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ASYMPTOTIC BEHAVIOR OF THE TRANSITION DENSITY FOR JUMP TYPE PROCESSES IN SMALL TIME

YASUSHI ISHIKAWA

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Abstract. The Markov process of pure jump type given by S.D.E. has a smooth density under non-degeneracy conditions both on the coefficient and on the Lévy measure of the driving Lévy process. In this case we obtain an estimate of this density when the time parameter is small. In this way we extend the Léandre estimate of the density for pure jump processes.

Introduction. Consider the stochastic differential equation (S.D.E.):

$$x_t(x)=x+\sum_{s\leq t}\gamma(x_{s-}(x), \Delta z(s)), \tag{0.1}$$

where $z(t)$ denotes an $\boldsymbol{R}^d$-valued Lévy process (semimartingale) of pure jump type with the Lévy measure $h(d\zeta)$, $\Delta z(t)=z(t)-z(t-)$ and $\gamma(x, \zeta)$ denotes a non-degenerate bounded function from $\boldsymbol{R}^d\times\boldsymbol{R}^d$ to $\boldsymbol{R}^d$. It is known that the process $x_t(x)$ has a generator L of the corresponding semigroup which is of the form

$$Lf(x)=\int_{\boldsymbol{R}^d\setminus\{x\}}[f(z)-f(x)]g(x, dz),$$

for a function f in a certain class. Here $g(x, A)=\int_{\boldsymbol{R}^d}1_{A\setminus\{x\}}(x+\gamma(x, \zeta))h(d\zeta)$ is the *Lévy measure* of $x_t(x)$.

Under certain conditions (including $g(x, dz)=g(x, z)dz$), Léandre [8] studied the asymptotic behavior of the transition density $p_t(x, y)$ of this process, and showed that

$$p_t(x, y)\sim g(x, y)t \quad \text{as} \quad t\to 0 \quad \text{if} \quad g(x, y)\neq 0. \tag{0.2}$$

Here we note that the condition "$g(x, y)\neq 0$" implies that the process can reach y from x by a single jump. The object of this paper is to give a refinement of Léandre's result in the following form:

$$p_t(x, y)\sim C(x, y, \alpha(x, y))t^{\alpha(x, y)} \quad \text{as} \quad t\to 0, \tag{0.3}$$

where $\alpha(x, y)$ can be interpreted as, roughly speaking, the minimum number of jumps by which the trajectory can reach y from x ($y\neq x$).

The proof heavily depends on Léandre's method and results in [8], and we make

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use of the recently developed theory of Malliavin calculus of jump type (cf. [1], [2], [7], [9], [10], [12]).

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1. Notation and results. Let $z(t)$ be a d-dimensional Lévy process of the form

$$z(t)=\int_0^{t+}\int_{\boldsymbol{R}^d\setminus\{0\}}\zeta N(ds\,d\zeta)\,,$$

where $N(ds\,d\zeta)$ is a Poisson random measure on $[0,+\infty)\times(\boldsymbol{R}^d\setminus\{0\})$ with the mean measure $ds\times h(d\zeta)$. We put the following assumptions on $h(d\zeta)$:

$h(d\zeta)$ has a C^∞-density $h(\zeta)$ on $\boldsymbol{R}^d\setminus\{0\}$ such that

$$\text{supp}\,h(\cdot)\subset\{\zeta\in\boldsymbol{R}^d;|\zeta|\le c\}\,,\quad\text{where}\quad 0<c\le+\infty\,, \tag{1.1}$$

$$\int_{\boldsymbol{R}^d\setminus\{0\}}\min(|\zeta|,1)h(\zeta)d\zeta<+\infty$$

$$\int_{|\zeta|\ge\varepsilon}\left(\frac{|\partial h/\partial\zeta|^2}{h(\zeta)}\right)d\zeta<+\infty\qquad\text{for}\quad\varepsilon>0\,,$$

and that

$$h(\zeta)=a\left(\frac{\zeta}{|\zeta|}\right)\cdot|\zeta|^{-d-\alpha} \tag{1.2}$$

in a neighborhood of the origin for some $\alpha\in(0,1)$, and some strictly positive function $a(\cdot)\in C^\infty(S^{d-1})$.

In what follows we carry out our study in the usual probability space $(\Omega,(\mathscr{F}_t)_{t\ge0},\Pi)$, where $\Omega=D[\boldsymbol{R}^+,\boldsymbol{R}^d]$ denotes the Skorohod space, $\mathscr{F}_t$ is the filtration generated by $z(t)$ and Π is the probability measure on Ω of $z(t)$. Consider the following S.D.E.:

$$x_t(x)=x+\sum_{s\le t}\gamma(x_{s-}(x),\Delta z(s)) \tag{1.3}$$

or

$$x_t(x)=x+\int_0^t\int\gamma(x_{s-}(x),\zeta)N(ds\,d\zeta)\,.$$

Throughout this paper $\gamma(x, \zeta)$ is assumed to be a bounded C^∞-function from $\boldsymbol{R}^d \times \boldsymbol{R}^d$ to $\boldsymbol{R}^d$, whose derivatives of all orders are bounded, which satisfies $\gamma(x, 0) = 0$ and

$$(1.4) \qquad \inf\left\{ {}^t y \frac{\partial \gamma}{\partial \zeta}(x, \zeta) {}^t\left(\frac{\partial \gamma}{\partial \zeta}(x, \zeta) \right) y \,;\, x \in \boldsymbol{R}^d, \zeta \in \operatorname{supp} h \right\} \geq \delta |y|^2$$

$$\text{on} \quad \boldsymbol{R}^d \quad \text{for some} \quad \delta > 0 .$$

The S.D.E. (1.3) has a unique solution for $\gamma(x, \zeta)$ above, which we shall denote by $x_t(x)$ (or more precisely $(x_t(x), z(t))$), and we shall denote the law of $x_t(x)$ under Π by P. The following result of Léandre plays a fundamental role.

PROPOSITION 1.1 (cf. Léandre [8, (1.6)]). *Suppose that there exists* $C > 0$ *such that*

$$(1.5) \qquad \inf_{\substack{x \in \boldsymbol{R}^d \\ \zeta \in (\operatorname{supp} h)}} \left| \det\left(I + \frac{\partial \gamma}{\partial x}(x, \zeta) \right) \right| > C$$

and

$$(1.6) \qquad \liminf_{\eta \to 0} \eta^\alpha \int_{|\zeta| > \eta} h(\zeta) d\zeta > 0 \qquad \textit{for some} \quad \alpha \in (0, 2) .$$

Then there exists a subset of probability 1 *such that the mapping*

$$\varphi_t(x)(\omega) : \boldsymbol{R}^d \to \boldsymbol{R}^d , \qquad x \mapsto x_t(x)(\omega)$$

defines a diffeomorphism for all t, and the law of $x_t(x)$ *possesses a density* $p_t(x, y)$ *of class* C^∞ *with respect to y for each* $t > 0$ *and x.*

Note that (1.6) follows from (1.2). We also remark that $x \mapsto p_t(x, y)$ is also of class C^∞, by Theorem 2.9 and lemma 2.3, (iii) of Norris [12].

Let us introduce crucial notation in this paper. Set

$$P_x = x + \gamma(x, \operatorname{supp} h) = x + \{\gamma(x, \zeta) \,;\, \zeta \in \operatorname{supp} h\} .$$

Then each P_x is a compact set in $\boldsymbol{R}^d$ since γ is bounded. For each $y \in \boldsymbol{R}^d$ $(y \neq x)$, we put

$$\alpha(x, y) \equiv l_0(x, y) + 1 .$$

Here $l_0(x, y)$ denotes the minimum number of distinct points $z_1, \dots, z_l \in \boldsymbol{R}^d$ such that

$$(1.7) \qquad z_1 \in P_x, \ z_i \in P_{z_{i-1}}, \ i = 2, \dots, l \quad \text{and} \quad y \in P_{z_l} \quad (z_0 = x) .$$

We always have $\alpha(x, y) < \infty$ for each given $x, y \in \boldsymbol{R}^d$ $(x \neq y)$ by (1.2), and (1.4).

We set

$$(1.8) \qquad g(x, z) = h(H_x^{-1}(z)) \,|\, [J\gamma]^{-1}(x, H_x^{-1}(z)) \,| \qquad \text{for} \quad z \in P_x \setminus \{x\}$$

and $g(x, z) = 0$ otherwise. Here we put $H_x : \operatorname{supp} h \to P_x$, $\zeta \mapsto x + \gamma(x, \zeta)$ $(= z)$, and $J\gamma = (\partial \gamma / \partial \zeta)(x, \zeta)$ is the Jacobian matrix of γ. The kernel $g(x, z)$ is well defined and satisfies

$$\int f(z)g(x, z)dz = \int f(x+\gamma(x, \zeta))h(\zeta)d\zeta\,, \qquad \text{for} \quad f \in C(P_x)\,. \tag{1.9}$$

That is, $g(x, dz) = g(x, z)dz$ is the Lévy measure of $x_t(x)$ mentioned in the introduction. Note that $\operatorname{supp} g(x, \cdot) \subset P_x$ by definition.

Now we can state our main result.

THEOREM. *Under the assumptions* (1.1), (1.2) *and* (1.4), (1.5) *we have, for each distinct pair* $x, y \in \boldsymbol{R}^d$ *for which* $\kappa = \alpha(x, y)$ $(< +\infty)$,

$$\lim_{t \to 0} \frac{p_t(x, y)}{t^\kappa} = C\,,$$

where

(1.10)

$$C = C(x, y, \kappa) = \begin{cases} (1/\kappa!)\left\{\int_{P_x} dz_1 \cdots \int_{P_{z_{\kappa-2}}} dz_{\kappa-1} g(x, z_1) \cdots g(z_{\kappa-1}, y)\right\}, & \kappa \geq 2\,, \\ g(x, y)\,, & \kappa = 1\,. \end{cases}$$

The proof of this Theorem will be given at the end of Section 4.

EXAMPLE. We give here a very simple example of the Theorem. Let $d=2$, and let a smooth radial function η satisfy $\operatorname{supp} \eta = \{x; |x| \leq 1\}$ and $\eta(x) \equiv 1$ in $\{x; |x| \leq 1/2\}$. Put

$$h(\zeta) = \eta(\zeta)|\zeta|^{-d-\alpha}\,, \quad \alpha \in (0, 1)\,, \quad \text{for} \quad \zeta \in \boldsymbol{R}^d \setminus \{0\}\,, \tag{1.11}$$

that is, $h(\zeta)d\zeta$ is the Lévy measure for a *truncated stable process* (cf. [4, Section 3]) with index $\alpha \in (0, 1)$. Then h satisfies (1.1) with $c=1$ and (1.2). Let $\gamma(x, \zeta) = \zeta$, and let $x_t(x)$ be given by

$$x_t(x) = x + \sum_{s \leq t} \Delta z(s)\,. \tag{1.12}$$

Then $P_x = x + \operatorname{supp} h = x + B(1)$ $(B(1) \equiv \{x; |x| \leq 1\})$, and $g(x, z)$ is reduced to $g(x, z) = h(z-x)$ (cf. (1.8)).

Let $x_0 = (0, 0)$ and choose $y_0 = (e, 0)$ so that $1 < e < 2$. We then have $\alpha(x_0, y_0) = 2$. The constant $C(x_0, y_0, 2)$ is calculated as follows:

$$C(x_0, y_0, 2) = \int_{P_{x_0}} g(x_0, z)g(z, y_0)dz = \int_{B(1)} h(z-x_0)h(y_0-z)dz\,. \tag{1.13}$$

The integral (1.13) makes sense. Indeed, if $z=0$ then $h(y_0-0) = h(y_0) = 0$. Since $y_0 \in \{x; |x| > 1\}$, by the continuity of $x \mapsto P_x$, there exists $\delta > 0$ such that if $|z-x_0| < \delta$ then $y_0 \notin \operatorname{supp}(g(z, \cdot))$. That is, $g(z, y_0) = 0$ for $z \in \{z; |z-x_0| < \delta\}$, and this implies that the integral exists.

2. Decomposition of transition density. In this section we give a decomposition of $p_t(x, y)$, which plays a crucial role in Léandre's paper [8]. We will state it in a little detail, since it is also essential in the proof of our theorem.

Given $\varepsilon>0$, let ϕ_ε: $\boldsymbol{R}^d \to \boldsymbol{R}$ be a non-negative C^∞-function such that $\phi_\varepsilon(\zeta)=1$ if $|z|\geq\varepsilon$ and $\phi_\varepsilon(\zeta)=0$ if $|z|\leq\varepsilon/2$. Let $z_t(\varepsilon)$ and $z_t'(\varepsilon)$ be two independent Lévy processes whose Lévy measures are given by $\phi_\varepsilon(\zeta)h(\zeta)d\zeta$ and $(1-\phi_\varepsilon(\zeta))h(\zeta)d\zeta$, respectively. Then the process $z(t)$ has the same law as that of $z_t(\varepsilon)+z_t'(\varepsilon)$. Since the process $z_t(\varepsilon)$ has finite Lévy measure on $\boldsymbol{R}^d\setminus\{0\}$, the corresponding Poisson point process $N_s(\varepsilon)$ on $[0, \infty)\times(\boldsymbol{R}^d\setminus\{0\})$ jumps only finite times in each finite interval $[0, t]$. We set $\mathscr{S}_{t,k}=([0, t]^k/\sim)$ and $\mathscr{S}_t=\coprod_{k\geq 0}\mathscr{S}_{t,k}$, where $\coprod_{k\geq 0}$ denotes the disjoint sum and $\sim$ means the identification of the coordinates on the product space $[0, t]^k$ by the permutation. The distribution $\tilde{P}_{t,\varepsilon}$ of the moments (instants) of jumps related to $N_s(\varepsilon)$ in $[0, t]$ is given by

$$(2.1)\qquad \int_{\{\#S_t=k\}} f(S_t)d\tilde{P}_{t,\varepsilon}(S_t)=\left\{\left(t\int\phi_\varepsilon(\zeta)h(\zeta)d\zeta\right)^k\left(\frac{1}{k!}\right)\exp\left(-t\int\phi_\varepsilon(\zeta)h(\zeta)d\zeta\right)\right\}$$
$$\times\frac{1}{t^k}\int_0^t\cdots\int_0^t f(s_1, \dots, s_k)ds_1\cdots ds_k\,,$$

where f is a function on $\mathscr{S}_{t,k}$ (a symmetric function on $[0, t]^k$).

Let $J(\varepsilon)$ be a random variable whose law is $\phi_\varepsilon(\zeta)h(\zeta)d\zeta/(\int\phi_\varepsilon(\zeta)h(\zeta)d\zeta)$, and choose a family of independent random variables $J_i(\varepsilon)$, $i=1, 2, \dots$, having the same law as $J(\varepsilon)$. Choose $0\leq s\leq t<+\infty$. For a fixed $S_t\in\mathscr{S}_t$, we consider the solution of the following S.D.E. of jump type:

$$(2.2)\qquad x_s(\varepsilon, S_t, x)=x+\sum_{u\leq s}\gamma(x_{u-}(\varepsilon, S_t, x), \Delta z_u'(\varepsilon))+\sum_{s_i\in S_t, s_i\leq s}\gamma(x_{s_i-}(\varepsilon, S_t, x), J_i(\varepsilon))\,.$$

Then the law of $x_s(\varepsilon, S_t, x)$ has a smooth density, denoted by $p_s(\varepsilon, S_t, x, y)$ (cf. [8, p. 87]). Then

$$(2.3)\qquad p_s(x, y)=\int_{S_t} p_s(\varepsilon, S_t, x, y)d\tilde{P}_{t,\varepsilon}(S_t)\,,$$

because $z_t(\varepsilon)$ and $z_t'(\varepsilon)$ are independent, which is written in a *decomposed form* (by putting $s=t$)

$$(2.4)\qquad p_t(x, y)=\sum_{i=0}^{N-1} p_t(i, \varepsilon, x, y)+\bar{p}_t(N, \varepsilon, x, y)\,,$$

where

$$p_t(i, \varepsilon, x, y)=\int_{\mathscr{S}_{t,i}} p_t(\varepsilon, S_t, x, y)d\tilde{P}_{t,\varepsilon}(S_t)$$

and $\bar{p}_t(N, \varepsilon, x, y)$ is the remaining term. Then it follows from the definition and (2.1) that

$$p_t(k, \varepsilon, x, y) = (1/k!) \cdot \exp\left(-t\int \phi_\varepsilon(\zeta)h(\zeta)d\zeta\right) \cdot \left(\int \phi_\varepsilon(\zeta)h(\zeta)d\zeta\right)^k \times \int_0^t \cdots \int_0^t p_t(\varepsilon, \{s_1, \ldots, s_k\}, x, y) ds_k \ldots ds_1 . \tag{2.5}$$

We denote by $x_s(\varepsilon, \varnothing, x)$ the process in (2.2) with $S_t \in \mathcal{S}_{t,0}$, and by $p_s(\varepsilon, \varnothing, x, y)$ its density.

For the random variable $J(\varepsilon)$ introduced above, there exists a density $g_\varepsilon(x, z)$ such that $P(x + \gamma(x, J(\varepsilon)) \in dz) = g_\varepsilon(x, z)dz$. Indeed, $g_\varepsilon(x, z)$ is given by

$$g_\varepsilon(x, z) = \frac{\phi_\varepsilon(H_x^{-1}(z))g(x, z)}{\int \phi_\varepsilon(\zeta)h(\zeta)d\zeta} , \tag{2.6}$$

where $g(x, z)$ is the density of the Lévy measure of $x_t(x)$ (cf. (1.8)). Note that, by definition, $\operatorname{supp} g_\varepsilon(x, \cdot) \subset P_x$, and that $g_\varepsilon(x, z)$ is of class C^∞ whose derivatives are uniformly bounded (since $g(x, z)$ has an only singularity at $x = z$). Now we have, for each $s_1 < \cdots < s_k < t$,

$$\begin{aligned} p_t(\varepsilon, \{s_1, \ldots, s_k\}, x, y) = \int dz'_0 \int_{P_{z'_0}} dz_1 \cdots \int dz'_{k-1} \int_{P_{z'_{k-1}}} dz_k \\ \times \{p_{s_1}(\varepsilon, \varnothing, x, z'_0) g_\varepsilon(z'_0, z_1) p_{s_2 - s_1}(\varepsilon, \varnothing, z_1, z'_1) g_\varepsilon(z'_1, z_2) \\ \times p_{s_3 - s_2}(\varepsilon, \varnothing, z_2, z'_2) \cdots g_\varepsilon(z'_{k-1}, z_k) p_{t - s_k}(\varepsilon, \varnothing, z_k, y)\} . \end{aligned} \tag{2.7}$$

Indeed, the increment $x_{s_i + u}(\varepsilon, \{s_1, \ldots, s_k\}, x) - x_{s_i}(\varepsilon, \{s_1, \ldots, s_k\}, x)$ has the same law as that of $x_u(\varepsilon, \varnothing, x) - x$ on $(0, s_{i+1} - s_i)$ for $i = 0, \ldots, k$ $(s_0 = 0, s_{k+1} = t)$, and $x_{(s_i + u)-}(\varepsilon, \{s_1, \ldots, s_k\}, x)$ is going to make a "big jump" (i.e., a jump derived from $J_{i+1}(\varepsilon)$) at $u = s_{i+1} - s_i$ according to the law $g_\varepsilon(x_{(s_{i+1})-}(\varepsilon, \{s_1, \ldots, s_k\}, x), z)dz$.

3. The proof of the Theorem, (I) lower bound. Let $\varepsilon_c \equiv \sup\{\varepsilon > 0 ; \{|\zeta| < \varepsilon\} \subset \operatorname{supp} h\} > 0$, and choose $0 < \varepsilon < \varepsilon_c$. First we note that, for each $\eta > 0$ and a compact set K, we have uniformly in $y \in K$,

$$\lim_{s \to 0} \int_{\{z ; |z - y| \le \eta\}} p_s(\varepsilon, \varnothing, z, y) dz = 1 , \tag{3.1}$$

by Proposition I.2 in Léandre [8]. Then we have:

LEMMA 3.1. *Let $\mathcal{K}$ be a class of non-negative, equi-continuous, uniformly bounded functions whose supports are contained in a fixed compact set K. Then, given $\delta > 0$, there exists $t_0 > 0$ such that*

$$\inf_{s \le t} \int f(z) p_{t-s}(\varepsilon, \varnothing, z, y) dz \ge f(y) - \delta \tag{3.2}$$

for every $f\in\mathcal{K}$, $y\in K$ *and every* $t\in(0, t_0)$.

PROOF. Given $\delta>0$, choose $\eta>0$ so that $|f(z)-f(y)|<\delta/2$ for each $|z-y|<\eta$. Then,

$$\int f(z)p_{t-s}(\varepsilon, \varnothing, z, y)dz \geq \int_{\{z;|z-y|\leq\eta\}} f(z)p_{t-s}(\varepsilon, \varnothing, z, y)dz$$

$$=\int_{\{z;|z-y|\leq\eta\}} (f(z)-f(y))p_{t-s}(\varepsilon, \varnothing, z, y)dz+f(y)\int_{\{z;|z-y|\leq\eta\}} p_{t-s}(\varepsilon, \varnothing, z, y)dz$$

$$\geq(f(y)-\delta/2)\int_{\{z;|z-y|\leq\eta\}} p_{t-s}(\varepsilon, \varnothing, z, y)dz\,.$$

By (3.1) we can choose $t_0>0$ so that, if $t<t_0$ then

$$\left\{\inf_{y\in K, s\leq t}\int_{\{z;|z-y|\leq\eta\}} p_{t-s}(\varepsilon, \varnothing, z, y)dz\right\}(f(y)-\delta/2)\geq f(y)-\delta \qquad \text{for all} \quad y\in K\,.$$

q.e.d.

Now we choose an arbitrary compact neighborhood $\bar{U}(x)$ of x and arbitrary compact sets $K_1, \ldots, K_{k-1}$ of $\boldsymbol{R}^d$, and set

$$\mathcal{K}=\{g_\varepsilon(z_0', \cdot), g_\varepsilon(z_1', \cdot), \ldots, g_\varepsilon(z_{k-1}', \cdot); z_0'\in\bar{U}(x), z_1'\in K_1, \ldots, z_{k-1}'\in K_{k-1}\}\,.$$

Since $\mathcal{K}$ has the property in Lemma 3.1 (cf. (2.6)), it follows from (2.7) that, for every $\delta>0$, there exists $t_0>0$ such that for every $0<t<t_0$

$$\text{(3.3)} \qquad p_t(\varepsilon, \{s_1, \ldots, s_k\}, x, y) \geq \int_{\bar{U}(x)} p_{s_1}(\varepsilon, \varnothing, x, z_0')dz_0'$$
$$\times\int_{K_1} dz_1'\cdots\int_{K_{k-1}} dz_{k-1}'(g_\varepsilon(z_0', z_1')-\delta)\cdots(g_\varepsilon(z_{k-1}', y)-\delta)\,.$$

But we have, for each fixed $\eta>0$,

$$\text{(3.4)} \qquad \lim_{t\to0}\ \inf_{s_1\leq t, x\in\boldsymbol{R}^d}\int_{\{|x-z_0'|\leq\eta\}} p_{s_1}(\varepsilon, \varnothing, x, z_0')dz_0'=1\,,$$

by (2.13) in [8]. So it holds, for sufficiently small every $t>0$, that

$$\text{(3.5)} \qquad p_t(\varepsilon, \{s_1, \ldots, s_k\}, x, y)\geq(1-\delta)\int_{K_1} dz_1'\int_{K_2} dz_2'\cdots$$
$$\times\int_{K_{k-1}} dz_{k-1}'(g_\varepsilon(z_0', z_1')-\delta)(g_\varepsilon(z_1', z_2')-\delta)\cdots(g_\varepsilon(z_{k-1}', y)-\delta)\,.$$

Combining (2.5) with (3.5), we have

$$\liminf_{t\to 0}\left(\frac{1}{t^k}\right)p_t(k,\varepsilon,x,y)\geq(1/k!)\cdot(1-\delta)\cdot\left(\int\phi_\varepsilon(\zeta)h(\zeta)d\zeta\right)^k \tag{3.6}$$
$$\times\int_{K_1}dz'_1\int_{K_2}dz'_2\cdots\int_{K_{k-1}}dz'_{k-1}(g_\varepsilon(x,z'_1)-\delta)(g_\varepsilon(z'_1,z'_2)-\delta)\cdots(g_\varepsilon(z'_{k-1},y)-\delta)\,.$$

Since $\delta>0$ and $K_1,\dots,K_{k-1}$ are arbitrary, and since $\operatorname{supp} g_\varepsilon(z'_{i-1},\cdot)\subset P_{z'_{i-1}}$, $i=1,\dots,k-1$, we have

$$\liminf_{t\to 0}\left(\frac{1}{t^k}\right)p_t(k,\varepsilon,x,y)\geq(1/k!)\cdot\left(\int\phi_\varepsilon(\zeta)h(\zeta)d\zeta\right)^k \tag{3.7}$$
$$\times\int_{P_x}dz_1\int_{P_{z_1}}dz_2\cdots\int_{P_{z_{k-2}}}dz_{k-1}g_\varepsilon(x,z_1)g_\varepsilon(z_1,z_2)\cdots g_\varepsilon(z_{k-1},y)\,.$$

Hence it follows from (2.4) that

$$\liminf_{t\to 0}\left(\frac{1}{t^{\alpha(x,y)}}\right)p_t(x,y)\geq(1/\alpha(x,y)!)\cdot\left(\int\phi_\varepsilon(\zeta)h(\zeta)d\zeta\right)^k \tag{3.8}$$
$$\times\int_{P_x}dz_1\int_{P_{z_1}}dz_2\cdots\int_{P_{z_{k-2}}}dz_{k-1}g_\varepsilon(x,z_1)g_\varepsilon(z_1,z_2)\cdots g_\varepsilon(z_{k-1},y)\,.$$

Since $\varepsilon>0$ is arbitrary, we have in view of (2.6)

$$\liminf_{t\to 0}\left(\frac{1}{t^{\alpha(x,y)}}\right)p_t(x,y)\geq(1/\alpha(x,y)!) \tag{3.9}$$
$$\times\int_{P_x}dz_1\int_{P_{z_1}}dz_2\cdots\int_{P_{z_{k-2}}}dz_{k-1}\{g(x,z_1)g(z_1,z_2)\cdots g(z_{k-1},y)\}\,.$$

The proof for the lower bound is now complete.

4. The proof of the Theorem, (II) upper bound. The proof of the upper bound of $\limsup_{t-0}(1/t^{\alpha(x,y)})p_t(x,y)$ is rather delicate and is carried out in the same way as in Léandre [8], but it is a little more tedious in our case.

First we choose and fix $N\geq\alpha(x,y)+1$. Noting that $\sup\{p_t(\varepsilon,S_t,x,y)\,;\,\#S_t\geq 2,\,x,y\in\boldsymbol{R}^d\}\leq\tilde{C}(\varepsilon)$ (cf. [8, (3.23)]), we have

LEMMA 4.1 (cf. Léandre [8, Proposition III.3]). *For every $\varepsilon>0$ and $t>0$, we have*

$$\sup_{y\in\boldsymbol{R}^d}\bar{p}_t(N,\varepsilon,x,y)\leq C(\varepsilon)t^N\,. \tag{4.1}$$

PROOF. Recall that

$$\bar{p}_t(N, \varepsilon, x, y) = \int_{\mathscr{S}_t \setminus (\bigcup_{i=0}^{N-1} \mathscr{S}_{t,i})} p_t(\varepsilon, S_t, x, y) d\tilde{P}_{t,\varepsilon}(S_t) .$$

Since

$$\tilde{P}_{t,\varepsilon}(\# S_t \geq N) \leq C(\varepsilon) t^N$$

by the Poisson law, we have

$$\begin{aligned} \bar{p}_t(N, \varepsilon, x, y) &\leq \sup_{\# S_t \geq N} p_t(\varepsilon, S_t, x, y) \tilde{P}_{t,\varepsilon}(\# S_t \geq N) \\ &\leq \sup\{p_t(\varepsilon, S_t, x, y);\ \# S_t \geq 2,\ x, y \in \boldsymbol{R}^d\} C(\varepsilon) t^N \\ &\leq \tilde{C}(\varepsilon) C(\varepsilon) t^N . \end{aligned}$$

q.e.d.

LEMMA 4.2 (cf. Léandre [8, Proposition III.2]). *For every* $p > 1$ *and any* $\eta > 0$, *there exists* $\varepsilon > 0$ *such that for all* $t \leq 1$ *and every* $\varepsilon' \in (0, \varepsilon)$,

$$\sup_{|x-y| \geq \eta} p_t(0, \varepsilon', x, y) \leq C(\varepsilon', \eta) t^p . \tag{4.2}$$

Combining Lemma 4.1 with Lemma 4.2, we see that it is sufficient to study $p_t(k, \varepsilon, x, y)$ for $1 \leq k \leq \alpha(x, y)$. For a given $\eta > 0$, make a subdivision of the space

$$A \equiv \{(z'_0, z_1, \dots, z'_{k-1}, z_k) \in \boldsymbol{R}^d \times \cdots \times \boldsymbol{R}^d;\ z_1 \in P_{z'_0}, z_2 \in P_{z'_1}, \dots, z_k \in P_{z'_{k-1}}\}$$

as $A = \bigcup_{i=1}^{2^{k+1}} A_i(\eta)$, where

$$\begin{aligned} A_1(\eta) = \{&(z'_0, z_1, \dots, z'_{k-1}, z_k) \in \boldsymbol{R}^d \times \cdots \times \boldsymbol{R}^d;\ z_1 \in P_{z'_0}, z_2 \in P_{z'_1}, \dots, z_k \in P_{z'_{k-1}} \\ &\text{and } |x - z'_0| \leq \eta, |z_1 - z'_1| \leq \eta, |z_2 - z'_2| \leq \eta, \dots, |z_k - y| \leq \eta\} , \\ A_2(\eta) = \{&(z'_0, z_1, \dots, z'_{k-1}, z_k) \in \boldsymbol{R}^d \times \cdots \times \boldsymbol{R}^d;\ z_1 \in P_{z'_0}, z_2 \in P_{z'_1}, \dots, z_k \in P_{z'_{k-1}} \\ &\text{and } |x - z'_0| \leq \eta, |z_1 - z'_1| > \eta, |z_2 - z'_2| \leq \eta, \dots, |z_k - y| \leq \eta\} , \end{aligned}$$

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and

$$\begin{aligned} A_{2^{k+1}}(\eta) = \{&(z'_0, z_1, \dots, z'_{k-1}, z_k) \in \boldsymbol{R}^d \times \cdots \times \boldsymbol{R}^d;\ z_1 \in P_{z'_0}, z_2 \in P_{z'_1}, \dots, z_k \in P_{z'_{k-1}} \\ &\text{and } |x - z'_0| > \eta, |z_1 - z'_1| > \eta, |z_2 - z'_2| > \eta, \dots, |z_k - y| > \eta\} , \end{aligned}$$

and we shall classify those divisions into four cases:

$$[A] = \{A_i(\eta);\ |z_k - y| > \eta\} ,$$

$$[B] = \{A_i(\eta);\ |x - z'_0| > \eta \text{ and } |z_k - y| \leq \eta\} ,$$

$$[C] = \{A_1(\eta)\} \equiv \{\{|x - z'_0| \leq \eta, |z_1 - z'_1| \leq \eta, |z_2 - z'_2| \leq \eta, \dots, |z_k - y| \leq \eta\}\} ,$$

and

$[D]=\{A_i(\eta);|x-z'_0|\leq\eta \text{ and } |z_k-y|\leq\eta\}\setminus\{A_1(\eta)\}$.

We put

$$I_{[N],t,\eta}(\varepsilon,\{s_1,\dots,s_k\},x,y) \tag{4.3}$$
$$=\sum_{A_i(\eta)\in[N]}\int_{A_i(\eta)}\{p_{s_1}(\varepsilon,\emptyset,x,z'_0)g_\varepsilon(z'_0,z_1)p_{s_2-s_1}(\varepsilon,\emptyset,z_1,z'_1)$$
$$\times g_\varepsilon(z'_1,z_2)p_{s_3-s_2}(\varepsilon,\emptyset,z_2,z'_2)\times\cdots\times g_\varepsilon(z'_{k-1},z_k)p_{t-s_k}(\varepsilon,\emptyset,z_k,y)\}$$
$$\times dz_k dz'_{k-1}\cdots dz_1 dz'_0,\qquad [N]=[A],[B],[C],[D].$$

Then in view of (2.7) we have

$$p_t(\varepsilon,\{s_1,\dots,s_k\},x,y)=(I_{[A],t,\eta}+I_{[B],t,\eta}+I_{[C],t,\eta}+I_{[D],t,\eta})(\varepsilon,\{s_1,\dots,s_k\},x,y), \tag{4.4}$$

since $\operatorname{supp}(g_\varepsilon(z'_{i-1},\cdot))\subset P_{z'_{i-1}}$ for $i=1,\dots,k$.

LEMMA 4.3. *For any $(x,y)\in(\boldsymbol{R}^d\times\boldsymbol{R}^d)\setminus\Delta$ $(\Delta\equiv\{(x,x);x\in\boldsymbol{R}^d\})$, any $\{s_1,\dots,s_k\}$, any $\eta>0$ and any $p>1$, there exists $\varepsilon>0$ such that if $0<\varepsilon'<\varepsilon$ and $t\leq1$, then*

$$(I_{[A],t,\eta}+I_{[B],t,\eta}+I_{[D],t,\eta})(\varepsilon',\{s_1,\dots,s_k\},x,y)\leq C(\varepsilon',\eta,k,p)t^p.$$

PROOF. The proof is essentially the same as that in Léandre [8, Proposition III.4], but is a little more complicated.

First note that there exists $\varepsilon>0$ such that if $0<\varepsilon'<\varepsilon$ and $t\leq1$, then

$$P\left\{\sup_{0\leq s\leq t}|x_s(\varepsilon',\emptyset,x)-x|>\eta\right\}\leq c(\varepsilon',\eta,p)t^p \tag{4.5}$$

(see [8, Proposition I.4] and Lepeltier-Marchal [11, Lemme 17]). Then we observe

$$I_{[B],t,\eta}(\varepsilon',\{s_1,\dots,s_k\},x,y)\leq C_1(\varepsilon',\eta,k,p)t^p.$$

Next, let $G_\zeta:\boldsymbol{R}^d\to\boldsymbol{R}^d$, $x\mapsto x+\gamma(x,\zeta)$. If $\zeta\in\operatorname{int}(\operatorname{supp}h)$, then G_ζ gives a diffeomorphism by (1.5). Let G_ζ^{-1} denote its inverse mapping, and put $\tilde{\gamma}(x,\zeta)\equiv G_\zeta^{-1}(x)-x$, $\zeta\in\operatorname{int}(\operatorname{supp}h)$. Since $\gamma(x,\zeta)$ is a bounded, C^∞-function both in x and ζ (cf. Section 1), (1.4) and (1.5) imply that $\tilde{\gamma}(x,\zeta)$ is also bounded, C^∞ in $x\in\boldsymbol{R}^d$ and $\zeta\in\operatorname{int}(\operatorname{supp}h)$. Note that $\tilde{\gamma}(x,0)=0$ since $G_0(x)=x$. We put, for fixed $\varepsilon>0$,

$$\tilde{S}_\varepsilon=\sup\{|\tilde{\gamma}(x,\zeta)|;x\in\boldsymbol{R}^d,\zeta\in\operatorname{int}(\operatorname{supp}(1-\phi_\varepsilon)\cdot h)\}.$$

The following estimate also obtained by Léandre [8, Proposition I.3] is used in the estimate of $I_{[A],t,\eta}(\varepsilon',\{s_1,\dots,s_k\},x,y)$: for every $p>1$, and every η with $\tilde{S}_\varepsilon<\eta$,

$$\limsup_{s\to0}\sup_{y\in\boldsymbol{R}^d}(1/s^p)\int_{\{x;|x-y|>\eta\}}p_s(\varepsilon,\emptyset,x,y)dx<+\infty. \tag{4.6}$$

Since $\tilde{S}_\varepsilon\to0$ as $\varepsilon\to0$, it follows from (4.3), (4.6) that

$$I_{[A],t,\eta}(\varepsilon', \{s_1, \dots, s_k\}, x, y) \le C_2(\varepsilon', \eta, k, p)t^p .$$

Using the inequality (4.5), we can prove a similar estimate for

$$I_{[D],t,\eta}(\varepsilon', \{s_1, \dots, s_k\}, x, y) .$$

q.e.d.

Noting Lemma 4.3 we have only to study $I_{[C],t,\eta}(\varepsilon', \{s_1, \dots, s_k\}, x, y)$ for each small $\eta>0$ and $0<\varepsilon'<\varepsilon$. Put $\alpha(x, y)=\kappa$ $(1\le\kappa<+\infty)$. Then we have:

PROPOSITION 4.4. *If η is small and $1\le i<\kappa=\alpha(x, y)$, then*

$$I_{[C],t,\eta}(\varepsilon', S_t, x, y)=0 \qquad \text{for} \quad S_t\in\mathscr{S}_{t,i} , \tag{4.7}$$

hence

$$\int_{\mathscr{S}_{t,i}} I_{[C],t,\eta}(\varepsilon', S_t, x, y)\, d\tilde{P}_{t,\varepsilon'}(S_t)=0 . \tag{4.8}$$

PROOF. Let $Q_{x,\eta,i}$ and $Q_{x,i}$ be as follows:

$$\begin{aligned} Q_{x,\eta,i}\equiv\{z\in \boldsymbol{R}^d\,;\ &\exists(z_0', z_1, z_1', \dots, z_{i-1}', z_i)\in \boldsymbol{R}^d\times\cdots\times \boldsymbol{R}^d,\ z_1\in P_{z_0'},\ z_2\in P_{z_1'}, \dots,\\ &z_i\in P_{z_{i-1}'},\ |x-z_0'|\le\eta,\ |z_1-z_1'|\le\eta,\ |z_2-z_2'|\le\eta, \dots, |z_i-z|\le\eta\} ,\\ Q_{x,i}\equiv\{z\in \boldsymbol{R}^d\,;\ &\exists(z_1, \dots, z_{i-1})\in \boldsymbol{R}^d\times\cdots\times \boldsymbol{R}^d,\ z_1\in P_x,\ z_2\in P_{z_1}, \dots,\\ &z_{i-1}\in P_{z_{i-2}},\ z\in P_{z_{i-1}}\}\\ =\bigcup\{P_{z_{i-1}}\,;\ &z_1\in P_x, \dots, z_{i-1}\in P_{z_{i-2}}\} . \end{aligned}$$

Here we put $z_0=x$. Then $Q_{x,i}$ is a closed set in $\boldsymbol{R}^d$, since each P_{z_j} is compact and $z_j\mapsto P_{z_j}$ is continuous. Observe that $Q_{x,\eta,i}\supset Q_{x,i}$ for all $\eta>0$, and $\bigcap_{\eta>0} Q_{x,\eta,i}=Q_{x,i}$. That is, $y\in Q_{x,i}$ if and only if $y\in Q_{x,\eta,i}$ for all $\eta>0$. Since $\operatorname{supp}(g_{\varepsilon'}(z_j', \cdot))\subset P_{z_j'}$ for $j=0, \dots, i-1$, we observe

$$\begin{aligned} I_{[C],t,\eta}(\varepsilon', S_t, x, y)\equiv\int_{A_1(\eta)} &\{p_{s_1}(\varepsilon', \varnothing, x, z_0')g_{\varepsilon'}(z_0', z_1)p_{s_2-s_1}(\varepsilon', \varnothing, z_1, z_1')\\ &\times g_{\varepsilon'}(z_1', z_2)p_{s_3-s_2}(\varepsilon', \varnothing, z_2, z_2')\times\cdots\times g_{\varepsilon'}(z_{i-1}', z_i)p_{t-s_i}(\varepsilon', \varnothing, z_i, y)\}\\ &\times dz_i dz_{i-1}'\cdots dz_1 dz_0' . \end{aligned} \tag{4.9}$$

In view of the condition in $Q_{x,\eta,i}$, we see that

$$\text{if} \quad y\notin Q_{x,\eta,i} \quad \text{then} \quad I_{[C],t,\eta}(\varepsilon', S_t, x, y)=0 . \tag{4.10}$$

Recall that $\alpha(x, y)=\kappa$ and $i<\kappa$. By the definition of $\alpha(x, y)$ in Section 1, (1.7), we have that $y\notin Q_{x,i}$, which implies $y\notin Q_{x,\eta,i}$ for every sufficiently small $\eta>0$. Hence, $I_{[C],t,\eta}(\varepsilon', S_t, x, y)=0$ for $S_t\in\mathscr{S}_{t,i}$ if $i<\kappa=\alpha(x, y)$. Thus, $\int_{\mathscr{S}_{t,i}} I_{[C],t,\eta}(\varepsilon', S_t, x, y)d\tilde{P}_{t,\varepsilon'}(S_t)=0$.

q.e.d.

LEMMA 4.5. *Let $\mathscr{K}$ be a class of non-negative, equi-continuous, uniformly bounded functions whose supports are contained in a fixed compact set K. Let the constants $\eta>0$ and $\varepsilon'>0$ be those appearing in $I_{[C],t,\eta}(\varepsilon', S_t, x, y)$ and (4.2), respectively. Then, for every $\delta>0$, there exists $t_0>0$ such that*

$$\sup_{s\leq t}\int_{\{z;|z-y|\leq\eta\}} f(z)p_{t-s}(\varepsilon', \emptyset, z, y)dz \leq f(y)+\delta \tag{4.11}$$

for every $f\in\mathscr{K}$, $y\in K$ and every $t\in(0, t_0)$.

PROOF. For given $\delta>0$, there exists $\eta_1=\eta_1(K, \mathscr{K}, \delta)>0$ such that $|f(z)-f(y)|<\delta/4$ for each $|z-y|<\eta_1$. We may assume $\eta_1\leq\eta$ by choosing small δ. Then we have

$$\begin{aligned}(4.12)\qquad &\sup_{s\leq t}\int_{\{z;|z-y|\leq\eta\}} f(z)p_{t-s}(\varepsilon', \emptyset, z, y)dz \\ &\leq \sup_{s\leq t}\int_{\{z;|z-y|\leq\eta_1\}} f(z)p_{t-s}(\varepsilon', \emptyset, z, y)dz + \sup_{s\leq t}\int_{\{z;\eta_1<|z-y|\leq\eta\}} f(z)p_{t-s}(\varepsilon', \emptyset, z, y)dz\,.\end{aligned}$$

The first term can be estimated as in Lemma 3.1 by $f(y)+\delta/2$ for $t\in(0, t_1)$ for some $t_1>0$. As for the second term, since $\eta>0$ is arbitrary in (3.1) and since all f's in $\mathscr{K}$ are uniformly bounded, there exists $t_2>0$ such that

$$\sup_{s\leq t}\int_{\{z;\eta_1<|z-y|\leq\eta\}} f(z)p_{t-s}(\varepsilon', \emptyset, z, y)dz \leq \delta/2$$

for every $f\in\mathscr{K}$, $y\in K$ and $t\in(0, t_2)$.

Letting $t_0\equiv\min\{t_1, t_2\}>0$, and we have the assertion. q.e.d.

Choose an arbitrary compact neighborhood $\bar{U}(x)$ of x and arbitrary compact sets $K_1, \dots, K_{\kappa-1}$ of $\boldsymbol{R}^d$ such that $\{z;|z-x|\leq\eta\}\subset\bar{U}(x)$ and that $Q_{x,\eta,i}\subset K_i$, $i=1, \dots, \kappa-1$. Set

$$\mathscr{K}=\{g_{\varepsilon'}(z'_0, \cdot), g_{\varepsilon'}(z'_1, \cdot), \dots, g_{\varepsilon'}(z'_{\kappa-1}, \cdot); z'_0\in\bar{U}(x), z'_1\in K_1, \dots, z'_{\kappa-1}\in K_{\kappa-1}\}\,.$$

To apply Lemma 4.5, we should be a little more careful, since $\varepsilon'>0$ depends on the choice of $\eta>0$ by Lemma 4.3. Since $\mathscr{K}$ has the property in Lemma 4.5, for given $\delta>0$, $\eta>0$, $\varepsilon'>0$, there exists $t_0>0$ such that for every $0<t<t_0$

$$\sup_{s\leq t}\int_{\{|z_i-z'_i|\leq\eta\}} g_{\varepsilon'}(z'_{i-1}, z_i)p_{t-s}(\varepsilon', \emptyset, z_i, z'_i)dz_i \leq g_{\varepsilon'}(z'_{i-1}, z'_i)+\delta\,, \tag{4.13}$$

for $z'_0\in\bar{U}(x)$, $z'_{i-1}\in K_{i-1}$, $i=2, \dots, \kappa$ $(z'_\kappa=y)$.

From (4.13) we have in view of (4.9), for $0<t<t_0$,

$$I_{[C],t,\eta}(\varepsilon', \{s_1, \dots, s_\kappa\}, x, y) \leq \int_{\bar{U}(x)} p_{s_1}(\varepsilon', \emptyset, x, z'_0)dz'_0 \int_{K_1} dz'_1 \cdots \int_{K_{\kappa-1}} dz'_{\kappa-1} \tag{4.14}$$

$$\times\{(g_{\varepsilon'}(z'_0, z'_1)+\delta)\cdots(g_{\varepsilon'}(z'_{\kappa-1}, y)+\delta)\}\,.$$

Hence by (3.4)

$$\text{(4.15)}\qquad \limsup_{t\to 0}(1/t^{\kappa})\int_0^t\cdots\int_0^t\left(\int\phi_{\varepsilon'}(\zeta)h(\zeta)d\zeta\right)^{\kappa}I_{[C],t,\eta}(\varepsilon', \{s_1, \ldots, s_{\kappa}\}, x, y)ds_{\kappa}\cdots ds_1$$
$$\le\left(\int\phi_{\varepsilon'}(\zeta)h(\zeta)d\zeta\right)^{\kappa}\int_{K_1}dz_1\int_{K_2}dz_2\cdots\int_{K_{\kappa-1}}dz_{\kappa-1}$$
$$\times\{(g_{\varepsilon'}(x, z_1)+\delta)(g_{\varepsilon'}(z_1, z_2)+\delta)\cdots(g_{\varepsilon'}(z_{\kappa-1}, y)+\delta)\}\,.$$

Since $\delta>0$ and $K_1, \ldots, K_{\kappa-1}$ are arbitrary, we have in view of Lemma 4.1 (with $N=\kappa+1$), Lemmas 4.2 and 4.3, Proposition 4.4 and (2.4), (2.5), (4.4), that

$$\text{(4.16)}\qquad \limsup_{t\to 0}\left(\frac{1}{t^{\kappa}}\right)p_t(x, y)\le(1/\kappa!)\cdot\left(\int\phi_{\varepsilon'}(\zeta)h(\zeta)d\zeta\right)^{\kappa}$$
$$\times\int_{P_x}dz_1\int_{P_{z_1}}dz_2\cdots\int_{P_{z_{\kappa-2}}}dz_{\kappa-1}\{g_{\varepsilon'}(x, z_1)g_{\varepsilon'}(z_1, z_2)\cdots g_{\varepsilon'}(z_{\kappa-1}, y)\}\,.$$

Letting $\varepsilon'\to 0$, we have in view of (2.6)

$$\text{(4.17)}\qquad \limsup_{t\to 0}\left(\frac{1}{t^{\kappa}}\right)p_t(x, y)$$
$$\le(1/\kappa!)\int_{P_x}dz_1\int_{P_{z_1}}dz_2\cdots\int_{P_{z_{\kappa-2}}}dz_{\kappa-1}\{g(x, z_1)g(z_1, z_2)\cdots g(z_{\kappa-1}, y)\}\,.$$

The proof for the upper bound is now complete.

Proof of Theorem. The statement of the Theorem is immediate from (3.9) (with $k=\alpha(x, y)$) and (4.17). q.e.d.

5. Concluding remark. In Section 1 we confined ourselves to the simple case where the Lévy measure of the driving Lévy process has exponent $\alpha\in(0, 1)$ near the origin. Although this condition is not indispensable, our method heavily depends on the fact that the process $x_t(x)$ is of pure jump type.

Apart from our case, Duflo [3] studied the upper bound of the semigroup of measures on a locally compact Lie group G generated by an operator satisfying the maximum principle (called χ-generalized Laplacian), which may correspond to jump processes on G. He obtained a general result which implies, when reduced to the case $G=\boldsymbol{R}^d$ and for the truncated stable process given in the example in Section 1, the following proposition:

Proposition 5.1 (Duflo [3, p. 239]). *Let $p_t^1(x, y)=p_t^1(y-x)$ denote the transition density corresponding to the semigroup generated by $h(d\zeta)=h(\zeta)d\zeta$, where $h(\zeta)$ is given by*

(1.11). *Then, for* $n=1, 2, \dots$, *there exists constant* C_n *such that*

$$p_t^1(x) \le C_n t^{n+1} \quad \textit{for} \quad t \in (0, 1) \quad \textit{and} \quad x \in \{x; |x| > n\}. \tag{5.1}$$

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DEPARTMENT OF INFORMATION SCIENCE
FACULTY OF SCIENCE
HIROSAKI UNIVERSITY
HIROSAKI, AOMORI 036
JAPAN