u''(t)=d_1>0;$$

and

(c) $$\lim_{t\to\infty} u''(t)=\infty .$$

LEMMA 4.1. If $2n+\sigma+1\geqslant 0$, $(w\leqslant 2)$ then Eq. (3.1) does not have a positive, proper solution $u(t)$ such that $\lim_{t\to\infty} u''(t)=d_1>0$.

PROOF. – Suppose there exists such a solution. Then $u(t)=dt^2(1+o(1))$, where $d=d_1/2$. From (3.1) we have

$$u'''=d^n t^{\sigma+2n}(1+o(1)) .$$

Since $2n+\sigma\geqslant -1$, we have by integrating that $\lim_{t\to\infty} u''(t)=\infty$. This is a contradiction.

LEMMA 4.2. – If $n+\sigma+2\geqslant 0$, $(w\leqslant 1)$ then the Eq. (3.1) does not have a positive, proper solution $u(t)$ such that $\lim_{t\to\infty} u'(t)=b>0$.

LEMMA 4.3. – If $\sigma+3\geqslant 0$, $(w\leqslant 0)$ then the Eq. (3.1) does not have a positive, proper solution $u(t)$ such that $\lim_{t\to\infty} u(t) = a > 0$.

The proofs of Lemmas 4.2 and 4.3 are similar to the proof of Lemma 4.1.

For the case $2n+\sigma+1=0$ $(w=2)$ we eliminate case (b) by Lemma 4.1 and apply Theorem 3.1 to case (c) to get the possible asymptotic behavior (3.11). For case (a) since $u''(t)<0$ and $u(t)>0$ we have

$$(a_1) \qquad \lim_{t\to\infty} u'(t) = b > 0$$

or

$$(a_2) \qquad \lim_{t\to\infty} u'(t) = 0\,.$$

By substituting $u(t)=bt(1+o(1))$ into (3.1) and integrating case (a_1) yields either (3.8) or (3.9). In the case (a_2) if $\lim_{t\to\infty} u(t) = a$, $0<a<\infty$, we substitute $u(t)= = a(1+o(1))$ into (3.1) and integrate from t to infinity three times to obtain the asymptotic expansion (3.10). If $\lim_{t\to\infty} u(t)=\infty$, then we have the possible asymptotic behavior (4.2).

At this point we observe that for all remaining cases, as indicated by the chart, Corollary 3.1 eliminates case (c) and Lemma 4.1 eliminates case (b).

For $2n+\sigma+1>0$, $\sigma+3<0$, $n+\sigma+2<0$ $(1<w<2)$ we note that c is not a positive number, hence we do not have the behavior (3.4). For the case (a) we have that $\lim_{t\to\infty} u'(t) = b\geqslant 0$. Setting $v=u/t^w$ we obtain by L'Hospital's Rule $\lim v(t) = = \lim_{t\to\infty}[u'/(wt^{w-1})]=0$. Performing the transformations $v=u/t^w$ and $t=e^s$ in Eq. (3.1) we have

$$(4.3) \qquad v''' + \alpha v'' + \beta v + \gamma v = v^n$$

where

$$(4.4) \qquad \alpha = 3(w-1)\,, \quad \beta = 3w^2-6w+2 \quad \text{and} \quad \gamma = w(w-1)(w-2)\,.$$

Changing (4.3) into system form and applying Lemma 2.4 in a manner analogous to that in the paragraph containing Eq. (3.14) we have

$$v(s) = k\exp[\lambda s] + o(\exp[\lambda s])$$

where $k\neq 0$ and λ is one of the negative characteristic roots of the linear part of (4.3); $-w$ or $1-w$. Transforming these two forms to their corresponding forms for $u(t)$, we see that

$$u(t) = kt^{\lambda+w} + o(t^{\lambda+w})\,.$$

For $\lambda=-w$ and $\lambda=1-w$ this yields the asymptotic behaviors (3.10) and (3.8) or (3.9) respectively.

For $n+\sigma+2=0$ $(w=1)$ we handle case (a) in the same manner as we did for $w=2$ in the argument following Lemma 4.3.

For $2n+\sigma+1>0$, $n+\sigma+2>0$, $\sigma+3<0$ $(0<w<1)$ we have from $\lim_{t\to\infty} u(t)=0$

$$(a_1) \qquad \lim_{t\to\infty} u'(t)=0$$

or

$$(a_2) \qquad \lim_{t\to\infty} u'(t)=a\,.$$

Since Lemma 4.2 eliminates case (a_2) we have either $\lim_{t\to\infty} u(t)=a$ or $\lim_{t\to\infty} u(t)=\infty$. The first limit yields the asymptotic expansion (3.10). The second condition is satisfied by the solution $u(t)=ct^w$. To investigate the possibility of other solutions of this form, we again make the transformations $u=ct^w$ and $t=e^s$. We obtain as in the past

$$v'''+\alpha v''+\beta v'+\gamma v=\alpha v^n\,, \tag{4.5}$$

where α, β and γ are defined in (4.4). If $\overline{\lim_{t\to\infty}}\, k(t)=k$, a nonzero finite number, then it follows from L'Hospital's Rule, Eq. (3.1), and the definition of w that

$$\begin{aligned} k &= \overline{\lim_{t\to\infty}}\, u/(ct^w) \leqslant \overline{\lim_{t\to\infty}}\, u'''/[cw(w-1)(w-2)t^{w-3}] \\ &= \overline{\lim_{t\to\infty}}\, tu^n/[cw(w-1)(w-2)t^{w-3}] = \overline{\lim_{t\to\infty}}\, [u/(ct^w)]^n = k^n\,. \end{aligned} \tag{4.6}$$

Therefore, we have $k\geqslant 1$. Likewise, if $\lim_{t\to\infty} v(t)=m$, not zero or infinity, an analogous argument shows that $m\leqslant 1$. From this we see that one of the following cases must occur:

(a_{11}) $\lim_{t\to\infty} v(t)=1$;

(a_{12}) $\lim_{t\to\infty} v(t)=0$;

(a_{13}) $\lim_{t\to\infty} v(t)=\infty$;

(a_{14}) $v(t)$ intersects the line $v=1$ infinity often and has no limit as t tends to infinity.

In case (a_{12}) we can proceed as was done in the paragraph containing Eq. (3.14). This would lead to

$$v(s)=k\exp[-ws]+o(\exp[-ws])\,, \qquad k\neq 0\,.$$

This leads to $u(t)=ck(1+o(1))$ which is the asymptotic expansion (3.10).

To eliminate $v(t)$ tending to infinity as t tends to infinity we note that for our choice of w

$$\alpha < 0\,, \quad \beta < 0\,, \quad \gamma > 0\,.$$

If $v(s)$ is not eventually monotone increasing, then $v'(s)$ is oscillatory. By applying Lemma 2.5 first to the function $v'(s)$ and then to the function

$$f(s) = \mu[v''(s) + (\beta/\mu)v'(s)]$$

where

$$\mu = \tfrac{1}{2}[\alpha + (\alpha^2 - 4\beta)^{\frac{1}{2}}]$$

we have that the function

$$v''' + \alpha v'' + \beta v' = [v''' + (\beta/\mu)v''] + \mu[v'' + (\beta/\mu)v']$$

is oscillatory. Using Eq. (4.5) $\lim_{s\to\infty} v(s) = \infty$ implies that this is impossible, hence $v'(s)$ is eventually positive. It follows that

$$v''' + \alpha v'' = \gamma(v^n - v) - \beta v' > 0\,.$$

By Lemma 2.5 $v''(s)$ does not oscillate for otherwise the function $f(s) = v'''(s) + v''(s)$ would also oscillate. If $v''(s) \geqslant 0$ or if $\lim_{s\to\infty} v''(s) = 0$ we have for all s sufficiently large

$$v'''' = \gamma(v^n - v) - \beta v' - \alpha v'' > 0\,.$$

With $v'''(s)$ eventually positive and $\lim_{s\to\infty} v(s) = \infty$ the following possibilities exist:

$$\lim_{s\to\infty} v''(s) = \lim_{s\to\infty} v'(s) = \lim_{s\to\infty} v(s) = \infty\,,$$

or

$$\lim_{s\to\infty} v''(s) = k \geqslant 0\,, \quad \lim_{s\to\infty} v'(s) = \lim_{s\to\infty} v(s) = \infty\,,$$

or

$$\lim_{s\to\infty} v''(s) = k_1 \geqslant 0\,, \quad \lim_{s\to\infty} v'(s) = k_2 \geqslant 0\,, \quad \lim_{s\to\infty} v(s) = \infty\,.$$

If $\lim_{s\to\infty} v^{(i)}(s)$, $i = 1, 2$ is finite then we have $\lim_{s\to\infty} v^{(i)}(s)/v^n(s) = 0$ since $\lim_{s\to\infty} v(s) = \infty$ and from Lemma 2.9 $v^{(i-1)}(s)/v^n(s)$ is arbitrary small except possibly on a finite set. If $\lim_{s\to\infty} v^{(i)}(s) = \infty$, $i = 0, 1, 2$, then since $v^{(i+1)}(s) \geqslant 0$ eventually we have by Lemma 2.1 $v^{(i)} \leqslant [v^{(i-2)}]^{(1+\varepsilon)} \leqslant v^{(n-1)/2}$ for an appropriate $\varepsilon > 0$ which implies that $v^{(i)}(s)/v^n(s)$ is arbitrarily small except possibly on some finite set. In any case

$$[v''' + \alpha v'' + \beta v' + kv]/v^n$$

can be made arbitrary small except on a finite set. This contradicts equation (4.5). If $\lim_{s\to\infty} v''(s) \neq 0$ but $v''(s) \leqslant 0$ then we have $\lim_{s\to\infty} v'(s) = k = 0$ which implies by Lemma 2.9 that for arbitrary $\varepsilon > 0$

$$-\varepsilon < v''(s) \leqslant 0$$

except possibly on a set of finite measure. Let $\{s_i\}_{i=1}^{\infty}$ be a sequence of points where $v''(s)$ is maximum. Then for points sufficiently large

$$v^n(s_i) - \gamma v(s_i) - \beta v'(s_i) - \alpha v''(s_i) > 0 \tag{4.7}$$

but by equation (4.5) statement (4.7) is equal to $v'''(s_i) = 0$.

Cases (a_{11}) and (a_{1q}) yield the asymptotic behaviors (3.4) and (4.1) respectively.

If $\sigma + 3 \geqslant 0$ $(w \leqslant 0)$ Lemma 4.2 rules out $\lim_{t\to\infty} u'(t) = b > 0$ and Lemma 4.3 eliminates $\lim_{t\to\infty} u(t) = a \geqslant 0$. The only remaining possibility is for $\lim_{t\to\infty} u(t) = \infty$ with $\lim_{t\to\infty} u'(t) = 0$. To eliminate this possibility we first observe that $u''(t) < 0$ and $u'(t) > 0$ which with L'Hospital's Rule implies

$$0 \leqslant \overline{\lim_{t\to\infty}}\, tu'/u \leqslant \overline{\lim_{t\to\infty}}\, [1 + tu''/u'] \leqslant 1\,. \tag{4.8}$$

In equation (3.1) we substitute $t = e^s$ whenever $w = 0$ and $u = t^w v$, $t = e^s$ whenever $w < 0$. We get respectively

$$u'''(s) - 3u''(s) + 2u'(s) = u^n(s) \tag{4.9}$$

and

$$v''' + \alpha v'' + \beta v' + \gamma v = v^n \tag{4.10}$$

where α, β, γ are defined by (4.4).

From (4.8) we see that

$$\begin{aligned} u''' - 3u'' &= u^n(s) - 2u'(s) = u^n(t) - 2tu'(t) \\ &= u^n(t)[1 - 2tu'(t)/u^n(t)] > 0\,, \end{aligned} \tag{4.11}$$

and since

$$v'(s)/v(s) = (tu' - wu)t^{-w}/(ut^{-w}) = tu'/u - w < 1 - w\,,$$

$$v''' - \alpha v'' = v^n - \gamma v - \beta v' > 0\,. \tag{4.12}$$

By Lemma 2.5 statements (4.11) and (4.12) imply respectively that the functions $u''(s)$ and $v''(s)$ do not oscillate, which, in turn, implies that the functions $u'(s)$ and $v'(s)$ do not oscillate.

For both $w=0$ and $w<0$ we can proceed as we did for $0<w<1$ in the above paragraph. Thus, we are able to conclude that for $w\leqslant 0$, there are no positive, proper solutions for Eq. (3.1).

Theorem 3.3 shows the existence of the asymptotic expansions (3.5)-(3.10) for all values of σ and n that permit them to exist. Solutions with a finite escape time can be shown to exist for values of $w\leqslant 2$ by methods analogous to Theorem 3.4.

EXAMPLE. – If $n=-\frac{7}{2}$, then $u(t)=\left(1-2/\sqrt[3]{15}t\right)^{-\frac{2}{3}}$ is a solution of the differential equation $u'''-u^n=0$ with a finite escape time.

5. – Notes.

i) If in Eq. (3.1) t^σ is replaced by $q(t)$, we can use the techniques of Theorem 3.1 to show the following.

THEOREM 5.1. – If for $t\geqslant t_0$, $0<\alpha t^{-\varepsilon k}\leqslant q(t)$ for some real numbers α, k and for some ε, $0<\varepsilon<n-1$, then there exists a constant $\beta>0$ such that for any positive, proper solution $u(t)$ of the equation

$$u'''-q(t)u^n=0\,,\qquad n>1$$

satisfies $u(t)\leqslant\beta t^k$ for t sufficiently large.

ii) Because the discussion of the nonoscillatory solutions of the equation

$$u'''+t^\sigma u^n=0\,,\qquad n>1\,,\qquad \sigma\in R \tag{5.1}$$

is similar to the work in Sections 3 and 4 we will state our results without proof.

THEOREM 5.2. – All proper, positive solutions $u(t)$ of Eq. (5.1) have the following asymptotic behavior:

σ, n	w	asymptotic behavior
$2n+\sigma+1<0$	$w>2$	(5.3) - (5.8)
$2n+\sigma+1=0$	$w=2$	(5.6) (5.7) (5.8) (5.10)
$2n+\sigma+1>0$, $\sigma+3<0$, $n+\sigma+2<0$	$1<w<2$	(5.2) (5.5) (5.7) (5.8) (5.9)
$n+\sigma+2=0$	$w=1$	(5.8)
$2n+\sigma+1>0$, $\sigma+3<0$, $n+\sigma+2>0$	$0<w<1$	(5.8)
$\sigma+3=0$	$w=0$	(5.11)
$\sigma+3>0$	$w<0$	(5.2) (5.11)

In the following expressions a, b, d, and t_0 are some constants.

(5.2) $$u(t) = ct^w(1 + o(1))$$

(5.3) $$u(t) = a + b(t - t_0) + d(t - t_0)^2 - \frac{d^n(t - t_0)^{2n+\sigma+3}}{(2n + \sigma + 3)(2n + \sigma + 2)(2n + \sigma + 1)}(1 + o(1))$$

(5.4) $$u(t) = a + b(t - t_0) + d(t - t_0)^2 - d^n[t \ln(t/t_0) - t - t_0](1 + o(1))$$

(5.5) $$u(t) = a + b(t - t_0) + \tfrac{1}{2}d(t - t_0)^2 - \tfrac{1}{2}d^n \ln(t/t_0)(1 + o(1))$$

(5.6) $$u(t) = a + b(t - t_0) - \frac{b^n(t - t_0)^{n+\sigma+3}(1 + o(1))}{(n + \sigma + 1)(n + \sigma + 2)(n + \sigma + 3)}$$

(5.7) $$u(t) = a + b(t - t_0) - b^n \ln(t/t_0)(1 + o(1))$$

(5.8) $$u(t) = a - \frac{a^n(t - t_0)^{\sigma+3}}{(\sigma + 1)(\sigma + 2)(\sigma + 3)}(1 + o(1))$$

(5.9) The solution $u(t)$ intersects with the function $v(t) = ct^w$ infinity often .

(5.10) $$\lim_{t\to\infty} u''(t) = 0;\ \lim_{t\to\infty} u'(t) = \lim_{t\to\infty} u(t) = \infty\,, \qquad u \neq ct^w\,.$$

(5.11) $$\lim_{t\to\infty} u''(t) = \lim_{t\to\infty} u'(t) = \lim_{t\to\infty} u(t) = 0\,, \qquad u \neq ct^w\,.$$

In the case $w < 0$ we point out that for $u(t)$ a solution of Eq. (5.1) that $\lim_{t\to\infty} u(t) = \lim_{t\to\infty} u'(t) = \lim_{t\to\infty} u''(t) = 0$. The solution $u(t) = ct^w$ is of this form. To investigate the possibility of other asymptotic expressions possessing this behavioral condition, we set $u = ct^w v$, $t = e^s$ in Eq. (5.1) and obtain as before

(5.12) $$v''' + \alpha v'' + \beta v' + \gamma v = \gamma v^n\,.$$

Since the characteristic roots of the linear part of Eq. (5.12) are $-w$, $1 - w$, and $2 - w$, all positive, we have by Lemma 2.4 that no positive solution of (5.12) tends to zero as s tends to infinity. Hence, any proper solution of (5.1) for $w < 0$ tends to zero monotonically but not « faster » than ct^n.

In a theorem analogous to Theorem 3.2, we can prove that the asymptotic forms (5.3)-(5.8) are valid asymptotic behaviors for the values of σ and n that allow them to exist. By integrating

$$u(t) = u(t_0) + u'(t_0)(t - t_0) + \tfrac{1}{2}u''(t_0)(t - t_0)^2 - \tfrac{1}{2}\int_t^{t_0}(s - t)^2 s^\sigma u^n(s)\,ds \leqslant u(t_0) + u'(t_0)(t - t_0) + \tfrac{1}{2}u''(t_0)(t - t_0)^2 .$$

Therefore, any positive–solution of Eq. (5.1) can be continued to the entire interval $[t_0, \infty)$.

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