QUARTERLY OF APPLIED MATHEMATICS
VOLUME LXXIII, NUMBER 1
MARCH 2015, PAGES 23–54
S 0033-569X(2014)01382-4
Article electronically published on October 17, 2014

ASYMPTOTIC BEHAVIOR TO BRESSE SYSTEM WITH PAST HISTORY

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Abstract. In this paper we consider the Bresse system with past history acting in the shear angle displacement. We show the exponential decay of the solution if and only if the wave speeds are the same. On the contrary, we show that the Bresse system is polynomial stable with optimal decay rate. The systems of equations considered here introduce new mathematical difficulties in order to determine the asymptotic behavior. As far as the authors know, there have been no contributions made in this sense.

1. Introduction.

For the last several decades, various types of equations have been employed as some mathematical model describing physical, chemical, biological and engineering systems. Among them, the mathematical models of vibrating, flexible structures have been considerably stimulated in recent years by an increasing number of questions of practical concern. Research on stabilization of distributed parameter systems has largely focused on the stabilization of dynamic models of individual structural members such as strings, membranes and beams (see [18]).

In this work we study the circular arch problem also known as the Bresse system (see [7] for details). Elastic structures of the arcs type are objects of study widely explored in engineering, architecture, marine engineering, aeronautics and others. In particular, the free vibrations about elastic structures is a function of their natural properties and is an

Received December 14, 2012.
2010 *Mathematics Subject Classification.* Primary 74H40.
Key words and phrases. Bresse system, optimal result.
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important subject of investigation in the engineering field and also in the mathematics field. In the analysis mathematical field it is interesting to know the properties which relate the behavior of the energy associated with solutions of the respective dynamic model. For feedback laws, for example, we can ask what conditions about the dynamic model must be ensured to obtain the decay of the energy of solutions in the time t. In this sense, the property of stabilization has been extensively studied for dynamic problems in elastic structures translated in terms of partial differential equations, and an interesting property determines that the exponential decay with few feedback laws occurs only in a particular situation (see [7]).

Following the main idea about deformation in elastic structures, we consider the Bresse system given by the equations of motion

$$\rho_1\varphi_{tt} = Q_x + lN, \tag{1.0.1}$$

$$\rho_2\psi_{tt} = M_x - Q, \tag{1.0.2}$$

$$\rho_1 w_{tt} = N_x - lQ, \tag{1.0.3}$$

where

$$N = \kappa_0(w_x - l\varphi), \tag{1.0.4}$$

$$Q = \kappa(\varphi_x + lw + \psi), \tag{1.0.5}$$

$$M = b\psi_x - \int_0^\infty g(s)\psi_x(t-s)\,ds \tag{1.0.6}$$

are the stress-strain relations for elastic behavior. Here $\rho_1 = \rho A$, $\rho_2 = \rho I$, $\kappa = k'GA$, $\kappa_0 = EA$, $b = EI$, $l = R^{-1}$ where ρ is the density of material, E is the modulus of elasticity, G is the shear modulus, k' is the shear factor, A is the cross-sectional area, I is the second moment of area of the cross-section and R is the radius of curvature. The functions w, φ and ψ are the longitudinal, vertical and shear angle displacements, respectively. Here g represents the memory effect acting only on the shear angle displacement.

From coupled equations (1.0.1)–1.0.6) we obtain the Bresse system with past memory given by

$$\rho_1\varphi_{tt} - \kappa(\varphi_x + \psi + lw)_x - \kappa_0 l(w_x - l\varphi) = 0 \text{ in } \Omega, \tag{1.0.7}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + \int_0^\infty g(s)\psi_{xx}(t-s)\,ds + \kappa(\varphi_x + \psi + lw) = 0 \text{ in } \Omega, \tag{1.0.8}$$

$$\rho_1 w_{tt} - \kappa_0(w_x - l\varphi)_x + \kappa l(\varphi_x + \psi + lw) = 0 \text{ in } \Omega, \tag{1.0.9}$$

with initial conditions given by

$$\begin{aligned} \varphi(\cdot,0) = \varphi_0,\ \varphi_t(\cdot,0) &= \varphi_1,\ \psi(\cdot,0) = \psi_0,\ \psi_t(\cdot,0) = \psi_1, \\ w(\cdot,0) &= w_0,\ w_t(\cdot,0) = w_1 \text{ in }]0,L[\end{aligned} \tag{1.0.10}$$

where $\Omega = (0,L)\times(0,\infty)$. When $g \equiv 0$, (1.0.7)-(1.0.9) are the governing equations for the theory of circular arch. For more details see [7].

Remark 1.1. If $R \to \infty$ and $g = 0$, then $l \to 0$ and this model reduces to the well-known Timoshenko beam equations (see [6] and [7] for details). If $R \to \infty$ and $g \neq 0$, then $l \to 0$ and this model reduces to the well-known Timoshenko beam equations with past history (see Rivera [16] for details).

REMARK 1.2. Many interesting physical phenomena (such as viscoelasticity, hereditary polarization in dielectrics, population dynamics or heat flow in real conductors, to name some) are modeled by differential equations which are influenced by the past values of one or more variables in play, so-called equations with memory. The main problem in the analysis of equations of this kind lies in their nonlocal character, due to the presence of the memory term (in general, the time convolution of the unknown function against a suitable memory kernel).

In this work we will examine the issues concerning the asymptotic stabilization of the Bresse system with past memory. Our main tool is Prüss's result on the exponential stability of semigroups (see [5, 9, 11]). So to use these results, it is necessary to put the problem in the context of semigroups; thus some modifications to the original problem (1.0.7)-(1.0.10) should be made.

In fact, following the approach of Dafermos [3] and Fabrizio [4], we consider $\eta = \eta^t(s)$, the relative history of ψ, defined as

$$\eta = \eta^t(s) = \psi(t) - \psi(t-s). \tag{1.0.11}$$

Hence, putting

$$\beta_0 = b - b_0 > 0, \text{ with } b_0 = \int_0^\infty g(s)\, ds, \tag{1.0.12}$$

the system (1.0.7)-(1.0.9) and (1.0.10) turns into the system

$$\rho_1\varphi_{tt} - \kappa(\varphi_x + \psi + lw)_x - \kappa_0 l(w_x - l\varphi) = 0 \text{ in } \Omega, \tag{1.0.13}$$

$$\rho_2\psi_{tt} - \beta_0\psi_{xx} - \int_0^\infty g(\tau)\eta_{xx}(\tau)\, d\tau + \kappa(\varphi_x + \psi + lw) = 0 \text{ in } \Omega, \tag{1.0.14}$$

$$\rho_1 w_{tt} - \kappa_0(w_x - l\varphi)_x + \kappa l(\varphi_x + \psi + lw) = 0 \text{ in } \Omega, \tag{1.0.15}$$

$$\eta_t + \eta_s - \psi_t = 0 \text{ in } \Omega, \tag{1.0.16}$$

$$\varphi(\cdot, 0) = \varphi_0(x),\ \varphi_t(\cdot, 0) = \varphi_1(x) \text{ in } (0, L), \tag{1.0.17}$$

$$\psi(\cdot, 0) = \psi_0(x), \psi_t(\cdot, 0) = \psi_1(x), w(\cdot, 0) = w_0(x),\ w_t(\cdot, 0) = w_1(x) \text{ in } (0, L), \tag{1.0.18}$$

$$\eta_0(\cdot, s) = \psi_0(\cdot, 0) - \psi_0(\cdot, -s) \quad \text{in} \quad (0, L) \times (0, \infty), \tag{1.0.19}$$

where the fourth equation is obtained differentiating (1.0.11) with respect to s, and the condition (1.0.19) means that the history is considered an initial value. We consider the Dirichlet boundary conditions

$$\begin{aligned} \varphi(0,t) &= \varphi(L,t) = \psi(0,t) = \psi(L,t) = w(0,t) = w(L,t) = 0, \\ \eta^t(0,s) &= \eta^t(L,s) = 0, \quad s, t \geq 0, \end{aligned} \tag{1.0.20}$$

or Dirichlet-Neumann–Neumann–Neumann boundary conditions

$$\begin{aligned} \varphi(0,t) &= \varphi(L,t) = \psi_x(0,t) = \psi_x(L,t) = w_x(0,t) = w_x(L,t) = 0, \\ \eta_x^t(0,s) &= \eta_x^t(L,s) = 0 \quad s, t \geq 0. \end{aligned} \tag{1.0.21}$$

There exist only a few results about the asymptotic behavior to the Bresse system. The most important, from our point of view, is given by Liu and Rao [15]. In that paper, the authors consider the Bresse system with two different dissipative mechanisms, given by two temperatures coupled to the system. The authors showed the same result concerning the exponential stability, but concerning the polynomial decay, they found rates that depend on the boundary condition. When the system has a Dirichlet–Neumann boundary condition, they show that the system decays as t^{-4} and for a fully Dirichlet boundary condition, they proved that the solution decays as t^{-8}. An important problem in the Bresse system is to find a minimum dissipation by which their solutions decay uniformly to zero in time. In this direction we have the paper of Fatori and Rivera [13], which improved the paper by Liu and Rao [15]. They showed that, in general, the Bresse system is not exponentially stable but that there exists polynomial stability with rates that depend on the wave propagations and the regularity of the initial data. Moreover, they introduced a necessary condition to dissipative semigroup decay polynomially. This result allowed them to show some optimality to the polynomial rate of decay. The Bresse system with frictional damping was considered by Fatiha Alabau Boussouira et al. [10]. In that paper the authors showed that the Bresse system is exponentially stable if and only if the velocities of waves propagations are the same. Also, they showed that when the velocities are not the same, the system is not exponentially stable, and they proved that the solution in this case goes to zero polynomially, with rates that can be improved by taking more regular initial data. This rate of polynomial decay was improved by Luci Harue Fatori and Rodrigo Nunes Monteiro [12]. The indefinite damping acting on the shear angle displacement was considered by Juan A. Palomino et al. [14]. In [19] Nahla Noun and Ali Wehbe extended the results of Alabau-Boussouira et al. [10] and considered the important case when the dissipation law is locally distributed.

In this paper we study the Bresse system with past history. We show that the system is exponentially stable if and only if the wave speeds are the same. When in general the wave speeds are not the same we prove that the Bresse system is polynomially stable with optimal decay rate. The systems of equations considered here introduce new mathematical difficulties in order to determine the asymptotic behavior. As far as the authors know, there have been no contributions made in this sense.

The paper is organized as follows: in section 2 we establish the existence, regularity and uniqueness of global solutions of the problem (1.0.13)-(1.0.20) and also of system (1.0.13)-(1.0.19) with boundary conditions (1.0.21). We use the semigroup technique (see [8, 20]). In section 3 we study the exponential decay of the strong solutions of the system (1.0.13)-(1.0.20). We show the uniform decay of the solution by using a multipliers method. In section 4 we study the lack of exponential decay using Prüss's results [9] (see also [5, 11]). Finally in section 5 we show that the system (1.0.13)-(1.0.19) with boundary (1.0.21) is polynomially stable with optimal decay rate. For this we use the recent result due to Borichev and Tomilov [1].

2. The semigroup setting. In this section we will study the existence and uniqueness of strong solutions for the system (1.0.13)-(1.0.20) and also of system (1.0.13)-(1.0.19) with boundary conditions (1.0.21) using semigroup techniques. For this, we consider that the kernel g satisfies the following hypotheses (as in [17]):

$$g \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+),\ g(t) > 0,\ \exists q_0,\ q_1 > 0 :\ -q_0 g(t) \le g'(t) \le -q_1 g(t),\ \forall t \ge 0. \tag{2.0.1}$$

In view of (2.0.1), let $L^2_g(\mathbb{R}^+, H^1_0(0,L))$ be the Hilbert space of $H^1_0(0,L)$-value functions on $\mathbb{R}^+$, endowed with the inner product

$$(f,h)_{L^2_g(\mathbb{R}^+,H^1_0(0,L))} = \int_0^\infty g(s) \int_0^L f_x(x,s)\overline{h_x}(x,s)\,dx\,ds.$$

To give an accurate formulation of the evolution problem we are introducing the product Hilbert space

$$\mathcal{H}_1 := H^1_0(0,L)\times L^2(0,L)\times H^1_0(0,L)\times L^2(0,L)\times H^1_0(0,L)\times L^2(0,L)\times L^2_g(\mathbb{R}^+,H^1_0(0,L)) \tag{2.0.2}$$

and

$$\mathcal{H}_2 := H^1_0(0,L)\times L^2(0,L)\times H^1_*(0,L)\times L^2_*(0,L)\times H^1_*(0,L)\times L^2_*(0,L)\times L^2_g(\mathbb{R}^+,H^1_*(0,L)) \tag{2.0.3}$$

with inner product given by

$$\begin{aligned}(U_1,U_2)_{\mathcal{H}_i} &= \rho_1\int_0^L \widetilde{\varphi_1}\overline{\widetilde{\varphi_2}}\,dx + \rho_2\int_0^L \widetilde{\psi_1}\overline{\widetilde{\psi_2}}\,dx + \rho_1\int_0^L \widetilde{w_1}\overline{\widetilde{w_2}}\,dx\\ &+ \beta_0\int_0^L \psi_{1x}\overline{\psi_{2x}}\,dx + \kappa\int_0^L(\varphi_{1x}+\psi_1+lw_1)(\overline{\varphi_{2x}}+\overline{\psi_2}+l\overline{w_2})\,dx\\ &+ \kappa_0\int_0^L(w_{1x}-l\varphi_1)(\overline{w_{2x}}-l\overline{\varphi_2})\,dx + \int_0^\infty g(s)\int_0^L \eta_{1x}\overline{\eta_{2x}}\,dx\,ds, \end{aligned} \tag{2.0.4}$$

where

$$U_1 = (\varphi_1,\widetilde{\varphi_1},\psi_1,\widetilde{\psi_1},w_1,\widetilde{w_1},\eta_1)^T,\ U_2 = (\varphi_2,\widetilde{\varphi_2},\psi_2,\widetilde{\psi_2},w_2,\widetilde{w_2},\eta_2)^T \in \mathcal{H}_i$$

and norm given by

$$\begin{aligned}&\|U\|^2_{\mathcal{H}_i} = \|(\varphi,\widetilde{\varphi},\psi,\widetilde{\psi},w,\widetilde{w},\eta)^T\|^2_{\mathcal{H}_i}\\ &= \int_0^L \rho_1|\widetilde{\varphi}|^2 + \rho_2|\widetilde{\psi}|^2 + \rho_1|\widetilde{w}|^2 + \beta_0|\psi_x|^2 + \kappa|\varphi_x+\psi+lw|^2 + \kappa_0|w_x-l\varphi|^2\,dx\\ &+ \int_0^\infty g(s)\int_0^L |\eta_x(x,s)|^2\,dx\,ds. \end{aligned} \tag{2.0.5}$$

Here we consider

$$L^2_*(0,L) = \{f\in L^2(0,L) : \int_0^L f(x)\,dx = 0\},\quad H^1_*(0,L) = H^1(0,L)\cap L^2_*(0,L). \tag{2.0.6}$$

Let $U = (\varphi,\varphi_t,\psi,\psi_t,w,w_t,\eta)^T$ exist and define the operator $\mathcal{A}_i : D(\mathcal{A}_i)\subset\mathcal{H}_i\to\mathcal{H}_i$ given by

$$\mathcal{A}_i = \begin{pmatrix} 0 & I & 0 & 0 & 0 & 0 & 0\\ \frac{\kappa}{\rho_1}\partial_x^2 - \frac{\kappa_0 l^2}{\rho_1}I & 0 & \frac{\kappa}{\rho_1}\partial_x & 0 & \frac{(\kappa+\kappa_0)l}{\rho_1}\partial_x & 0 & 0\\ 0 & 0 & 0 & I & 0 & 0 & 0\\ -\frac{\kappa}{\rho_2}\partial_x & 0 & \frac{\beta_0}{\rho_2}\partial_x^2 - \frac{\kappa}{\rho_2}I & 0 & -\frac{\kappa l}{\rho_2}I & 0 & \frac{1}{\rho_2}\int_0^\infty g(s)\partial_x^2(\cdot,s)ds\\ 0 & 0 & 0 & 0 & 0 & I & 0\\ -\frac{(\kappa_0+\kappa)l}{\rho_1}\partial_x & 0 & -\frac{l\kappa}{\rho_1}I & 0 & \frac{\kappa_0}{\rho_1}\partial_x^2 - \frac{l^2\kappa}{\rho_1}I & 0 & 0\\ 0 & 0 & 0 & I & 0 & 0 & -T \end{pmatrix} \tag{2.0.7}$$

with domain

$$\begin{aligned} D(\mathcal{A}_1) &= \{(\varphi, \widetilde{\varphi}, \psi, \widetilde{\psi}, w, \widetilde{w}, \eta)^T \in \mathcal{H}_1; \quad \varphi, w \in H_0^1(0,L) \cap H^2(0,L), \widetilde{\varphi}, \ \widetilde{\psi}, \widetilde{w} \in H_0^1(0,L), \\ & \beta_0 \psi + \int_0^\infty g(s)\eta(s)\, ds \in H_0^1(0,L) \cap H^2(0,L), \eta \in D_1(T)\}, \end{aligned} \tag{2.0.8}$$

where $T\eta = \eta_s$ with

$$D_1(T) = \{\eta \in L_g^2(\mathbb{R}^+; H_0^1(0,L)); \eta_s \in L_g^2(\mathbb{R}^+; H_0^1(0,L)), \eta(0) = 0\}$$

and

$$\begin{aligned} D(\mathcal{A}_2) &= \{(\varphi, \widetilde{\varphi}, \psi, \widetilde{\psi}, w, \widetilde{w}, \eta)^T \in \mathcal{H}_2; \quad \varphi \in H_0^1(0,L) \cap H^2(0,L), \\ & w \in H_*^1(0,L) \cap H^2(0,L), \ \widetilde{\varphi} \in H_0^1(0,L), \widetilde{\psi}, \widetilde{w} \in H_*^1(0,L), \\ & \beta_0 \psi + \int_0^\infty g(s)\eta(s)\, ds \in H_*^1(0,L) \cap H^2(0,L), \eta \in D_2(T)\}, \end{aligned} \tag{2.0.9}$$

where $T\eta = \eta_s$ with

$$D_2(T) = \{\eta \in L_g^2(\mathbb{R}^+; H_*^1(0,L)); \eta_s \in L_g^2(\mathbb{R}^+; H_*^1(0,L)), \eta(0) = 0\},$$

where η_s is the distributional derivative of η with respect to the internal variable s. Therefore, the system (1.0.13)-(1.0.19) with boundary conditions (1.0.20) or (1.0.21) is equivalent to

$$\left\{ \begin{array}{ll} U_t & = \mathcal{A}_i U \\ U(0) & = U_0 \end{array} \right. \tag{2.0.10}$$

where $U_0 := (\varphi_0, \varphi_1, \psi_0, \psi_1, w_0, w_1, \eta_0)^T$, $i = 1, 2$.

THEOREM 2.1. The operator $\mathcal{A}_i$ generates a C_0-semigroup $S(t)$ of contraction on $\mathcal{H}_i$. Thus, for any initial data $U_0 \in \mathcal{H}_i$, the problem (1.0.13)-(1.0.19) with boundary conditions (1.0.20) or (1.0.21) has a unique weak solution $U(t) \in C^0([0,\infty[, \mathcal{H}_i)$. Moreover, if $U_0 \in \mathcal{D}(\mathcal{A}_i)$, then $U(t)$ is a strong solution of (1.0.13)-(1.0.19), i.e., $U(t) \in C^1([0,\infty[, \mathcal{H}_i) \cap C^0([0,\infty[, \mathcal{D}(\mathcal{A}_i))$.

Proof. It is easy to see that $\mathcal{D}(\mathcal{A}_i)$ is dense in $\mathcal{H}_i$, $i = 1, 2$. Now, for $U = (\varphi, \varphi_t, \psi, \psi_t, w, w_t, \eta)^T$ using the inner product (2.0.4) we get

$$\begin{aligned} (\mathcal{A}_i U, U)_{\mathcal{H}_i} &= \rho_1 \int_0^L \Big(\frac{\kappa}{\rho_1}\varphi_{xx} - \frac{\kappa_0 l^2}{\rho_1}\varphi + \frac{\kappa}{\rho_1}\psi_x + \frac{(\kappa+\kappa_0)l}{\rho_1} w_x\Big)\overline{\varphi_t}\, dx \\ &+ \rho_2 \int_0^L \Big(-\frac{\kappa}{\rho_2}\varphi_x + \frac{\beta_0}{\rho_2}\psi_{xx} - \frac{\kappa}{\rho_2}\psi - \frac{\kappa l}{\rho_2} w + \frac{1}{\rho_2}\int_0^\infty g(s)\eta_{xx}(s)\, ds\Big)\overline{\psi_t}\, dx \\ &+ \rho_1 \int_0^L \Big(-\frac{(\kappa_0+\kappa)l}{\rho_1}\varphi_x + \frac{\kappa l}{\rho_1}\psi + \frac{\kappa_0}{\rho_1} w_{xx} - \frac{\kappa l^2}{\rho_1} w\Big)\overline{w_t}\, dx \\ &+ \beta_0 \int_0^L \psi_{tx}\overline{\psi_x}\, dx + \kappa \int_0^L (\varphi_{tx} + \psi_t + l w_t)(\overline{\varphi_x} + \overline{\psi} + l\overline{w})\, dx \\ &+ \kappa_0 \int_0^L (w_{tx} - l\varphi_t)(\overline{w_x} - l\overline{\varphi})\, dx + \int_0^\infty \int_0^L (\psi_{tx} - \eta_{sx})\overline{\eta_x}\, dx\, ds. \end{aligned}$$

Using integration by parts and the boundary conditions (1.0.20) or (1.0.21), after easy simplifications we can take the real parts to obtain

$$\mathcal{R}e(\mathcal{A}_iU,U)_{\mathcal{H}_i}=\frac{1}{2}\int_0^\infty g'(s)\int_0^L |\eta_x(x,s)|^2\,dx\,ds.$$

From hypothesis (2.0.1) on g we conclude that

$$\mathcal{R}e(\mathcal{A}_iU,U)_{\mathcal{H}_i}\le -\frac{q_1}{2}\int_0^\infty g(s)\int_0^L |\eta_x(x,s)|^2\,dx\,ds\le 0.$$

Therefore, $\mathcal{A}_i$ is a dissipative operator.

Next, we show that the operator $I-\mathcal{A}_i$ is onto. For this, let us consider the equation

$$(I-\mathcal{A}_i)U=F$$

where $U=(\varphi,\widetilde{\varphi},\psi,\widetilde{\psi},w,\widetilde{w},\eta)$ and $F=(f^1,f^2,f^3,f^4,f^5,f^6,f^7)^T\in\mathcal{H}_i$. Then, in terms of its components the above equation becomes

$$\varphi-\widetilde{\varphi}=f^1, \tag{2.0.11}$$

$$\widetilde{\varphi}-\frac{\kappa}{\rho_1}\varphi_{xx}+\frac{\kappa_0 l^2}{\rho_1}\varphi-\frac{\kappa}{\rho_1}\psi_x-\frac{(\kappa+\kappa_0)l}{\rho_1}w_x=f^2, \tag{2.0.12}$$

$$\psi-\widetilde{\psi}=f^3, \tag{2.0.13}$$

$$\widetilde{\psi}+\frac{\kappa}{\rho_2}\varphi_x-\frac{\beta_0}{\rho_2}\psi_{xx}+\frac{\kappa}{\rho_2}\psi+\frac{\kappa l}{\rho_2}w-\frac{1}{\rho_2}\int_0^\infty g(s)\eta_{xx}(s)\,ds=f^4, \tag{2.0.14}$$

$$w-\widetilde{w}=f^5, \tag{2.0.15}$$

$$\widetilde{w}+\frac{(\kappa_0+\kappa)l}{\rho_1}\varphi_x-\frac{\kappa l}{\rho_1}\psi-\frac{\kappa_0}{\rho_1}w_{xx}+\frac{\kappa l^2}{\rho_1}w=f^6, \tag{2.0.16}$$

$$\eta-\widetilde{\psi}+\eta_s=f^7. \tag{2.0.17}$$

Integrating (2.0.17), we obtain

$$\eta(\cdot,s)=\widetilde{\psi}(\cdot)(1-e^{-s})+\int_0^s e^{\tau-s}f^7(\cdot,\tau)\,d\tau. \tag{2.0.18}$$

Substituting $\widetilde{\psi}$ and η from (2.0.13) and (2.0.18) into (2.0.14), we have

$$\begin{aligned}&\rho_2\psi+\kappa\varphi_x-C_g\psi_{xx}+\kappa\psi+\kappa l w=\rho_2(f^3+f^4)\\&+\int_0^\infty g(s)\left[(e^{-s}-1)f^3_{xx}+\int_0^s e^{\tau-s}f^7_{xx}(\cdot,\tau)\,d\tau\right]ds,\end{aligned} \tag{2.0.19}$$

where

$$C_g=\beta_0+\int_0^\infty g(s)(1-e^{-s})\,ds.$$

Note that C_g is a positive constant by virtue of (2.0.1). Moreover, it can be shown that the right-hand side of (2.0.19) is in $H^{-1}(0,L)$.

On the other hand, substituting $\widetilde{\varphi}$ given by (2.0.11) and $\widetilde{w}$ given by (2.0.15) into (2.0.12) and (2.0.16), respectively, we obtain

$$\rho_1\varphi-\kappa\varphi_{xx}+\kappa_0 l^2\varphi-\kappa\psi_x-(\kappa+\kappa_0)lw_x=\rho_1(f^1+f^2), \tag{2.0.20}$$

$$\rho_1 w+(\kappa_0+\kappa)l\varphi_x-\kappa l\psi-\kappa_0 w_{xx}+\kappa l^2 w=\rho_1(f^5+f^6). \tag{2.0.21}$$

First we prove that $\varphi, \psi, w \in H_0^1(0,L)$, in the case of operator $\mathcal{A}_1$, or $\varphi \in H_0^1(0,L)$, $\psi, w \in H_*^1(0,L)$, in the case of operator $\mathcal{A}_2$.

To do this, let us consider the bilinear form

$$\begin{aligned} a(\Phi_1,\Phi_2) &= C_g \int_0^L \psi_{1x}\psi_{2x}\,dx + \kappa\int_0^L (\varphi_{1x}+\psi_1+lw_1)(\overline{\varphi_{2x}}+\overline{\psi_2}+l\overline{w_2})\,dx \\ &+\kappa_0\int_0^L (w_{1x}-l\varphi_1)(\overline{w_{2x}}-l\overline{\varphi})\,dx+\rho_1\int_0^L \varphi_1\overline{\varphi_2}\,dx \\ &+\rho_2\int_0^L \psi_1\overline{\psi_2}\,dx+\rho_1\int_0^L w_1\overline{w_2}\,dx, \end{aligned} \tag{2.0.22}$$

for $\Phi_1 = (\varphi_1,\psi_1,w_1)$, $\Phi_2 = (\varphi_2,\psi_2,w_2) \in (H_0^1(0,L))^3$ or $\Phi_1 = (\varphi_1,\psi_1,w_1)$, $\Phi_2 = (\varphi_2,\psi_2,w_2) \in H_0^1(0,L)\times H_*^1(0,L)\times H_*^1(0,L)$. Then, the Lax-Milgram theorem (see [2]) provides existence and uniqueness of the solutions

$$(\varphi,\psi,w) \in (H_0^1(0,L))^3$$

or

$$(\varphi,\psi,w) \in H_0^1(0,L)\times H_*^1(0,L)\times H_*^1(0,L)$$

of problem (2.0.19)-(2.0.21). As a consequence, $\eta \in L_g^2(\mathbb{R}^+;H_0^1(0,L))$ or $\eta \in L_g^2(\mathbb{R}^+; H_*^1(0,L))$. In fact, (2.0.13) yields $\widetilde{\psi} \in H_0^1(0,L)$ or $\widetilde{\psi} \in H_*^1(0,L)$, so that from (2.0.16) it easily follows that

$$\begin{aligned} \int_0^\infty g(s)\int_0^L |\eta_x(s)|^2\,dx\,ds &= \int_0^\infty g(s)\int_0^L \eta_x(s)\overline{\widetilde{\psi}_x}\,dx\,ds \\ +\frac{1}{2}\int_0^\infty g'(s)\int_0^L |\eta_x(s)|^2\,dx\,ds &+ \int_0^\infty g(s)\int_0^L \eta_x(s)f^7(s)\,dx\,ds, \end{aligned}$$

which in turn, by virtue of (2.0.1) and the Young inequality, yields

$$\frac{1}{2}||\eta||^2_{L_g^2(\mathbb{R}^+;H_0^1(0,L))} \le C\left(||\widetilde{\psi}_x||^2_{L^2}+||f^7||^2_{L_g^2(\mathbb{R}^+;H_0^1(0,L))}\right)$$

or

$$\frac{1}{2}||\eta||^2_{L_g^2(\mathbb{R}^+;H_*^1(0,L))} \le C\left(||\widetilde{\psi}_x||^2_{L^2}+||f^7||^2_{L_g^2(\mathbb{R}^+;H_*^1(0,L))}\right).$$

From (2.0.11) and (2.0.15), we can conclude that $\widetilde{\varphi}, \widetilde{w} \in H_0^1(0,L)$ or $\widetilde{\varphi} \in H_0^1(0,L)$, $\widetilde{w} \in H_*^1(0,L)$. Now, from (2.0.14) we get that

$$\beta_0\psi_{xx}+\int_0^\infty g(s)\eta_{xx}(s)\,ds \in L^2(0,L).$$

Furthermore, from (2.0.17), we have

$$||\eta_s||_{L_g^2(\mathbb{R}^+;H_0^1(0,L))} \le C\left(||\widetilde{\psi}_x||_{L^2}+||\eta||_{L_g^2(\mathbb{R}^+;H_0^1(0,L))}+||f^7||_{L_g^2(\mathbb{R}^+;H_0^1(0,L))}\right)$$

or

$$||\eta_s||_{L_g^2(\mathbb{R}^+;H_*^1(0,L))} \le C\left(||\widetilde{\psi}_x||_{L^2}+||\eta||_{L_g^2(\mathbb{R}^+;H_*^1(0,L))}+||f^7||_{L_g^2(\mathbb{R}^+;H_*^1(0,L))}\right).$$

Hence, $\eta_s \in L_g^2(\mathbb{R}^+;H_0^1(0,L))$ or $\eta_s \in L_g^2(\mathbb{R}^+;H_*^1(0,L))$. On the other hand, from (2.0.18) we have

$$\eta(0) = \eta(\cdot,0) = 0.$$

Thus, $I-\mathcal{A}_i$ is onto. Then, thanks to the Lumer-Phillips theorem (see [8], Theorem 1.4.3), the operator $\mathcal{A}_i$ generates a C_0-semigroup of contractions $e^{-t\mathcal{A}_i}$ on $\mathcal{H}_i$. □

3. Uniform exponential decay. In this section we are assuming the boundary conditions (1.0.20) and we are denoting by $\mathcal{A}$ the operator $\mathcal{A}_1$ and by $\mathcal{H}$ the Hilbert space $\mathcal{H}_1$. However, all results in this section remain valid with boundary conditions (1.0.21) with slight modifications.

Remark 3.1. Note that

$$\mathcal{V} := H_0^1(0,L) \times H_0^1(0,L) \times H_0^1(0,L)$$

with the norm given by

$$||(\varphi,\psi,w)||_1^2 = ||\varphi|_{H_0^1(0,L)}^2 + ||\psi||_{H_0^1(0,L)}^2 + ||w||_{H_0^1(0,L)}^2$$

is a Hilbert space. On the other hand, it easy to see that $\mathcal{V}$ with the norm given by

$$||(\varphi,\psi,w)||_2^2 = \kappa||\varphi_x+\psi+lw||_{L^2}^2 + \kappa_0||w_x - l\varphi||_{L^2}^2 + b||\psi_x||_{L^2}^2$$

is a Banach space. Indeed, using triangle and Poincaré inequalities, we have

$$||(\varphi,\psi,w)||_2^2 \leq C_1||(\varphi,\psi,w)||_1^2,$$

where C_1 is a positive constant. Now, as a consequence of the open mapping theorem, we conclude that there exists a positive constant C_2 such that

$$||(\varphi,\psi,w)||_1^2 \leq C_2||(\varphi,\psi,w)||_2^2.$$

From the above considerations, we suppose that the system (1.0.13)-(1.0.20) and hypotheses (1.0.12), (2.0.1) over the kernel g hold. We shall demonstrate that the energy

$$\begin{aligned} E(t) &= \frac{1}{2}\int_0^L \left(\rho_1|\varphi_t|^2 + \rho_2|\psi_t|^2 + \rho_1|w_t|^2 + \beta_0|\psi_x|^2 + \kappa|\varphi_x+\psi+lw|^2 + \kappa_0|w_x - l\varphi|^2\right)\, dx \\ &\quad + \frac{1}{2}\int_0^\infty g(s)\int_0^L |\eta_x(x,s)|^2\, dx\, ds \end{aligned} \tag{3.0.1}$$

decays to zero exponentially as time goes to infinity provided conditions

$$\frac{\rho_1}{\rho_2} = \frac{\kappa}{b} \quad \text{and} \quad \kappa = \kappa_0 \tag{3.0.2}$$

hold. We shall use Prüss's result [9], which states that a semigroup $e^{\mathcal{A}t}$ is exponentially stable if and only if the following conditions hold:

$$i\mathbb{R} \subset \rho(\mathcal{A}) \quad \text{resolvent set} \tag{3.0.3}$$

and

$$\exists K > 0, \quad \forall U \in D(\mathcal{A}), \quad \forall \lambda \in \mathbb{R}: \quad ||(i\lambda I - \mathcal{A})^{-1}||_{\mathcal{H}} \leq K. \tag{3.0.4}$$

In fact, note that the resolvent equation $(i\lambda I - \mathcal{A})U = F$ is given by

$$\begin{aligned} i\lambda u^1 - u^2 &= f^1, && (3.0.5)\\ i\lambda\rho_1 u^2 - \kappa(u^1_x + u^3 + lu^5)_x - \kappa_0 l(u^5_x - lu^1) &= \rho_1 f^2, && (3.0.6)\\ i\lambda u^3 - u^4 &= f^3, && (3.0.7)\\ i\lambda\rho_2 u^4 - \beta_0 u^3_{xx} + \kappa(u^1_x + u^3 + lu^5) - \int_0^\infty g(s)\eta_{xx}(s)\,ds &= \rho_2 f^4, && (3.0.8)\\ i\lambda u^5 - u^6 &= f^5, && (3.0.9)\\ i\lambda\rho_1 u^6 - \kappa_0(u^5_x - lu^1)_x + \kappa l(u^1_x + u^3 + lu^5) &= \rho_1 f^6, && (3.0.10)\\ i\lambda\eta + \eta_s - u^4 &= f^7. && (3.0.11) \end{aligned}$$

To prove condition (3.0.4) we will use a series of lemmas.

LEMMA 3.2. Let $\mathcal{A}$ be defined in (2.0.7) and let us suppose that conditions (1.0.12) and (2.0.1) on g hold. Then set $i\mathbb{R} = \{i\lambda : \ \lambda \in \mathbb{R}\}$ is contained in $\rho(\mathcal{A})$.

Proof. Then following the arguments given by Liu and Zheng [20], the proof consists of the following steps:

Step 1. In this lemma, we will use $||\cdot||$ to denote the norm in the space $\mathcal{L}(\mathcal{H})$. Since $0 \in \rho(\mathcal{A})$, for any real number λ with $||\lambda\mathcal{A}^{-1}|| < 1$, the linear bounded operator $(i\lambda\mathcal{A}^{-1} - I)$ is invertible; therefore $i\lambda I - \mathcal{A} = \mathcal{A}(i\lambda\mathcal{A}^{-1} - I)$ is invertible and its inverse belongs to $\mathcal{L}(\mathcal{H})$, that is, $i\lambda \in \rho(\mathcal{A})$. Moreover, $||(i\lambda I - \mathcal{A})^{-1}||$ is a continuous function of λ in the interval $(-||\mathcal{A}^{-1}||^{-1}, ||\mathcal{A}^{-1}||^{-1})$.

Step 2. If $\sup\{||(i\lambda I - \mathcal{A})^{-1}|| : \ |\lambda| < ||\mathcal{A}^{-1}||^{-1}\} = M < \infty$, then for $|\lambda_0| < ||\mathcal{A}^{-1}||^{-1}$ and $\lambda \in \mathbb{R}$ such that $|\lambda - \lambda_0| < M^{-1}$, we have $||(\lambda - \lambda_0)(i\lambda_0 I - \mathcal{A})^{-1}|| < 1$; therefore the operator $i\lambda I - \mathcal{A} = (i\lambda_0 I - \mathcal{A})(I + i(\lambda - \lambda_0)(i\lambda_0 I - \mathcal{A})^{-1})$ is invertible with its inverse in $\mathcal{L}(\mathcal{H})$, that is, $i\lambda \in \rho(\mathcal{A})$. Since λ_0 is arbitrary we can conclude that $\{i\lambda : \ |\lambda| < ||\mathcal{A}^{-1}||^{-1} + M^{-1}\} \subset \rho(\mathcal{A})$ and the function $||(i\lambda I - \mathcal{A})^{-1}||$ is continuous in the interval $(-||\mathcal{A}^{-1}||^{-1} - M^{-1}, ||\mathcal{A}^{-1}||^{-1} + M^{-1})$.

Step 3. Thus, it follows by (3.0.4) that if $i\mathbb{R} \subset \rho(\mathcal{A})$ is not true, then there exists $\omega \in \mathbb{R}$ with $||\mathcal{A}^{-1}||^{-1} \leq |\omega|$ such that $\{i\lambda : \ |\lambda| < |\omega|\} \subset \rho(\mathcal{A})$ and $\sup\{||(i\lambda I - \mathcal{A})^{-1}|| : |\lambda| < |\omega|\} = \infty$. Therefore, there exist a sequence λ_n in $\mathbb{R}$ with $\lambda_n \to \omega$, $|\lambda_n| < |\omega|$ and sequences of vector functions $U_n = (u^1_n, u^2_n, u^3_n, u^4_n, u^5_n, u^6_n, \eta) \in \mathcal{D}(\mathcal{A})$, with $||U_n||_{\mathcal{H}} = 1$, and $F_n = (f^1_n, f^2_n, f^3_n, f^4_n, f^5_n, f^6_n, f^7_n) \in \mathcal{H}$, such that $(i\lambda I - \mathcal{A})U_n = F_n$ and $F_n \to 0$ in $\mathcal{H}$ when $n \to \infty$.

Since

$$\mathcal{R}e\langle(i\lambda I - \mathcal{A})U_n, U_n\rangle_{\mathcal{H}} \to 0, \text{ when } n \to \infty$$

we get

$$\eta_n \to 0, \text{ in } L^2_g(\mathbb{R}^+; \ H^1_0(0, L)). \tag{3.0.12}$$

From (3.0.11) and (3.0.7) we conclude that

$$u_n^4 \to 0, \text{ in } H_0^1(0,L), \tag{3.0.13}$$

$$u_n^3 \to 0, \text{ in } H_0^1(0,L) \tag{3.0.14}$$

because $\omega \neq 0$. Now, multiplying (3.0.8) by $\overline{(u_{nx}^1 + u_n^3 + lu_n^5)}$ and performing integration by parts, we have

$$\begin{aligned}
\kappa \int_0^L |u_{nx}^1 + u_n^3 + lu_n^5|^2 \, dx &= -i\lambda_n \rho_2 \int_0^L u_n^4 \overline{(u_{nx}^1 + u_n^3 + lu_n^5)} \, dx \\
&\underbrace{- \beta_0 \int_0^L u_{nx}^3 \overline{(u_{nx}^1 + u_n^3 + lu_n^5)}_x \, dx}_{:=M_1} \\
&\underbrace{- \int_0^\infty g(s) \int_0^L \eta_{nx}(s) \overline{(u_{nx}^1 + u_n^3 + lu_n^5)}_x \, dx \, ds}_{:=M_2} \\
&+ \rho_2 \int_0^L f_n^4 \overline{(u_{nx}^1 + u_n^3 + lu_n^5)} \, dx \\
&+ \left[\left(\beta_0 u_{nx}^3 + \int_0^\infty g(s) \eta_{nx}(s) \, ds \right) u_{nx}^1 \right]_{x=0}^{x=L}.
\end{aligned} \tag{3.0.15}$$

Now, substituting $(u_{nx}^1 + u_n^3 + lu_n^5)_x$ given in (3.0.6) into M_1 and M_2, respectively, we obtain

$$\begin{aligned}
\kappa \int_0^L |u_{nx}^1 + u_n^3 + lu_n^5|^2 \, dx &= -i\lambda_n \rho_2 \int_0^L u_n^4 \overline{(u_{nx}^1 + u_n^3 + lu_n^5)} \, dx \\
&+ \beta_0 \int_0^L u_{nx}^3 \overline{(-i\lambda_n \rho_1 u_n^2 + \kappa_0 l (u_{nx}^5 - lu_n^1) + \rho_1 f_n^2)} \, dx \\
&+ \int_0^\infty g(s) \int_0^L \eta_{nx}(s) \overline{(-i\lambda_n \rho_1 u_n^2 + \kappa_0 l (u_{nx}^5 - lu_n^1) + \rho_1 f_n^2)} \, dx \, ds \\
&+ \rho_2 \int_0^L f_n^4 \overline{(u_{nx}^1 + u_n^3 + lu_n^5)} \, dx \\
&+ \underbrace{\left[\left(\beta_0 u_{nx}^3 + \int_0^\infty g(s) \eta_{nx}(s) \, ds \right) u_{nx}^1 \right]_{x=0}^{x=L}}_{:=M_3}.
\end{aligned} \tag{3.0.16}$$

We analyze each term on the right side of the above equality. Then, using Cauchy-Schwarz and Poincaré inequalities and noting that $|\lambda_n| < |\omega|$, we have

$$\left| i\lambda_n \rho_2 \int_0^L u_n^4 \overline{(u_{nx}^1 + u_n^3 + lu_n^5)} \, dx \right| \leq C|\omega| ||u_{nx}^4||_{L^2} ||u_{nx}^1 + u_n^3 + lu_n^5||_{L^2}, \tag{3.0.17}$$

$$\begin{aligned}
&\left| \beta_0 \int_0^L u_{nx}^3 \overline{(-i\lambda_n \rho_1 u_n^2 + \kappa_0 l (u_{nx}^5 - lu_n^1) + \rho_1 f_n^2)} \, dx \right| \\
&\leq C ||u_{nx}^3||_{L^2} \left(|\omega| ||u_n^2||_{L^2} + ||u_{nx}^5 - lu_n^1||_{L^2} + ||f_n^2||_{L^2} \right),
\end{aligned} \tag{3.0.18}$$

$$\left|\int_0^\infty g(s)\int_0^L \eta_{nx}(s)\overline{(-i\lambda_n\rho_1 u_n^2+\kappa_0 l(u_{nx}^5-lu_n^1)+\rho_1 f_n^2)}\,dx\,ds\right|$$
$$\leq C||\eta||_{L^2(\mathbb{R}^+;H_0^1(0,L))}\left(|\omega|||u_n^2||_{L^2}+||u_{nx}^5-lu_n^1||_{L^2}+||f_n^2||_{L^2}\right) \tag{3.0.19}$$

and

$$\left|\rho_2\int_0^L f_n^4\overline{(u_{nx}^1+u_n^3+lu_n^5)}\,dx\right|\leq C||f_n^4||_{L^2}||u_{nx}^1+u_n^3+lu_n^5||_{L^2} \tag{3.0.20}$$

where C is a positive constant.

In what follows we analyze the term M_3. For this let us take $q\in C^1([0,L])$ such that $q(0)=-q(L)=1$. Then, multiplying equation (3.0.8) by $q(x)\overline{(\beta_0 u_{nx}^3+\int_0^\infty g(s)\eta_{nx}(s)\,ds)}$ and integrating by parts in $L^2(0,L)$, we have

$$-\left[\frac{q(x)}{2}\left(\beta_0 u_{nx}^3+\int_0^\infty g(s)\eta_{nx}(s)\,ds\right)^2\right]_{x=0}^{x=L}$$
$$=-\frac{1}{2}\int_0^L q'(x)\left(\beta_0 u_{nx}^3+\int_0^\infty g(s)\eta_{nx}(s)\,ds\right)^2\,dx$$
$$-i\lambda_n\rho_2\int_0^L u_n^4 q(x)\overline{(\beta_0 u_{nx}^3+\int_0^\infty g(s)\eta_{nx}(s)\,ds)}\,dx$$
$$-\kappa\int_0^L (u_{nx}^1+u_n^3+lu_n^5)q(x)\overline{(\beta_0 u_{nx}^3+\int_0^\infty g(s)\eta_{nx}(s)\,ds)}\,dx$$
$$+\rho_2\int_0^L f_n^4 q(x)\overline{(\beta_0 u_{nx}^3+\int_0^\infty g(s)\eta_{nx}(s)\,ds)}\,dx.$$

Using Young and Cauchy-Schwarz inequalities, the above equation can be rewritten as

$$-\left[\frac{q(x)}{2}\left|\beta_0 u_{nx}^3+\int_0^\infty g(s)\eta_{nx}(s)\,ds\right|^2\right]_{x=0}^{x=L}$$
$$\leq C\left(||u_n^4||_{L^2}^2+||u_{nx}^3||_{L^2}^2+||\eta||^2_{L^2(\mathbb{R}^+;H_0^1(0,L))}+||f_n^4||^2_{L^2}\right)$$
$$+C||u_{nx}^1+u_n^3+lu_n^5||_{L^2}\left(||u_{nx}^3||_{L^2}+||\eta||_{L^2(\mathbb{R}^+;H_0^1(0,L))}\right). \tag{3.0.21}$$

From the convergences (3.0.12), (3.0.13), and (3.0.14), noting that

$$F_n\to 0,\quad \text{in}\quad \mathcal{H}\quad\text{when}\quad n\to\infty$$

and taking into account the estimates (3.0.17)-(3.0.21), we can conclude that

$$\left| i\lambda_n \rho_2 \int_0^L u_n^4 \overline{(u_{nx}^1 + u_n^3 + l u_n^5)}\, dx \right| \to 0, \tag{3.0.22}$$

$$\left| \beta_0 \int_0^L u_{nx}^3 \overline{(-i\lambda_n \rho_1 u_n^2 + \kappa_0 l (u_{nx}^5 - l u_n^1) + \rho_1 f_n^2)}\, dx \right| \to 0, \tag{3.0.23}$$

$$\left| \int_0^\infty g(s) \int_0^L \eta_{nx}(s) \overline{(-i\lambda_n \rho_1 u_n^2 + \kappa_0 l (u_{nx}^5 - l u_n^1) + \rho_1 f_n^2)}\, dx\, ds \right| \to 0, \tag{3.0.24}$$

$$\left| \rho_2 \int_0^L f_n^4 \overline{(u_{nx}^1 + u_n^3 + l u_n^5)}\, dx \right| \to 0, \tag{3.0.25}$$

$$|M_3| \to 0, \quad \text{when} \quad n \to \infty \tag{3.0.26}$$

because $\omega \neq 0$ and $||U_n||_{\mathcal{H}} = 1$. Therefore, from (3.0.16)

$$\kappa \int_0^L |u_{nx}^1 + u_n^3 + l u_n^5|^2\, dx \to 0, \quad \text{when} \quad n \to \infty,$$

from which it follows that

$$(u_{nx}^1 + u_n^3 + l u_n^5) \to 0, \text{ in } L^2(0, L). \tag{3.0.27}$$

Thus, from (3.0.5), (3.0.9), of the convergence obtained and Remark 3.1, we get

$$u_n^2 \to 0, \text{ in } L^2(0, L), \tag{3.0.28}$$

$$u_n^6 \to 0, \text{ in } L^2(0, L), (u_{nx}^5 - l u_n^1) \to 0, \text{ in } L^2(0, L). \tag{3.0.29}$$

Since $||U_n||_{\mathcal{H}} = 1$, for all $n \in \mathbb{N}$, we have a contradiction and the proof of the lemma is complete. □

REMARK 3.3. In particular this result implies that the semigroup is strongly stable, that is

$$S(t)U_0 \quad \to \quad 0,$$

where $S(t) := e^{\mathcal{A}t}$ is the C_0-semigroup of contractions on Hilbert space $\mathcal{H}_2$ and Ψ_0 is the initial data.

LEMMA 3.4. *Let us suppose that conditions (1.0.12) and (2.0.1) on g hold. Then there exists a positive constant $K > 0$ being independent of $F \in \mathcal{H}$, such that*

$$\int_0^L \int_0^\infty g(s) |\eta_x|^2\, ds\, dx \leq K ||U||_{\mathcal{H}} ||F||_{\mathcal{H}}.$$

Proof. Multiplying equation (3.0.6) by $\overline{u^2}$ (in $L^2(0, L)$) we get

$$i\lambda \rho_1 \int_0^L |u^2|^2\, dx + \kappa \int_0^L (u_x^1 + u^3 + l u^5) \overline{u_x^2}\, dx - \kappa_0 l \int_0^L (u_x^5 - l u^1) \overline{u^2}\, dx = \rho_1 \int_0^L f^2 \overline{u^2}\, dx$$

and, using equation (3.0.5) we arrive at

$$\begin{aligned}&i\lambda\rho_1\int_0^L |u^2|^2\,dx - i\lambda\kappa\int_0^L (u_x^1+u^3+lu^5)\overline{u_x^1}\,dx + i\lambda\kappa_0 l\int_0^L (u_x^5-lu^1)\overline{u^1}\,dx\\&=\rho_1\int_0^L f^2\overline{u^2}\,dx+\kappa\int_0^L (u_x^1+u^3+lu^5)\overline{f_x^1}\,dx\\&-\kappa_0 l\int_0^L (u_x^5-lu^1)\overline{f^1}\,dx.\end{aligned}\tag{3.0.30}$$

On the other hand, multiplying equation (3.0.8) by $\overline{u^4}$ (in $L^2(0,L)$) we get

$$\begin{aligned}&i\lambda\rho_2\int_0^L |u^4|^2\,dx+\underbrace{\beta_0\int_0^L u_x^3\overline{u_x^4}\,dx}_{:=I_1}+\underbrace{\kappa\int_0^L (u_x^1+u^3+lu^5)\overline{u^4}\,dx}_{:=I_2}\\&+\underbrace{\int_0^L\int_0^\infty g(s)\eta_x\overline{u_x^4}\,ds\,dx}_{:=I_3}=\rho_2\int_0^L f^4\overline{u^4}\,dx.\end{aligned}$$

Substituting u^4 given by (3.0.7) and (3.0.11) into I_1, I_2 and I_3, respectively, we get

$$\begin{aligned}&i\lambda\rho_2\int_0^L |u^4|^2\,dx - i\beta_0\lambda\int_0^L |u_x^3|^2\,dx - i\lambda\kappa\int_0^L (u_x^1+u^3+lu^5)\overline{u^3}\,dx\\&-i\lambda\int_0^L\int_0^\infty g(s)|\eta_x|^2\,ds\,dx+\int_0^L\int_0^\infty g(s)\eta_x\overline{\eta_{xs}}\,ds\,dx=\rho_2\int_0^L f^4\overline{u^4}\,dx\\&+\beta_0\int_0^L f_x^3\overline{u_x^3}\,dx+\kappa\int_0^L (u_x^1+u^3+lu^5)\overline{f^3}\,dx\\&+\int_0^L\int_0^\infty g(s)\eta_x\overline{f_x^7}\,ds\,dx.\end{aligned}\tag{3.0.31}$$

Now, multiplying the equality (3.0.10) by $\overline{u^6}$ (in $L^2(0,L)$) we get

$$\begin{aligned}&i\lambda\rho_1\int_0^L |u^6|^2\,dx+\underbrace{\kappa_0\int_0^L (u_x^5-lu^1)\overline{u_x^6}\,dx}_{:=I_4}\\&+\underbrace{\kappa l\int_0^L (u_x^1+u^3+lu^5)\overline{u^6}\,dx}_{:=I_5}=\rho_1\int_0^L f^6\overline{u^6}\,dx.\end{aligned}$$

Using the equality (3.0.9) in I_4 and I_5 we have

$$\begin{aligned}&i\lambda\rho_1\int_0^L |u^6|^2\,dx - i\lambda\kappa_0\int_0^L (u_x^5-lu^1)\overline{u_x^5}\,dx - i\lambda\kappa l\int_0^L (u_x^1+u^3+lu^5)\overline{u^5}\,dx\\&=\kappa_0\int_0^L (u_x^5-lu^1)\overline{f_x^5}\,dx+\kappa l\int_0^L (u_x^1+u^3+lu^5)\overline{f^5}\,dx+\rho_1\int_0^L f^6\overline{u^6}\,dx.\end{aligned}\tag{3.0.32}$$

Adding (3.0.30), (3.0.31) and (3.0.32) we get

$$
\begin{aligned}
& i\lambda\rho_1 \int_0^L |u^2|^2\,dx + i\lambda\rho_2 \int_0^L |u^4|^2\,dx + i\lambda\rho_1 \int_0^L |u^6|^2\,dx - i\lambda\kappa \int_0^L |u_x^1 + u^3 + lu^5|^2\,dx \\
& - i\lambda\kappa_0 \int_0^L |u_x^5 - lu^1|^2\,dx - i\beta_0\lambda \int_0^L |u_x^3|^2\,dx - i\lambda \int_0^L \int_0^\infty g(s)|\eta_x|^2\,ds\,dx \\
& + \int_0^L \int_0^\infty g(s)\eta_x \overline{\eta_{xs}}\,ds\,dx = \rho_1 \int_0^L f^2 \overline{u^2}\,dx + \kappa \int_0^L (u_x^1 + u^3 + lu^5)\overline{f_x^1}\,dx \\
& - \kappa_0 l \int_0^L (u_x^5 - lu^1)\overline{f^1}\,dx + \rho_2 \int_0^L f^4 \overline{u^4}\,dx + \beta_0 \int_0^L f_x^3 \overline{u_x^3}\,dx \\
& + \kappa \int_0^L (u_x^1 + u^3 + lu^5)\overline{f^3}\,dx + \int_0^L \int_0^\infty g(s)\eta_x \overline{f_x^7}\,ds\,dx + \kappa_0 \int_0^L (u_x^5 - lu^1)\overline{f_x^5}\,dx \\
& + \kappa l \int_0^L (u_x^1 + u^3 + lu^5)\overline{f^5}\,dx + \rho_1 \int_0^L f^6 \overline{u^6}\,dx.
\end{aligned}
$$

Using integrating by parts, we have

$$
\int_0^L \int_0^\infty g(s)\eta_x \overline{\eta_{xs}}\,dx = -\frac{1}{2} \int_0^L \int_0^\infty g'(s)|\eta_x|^2\,ds\,dx. \tag{3.0.33}
$$

Substituting equation (3.0.33) into the above equation, we get

$$
\begin{aligned}
& i\lambda\rho_1 \int_0^L |u^2|^2\,dx + i\lambda\rho_2 \int_0^L |u^4|^2\,dx + i\lambda\rho_1 \int_0^L |u^6|^2\,dx - i\lambda\kappa \int_0^L |u_x^1 + u^3 + lu^5|^2\,dx \\
& - i\lambda\kappa_0 \int_0^L |u_x^5 - lu^1|^2\,dx - i\beta_0\lambda \int_0^L |u_x^3|^2\,dx - i\lambda \int_0^L \int_0^\infty g(s)|\eta_x|^2\,ds\,dx \\
& - \frac{1}{2} \int_0^L \int_0^\infty g'(s)|\eta_x(s)|^2\,ds\,dx = \rho_1 \int_0^L f^2 \overline{u^2}\,dx + \kappa \int_0^L (u_x^1 + u^3 + lu^5)\overline{f_x^1}\,dx \\
& - \kappa_0 l \int_0^L (u_x^5 - lu^1)\overline{f^1}\,dx + \rho_2 \int_0^L f^4 \overline{u^4}\,dx + \beta_0 \int_0^L f_x^3 \overline{u_x^3}\,dx \\
& + \kappa \int_0^L (u_x^1 + u^3 + lu^5)\overline{f^3}\,dx + \int_0^L \int_0^\infty g(s)\eta_x \overline{f_x^7}\,ds\,dx + \kappa_0 \int_0^L (u_x^5 - lu^1)\overline{f_x^5}\,dx \\
& + \kappa l \int_0^L (u_x^1 + u^3 + lu^5)\overline{f^5}\,dx + \rho_1 \int_0^L f^6 \overline{u^6}\,dx.
\end{aligned}
$$

Taking the real part on the left side of the above equality and using the conditions (2.0.1) on g, our conclusion follows. □

LEMMA 3.5. With the same hypotheses as in Lemma 3.4 there exists $K > 0$ such that

$$
\rho_2 \int_0^L |u^4|^2\,dx \le K||U||_{\mathcal{H}}||F||_{\mathcal{H}} + K||U||_{\mathcal{H}}^{\frac{1}{2}}||F||_{\mathcal{H}}^{\frac{1}{2}}(||u_x^3||_{L^2} + ||u_x^1 + u^3 + lu^5||_{L^2}).
$$

Proof. Multiplying (3.0.8) by $\int_0^\infty g(s)\overline{\eta}\,ds$ in $L^2(0,L)$ we get

$$\underbrace{i\lambda\rho_2\int_0^L\int_0^\infty g(s)\overline{\eta}u^4\,ds\,dx}_{:=I_6}+\beta_0\int_0^L\int_0^\infty g(s)\overline{\eta_x}u_x^3\,ds\,dx$$
$$+\kappa\int_0^L\int_0^\infty g(s)(u_x^1+u^3+lu^5)\overline{\eta}\,ds\,dx+\int_0^L\left|\int_0^\infty g(s)\eta_x\,ds\right|^2\,dx$$
$$=\rho_2\int_0^L\int_0^\infty g(s)\overline{\eta}f^4\,ds\,dx.$$

Substituting η given in (3.0.11) into I_6 we get

$$\rho_2 b_0\int_0^L|u^4|^2\,dx=-\rho_2\int_0^L\int_0^\infty g(s)u^4\overline{f^7}(s)\,ds\,dx+\rho_2\int_0^\infty g(s)\int_0^L u^4\overline{\eta}_s(s)\,dx\,ds$$
$$+\beta_0\int_0^L\int_0^\infty g(s)\overline{\eta_x}(s)u_x^3\,ds\,dx+\kappa\int_0^L\int_0^\infty g(s)(u_x^1+u^3+lu^5)\overline{\eta}(s)\,ds\,dx$$
$$+\int_0^L\left|\int_0^\infty g(s)\eta_x(s)\,ds\right|^2\,dx-\rho_2\int_0^L\int_0^\infty g(s)\overline{\eta}(s)f^4\,ds\,dx \tag{3.0.34}$$

where $b_0=\int_0^\infty g(s)\,ds$. From Lemma 3.4 and using the Poincaré inequality we obtain

$$\int_0^L\left|\int_0^\infty g(s)\eta_x(s)\,ds\right|^2dx\le\int_0^\infty g(s)\,ds\int_0^L\int_0^\infty g(s)|\eta_x|^2\,ds\,dx\le C||U||_{\mathcal{H}}||F||_{\mathcal{H}}, \tag{3.0.35}$$

$$\left|\kappa\int_0^L\int_0^\infty g(s)(u_x^1+u^3+lu^5)\overline{\eta}(s)\,ds\,dx\right|\le K||u_x^1+u^3+lu^5||_{L^2}||U||_{\mathcal{H}}^{\frac12}||F||_{\mathcal{H}}^{\frac12} \tag{3.0.36}$$

and

$$\left|\beta_0\int_0^\infty g(s)\int_0^L u_x^3\overline{\eta}_x(s)\,dx\,ds\right|\le K||u_x^3||_{L^2}||U||_{\mathcal{H}}^{\frac12}||F||_{\mathcal{H}}^{\frac12}. \tag{3.0.37}$$

Substituting (3.0.35)-(3.0.37) into (3.0.34) and noting that

$$\left|\int_0^L\int_0^\infty g(s)\overline{\eta_s}u^4\,ds\,dx\right|\le\frac{\rho_2}{2}\int_0^L|u^4|^2\,dx+K\int_0^L\int_0^\infty|g'(s)||\eta_x|^2\,ds\,dx,$$

our conclusion follows. □

LEMMA 3.6. With the same hypotheses as in Lemma 3.4 there exists $K>0$ such that

$$\left(\rho_1-\frac{\kappa_0 l^2}{|\lambda|^2}\right)\int_0^L|u^2|^2\,dx+\frac{\kappa_0 l^2}{2}\int_0^L|u^5|^2\,dx\le K\int_0^L|u_x^1+u^3+lu^5|^2\,dx$$
$$+K\int_0^L|u_x^3|^2\,dx+K||U||_{\mathcal{H}}||F||_{\mathcal{H}}+K||F||_{\mathcal{H}}^2$$

and

$$\beta_0\int_0^L|u_x^3|^2\,dx\le\kappa\int_0^L|u_x^1+u^3+lu^5|^2\,dx+K||U||_{\mathcal{H}}||F||_{\mathcal{H}}$$
$$+K||U||_{\mathcal{H}}^{\frac12}||F||_{\mathcal{H}}^{\frac12}\left(||u_x^3||_{L^2}+||u_x^1+u^3+lu^5||_{L^2}\right)$$

for $|\lambda|>1$ large enough.

Proof. Multiplying (3.0.6) by $\overline{u^1}$ in $L^2(0,L)$ and using (3.0.5), we arrive at

$$\begin{aligned}
\rho_1 \int_0^L |u^2|^2\,dx &= \kappa \int_0^L (u^1_x + u^3 + lu^5)\overline{u^1_x}\,dx - \kappa_0 l \int_0^L (u^5_x - lu^1)\overline{u^1}\,dx \\
&- \rho_1 \int_0^L (f^2\overline{u^1} + u^2\overline{f^1})\,dx \\
&= \kappa \int_0^L |u^1_x + u^3 + lu^5|^2\,dx - \kappa \int_0^L (u^1_x + u^3 + lu^5)\overline{(u^3 + lu^5)}\,dx \\
&+ \kappa_0 l \int_0^L u^5\overline{u^1_x}\,dx + \kappa_0 l^2 \int_0^L |u^1|^2\,dx \\
&- \rho_1 \int_0^L (f^2\overline{u^1} + u^2\overline{f^1})\,dx \\
&= \kappa \int_0^L |u^1_x + u^3 + lu^5|^2\,dx - \kappa \int_0^L (u^1_x + u^3 + lu^5)\overline{(u^3 + lu^5)}\,dx \\
&+ \kappa_0 l \int_0^L u^5(\overline{u^1_x + u^3 + lu^5})\,dx - \kappa_0 l \int_0^L u^5\overline{u^3}\,dx - \kappa_0 l^2 \int_0^L |u^5|^2\,dx \\
&+ \kappa_0 l^2 \int_0^L |u^1|^2\,dx - \rho_1 \int_0^L (f^2\overline{u^1} + u^2\overline{f^1})\,dx.
\end{aligned}$$

From (3.0.5), we have

$$|u^1| \le \frac{|u^2|}{|\lambda|} + \frac{|f^1|}{|\lambda|}. \tag{3.0.38}$$

On the other hand, using the Poincaré inequality and Remark 3.1, we can conclude that

$$||f^1||_{L^2} \le C||f^1_x||_{L^2} \le C||F||_{\mathcal{H}}$$

where C is a positive constant.

Therefore, substituting the inequality (3.0.38) into the above equation, we obtain

$$\begin{aligned}
\rho_1 \int_0^L |u^2|^2\,dx &\le \kappa \int_0^L |u^1_x + u^3 + lu^5|^2\,dx - \underbrace{\kappa \int_0^L (u^1_x + u^3 + lu^5)\overline{u^3}\,dx}_{:=a_1} \\
&- \underbrace{\kappa l \int_0^L (u^1_x + u^3 + lu^5)\overline{u^5}\,dx}_{:=a_2} + \underbrace{\kappa_0 l \int_0^L u^5\overline{(u^1_x + u^3 + lu^5)}\,dx}_{:=a_3} \\
&- \underbrace{\kappa_0 l \int_0^L u^5\overline{u^3}\,dx}_{:=a_4} - \kappa_0 l^2 \int_0^L |u^5|^2\,dx + \frac{\kappa_0 l^2}{|\lambda|^2} \int_0^L |u^2|^2\,dx \\
&+ \frac{1}{|\lambda|^2}||F||^2_{\mathcal{H}} + \frac{2}{|\lambda|^2}||U||_{\mathcal{H}}||F||_{\mathcal{H}} + \rho_1||U||_{\mathcal{H}}||F||_{\mathcal{H}}.
\end{aligned} \tag{3.0.39}$$

Applying the Young inequality in a_1, a_2, a_3 and a_4, the inequality (3.0.39) becomes

$$\begin{aligned}
&\left(\rho_1 - \frac{\kappa_0 l^2}{|\lambda|^2}\right) \int_0^L |u^2|^2\,dx + \frac{\kappa_0 l^2}{2} \int_0^L |u^5|^2\,dx \le K \int_0^L |u^1_x + u^3 + lu^5|^2\,dx \\
&+K \int_0^L |u^3_x|^2\,dx + K||U||_{\mathcal{H}}||F||_{\mathcal{H}} + K||F||^2_{\mathcal{H}}.
\end{aligned} \tag{3.0.40}$$

Now, multiplying (3.0.8) by $\overline{u^3}$ we get

$$\underbrace{i\lambda\rho_2\int_0^L u^4\overline{u^3}\,dx}_{:=a_5} + \beta_0\int_0^L |u_x^3|^2\,dx + \kappa\int_0^L (u_x^1+u^3+lu^5)\overline{u^3}\,dx$$
$$+ \int_0^L\int_0^\infty g(s)\eta_x(s)\overline{u_x^3}\,ds\,dx = \rho_2\int_0^L f^4\overline{u^3}\,dx.$$

Substituting u^3 given in (3.0.7) into a_5, we have

$$\beta_0\int_0^L |u_x^3|^2\,dx = \rho_2\int_0^L |u^4|^2\,dx - \kappa\int_0^L (u_x^1+u^3+lu^5)\overline{u^3}\,dx$$
$$-\int_0^L\int_0^\infty g(s)\eta_x(s)\overline{u_x^3}\,ds + \rho_2\int_0^L f^4\overline{u^3}\,dx + \rho_2\int_0^L u^4\overline{f^3}\,dx. \quad (3.0.41)$$

Now we can consider the Poincaré and Young inequalities, then take into account Lemmas 3.4 and 3.5 to obtain the following inequality:

$$\beta_0\int_0^L |u_x^3|^2\,dx \le \kappa\int_0^L |u_x^1+u^3+lu^5|^2\,dx + K||U||_{\mathcal{H}}||F||_{\mathcal{H}}$$
$$+K||U||_{\mathcal{H}}^{\frac{1}{2}}||F||_{\mathcal{H}}^{\frac{1}{2}}\left(||u_x^3||_{L^2} + ||u_x^1+u^3+lu^5||_{L^2}\right), \quad (3.0.42)$$

from which the second part of the lemma follows. The proof is now complete. □

Our next step is to estimate the term $||u_x^1+u^3+lu^5||_{L^2}$. Here we shall use the condition (3.0.2).

LEMMA 3.7. For any strong solutions of system (1.0.13)-(1.0.20) and for any $\varepsilon_1 > 0$ there exists $K_{\varepsilon_1} > 0$ such that

$$\kappa\left(1-\frac{\beta_0 l^2}{2\kappa|\lambda|}\right)\int_0^L |u_x^1+u^3+lu^5|^2\,dx \le K_{\varepsilon_1}||U||_{\mathcal{H}}||F||_{\mathcal{H}}$$
$$+\mathcal{R}e\left(\left[\beta_0 u_x^3 + \int_0^\infty g(s)\eta_x\,ds\right]\overline{u_x^1}\right)_{x=0}^{x=L}$$
$$+\varepsilon_1||u^2||_{L^2(0,L)}^2 + \varepsilon_1||u_x^5 - lu^1||_{L^2(0,L)}^2$$
$$+\left(\rho_2 + 2K_{\varepsilon_1} + \frac{\beta_0 l^2}{2|\lambda|}\right)\int_0^L |u^4|^2\,dx + \varepsilon_1\rho_1\int_0^L |u^6|^2\,dx$$
$$+|\lambda b|\left|\frac{\rho_1}{\kappa} - \frac{\rho_2}{b}\right|\int_0^L |u_x^3||u^2|\,dx$$

with $|\lambda| > 1$ large enough.

Proof. Multiplying equation (3.0.8) by $(\overline{u_x^1+u^3+lu^5})$ in $L^2(0,L)$ and integrating by parts we get

$$i\lambda\rho_2\int_0^L u^4(\overline{u_x^1+u^3+lu^5})\,dx-\left(\left[\beta_0u_x^3+\int_0^\infty g(s)\eta_x\,ds\right]\overline{u_x^1}\right)_{x=0}^{x=L}$$
$$+\underbrace{\int_0^L\left(\left[\beta_0u_x^3+\int_0^\infty g(s)\eta_x\,ds\right]\right)(\overline{u_x^1+u^3+lu^5})_x\,dx}_{:=I_7}+\kappa\int_0^L|u_x^1+u^3+lu^5|^2\,dx$$
$$=\rho_2\int_0^L f^4(\overline{u_x^1+u^3+lu^5})\,dx.$$

Substituting $(u_x^1+u^3+lu^5)_x$ given by (3.0.6) into I_7 we arrive at

$$\underbrace{i\lambda\rho_2\int_0^L u^4\overline{u_x^1}\,dx}_{:=I_8}+\underbrace{i\lambda\rho_2\int_0^L u^4\overline{u^3}\,dx}_{:=I_9}+\underbrace{i\lambda\rho_2 l\int_0^L u^4\overline{u^5}\,dx}_{:=I_{10}}$$
$$-\left(\left[\beta_0u_x^3+\int_0^\infty g(s)\eta_x\,ds\right]\overline{u_x^1}\right)_{x=0}^{x=L}-\frac{i\lambda\rho_1\beta_0}{\kappa}\int_0^L u_x^3\overline{u^2}\,dx$$
$$\underbrace{-\frac{\beta_0\kappa_0 l}{\kappa}\int_0^L u_x^3(\overline{u_x^5-lu^1})\,dx}_{:=I_{11}}-\frac{\beta_0\rho_1}{\kappa}\int_0^L u_x^3\overline{f^2}\,dx$$
$$\underbrace{-\frac{i\lambda\rho_1}{\kappa}\int_0^L\int_0^\infty g(s)\eta_x\overline{u^2}\,ds\,dx}_{:=I_{12}}-\frac{\kappa_0 l}{\kappa}\int_0^L\int_0^\infty g(s)\eta_x(\overline{u_x^5-lu^1})\,ds\,dx$$
$$+\kappa\int_0^L|u_x^1+u^3+lu^5|^2\,dx=\rho_2\int_0^L f^4\overline{(u_x^1+u^3+lu^5)}\,dx$$
$$+\frac{\rho_1}{\kappa}\int_0^L\int_0^\infty g(s)\eta_x\overline{f^2}\,ds\,dx. \tag{3.0.43}$$

Substituting u^1 given by (3.0.5) and u^4 given by (3.0.7) into I_8, we obtain

$$I_8=-i\lambda\rho_2\int_0^L u^3\overline{u_x^2}\,dx-\rho_2\int_0^L u^4\overline{f_x^1}\,dx+\rho_2\int_0^L f^3\overline{u_x^2}\,dx. \tag{3.0.44}$$

Substituting u^3 given in (3.0.7) into I_9 we get

$$I_9=-\rho_2\int_0^L|u^4|^2\,dx-\rho_2\int_0^L u^4\overline{f^3}\,dx. \tag{3.0.45}$$

On the other hand, substituting u^5 given by (3.0.9) into I_{10} and using the Young inequality, we obtain

$$|I_{10}|\le K_{\varepsilon_1}\int_0^L|u^4|^2\,dx+\frac{\varepsilon_1\rho_1}{2}\int_0^L|u^6|^2\,dx+K||U||_{\mathcal H}||F||_{\mathcal H}, \tag{3.0.46}$$

where ε_1 is a small positive constant. From (3.0.7), we get

$$|u^3| \leq \frac{|u^4|}{|\lambda|} + \frac{|f^3|}{|\lambda|}. \tag{3.0.47}$$

Performing an integration by parts in I_{11}, we get

$$I_{11} = \frac{\beta_0 \kappa_0 l}{\kappa} \int_0^L u^3 \overline{(u_x^5 - lu^1)_x}\, dx. \tag{3.0.48}$$

Substituting $(u_x^5 - lu^1)_x$ given by (3.0.10) into (3.0.48), we have

$$I_{11} = \frac{\beta_0 l}{\kappa} \int_0^L u^3(-i\lambda\rho_1 \overline{u^6} + \kappa l \overline{(u_x^1 + u^3 + lu^5)} - \rho_1 \overline{f^6})\, dx. \tag{3.0.49}$$

Again, substituting u^3 given by (3.0.7) and using (3.0.47) into (3.0.49), and applying the Young inequality we get

$$\begin{aligned} |I_{11}| \quad &\leq \quad K_{\varepsilon_1} \int_0^L |u^4|^2\, dx + \frac{\varepsilon_1 \rho_1}{2} \int_0^L |u^6|^2\, dx + K||U||_{\mathcal{H}}||F||_{\mathcal{H}} \\ &+ \quad \frac{\beta_0 l^2}{2|\lambda|} \int_0^L |u^4|^2\, dx + \frac{\beta_0 l^2}{2|\lambda|} \int_0^L |u_x^1 + u^3 + lu^5|^2\, dx. \end{aligned} \tag{3.0.50}$$

Finally, substituting η given by (3.0.11) into I_{12} yields

$$I_{12} = \frac{\rho_1}{\kappa} \int_0^L \int_0^\infty g(s) \eta_{xs} \overline{u^2}\, ds\, dx - \underbrace{\frac{\rho_1 b_0}{\kappa} \int_0^L u_x^4 \overline{u^2}\, dx}_{:=I_{13}} - \frac{\rho_1}{\kappa} \int_0^L \int_0^\infty g(s) f_x^7 \overline{u^2}\, ds\, dx.$$

Now, substituting u^4 given by (3.0.7) into I_{13}, we can rewrite I_{12} as

$$\begin{aligned} I_{12} \quad &= \quad -\frac{\rho_1}{\kappa} \int_0^L \int_0^\infty g'(s) \eta_x \overline{u^2}\, ds\, dx - \frac{i\lambda \rho_1 b_0}{\kappa} \int_0^L u_x^3 \overline{u^2}\, dx \\ &+ \quad \frac{\rho_1 b_0}{\kappa} \int_0^L f_x^3 \overline{u^2}\, dx - \frac{\rho_1}{\kappa} \int_0^L \int_0^\infty g(s) f_x^7 \overline{u^2}\, ds\, dx. \end{aligned} \tag{3.0.51}$$

Substituting (3.0.44), (3.0.45), (3.0.46), (3.0.50) and (3.0.51) into (3.0.43), we obtain

$$\begin{aligned}
&\kappa \int_0^L |u_x^1 + u^3 + lu^5|^2\, dx \le |\lambda| b \left|\frac{\rho_1}{\kappa} - \frac{\rho_2}{b}\right| \int_0^L |u_x^3||\overline{u^2}|\, dx \\
&+ \left(\left[\beta_0 u_x^3 + \int_0^\infty g(s)\eta_x\, ds\right] \overline{u_x^1}\right)_{x=0}^{x=L} + \left(\rho_2 + 2K_{\varepsilon_1} + \frac{\beta_0 l^2}{2|\lambda|}\right) \int_0^L |u^4|^2\, dx \\
&+\rho_2 \int_0^L u^4 \overline{f^3}\, dx + \rho_2 \int_0^L u^4 \overline{f_x^1}\, dx \\
&+\frac{\rho_1}{\kappa} \int_0^L \int_0^\infty g(s) f_x^7 \overline{u^2}\, ds\, dx + \frac{\rho_1}{\kappa} \int_0^L \int_0^\infty g'(s)\eta_x \overline{u^2}\, ds\, dx \\
&+\varepsilon_1 \rho_1 \int_0^L |u^6|^2\, dx + \frac{\beta_0 \rho_1}{\kappa} \int_0^L u_x^3 \overline{f^2}\, dx + \frac{\beta_0 l^2}{2|\lambda|} \int_0^L |u_x^1 + u^3 + lu^5|^2\, dx \\
&+\frac{\kappa_0 l}{\kappa} \int_0^L \int_0^\infty g(s)\eta_x (\overline{u_x^5 - lu^1})\, ds\, dx + \rho_2 \int_0^L f^4 (\overline{u_x^1 + u^3 + lu^5})\, dx \\
&+\frac{\rho_1}{\kappa} \int_0^L \int_0^\infty g(s)\eta_x \overline{f^2}\, ds\, dx + \left(\rho_2 - \frac{\rho_1 b_0}{\kappa}\right) \int_0^L f_x^3 \overline{u^2}\, dx \\
&+K||U||_{\mathcal{H}}||F||_{\mathcal{H}}. \qquad (3.0.52)
\end{aligned}$$

Applying the Young inequality in the above inequality, considering the previous lemmas and taking the real parts, we obtain the conclusion of the lemma. The proof is now complete. □

LEMMA 3.8. Under the above notation, let us take $q \in C^1([0, L])$ such that $q(0) = -q(L) = 1$. Then there exist $K,\ K_q > 0$ such that

$$\begin{aligned}
(i) \qquad & -\left(\frac{q(x)}{2}\left|\beta_0 u_x^3 + \int_0^\infty g(s)\eta_x\, ds\right|\right)_{x=0}^{x=L} \le K||U||_{\mathcal{H}}||F||_{\mathcal{H}} \\
& +K||U||_{\mathcal{H}}^{\frac{1}{2}}||F||_{\mathcal{H}}^{\frac{1}{2}}||u_x^1 + u^3 + lu^5||_{L^2(0,L)} + \varepsilon_1 K_{\rho_1}||u^2||_{L^2(0,L)}^2
\end{aligned}$$

and

$$\begin{aligned}
(ii) \qquad & -\left(\frac{q(x)}{2}|u_x^1|^2\right)_{x=0}^{x=L} \le K||U||_{\mathcal{H}}||F||_{\mathcal{H}} \\
& +K||U||_{\mathcal{H}}^{\frac{1}{2}}||F||_{\mathcal{H}}^{\frac{1}{2}}(||u_x^1 + u^3 + lu^5||_{L^2(0,L)} + ||u_x^3||_{L^2(0,L)}) \\
& +K_q(||u_x^5 - lu^1||_{L^2(0,L)}^2 + \varepsilon_1||u^2||_{L^2(0,L)}^2).
\end{aligned}$$

Proof. To prove (i), we multiply (3.0.8) by $q(x)\overline{(\beta_0 u_x^3 + \int_0^\infty g(s)\eta_x\, ds)}$ in $L^2(0,L)$ to obtain

$$\underbrace{i\lambda\rho_2 \int_0^L u^4 q(x)\overline{(\beta_0 u_x^3 + \int_0^\infty g(s)\eta_x\, ds)}dx}_{:=I_{14}}$$
$$-\underbrace{\int_0^L (\beta_0 u_{xx}^3 + \int_0^L g(s)\eta_{xx}\, ds)q(x)\overline{(\beta_0 u_x^3 + \int_0^\infty g(s)\eta_x\, ds)}dx}_{:=I_{15}}$$
$$+\kappa\int_0^L (u_x^1 + u^3 + lu^5)q(x)\overline{(\beta_0 u_x^3 + \int_0^\infty g(s)\eta_x\, ds)}dx$$
$$= \rho_2 \int_0^L f^4 q(x)\overline{(\beta_0 u_x^3 + \int_0^\infty g(s)\eta_x\, ds)}dx. \tag{3.0.53}$$

From (3.0.7) and (3.0.11), we have

$$I_{14} = \frac{\rho_2 b_0}{2}\int_0^L q'(x)|u^4|^2\, dx - \rho_2\beta_0 \int_0^L u^4 q(x)\overline{f_x^3}\, dx$$
$$-\rho_2 \int_0^L \int_0^\infty g'(s)q(x)u^4\overline{\eta_x}\, ds\, dx - \rho_2 \int_0^L \int_0^\infty g(s)q(x)u^4\overline{f_x^7}\, ds\, dx. \tag{3.0.54}$$

Now, we can note that

$$\begin{aligned} I_{15} &= -\left(\frac{q(x)}{2}\left|\beta_0 u_x^3 + \int_0^\infty g(s)\eta_x\, ds\right|\right)_{x=0}^{x=L} \\ &+ \frac{1}{2}\int_0^L q'(x)\left|\beta_0 u_x^3 + \int_0^\infty g(s)\eta_x\, ds\right|^2 dx. \end{aligned} \tag{3.0.55}$$

Substituting (3.0.54) and (3.0.55) into (3.0.53), taking the real part and using the previous lemmas, we get the conclusion of the first inequality.

To prove the inequality (ii) we multiply (3.0.6) by $q(x)\overline{u_x^1}$ in $L^2(0,L)$ to obtain

$$\underbrace{i\lambda\rho_1 \int_0^L u^2 q(x)\overline{u_x^1}\, dx}_{I_{16}} - \kappa\int_0^L (u_x^1 + u^3 + lu^5)_x q(x)\overline{u_x^1}\, dx$$
$$-\kappa_0 l\int_0^L (u_x^5 - lu^1)q(x)\overline{u_x^1}\, dx = \rho_1 \int_0^L f^2 q(x)\overline{u_x^1}\, dx.$$

Substituting u^1 given in (3.0.5) into I_{16} and integrating by parts, we have

$$-\kappa\left(\frac{q(x)}{2}|u_x^1|^2\right)_{x=0}^{x=L} = \rho_1\int_0^L q'(x)|u^2|^2\,dx$$
$$-\frac{\kappa}{2}\int_0^L q'(x)|u_x^1|^2\,dx + \kappa\int_0^L u_x^3 q(x)\overline{u_x^1}\,dx$$
$$+\kappa l\int_0^L u_x^5 q(x)\overline{u_x^1}\,dx + \kappa_0 l\int_0^L (u_x^5 - lu^1)q(x)\overline{u_x^1}\,dx$$
$$+\rho_1\int_0^L u^2 q(x)\overline{f_x^1}\,dx + \rho_1\int_0^L f^2 q(x)\overline{u_x^1}\,dx.$$

Using the previous lemmas and taking the real part, our conclusion follows. The proof is now complete. □

The next lemma gives an estimate to $||u_x^5 - lu^1||_{L^2}$.

LEMMA 3.9. For any $\varepsilon_2 > 0$ there exists $K_{\varepsilon_2} > 0$ such that

$$\kappa l\int_0^L |u_x^5 - lu^1|^2\,dx + \frac{\kappa\rho_1 l}{2\kappa_0}\int_0^L |u^6|^2\,dx \le i\lambda\rho_1\left(1-\frac{\kappa}{\kappa_0}\right)\int_0^L u^2 u_x^5\,dx$$
$$+\rho_1 l\int_0^L |u^2|^2\,dx + \frac{\kappa\rho_1}{2\kappa_0 l}\int_0^L |u^4|^2\,dx + K||U||_{\mathcal{H}}||F||_{\mathcal{H}}$$
$$+K_{\varepsilon_2}||U||_{\mathcal{H}}||F||_{\mathcal{H}} + (\frac{\kappa^2 l}{\kappa_0} + \varepsilon_2)||u_x^1 + u^3 + lu^5||_{L^2}^2.$$

Proof. Multiplying equation (3.0.6) by $(\overline{u_x^5 - lu^1})$ in $L^2(0,L)$, we have

$$\kappa l\int_0^L |u_x^5 - lu^1|^2\,dx = i\lambda\rho_1\int_0^L u^2\overline{u_x^5}\,dx - \underbrace{i\lambda\rho_1 l\int_0^L u^2\overline{u^1}\,dx}_{:=I_{14}}$$
$$\underbrace{+\,\kappa\int_0^L (u_x^1 + u^3 + lu^5)(\overline{u_x^5 - lu^1})_x\,dx}_{:=I_{15}}$$
$$-\rho_1\int_0^L f^2(\overline{u_x^5 - lu^1})\,dx - \kappa\left[u_x^1\overline{u_x^5}\right]_{x=0}^{x=L}.$$

Substituting the equations (3.0.5) and (3.0.10) into I_{14} and I_{15}, respectively, we arrive at

$$\kappa l\int_0^L |u_x^5 - lu^1|^2\,dx = i\lambda\rho_1\int_0^L u^2\overline{u_x^5}\,dx + \rho_1 l\int_0^L |u^2|^2\,dx + \rho_1 l\int_0^L u^2\overline{f^1}\,dx$$
$$+\frac{\kappa}{\kappa_0}\int_0^L (u_x^1 + u^3 + lu^5)(\overline{i\lambda\rho_1 u^6 + \kappa l(u_x^1 + u^3 + lu^5) - \rho_1 f^6})$$
$$-\rho_1\int_0^L f^2(\overline{u_x^5 - lu^1})\,dx - \kappa\left[u_x^1\overline{u_x^5}\right]_{x=0}^{x=L}, \tag{3.0.56}$$

from which it follows that

$$\kappa l\int_0^L |u_x^5 - lu^1|^2\,dx = i\lambda\rho_1\int_0^L u^2\overline{u_x^5}\,dx + \rho_1 l\int_0^L |u^2|^2\,dx + \rho_1 l\int_0^L u^2\overline{f^1}\,dx$$
$$\underbrace{-\frac{\kappa}{\kappa_0}i\lambda\rho_1\int_0^L u_x^1\overline{u^6}\,dx}_{:=I_{17}}\underbrace{-\frac{\kappa}{\kappa_0}i\lambda\rho_1\int_0^L u^3\overline{u^6}\,dx}_{:=I_{18}}$$
$$\underbrace{-\frac{\kappa}{\kappa_0}i\lambda\rho_1 l\int_0^L u^5\overline{u^6}\,dx}_{:=I_{19}}+\frac{\kappa^2 l}{\kappa_0}\int_0^L |u_x^1+u^3+lu^5|^2\,dx$$
$$-\frac{\kappa}{\kappa_0}\rho_1\int_0^L (u_x^1+u^3+lu^5)\overline{f^6}\,dx - \rho_1\int_0^L f^2\overline{(u_x^5-lu^1)}\,dx - \kappa\left[u_x^1\overline{u_x^5}\right]_{x=0}^{x=L}. \quad (3.0.57)$$

Substituting u^1 given by (3.0.5) into I_{17}, we have

$$I_{17} = \underbrace{-\frac{\kappa}{\kappa_0}\rho_1\int_0^L u_x^2\overline{u^6}\,dx}_{I_{20}} - \frac{\kappa}{\kappa_0}\rho_1\int_0^L f_x^1\overline{u^6}\,dx.$$

Now, substituting u^6 given by (3.0.9) into I_{20}, I_{17} can be rewritten as

$$I_{17} = \underbrace{\frac{\kappa}{\kappa_0}\rho_1 i\lambda\int_0^L u_x^2\overline{u^5}\,dx}_{I_{21}}\underbrace{-\frac{\kappa}{\kappa_0}\rho_1\int_0^L u_x^2\overline{f^5}\,dx}_{I_{22}} - \frac{\kappa}{\kappa_0}\rho_1\int_0^L f_x^1\overline{u^6}\,dx. \quad (3.0.58)$$

Integrating by parts I_{21} and I_{22}, we get

$$I_{17} = -\frac{\kappa}{\kappa_0}\rho_1 i\lambda\int_0^L u^2\overline{u_x^5}\,dx + \frac{\kappa}{\kappa_0}\rho_1\int_0^L u^2\overline{f_x^5}\,dx - \frac{\kappa}{\kappa_0}\rho_1\int_0^L f_x^1\overline{u^6}\,dx. \quad (3.0.59)$$

Substituting u^3 given by (3.0.7) into I_{18}, we obtain

$$I_{18} = -\frac{\kappa}{\kappa_0}\rho_1\int_0^L u^4\overline{u^6}\,dx - \frac{\kappa}{\kappa_0}\rho_1\int_0^L f^3\overline{u^6}\,dx. \quad (3.0.60)$$

On the other hand, substituting u^5 given by (3.0.9) into I_{19}, we have

$$I_{19} = -\frac{\kappa}{\kappa_0}\rho_1 l\int_0^L |u^6|^2\,dx - \frac{\kappa}{\kappa_0}\rho_1 l\int_0^L f^5\overline{u^6}\,dx. \quad (3.0.61)$$

Using a similar argument as in Alabau ([10], Lemma 4.3), we obtain

$$\left|u_x^1\overline{u_x^5}\right|_{x=0}^{x=L} \le \varepsilon_2||u_x^1+u^3+lu^5||^2_{L^2(0,L)} + K_{\varepsilon_2}||U||_{\mathcal{H}}||F||_{\mathcal{H}}, \quad (3.0.62)$$

where ε_2 is a small positive constant. Substituting the inequalities (3.0.59)-(3.0.62) into (3.0.57), then using the Young inequality we can conclude that

$$\kappa l\int_0^L |u_x^5 - lu^1|^2\,dx \le i\lambda\rho_1\left(1-\frac{\kappa}{\kappa_0}\right)\int_0^L u^2u_x^5\,dx + \rho_1 l\int_0^L |u^2|^2\,dx$$
$$-\frac{\kappa\rho_1 l}{2\kappa_0}\int_0^L |u^6|^2\,dx + \frac{\kappa\rho_1}{2\kappa_0 l}\int_0^L |u^4|^2\,dx + K||U||_{\mathcal{H}}||F||_{\mathcal{H}}$$
$$+K_{\varepsilon_2}||U||_{\mathcal{H}}||F||_{\mathcal{H}} + (\frac{\kappa^2 l}{\kappa_0}+\varepsilon_2)||u_x^1+u^3+lu^5||^2_{L^2}. \quad (3.0.63)$$

The proof is now complete. □

Now, we can prove the main result of this section.

THEOREM 3.10. Let us assume hypotheses (1.0.12) and (2.0.1) on g and suppose that initial data satisfies

$$\varphi_0,\ \psi_0,\ w_0 \in H_0^1(0,L)\cap H^2(0,L),\ \ \eta_0\in L_g^2(\mathbb{R}^+, H^2(0,L)\cap H_0^1(0,L)) \text{ and } \varphi_1,\ \psi_1,\ w_1\in H_0^1(0,L).$$

If

$$\frac{\rho_1}{\rho_2} = \frac{\kappa}{b} \quad \text{and} \quad \kappa = \kappa_0, \tag{3.0.64}$$

then there exist positive constants C and α such that

$$E(t) \le CE(0)e^{-\alpha t}, \quad \forall t \ge 0.$$

Proof. From lemmas 3.4, 3.5, 3.6, 3.7, 3.8 and 3.9 we can conclude that

$$||U||_{\mathcal{H}}^2 \le K||U||_{\mathcal{H}}||F||_{\mathcal{H}} + K||F||_{\mathcal{H}}^2 \ \ \forall U \in D(\mathcal{A}),$$

from which it follows that

$$||U||_{\mathcal{H}} \le K||F||_{\mathcal{H}},\ \forall U \in D(\mathcal{A}).$$

Using Prüss's result [9] the conclusion of the theorem follows. □

4. Lack of polynomial decay. In this section we are assuming the boundary conditions (1.0.21) and we are denoting by $\mathcal{A}$ the operator $\mathcal{A}_2$ and by $\mathcal{H}$ the Hilbert space $\mathcal{H}_2$.

Our starting point is to show that the semigroup associated to the Bresse system is not exponential stable. To show this, we assume that $g(t) = e^{-\omega t}$, with $\omega \in \mathbb{R}^+$. We will use Prüss's theorem [9] to prove the lack of exponential stability; that is, we will show that there exists a sequence of values λ_μ such that

$$||(\lambda_\mu - \mathcal{A})^{-1}||_{\mathcal{L}(\mathcal{H})} \to \infty. \tag{4.0.1}$$

It is equivalent to proving that there exist a sequence of data $F_\mu \in \mathcal{H}$ and a sequence of complex numbers $\lambda_\mu \in i\mathbb{R}$, with $||F_\mu||_{\mathcal{H}} \le 1$, such that

$$||(\lambda_\mu I - \mathcal{A})^{-1}F_\mu||_{\mathcal{H}} \to \infty \tag{4.0.2}$$

where

$$\lambda_\mu U_\mu - \mathcal{A}U_\mu = F_\mu \tag{4.0.3}$$

with U_μ not bounded. Rewriting the spectral equation in terms of its components we have

$$\begin{aligned} i\lambda u^1 - u^2 &= f^1, & (4.0.4)\\ i\lambda\rho_1 u^2 - \kappa(u^1_x + u^3 + lu^5)_x - \kappa_0 l(u^5_x - lu^1) &= \rho_1 f^2, & (4.0.5)\\ i\lambda u^3 - u^4 &= f^3, & (4.0.6)\\ i\lambda\rho_2 u^4 - \beta_0 u^3_{xx} + \kappa(u^1_x + u^3 + lu^5) - \int_0^\infty g(s)\eta_{xx}(s)\, ds &= \rho_2 f^4, & (4.0.7)\\ i\lambda u^5 - u^6 &= f^5, & (4.0.8)\\ i\lambda\rho_1 u^6 - \kappa_0(u^5_x - lu^1)_x + \kappa l(u^1_x + u^3 + lu^5) &= \rho_1 f^6, & (4.0.9)\\ i\lambda\eta + \eta_s - u^4 &= f^7. & (4.0.10) \end{aligned}$$

THEOREM 4.1. Let us assume hypotheses (1.0.12) and (2.0.1) on g, suppose that the initial data satisfies

$$(\varphi_0, \varphi_1, \psi_0, \psi_1, w_0, w_1, \eta)^T \in D(\mathcal{A}),$$

and suppose that

$$\frac{\rho_1}{\rho_2} \neq \frac{\kappa}{b} \quad \text{or} \quad \kappa \neq \kappa_0. \qquad (4.0.11)$$

Then the semigroup associated to system (1.0.13)-(1.0.19) with boundary conditions (1.0.21) is not exponentially stable.

Proof. Let us take $f_1 = f_3 = f_5 = f_7 = 0$. Using equations (3.0.5), (3.0.7) and (3.0.9) to eliminate the terms u^2, u^4 and u^6 we get

$$\begin{aligned} -\rho_1\lambda^2 u^1 - \kappa(u^1_x + u^3 + lu^5)_x - \kappa_0 l(u^5_x - lu^1) &= \rho_1 f_2, & (4.0.12)\\ -\rho_2\lambda^2 u^3 - \beta_0 u^3_{xx} + \kappa(u^1_x + u^3 + lu^5) - \int_0^\infty g(s)\eta_{xx}\, ds &= \rho_2 f_4, & (4.0.13)\\ -\rho_1\lambda^2 u^5 - \kappa_0(u^5_x - lu^1)_x + \kappa l(u^1_x + u^3 + lu^5) &= \rho_1 f_6, & (4.0.14)\\ i\lambda\eta + \eta_s - i\lambda u^3 &= 0. & (4.0.15) \end{aligned}$$

Let us take

$$f_2(x) = \sin\left(\frac{\mu\pi}{L}x\right), \quad f_4(x) = \cos\left(\frac{\mu\pi}{L}x\right), \quad f_6(x) = 0. \qquad (4.0.16)$$

Then we can look for solutions of the form

$$\begin{aligned} &u^1 = A\ \sin\left(\frac{\mu\pi}{L}x\right), \quad u^3 = B\ \cos\left(\frac{\mu\pi}{L}x\right), \quad u^5 = C\ \cos\left(\frac{\mu\pi}{L}x\right),\\ &\eta(x,s) = \gamma(s)\cos\left(\frac{\mu\pi}{L}x\right), \qquad (4.0.17) \end{aligned}$$

where $A,\ B,\ C$ and $\gamma(s)$ depend on λ and will be determined explicitly in what follows. Consequently the system (4.0.12)-(4.0.15) is equivalent to

$$\left[-\lambda^2+\frac{\kappa}{\rho_1}\left(\frac{\mu\pi}{L}\right)^2+\frac{\kappa_0}{\rho_1}l^2\right]A+\frac{\kappa}{\rho_1}\left(\frac{\mu\pi}{L}\right)B$$
$$+\frac{l}{\rho_1}\left(\frac{\mu\pi}{L}\right)C(\kappa+\kappa_0)=1, \tag{4.0.18}$$
$$\frac{\kappa}{\rho_2}\left(\frac{\mu\pi}{L}\right)A+\left[-\lambda^2+\frac{\beta_0}{\rho_2}\left(\frac{\mu\pi}{L}\right)^2+\left(\frac{\mu\pi}{L}\right)^2\int_0^\infty g(s)\gamma(s)\,ds+\frac{\kappa}{\rho_2}\right]B$$
$$+\frac{\kappa}{\rho_2}lC=1, \tag{4.0.19}$$
$$\frac{l}{\rho_1}\left(\frac{\mu\pi}{L}\right)(\kappa+\kappa_0)A+\frac{\kappa}{\rho_1}lB\left[-\lambda^2+\frac{\kappa_0}{\rho_1}\left(\frac{\mu\pi}{L}\right)^2+\frac{\kappa}{\rho_1}l^2\right]C=0, \tag{4.0.20}$$
$$\gamma'(s)+i\lambda\gamma(s)-i\lambda B=0. \tag{4.0.21}$$

Solving (4.0.21) we get

$$\gamma(s)=C_1e^{-i\lambda s}+B. \tag{4.0.22}$$

Since $\eta(0)=0$ then $C_1=-B$, and (4.0.22) becomes

$$\gamma(s)=B-Be^{-i\lambda s}. \tag{4.0.23}$$

Then, from (4.0.23) we have

$$\int_0^\infty g(s)\gamma(s)\,ds=\int_0^\infty(B-Be^{-i\lambda s})\,ds=Bb_0-B\int_0^\infty g(s)e^{-i\lambda s}\,ds \tag{4.0.24}$$

where $b_0=\int_0^\infty g(s)\,ds$. Consequently from (4.0.18)-(4.0.20) we get

$$\left[-\lambda^2+\frac{\kappa}{\rho_1}\left(\frac{\mu\pi}{L}\right)^2+\frac{\kappa_0}{\rho_1}l^2\right]A+\frac{\kappa}{\rho_1}\left(\frac{\mu\pi}{L}\right)B+\frac{l}{\rho_1}\left(\frac{\mu\pi}{L}\right)C(\kappa+\kappa_0)=1, \tag{4.0.25}$$
$$\frac{\kappa}{\rho_2}\left(\frac{\mu\pi}{L}\right)A+\left[-\lambda^2+\frac{b}{\rho_2}\left(\frac{\mu\pi}{L}\right)^2-\left(\frac{\mu\pi}{L}\right)^2\int_0^\infty g(s)e^{-i\lambda s}\,ds+\frac{\kappa}{\rho_2}\right]B$$
$$+\frac{\kappa}{\rho_2}lC=1, \tag{4.0.26}$$
$$\frac{l}{\rho_1}\left(\frac{\mu\pi}{L}\right)(\kappa+\kappa_0)A+\frac{\kappa}{\rho_1}lB+\left[-\lambda^2+\frac{\kappa_0}{\rho_1}\left(\frac{\mu\pi}{L}\right)^2+\frac{\kappa}{\rho_1}l^2\right]C=0. \tag{4.0.27}$$

First, we assume that

$$\frac{\rho_1}{\rho_2}\neq\frac{k}{b},\quad \kappa=\kappa_0.$$

Then, choosing $\lambda=\lambda_\mu$ such that

$$-\lambda^2+\frac{\kappa}{\rho_1}\left(\frac{\mu\pi}{L}\right)^2+\frac{\kappa_0}{\rho_1}l^2=2\frac{\kappa}{\rho_1}l\frac{\mu\pi}{L}$$

the above system can be rewritten as

$$2\frac{\kappa}{\rho_1}l\left(\frac{\mu\pi}{L}\right)A+\frac{\kappa}{\rho_1}\left(\frac{\mu\pi}{L}\right)B+2\frac{\kappa}{\rho_1}l\left(\frac{\mu\pi}{L}\right)C=1, \tag{4.0.28}$$

$$\frac{\kappa}{\rho_2}\left(\frac{\mu\pi}{L}\right)A+\left[-\lambda^2+\frac{b}{\rho_2}\left(\frac{\mu\pi}{L}\right)^2-\left(\frac{\mu\pi}{L}\right)^2\int_0^\infty g(s)e^{-i\lambda s}\,ds+\frac{\kappa}{\rho_2}\right]B$$
$$+\frac{\kappa}{\rho_2}lC=1, \tag{4.0.29}$$

$$2\frac{\kappa}{\rho_1}l\left(\frac{\mu\pi}{L}\right)A+\frac{\kappa}{\rho_1}lB+2\frac{\kappa}{\rho_1}l\left(\frac{\mu\pi}{L}\right)C=0. \tag{4.0.30}$$

From (4.0.28) and (4.0.30) we have

$$B=\frac{\rho_1}{\kappa\left[\frac{\mu\pi}{L}-l\right]}. \tag{4.0.31}$$

Then from (4.0.30) and (4.0.31) we get

$$A=-C-\frac{l}{\left(\frac{2\kappa l}{\rho_1}\right)\left(\frac{\mu\pi}{L}\right)\left[\frac{\mu\pi}{L}-l\right]}. \tag{4.0.32}$$

Substituting (4.0.31) and (4.0.32) into (4.0.29) yields

$$\begin{aligned}\frac{\kappa}{\rho_2}\left(l-\frac{\mu\pi}{L}\right)C &= \frac{\kappa}{\rho_2}\left(\frac{\mu\pi}{L}\right)\frac{l}{\left(\frac{2\kappa l}{\rho_1}\right)\left(\frac{\mu\pi}{L}\right)\left[\frac{\mu\pi}{L}-l\right]}\\ &- \frac{\rho_1}{\kappa\left[\frac{\mu\pi}{L}-l\right]}\left[\left(\frac{b}{\rho_2}-\frac{\kappa}{\rho_1}\right)\left(\frac{\mu\pi}{L}\right)^2+\frac{\kappa}{\rho_1}l\left(\frac{\mu\pi}{L}-l\right)\right]\\ &- \frac{\rho_1}{\kappa\left[\frac{\mu\pi}{L}-l\right]}\left[\frac{\kappa}{\rho_2}-\left(\frac{\mu\pi}{L}\right)^2\int_0^\infty g(s)e^{-i\lambda s}\,ds\right],\end{aligned} \tag{4.0.33}$$

from which it follows that

$$C\to\underbrace{-\frac{\rho_1\rho_2}{\kappa^2}\left[\left(\frac{b}{\rho_2}-\frac{k}{\rho_1}\right)-\int_0^\infty g(s)e^{-i\lambda s}\,ds\right]}_{\text{bounded as }\mu\to\infty} \tag{4.0.34}$$

and

$$A\to\underbrace{\frac{\rho_1\rho_2}{\kappa^2}\left[\left(\frac{b}{\rho_2}-\frac{k}{\rho_1}\right)-\int_0^\infty g(s)e^{-i\lambda s}\,ds\right]}_{\text{bounded as }\mu\to\infty}. \tag{4.0.35}$$

Then, from (4.0.34) and (4.0.35) we have

$$||U_\mu||^2_{\mathcal{H}}\geq\kappa\int_0^L|u^5_x-lu^1|^2\,dx\geq\kappa\left|C\left(\frac{\mu\pi}{L}\right)-lA\right|^2\frac{L}{2}\to\infty \tag{4.0.36}$$

as $\mu\to\infty$. Therefore there is no exponential stability. Now let us assume that the coefficients satisfy

$$\frac{\rho_1}{\rho_2}=\frac{\kappa}{b},\quad \kappa\neq\kappa_0.$$

Multiplying the equality (4.0.27) by $\frac{\kappa}{l(\kappa+\kappa_0)}$ and choosing λ such that

$$-\lambda^2 + \frac{\kappa_0}{\rho_1}\left(\frac{\mu\pi}{L}\right)^2 = \frac{\kappa_0}{\rho_1} l^2,$$

the equalities (4.0.25), (4.0.26) and (4.0.27) become

$$\left[\left(\frac{\kappa-\kappa_0}{\rho_1}\right)\left(\frac{\mu\pi}{L}\right)^2 + 2\frac{\kappa_0}{\rho_1} l^2\right] A + \frac{\kappa}{\rho_1}\left(\frac{\mu\pi}{L}\right) B + \frac{l}{\rho_1}\left(\frac{\mu\pi}{L}\right) C(\kappa+\kappa_0) = 1, \tag{4.0.37}$$

$$\frac{\kappa}{\rho_2}\left(\frac{\mu\pi}{L}\right) A + \left[\frac{\kappa}{\rho_1}\left(1-\frac{\kappa_0}{\kappa}\right)\left(\frac{\mu\pi}{L}\right)^2 - \left(\frac{\mu\pi}{L}\right)^2 \int_0^\infty g(s) e^{-i\lambda s}\, ds + \frac{\kappa}{\rho_2} + \frac{\kappa_0}{\rho_1} l^2\right] B$$
$$+ \frac{\kappa}{\rho_2} l C = 1, \tag{4.0.38}$$

$$\frac{\kappa}{\rho_1}\left(\frac{\mu\pi}{L}\right) A + \frac{\kappa^2}{\rho_1(\kappa+\kappa_0)} B + \frac{\kappa}{\rho_1} l C = 0. \tag{4.0.39}$$

From equations (4.0.38) and (4.0.39), we have

$$\left[\frac{\kappa}{\rho_1}\left(1-\frac{\kappa_0}{\kappa}\right)\left(\frac{\mu\pi}{L}\right)^2 - \left(\frac{\mu\pi}{L}\right)^2 \int_0^\infty g(s) e^{-i\lambda s}\, ds + \frac{\kappa}{\rho_2} + \frac{\kappa_0}{\rho_1} l^2 - \frac{\kappa^2}{\rho_1(\kappa+\kappa_0)}\right] B = 1,$$

from which it follows that

$$\left[\frac{\kappa}{\rho_1}\left(1-\frac{\kappa_0}{\kappa}\right)\left(\frac{\mu\pi}{L}\right)^2 - \left(\frac{\mu\pi}{L}\right)^2 \int_0^\infty g(s) e^{-i\lambda s}\, ds\right] B \to 1$$

as $\mu \to \infty$. Substituting this expression into (4.0.37) and (4.0.39) we get

$$\left[\left(\frac{\kappa-\kappa_0}{\rho_1}\right)\left(\frac{\mu\pi}{L}\right)^2 + 2\frac{\kappa_0}{\rho_1} l^2\right] A + \frac{l}{\rho_1}\left(\frac{\mu\pi}{L}\right) C(\kappa+\kappa_0) = 1 - o(\frac{1}{\mu}), \tag{4.0.40}$$

$$\frac{\kappa}{\rho_1}\left(\frac{\mu\pi}{L}\right) A + \frac{\kappa}{\rho_1} l C = 1 + o(\frac{1}{\mu^2}). \tag{4.0.41}$$

So, we have

$$-2\frac{\kappa_0}{\rho_1}\left[\left(\frac{\mu\pi}{L}\right)^2 - l^2\right] A = -\frac{\kappa+\kappa_0}{\kappa\rho_1}\left(\frac{\mu\pi}{L}\right) + 1 - o(\frac{1}{\mu}),$$

from which it follows that

$$\left(\frac{\mu\pi}{L}\right) A \to \frac{\kappa+\kappa_0}{2\kappa\kappa_0\rho_1}.$$

From (4.0.39) we can conclude that

$$C \to -\frac{\kappa+\kappa_0}{2\kappa_0}.$$

Using the same argument as above we conclude that

$$||U_\mu||_{\mathcal{H}} \to \infty. \tag{4.0.42}$$

So we have no exponential stability. □

REMARK 4.2. Note that showing the lack of exponential decay of the problem (1.0.13)-(1.0.19) with boundary conditions (1.0.20) when

$$\frac{\rho_1}{\rho_2} \neq \frac{\kappa}{b} \quad \text{or} \quad \kappa \neq \kappa_0$$

is an important open problem.

5. Polynomial decay. In this section we are assuming the boundary conditions (1.0.21) and we are denoting by $\mathcal{A}$ the operator $\mathcal{A}_2$ and by $\mathcal{H}$ the Hilbert space $\mathcal{H}_2$. Our main result is proving the polynomial decay. Therefore, the next theorem gives the polynomial decay of our problem studied here.

THEOREM 5.1. Let us suppose that

$$\frac{\rho_1}{\rho_2} \neq \frac{\kappa}{b} \quad \text{or} \quad \kappa \neq \kappa_0.$$

Then the semigroup associated to system (1.0.13)-(1.0.19) with boundary conditions (1.0.21) is polynomially stable and

$$||e^{\mathcal{A}t}U_0||_{\mathcal{H}} \leq \frac{1}{\sqrt{t}}||U_0||_{D(\mathcal{A})}.$$

Moreover, this rate of decay is optimal.

Proof. To start, let us suppose that

$$\frac{\rho_1}{\rho_2} = \frac{\kappa}{b}, \quad \kappa \neq \kappa_0.$$

Then from Lemma 3.9

$$\kappa l \int_0^L |u_x^5 - lu^1|^2\, dx + \frac{\kappa\rho_1 l}{2\kappa_0}\int_0^L |u^6|^2\, dx \leq \underbrace{i\lambda\rho_1\left(1-\frac{\kappa}{\kappa_0}\right)\int_0^L u^2 u_x^5\, dx}_{I_{23}}$$
$$+\rho_1 l\int_0^L |u^2|^2\, dx + \frac{\kappa\rho_1}{2\kappa_0 l}\int_0^L |u^4|^2\, dx + K||U||_{\mathcal{H}}||F||_{\mathcal{H}}$$
$$+K_{\varepsilon_2}||U||_{\mathcal{H}}||F||_{\mathcal{H}} + \varepsilon_2||u_x^1 + u^3 + lu^5||_{L^2}^2. \tag{5.0.1}$$

Substituting u^2 given by (3.0.5) and u^5 given by (3.0.9) into I_{23} and applying the Young inequality, we get

$$\begin{aligned} |I_{23}| &\leq K|\lambda|^2\int_0^L |u_x^1 + u^3 + lu^5|^2\, dx + \frac{\kappa\rho_1 l}{4\kappa_0}\int_0^L |u^6|^2\, dx \\ &+ K|\lambda|^2\int_0^L |u_x^3|^2\, dx + K|\lambda|^2\int_0^L |u^5|^2\, dx. \end{aligned} \tag{5.0.2}$$

Substituting the inequality (5.0.2) into (5.0.1) and using Lemmas 3.4, 3.5, 3.6, 3.7, 3.8, we have

$$\frac{1}{|\lambda|^2}||U||_{\mathcal{H}} \leq K||F||_{\mathcal{H}}$$

with $|\lambda| > 1$ large enough. Then using Theorem 2.4 in [1] the conclusion of theorem follows. The other case follows using the same argument. So the polynomial decay holds.

To prove that the rate of decay is optimal, we will argue by contradiction. Suppose that the rate $t^{-\frac{1}{2}}$ can be improved, for example that the rate is $t^{-\frac{1}{2-\epsilon}}$ for some $0 < \epsilon < 2$. From Theorem 5.3 in [13], the operator

$$|\lambda|^{-2+\frac{\epsilon}{2}}||(\lambda I - \mathcal{A})^{-1}||_{\mathcal{L}(\mathcal{H})}$$

should be limited, but this does not happen. For this, let us suppose that there exist a sequence $(\lambda_\mu) \subset \mathbb{R}$ with $\lim_{\mu\to\infty} |\lambda_\mu| = \infty$ and $(U_\mu) \subset D(\mathcal{A})$ for $(F_\mu) \subset \mathcal{H}$ such that

$$(i\lambda_\mu I - \mathcal{A})U_\mu = F_\mu$$

is bounded in $\mathcal{H}$ and

$$\lim_{\mu\to\infty} |\lambda_\mu|^{-2+\frac{\epsilon}{2}} ||U_\mu||_{\mathcal{H}} = \infty.$$

Then, we can consider, for each $\mu \in \mathbb{N}$, $F_\mu = \left(0, \sin\left(\mu\frac{\pi x}{L}\right), 0, \cos\left(\mu\frac{\pi x}{L}\right), 0, 0, 0\right)^T$ and $U_\mu = (\varphi_\mu, \widetilde{\varphi_\mu}, \psi_\mu, \widetilde{\psi_\mu}, \omega_\mu, \widetilde{\omega_\mu}, \eta_\mu)^T$, where, due to the boundary conditions, U_μ are of the form $\varphi_\mu = A\sin\left(\mu\frac{\pi x}{L}\right)$, $\psi_\mu = B\sin\left(\mu\frac{\pi x}{L}\right)$, $\omega_\mu = C\cos\left(\mu\frac{\pi x}{L}\right)$ and $\eta_\mu(x,s) = \gamma(s)\cos\left(\mu\frac{\pi x}{L}\right)$.

Then following the same steps as in the proof of Theorem 4.1, we can conclude that

$$|\lambda_\mu|^{-2+\frac{\epsilon}{2}} ||U_\mu||_{\mathcal{H}} \geq O(\mu^{\frac{\epsilon}{2}}) \to \infty, \quad \text{as} \quad \mu \to \infty.$$

Therefore the rate cannot be improved. The proof is now complete. □

Acknowledgment. The authors are grateful to the referee for the valuable suggestions that improved this paper.

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