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## ASYMPTOTIC BEHAVIOUR IN QUANTUM FIELD THEORY

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## INTRODUCTION

The organizers have asked me to talk about the renormalization group and its applications. I shall concentrate on explaining the rules for short-distance behaviour in field theory and current algebra. Nothing will be said about applications to form factors, on-shell scattering amplitudes, charmonium, etc. ${ }^{1}$ ).

The renormalization group originated in papers by Stueckelberg and Petermann ${ }^{2}$ ) (SP) and Gell-Mann and Low ${ }^{3}$ ) (GML) in $1953 / 54$ and was revived, after a long quiescent period, by Wilson ${ }^{4}$ ) (operator-product expansions and broken scale invariance) and Callan and Symanzik ${ }^{5}$ (CS) ( $1968 / 70$ ). Many people prefer the CS approach because there is no need to introduce unfamiliar renormalization prescriptions, but I think that it is unwise to completely ignore the original ideas. For example, a comparison of the GML and CS methods led to the use of massindependent renormalization prescriptions [such as ${ }^{6}$ ) dimensional renormalization] which generate "improved" CS equations"). Consequently, all of these methods will be reviewed. Applications will be restricted to the results of assuming asymptotic freedom ${ }^{8}$ ) or broken scale invariance. I shall also report progress on an awkward technical problem, the properties of operator-product expansions for non-Abelian gauge theories.

## 1. RENORMALIZATION GROUP: CALLAN-SYMANZIK METHOD

Any renormalized amplitude A possesses a corresponding Callan-Symanzik (CS) equation ${ }^{5)}$. The main ingredient in the derivation is knowing the mode of renormalization of $A$ : is it multiplicative, subtractive, or something more complicated? So we begin with a mini-review of the renormalization procedure itself ${ }^{9}$ ).

### 1.1 Renormalization

Let $\Lambda$ represent a cut-off mass which regulates ultraviolet divergences in Feynman diagrams. Typically, these divergences appear as powers and logarithms of $\Lambda$ which blow up as $\Lambda$ tends to infinity. Divergences are caused by loop integrations, so they can be isolated as divergences in one-particle-irreducible (1PI) (or "proper") subdiagrams*). Therefore, we consider the Feynman integral for an $\ell$-loop 1PI amplitude with $L$ external legs:

$$
\begin{equation*}
\Gamma\left(q_{1}, \ldots, q_{L-1}\right)=\int^{1} d^{4} p_{1} \ldots d^{4} p_{l} I\left(p_{1} \ldots p_{l} ; q_{1} \ldots q_{L-1}\right) \tag{1.1}
\end{equation*}
$$

The integrand $I$ is a linear combination of products of vertices and internal propagators which depend on loop momenta $p_{1}, \ldots, p_{\ell}$ and external momenta $q_{1}, \ldots, q_{L-1}$.

[^0]Convergence is tested by applying power-counting arguments ${ }^{10,11 \text { ). Imagine }}$ that a subset $S=\left\{p_{1}^{\prime}, \ldots, p_{m}^{\prime}\right\}$ of the loop momenta $\left\{p_{1}, \ldots, p_{\ell}\right\}$ is scaled to infinity $\left[\right.$ all $p_{j}^{\prime}=O(\eta)$ with $\eta \rightarrow \infty$ ] with all other momenta held fixed*). The integrand $I$ develops a power-law behaviour in $\eta$ for which the formula

$$
|I(\eta)| \stackrel{\eta^{c(S)}}{\sim}\{\text { CONSTANT }\}
$$

provides an asymptotic bound. Here $c(S)$ is the sum of individual powers contributed by each vertex and propagator depending on $p_{1}^{\prime}, \ldots, p_{m}^{\prime}$; (for example, there is a power -2 for each boson propagator involved). Thus each subset $S$ can contribute cut-off dependence

$$
\left|\int^{\wedge} d^{4} p_{1}^{\prime} \ldots d^{4} p_{m}^{\prime} I\right| \lesssim \Lambda^{d(S)}\{\log s \text { of } \Lambda\}
$$

to $\Gamma$, where the power

$$
\begin{equation*}
d(S)=4 m(S)+c(S) \tag{1.2}
\end{equation*}
$$

is called the superficial degree of divergence of the subintegration over $S$. The integral (1.1) converges if $d(S)$ is negative for $a l l$.

Consider the special case in which the only divergence in $\Gamma$ is that caused by all of the loop momenta $p_{1}, \ldots, p_{\ell}$ growing large together; in other words, $\Gamma$ is superficially divergent $[d(\Gamma) \geq 0]$ but internally convergent $[$ e.g., $d(S)<0$ for all $\left.S \neq\left\{p_{1}, \ldots, p_{\ell}\right\}\right]$. Suppose that $\Gamma$ is differentiated with respect to one of the external momenta $q_{i}$. When $\partial / \partial q_{i}$ acts on the integrand $I$, the characteristic exponent $c(\Gamma)$ for its asymptotic behaviour as $p_{1}, \ldots, p_{\ell}$ become large is reduced by at least 1 , while no increase occurs in the other $c(S)$. Therefore, if $\partial / \partial q_{i}$ is applied a sufficient number of times to $\Gamma$, the result is completely convergent. Since $\Gamma_{\text {div }}$ (the divergent part of $\Gamma$ ) is annihilated by these differentiations, it must be a polynomial of degree $\leq d(\Gamma)$ in $q_{1}, \ldots, q_{L-1}$. In diagrammatic language (Fig. 1), this means that $\Gamma_{\text {div }}$ is a local vertex produced by shrinking the blob $\Gamma$ to a point.

An $\ell$-loop divergence of this type is easily cancelled by including a counterterm $\Delta \mathscr{L}_{\ell}$ in the Lagrangian $\mathscr{L}$ with bare vertex equal to $-\Gamma_{\text {div }}$. However, many 1PI graphs do not possess this property of being "primitively divergent"**). An arbitrary $\ell$-loop 1 PI graph may contain divergent $\ell^{\prime}-100 p 1 P I$ subgraphs ( $\ell^{\prime}<\ell$ ) which must first be made convergent by including suitable counterterms $\Delta \mathscr{L}_{\ell}$, in $\mathscr{L}$; i.e., it will be necessary to include internal counterterm vertices in the blob in Fig. 1. So we try the following prescription ${ }^{10,12 \text { ) : }}$

[^1]i) Start with $\mathscr{L}_{0}$, a Lagrangian from which propagators and vertices can be constructed.
ii) Construct counterterms $\Delta \mathscr{L}_{1}$ which remove all divergences in 1-loop 1PI graphs generated by $\mathscr{L}_{0}$.
iii) Use a new Lagrangian $\mathscr{L}_{1}=\mathscr{L}_{0}+\Delta \mathscr{L}_{1}$ to generate $2-100 p$ graphs and construct $\Delta \mathscr{P}_{2}$ to get rid of the resulting $1 P I$ divergences, and so on, to any finite number of loops and vertices ${ }^{13}$ ):
$$
\mathscr{L}_{\ell}=\mathscr{L}_{\ell-1}+\Delta \mathscr{L}_{\ell}
$$

Obviously, we obtain finite results for diagrams in which, for each pair of divergent $1 P I$ subgraphs, one subgraph is entirely contained within the other ("nested" divergences) or the two subgraphs are disjoint. The difficult step in the proof ${ }^{12}$ ) of convergence is to disentangle overlapping divergences (sets of divergent 1PI graphs which are neither disjoint nor nested). The result, valid for all polynomial interactions (renormalizable or otherwise), is that the above procedure renders all subintegrations convergent by power-counting*). In other words, the Lagrangian

$$
\begin{equation*}
\mathscr{L}=\mathscr{L}_{\infty}=\mathscr{L}_{0}+\Delta \mathscr{L} \tag{1.3}
\end{equation*}
$$

with counterterm Lagrangian $\triangle \mathscr{L}$ given by

$$
\Delta \mathscr{L}=\Delta \mathscr{L}_{1}+\Delta \mathscr{L}_{2}+\ldots+\Delta \mathscr{L}_{\ell}+\ldots
$$

generates cut-off independent perturbative amplitudes.
The form of $\Delta \mathscr{P}$ is restricted by the fact that the degree of the polynomial $\Gamma_{\text {div }}\left(q_{1}, \ldots, q_{L-1}\right)$ in Fig. 1 cannot be greater than $d(\Gamma)$. The set of allowed vertices is easily determined by counting mass dimensionalities. Each term in the Lagrangian has dimensionality 4, in four space-time dimensions. Boson and fermion fields $\phi$ and $\psi$ have dimensionalities 1 and $3 / 2$ corresponding to $\overparen{\phi \phi} \sim p^{-2}$ and $\psi \bar{\psi} \sim p^{-1}$ for the propagators at large momentum $p$. [Higher-spin fields obey this rule if special circumstances, such as gauge invariance, ensure that longitudinal terms like $i q_{\mu} q_{\nu} / q^{2} m^{2}$ in the spin-1 propagator $\widehat{B}_{\mu} B_{\nu}$ ( $m=$ mass) are absent ( $m=0$ ) or cancelled off.] These rules fix the effective dimensionality $\operatorname{dim}(g)$ of a coupling constant $g$; for example, for the vertex $g \phi^{6}, \operatorname{dim}\left(\phi^{6}\right)=6$ implies $\operatorname{dim}(\mathrm{g})=-2$. A typical term in $\Delta \mathscr{L}$ is
$\Delta \mathcal{L} \propto \Lambda^{j}(\ln \Lambda)^{p} m^{r} g^{N} O_{d}(\varphi, \psi, \partial)$

[^2]for integer powers $j, p, r \geq 0$ and $N>0$, where the dimensionality $d$ of the operator $\mathrm{O}_{\mathrm{d}}$ is given by
$$
4=j+r+N \operatorname{dim}(g)+d
$$
i.e.,
\[

$$
\begin{equation*}
d \leqslant 4-N \operatorname{dim}(g) \tag{1.5}
\end{equation*}
$$

\]

If $\operatorname{dim}(g)$ is negative, $d$ is unbounded as the order of perturbation $N$ increases. The inevitable result is a non-renormalizable theory in which the number of distinct vertices (and hence new coupling constants) is unbounded. Apart from the lack of predictive power of these theories ${ }^{14}$ ), the prescription (1.3) implies a quantization procedure for high-derivative counterterms which is not obviously unitary. So we shall ensure renormalizability ( $\mathrm{d} \leq 4$ ) by requiring

$$
\begin{equation*}
\operatorname{dim}(g) \geqslant 0 \tag{1.6}
\end{equation*}
$$

but frequently permit external composite operators ${ }^{13},{ }^{15}$ ) $Q$ to couple to $\mathscr{C}$ via sources $X$ (to a fixed order in $X$ ):

$$
\begin{equation*}
\mathscr{L} \longrightarrow \mathscr{L}(x)=\mathscr{L}+x(Q+\Delta Q)+O\left(x^{2}\right) \tag{1.7}
\end{equation*}
$$

Given the constraint (1.6), the rule for renormalization counterterms $\Delta Q$ generated by 1PI diagrams with a single Q-insertion is simply

$$
\begin{equation*}
\operatorname{dim}(\Delta Q) \leqslant \quad \operatorname{dim}(Q) \tag{1.8}
\end{equation*}
$$

For example, consider

$$
\begin{equation*}
\mathscr{L}_{0}=\frac{1}{2}(\partial \varphi)^{2}-\frac{1}{2} m^{2} \varphi^{2}-g \varphi^{4} \tag{1.9}
\end{equation*}
$$

as a trial Lagrangian. Only the two- and four-legged 1 PI amplitudes $\Gamma^{(2)}, \Gamma^{(4)}$ are superficially divergent. According to (1.4), we have $d \leq 4$, so the only possibilities are counterterms proportional to $(\partial \phi)^{2}$ and $\phi^{2}\left[\right.$ from $\left.\Gamma_{\text {div }}^{(2)}\right]$ and $\phi^{4}$ [from $\Gamma_{\text {div }}^{(4)}$ ]. Therefore, the most general expression for the complete Lagrangian is

$$
\begin{equation*}
\mathscr{L}=\frac{1}{2} Z_{3}(\partial \varphi)^{2}-\frac{1}{2}\left(m^{2}-\delta m^{2}\right) \varphi^{2}-g Z_{1} \varphi^{4} \tag{1.10}
\end{equation*}
$$

where $Z_{1}, Z_{3}$, and $\delta \mathrm{m}^{2}$ are $\Lambda$-dependent. Note that there can be vertices in $\mathscr{C}$ which do not appear in $\mathscr{L}_{0}$. For example, the Yukawa trial Lagrangian

$$
\begin{equation*}
\mathscr{L}_{0}=\frac{1}{2}(\partial \varphi)^{2}-\frac{1}{2} m^{2} \varphi^{2}+\bar{\psi}(i \gamma-M) \psi+g^{\prime} \bar{\psi} \gamma_{5} \psi \varphi \tag{1.11}
\end{equation*}
$$

produces an additional $\phi^{4}$ vertex in $\mathscr{L}$ because of divergences in four-point 1PI boson amplitudes $\Gamma^{(4)}$. The result is a theory with two coupling constants, $g$ and $g^{\prime}$ :

$$
\begin{align*}
\mathscr{L}= & \frac{1}{2} Z_{3}(\partial \varphi)^{2}-\frac{1}{2}\left(m^{2}-\delta m^{2}\right) \varphi^{2}-g Z_{1} \varphi^{4} \\
& +Z_{2} \bar{\psi} i \not \gamma \psi-(M-\delta M) \bar{\psi} \psi+g^{\prime} Z_{1}^{\prime} \bar{\psi} \gamma_{5} \psi \varphi \tag{1.12}
\end{align*}
$$

The Lagrangian $\mathscr{P}$ [Eq. (1.10) or (1.12)] produces finite 1 PI amplitudes $\Gamma^{(L)}$ and connected Green's functions $G^{(L)}$ (with L legs). We have no further use for $\mathscr{L}_{0}$; it is just a crutch which helps us to arrive at a suitable $\mathscr{L}$.

Sometimes it is convenient to introduce a new set of Feynman rules (labelled with a B for "bare" or "unrenormalized") for the same $\mathscr{L}$ in terms of new quantities $\phi_{B}, m_{B}^{2}, g_{B}, \ldots$. The substitution

$$
\begin{align*}
\varphi_{B} & =Z_{3}^{1 / 2} \varphi \\
m_{B}^{2} & =Z_{3}^{-1}\left(m^{2}-\delta m^{2}\right)  \tag{1.13}\\
g_{B} & =Z_{3}^{-2} Z_{1} g
\end{align*}
$$

in Eq. (1.10) produces a simple set of $B$ rules:

$$
\begin{equation*}
\mathscr{L}=\frac{1}{2}\left(\partial \varphi_{B}\right)^{2}-\frac{1}{2} m_{B}^{2} \varphi_{B}^{2}-g_{B} \varphi_{B}^{4} \tag{1.14}
\end{equation*}
$$

Let $G_{B}^{(L)}$ be the L-point Green's function computed using the $B$ rules. Numerically, the only difference between $G_{B}^{(L)}$ and $G^{(L)}$ is the normalization of each external line -- internally, everything cancels when propagators and vertices are changed according to (1.13), because the Lagrangian $\mathscr{L}$ remains the same. So we get:

$$
\begin{equation*}
G^{(L)}\left(q_{1}, \ldots, q_{L-1} ; g, m\right)=Z_{3}^{-L / 2} G_{B}^{(L)}\left(q_{1}, \ldots, q_{L-1} ; g_{B}, m_{B}, \Lambda\right) \tag{1.15}
\end{equation*}
$$

Therefore, if some cut-off dependence is suitably absorbed in coupling constant and mass renormalization, the remaining divergence of an unrenormalized Green's function can be segregated as a multiplicative constant ${ }^{10)} Z_{3}^{\mathrm{L} / 2}$. There is a wave-function renormalization $Z_{3}^{-1 / 2}$ for each external boson line, and a similar factor $\mathrm{Z}_{2}^{-1 / 2}$ for each fermion [if Eq. (3.12) is being considered] with

$$
\begin{equation*}
g^{\prime}=\left(z_{1}^{\prime}\right)^{-1} z_{2} z_{3}^{1 / 2} g_{B}^{\prime} \tag{1.16}
\end{equation*}
$$

### 1.2 Renormalization group

Prescriptions like (1.2), (1.10), and (1.12) still contain a $\Lambda$-independent ambiguity, because all we have done so far is to require that divergent parts of 1PI graphs be cancelled. The corresponding finite parts have to be fixed by
imposing a renormalization prescription (call it $R$ ), and quantities such as $\delta \mathrm{m}^{2}$, $\phi, \ldots$, must be regarded as $R$-dependent: $\delta m_{R}^{2}, \phi_{R}, g_{R}, Z_{i}(R), \Gamma_{R}^{(L)}, G_{R}^{(L)}$, and so on. In order to completely specify each distinct vertex appearing in $\Delta \mathscr{L}$, we must impose a normalization condition on the corresponding 1PI amplitude. The label R refers to this set of conditions. The points in momentum space to which the conditions refer are called subtraction points.

A popular choice for $R$ is to normalize on-shell. For example, boson mass and wave-function normalizations are fixed by choosing

$$
\Gamma_{R}^{(2)}(q)=0, \quad \partial \Gamma_{R}^{(2)} / \partial q^{2}=0, \quad\left(q^{2}=m_{R}^{2}\right)
$$

for the self-energy $\Gamma_{R}^{(2)}$, so that the dressed boson propagator

$$
\begin{equation*}
G_{R}^{(2)}(q)=i /\left(q^{2}-m_{R}^{2}-i \Gamma_{R}^{(2)}(q)\right) \tag{1.17}
\end{equation*}
$$

has a pole at $q^{2}=m_{R}^{2}$ with conventional residue. Another example is the charge of the electron $\sqrt{4 \pi \alpha}$ (with $\alpha^{-1}=137.036, \ldots$ ) in quantum electrodynamics ( $Q E D$ ), which refers to the amplitude for an on-shell electron to absorb a zero-frequency photon.

Alternatively, R can be specified by intermediate renormalization ${ }^{13}$, 16) in which the subtraction point is at the origin of momentum space; e.g., $R$ for Eq. (1.10) is fixed by

$$
\begin{align*}
\Gamma_{R}^{(2)}(q)= & \quad, i m_{R}^{2} \\
& \Gamma_{R}^{(4)}\left(q_{1}=q_{2}=q_{3}=0\right) \quad \partial q^{2}=i \quad(q=0) \tag{1.18}
\end{align*}
$$

It does not matter that $g_{R}$ and $m_{R}^{2}$ are not directly measurable; they parametrize the theory as effectively as on-shell variables do.

More generally, R can be a function of several fixed "reference momenta" $\lambda_{1}, \lambda_{2}, \ldots$, which parametrize the subtraction points: consider the prescription $R=R\left(\lambda_{1}, \ldots, \lambda_{5}\right)$ for Eq. (1.10):

$$
\begin{align*}
\Gamma_{R}^{(2)}\left(\lambda_{1}\right)= & -i m_{R}^{2}, \quad \partial \Gamma_{R}^{(2)} /\left.\partial q^{2}\right|_{q=\lambda_{2}}=i \\
& \Gamma_{R}^{(4)}\left(\lambda_{3}, \lambda_{4}, \lambda_{5}\right)=-i g_{R} 4! \tag{1.19}
\end{align*}
$$

## Note

i) To avoid trouble with unitarity, we want $g_{R}$ and $m_{R}^{2}$ to be real. However, for arbitrary $\lambda$ in Eq. (1.19), $\Gamma^{(2)}$ and $\Gamma^{(4)}$ may have absorptive parts. In that case, the normalization conditions (1.19) apply only to the real parts of the 1PI amplitudes at non-singular points $\lambda$. Additional conditions for
absorptive parts are not needed: a 1PI absorptive part converges superficially because it is obtained by cutting internal lines of the full 1PI amplitude. The $\lambda$-momenta should be zero or space-like if a continuation of Feynman amplitudes to Euclidean space is being attempted.
ii) Suppose we use (1.19) to specify $G^{(L)}$ and $Z_{3}$ in Eq. (1.15); then they depend on $\lambda$ as well as the variables shown explicitly. Of course, the B rules make no reference to $\lambda$, so $G_{B}^{(L)}$, when written as a function of $m_{B}$ and $g_{B}$, does not depend on $\lambda$.
iii) Do not confuse the momenta $\lambda_{3}, \lambda_{4}, \lambda_{5}$ at which $g_{R}$ is fixed with the momenta entering and leaving a vertex inside a Feynman diagram?

Now suppose that two theorists are thinking about the same physical situation (i.e. they use the same $\mathscr{L}$ ) but use different renormalization procedures, $R$ and $R^{\prime}$;

$$
\mathcal{L}=\mathcal{L}_{R}(R \text {-quantities })=\mathcal{L}_{R^{\prime}}\left(R^{\prime} \text {-quantities }\right)
$$

Transformation equations relating $R$ and $R^{\prime}$ are easily found by referring back to the $B$ rules, which do not depend on $R$ or $R^{\prime}$. The equations

$$
\varphi_{R}=Z_{3}(R)^{-1 / 2} \varphi_{B} \quad, \quad \varphi_{R^{\prime}}=Z_{3}\left(R^{\prime}\right)^{-1 / 2} \varphi_{B}
$$

etc., imply [e.g., for Eq. (1.12)]

$$
\begin{align*}
\varphi_{R^{\prime}} & =z_{3}\left(R^{\prime}, R\right)^{-1 / 2} \varphi_{R} \\
\psi_{R^{\prime}} & =z_{2}\left(R^{\prime}, R\right)^{-1 / 2} \psi_{R} \\
g_{R^{\prime}} & =z_{1}\left(R^{\prime}, R\right)^{-1} z_{3}\left(R^{\prime}, R\right)^{2} g_{R} \\
g_{R^{\prime}}^{\prime} & =z_{1}^{\prime}\left(R^{\prime}, R\right)^{-1} z_{2}\left(R^{\prime}, R\right) z_{3}\left(R^{\prime}, R\right)^{1 / 2} g_{R^{\prime}}  \tag{1.20}\\
m_{R^{\prime}}^{2} & =z_{3}\left(R^{\prime}, R\right) m_{R}^{2}+\delta\left(R^{\prime}, R\right) \\
M_{R^{\prime}} & =z_{2}\left(R^{\prime}, R\right) M_{R}+\Delta\left(R^{\prime}, R\right)
\end{align*}
$$

and

$$
\begin{align*}
& G_{R^{\prime}}^{(b, f)}=Z_{3}\left(R^{\prime}, R\right)^{-b / 2} Z_{2}\left(R^{\prime}, R\right)^{-f / 2} G_{R}^{(b, f)},  \tag{1.21}\\
&(b \text { bosons , f fermions }),
\end{align*}
$$

where the transformation coefficients $\tilde{3}_{i}, \delta, \Delta$, given by

$$
\begin{align*}
Z_{i}\left(R^{\prime}, R\right) & =Z_{i}\left(R^{\prime}\right) / Z_{i}(R), \quad\left(\text { ditto for } Z_{1}^{\prime}\right) \\
\delta\left(R^{\prime}, R\right) & =\delta m_{R^{\prime}}^{2}-Z_{3}\left(R^{\prime}, R\right) \delta m_{R}^{2}  \tag{1.22}\\
\Delta\left(R^{\prime}, R\right) & =\delta M_{R^{\prime}}-Z_{2}\left(R^{\prime}, R\right) \delta M_{R}
\end{align*}
$$

are all cut-off independent. The ambiguity mentioned at the beginning of this subsection is now completely characterized in terms of these coefficients.

The set $\mathcal{G}$ of all transformations $R \rightarrow R^{\prime}$ is called") the "Renormalization Group". Measurable quantities $\mathcal{M}$ like decay rates and cross-sections are renormalization-group invariant: $\mathcal{M}\left(R^{\prime}\right)=\mathcal{M}(R)$. In other words, $\mathcal{M}$ should not be theorist-dependent.

The group property is partially realized in the existence of an identity transformation $R \rightarrow R$, an inverse $R^{\prime} \rightarrow R$ to $R \rightarrow R^{\prime}$, and a product $R \rightarrow R^{\prime \prime}$ equivalent to successive transformations $R \rightarrow R^{\prime}$ and $R^{\prime} \rightarrow R^{\prime \prime}$, with

$$
\begin{aligned}
& Z_{i}(R, R)=1 \\
& Z_{i}\left(R, R^{\prime}\right)=Z_{i}\left(R^{\prime}, R\right)^{-1} \\
& Z_{i}\left(R^{\prime \prime}, R\right)=Z_{i}\left(R^{\prime \prime}, R^{\prime}\right) Z_{i}\left(R^{\prime}, R\right) \quad, \text { etc. }
\end{aligned}
$$

However, we do not have a group at this stage ${ }^{17 \text { ) , because there is no rule for }}$ multiplying two arbitrary transformations $R \rightarrow R^{\prime}$ and $R^{\prime \prime} \rightarrow R^{\prime \prime \prime}$. A satisfactory rule can be obtained for special subsets of prescriptions which can be explicitly parametrized: $R=R(\lambda)$ [e.g., $R=R\left(\lambda_{1}, \ldots, \lambda_{5}\right)$ discussed above]. Then it is sufficient to consider transformations in $\lambda$-space which possess the group property -- inversions, translations, rotations, scale transformations, etc. So the group property is certainly a feature of many subsets of $\mathcal{G}$, but it is not clear whether $\mathcal{G}$ itself can be characterized as a group or not. (Henceforth, I shall forget about this fine distinction.)

The idea of $\mathrm{SP}^{2}$ ) was to look at sets of prescriptions R depending on continuous parameters $\lambda$ from which a Lie group with associated Lie algebras can be obtained, and find equations giving the result of performing an infinitesimal $\lambda$-transformation. The only subgroup of any practical importance seems to be the trivial case $U(1)$ for scale transformations of the subtraction points $\lambda$ (with generator $\lambda \cdot \partial / \partial \lambda$ ). Non-Abelian cases are easy to construct, but the results appear to be uninteresting.

Eq. (1.21) means that $G(b, f)$ transforms multiplicatively under the group. The 1PI amplitudes transform similarly (with $3^{1 / 2}$ for each line instead of $3^{-1 / 2}$ )

$$
\begin{align*}
& \Gamma_{R^{\prime}}^{(b, f)=} z_{3}\left(R^{\prime}, R\right)^{b / 2} z_{2}\left(R^{\prime}, R\right)^{f / 2} \Gamma_{R}^{(b, f)}  \tag{1.23}\\
& {[(b, f) \neq(2,0),(0,2)] }
\end{align*}
$$

except for self-energy amplitudes, where there is an additional subtractive renormalization: Eqs. (1.17) (valid irrespective of R) and (1.21) imply

$$
\begin{equation*}
\Gamma_{R^{\prime}}^{(2,0)}=Z_{3}\left(R^{\prime}, R\right) \Gamma_{R}^{(2,0)}(q)+i\left(Z_{3}\left(R^{\prime}, R\right)-1\right) q^{2}+i \delta\left(R^{\prime}, R\right) \tag{1.24}
\end{equation*}
$$

for the boson self-energy $\left[\Gamma^{(2)}=\Gamma^{(2,0)}\right]$, and there is an analogous equation for fermions:

$$
\Gamma_{R^{\prime}}^{(0,2)}=Z_{2}\left(R^{\prime}, R\right) \Gamma_{R}^{(0,2)}(q)+i\left(Z_{2}\left(R^{\prime}, R\right)-1\right) \notin+i \Delta\left(R_{(1,25)}^{\prime}\right)+i
$$

### 1.3 Callan-Symanzik equation

The CS equation ${ }^{5}$ ) was the product of a search for the Ward identity of scale transformations in perturbation theory ${ }^{1 \theta}$ ). The generator of scale transformations $\left.{ }^{19}\right), D\left(x_{0}\right)=\int d^{3} x x^{\mu} \theta_{\mu 0}(x)$, has a time-variation given by the trace of the energy momentum tensor $\theta_{\mu \nu}$. In free-field theory, $\theta_{\mu}^{\mu}$ is the sum of mass terms

$$
\begin{equation*}
g_{\mu}^{\mu}=m^{2} \varphi^{2}+M \bar{\psi} \psi \tag{1.26}
\end{equation*}
$$

To obtain something which looks like a scaling Ward identity for an amplitude A, we need the corresponding amplitude $A(\Delta)$ with an extra operator insertion $\Delta$ where, for free-field theory, $\Delta$ reduces to the zero-momentum mass operator $\int d^{4} x \theta_{\mu}^{\mu}(x)$.

Consider the L-leg boson Green's function $G^{(L)}$ (connected), which is multiplicatively renormalized [Eqs. (1.15), (1.21)], and assume for convenience that there is only one mass $m$ and dimensionless coupling constant $g$. If we use the $B$ rules (1.14) to compute ${ }^{20)} G_{B}^{(L)}$, the only place in which $m_{B}$ appears is in undressed propagators $i /\left(p^{2}-m_{B}^{2}\right)$. Hence the identity

$$
m_{B} \frac{\partial}{\partial m_{B}} \frac{i}{p^{2}-m_{B}^{2}}=\frac{i}{p^{2}-m_{B}^{2}}\left(-2 i m_{B}^{2}\right) \frac{i}{p^{2}-m_{B}^{2}}
$$

implies that the operation $m_{B} \partial / \partial m_{B}$ (with external momenta and $g_{B}$, $\Lambda$ fixed) is equivalent to the insertion of a new vertex $-i m_{B}^{2} \phi_{B}^{2}$ at zero-momentum transfer (i.e. $-i \Delta$ ). So we define

$$
\begin{equation*}
G_{B}^{(L)}(\Delta)=i m_{B} \frac{\partial}{\partial m_{B}} G_{B}^{(L)}\left(q ; g_{B}, m_{B}, \Lambda\right) \tag{1.27}
\end{equation*}
$$

where $q$ is shorthand for $\left(q_{1}, \ldots, q_{L-1}\right)$. Of course, $G_{B}^{(L)}(\Delta)$ is cut-off dependent, but its renormalization is very simple. If the operator $Q$ in (1.7) is chosen to be $\phi^{2}$, the only counterterm operator $\Delta Q$ which can satisfy (1.8) is again $\phi^{2}[\phi$ does not appear because of $\mathscr{L}(\phi)=\mathscr{L}(-\phi)]$ :

$$
\mathscr{L}(X)=\mathscr{L}(0)+\chi Z_{\Delta}(\Lambda ; R) \varphi_{R}^{2}+\chi^{2} Z_{\Delta \Delta}
$$

i.e. the composite operator $\phi^{2}(x)$ is multiplicatively renormalized. Hence there is a wave-function renormalization $Z_{\Delta}^{-1}$ for the external line represented by the source $X$ in addition to a factor $Z_{3}^{-1 / 2}$ for each boson line:

$$
\begin{equation*}
G_{R}^{(L)}(\Delta)=Z_{\Delta}^{-1} Z_{3}^{-L / 2} G_{B}^{(L)}(\Delta) \tag{1.28}
\end{equation*}
$$

It is convenient (but not essential) to use the intermediate renormalization prescription (1.18), because then we do not have to worry about additional dependence on reference momenta $\lambda$. In order to fix the finite ambiguity in $Z_{\Delta}$, we have to choose a normalization condition for $\Gamma^{(2)}(\Delta)$, the 1PI amplitude with two boson legs and a $\Delta$ insertion:

$$
\Gamma_{R}^{(2)}(\Delta ; q=0)=2 m_{R}^{2}
$$

Note the relation

$$
\begin{equation*}
G^{(2)}(\Delta ; q)=\left\{G^{(2)}(q)\right\}^{2} \Gamma_{R}^{(2)}(\Delta ; q) \tag{1.29}
\end{equation*}
$$

Now we construct two identities, each the result of applying $m_{R} d / d m_{R}$ (fixed $\mathrm{q}, \mathrm{g}_{\mathrm{B}}, \Lambda$ ) to $\mathrm{G}_{\mathrm{R}}^{(\mathrm{L})}$ and changing variables:

$$
\begin{aligned}
& \left.m_{R} \frac{d}{d m_{R}} \right\rvert\, q, g_{B}, \Lambda \\
& G_{R}^{(L)}= {\left[\left.m_{R} \frac{\partial}{\partial m_{R}}+m_{R} \frac{d g_{R-}}{d m_{R}} \right\rvert\, \wedge, g_{B} \frac{\partial}{\partial g_{R}}\right] G_{R}^{(L)}\left(q ; g_{R}, m_{R}\right) } \\
&=-\frac{1}{2} L Z_{3}^{-1} G_{R}^{(L)}\left\{m_{R} \partial / \partial m_{R}\right\} Z_{3}\left(\Lambda / m_{R}, g_{B}\right) \\
&+Z_{3}^{-L / 2}\left(m_{R} / m_{B}\right)\left(d m_{B} /\left.d m_{R}\right|_{g_{B}, \Lambda}\left\{m_{B} \partial / \partial m_{B}\right\} G_{B}^{(L)}\left(q ; g_{B}, m_{B}, \Lambda\right)\right.
\end{aligned}
$$

The second identity depends on Eq. (1.15). The result of combining these dentities with (1.28) is the CS equation ${ }^{5}, 21$ )

$$
\begin{align*}
{\left[m_{R} \frac{\partial}{\partial m_{R}}+\beta_{R} \frac{\partial}{\partial g_{R}}\right.} & \left.+L \gamma_{R}\right] G_{R}^{(L)}\left(q ; g_{R}, m_{R}\right) \\
& =-i \delta_{R} G_{R}^{(L)}\left(\Delta ; q, g_{R}, m_{R}\right) \tag{1.30}
\end{align*}
$$

with

$$
\begin{align*}
& \beta_{R}=m_{R}\left(d g_{R} / d m_{R}\right)_{g_{B}, \Lambda} \\
& \gamma_{R}=-m_{R} \partial / \partial m_{R} \ln \left\{Z_{3}\left(\Lambda / m_{R}, g_{B}\right)^{-1 / 2}\right\}  \tag{1.31}\\
& \delta_{R}=Z_{\Delta}\left(m_{R} / m_{B}\right)\left(d m_{B} / d m_{R}\right)_{g_{B}}, \Lambda
\end{align*}
$$

The functions $\gamma, \delta$ are cut-off independent, because we can apply the conditions

$$
\begin{align*}
& G_{R}^{(2)}(q=0)=-i / m_{R}^{2} \\
& \left(\partial / \partial q^{2}\right) G_{R}^{(2)}(q=0)=-i / m_{R}^{4}  \tag{1.32}\\
& G_{R}^{(2)}(\Delta ; q=0)=-2 / m_{R}^{2}
\end{align*}
$$

[implied by Eqs. (1.17), (1.18), (1.18'), (1.29)] to Eq. (1.30) with $L=2$ :

$$
\begin{align*}
& \delta_{R}=1-\gamma_{R} \\
& \gamma_{R}=2+\frac{1}{2} m_{R}^{4} \delta_{R}\left(\partial / \partial q^{2}\right) G_{R}^{(L)}(\Delta ; q=0) \tag{1.33}
\end{align*}
$$

It follows from (1.30) that $\beta$ is also cutoff independent. Hence the dimensionless functions $\beta, \gamma, \delta$ can only depend on $g_{R}$, in this renormalization prescription. The formulas

$$
\begin{align*}
& \beta_{R}\left(g_{R}\right)=-g_{R} \Lambda \partial / \partial \Lambda \ln z_{g}\left(\Lambda / m_{R}, g_{B}\right), \quad\left(g_{R}=z_{g} g_{B}\right)  \tag{1.34}\\
& \gamma_{R}\left(g_{R}\right)=\Lambda \partial / \partial \Lambda \ln \left\{Z_{3}\left(\Lambda / m_{R}, g_{B}\right)\right\}^{-1 / 2}, \quad(\Lambda \rightarrow \infty)
\end{align*}
$$

[from (1.31)] show the connection between $\beta$ and $\gamma$ and infinities in couplingconstant and wave-function renormalization.

Let us introduce a parameter $\eta$ which scales all the momenta together:
$q \rightarrow n q$. Since $G_{R}^{(L)}$ has mass dimensionality $4-3 L$, purely dimensional arguments imply the identity

$$
\begin{equation*}
\left[\eta \partial / \partial \eta+m_{R} \partial / \partial m_{R}+3 L-4\right] G_{R}^{(L)}\left(\eta q ; g_{R}, m_{R}\right)=0 \tag{1.35}
\end{equation*}
$$

so the CS equation can be cast in a form which shows the connection with scale transformations:

$$
\begin{gather*}
{\left[\eta \partial / \partial \eta-\beta_{R}\left(g_{R}\right) \partial / \partial g_{R}+3 L-4-L \gamma_{R}\left(g_{R}\right)\right] G_{R}^{(L)}\left(\eta q_{i} ; g_{R}, m_{R}\right)} \\
=-i \delta_{R}\left(g_{R}\right) G_{R}^{(L)}\left(\Delta ; \eta q ; g_{R}, m_{R}\right) \tag{1.36}
\end{gather*}
$$

The term $\beta_{R} \partial / \partial g_{R}$, caused by coupling-constant renormalization, means that $\int d^{4} x \theta_{\mu}^{\mu}(x)$ effectively picks up a dimension 4 vertex $\left.{ }^{5}, 22\right)$-i $\beta_{R} \int d^{4} x \phi^{4}(x)$.

If there are several coupling constants $g_{R}(i)(i=1, \ldots, j)$, the $\beta_{R} \partial / \partial g_{R}$ term becomes $\sum_{i=1}^{j} \beta_{R}^{(i)} \partial / \partial g_{R}(i)$; i.e. there is a $\beta$-function $\beta_{R}^{(i)}$ for each $g_{R}(i)$, and it depends on all of the $g_{R}(i)$ 's:

$$
\begin{align*}
\vec{g}_{R}= & \left(g_{R}(1), \ldots, g_{R}(j)\right) \\
\vec{\beta}_{R}= & \left(\beta_{R}^{(1)}, \ldots ., \beta_{R}^{(j)}\right)  \tag{1.37}\\
\vec{\beta}_{R}= & \vec{\beta}_{R}\left(\vec{g}_{R}\right), \gamma_{R}=\gamma_{R}\left(\vec{g}_{R}\right) \\
& \vec{\beta}_{R} \cdot \partial / \partial \vec{g}_{R} \equiv \sum_{i=1}^{j}, \beta_{R}^{(i)} \partial / \partial g_{R}(i)
\end{align*}
$$

If there are also several masses $m_{R}(k)$, each mass can generate its own CS equation, $\beta$-functions $\beta_{R}^{(i, k)}$, and $\gamma$-function $\gamma_{R}^{(k)}$. Occasionally, these CS equations are separately useful ${ }^{23}$ ), but in most cases, it is better to sum them because then $\vec{m}_{R} \cdot \partial / \partial \vec{m}_{R}$ can be converted to $\eta \partial / \partial \eta$ using the generalization of (1.35).

Subscripts $R$ have been attached to the symbols $\beta, \gamma, \delta$ in Eq. (1.36) to indicate that their functional forms depend on the renormalization prescription. To see this, consider a prescription $R^{\prime}$ (such as on-shell normalization) in which
$g_{R} \prime, \quad \tilde{3}_{3}\left(R^{\prime}, R\right)$, and $m_{R} / m_{R}$ depend on $g_{R}$ alone; i.e. we avoid introducing dependene on a separate dimensionful quantity $\lambda$. Then a change of variables $\left(g_{R}, m_{R}\right)$ to ( $g_{R^{\prime}}^{\prime, m_{R}^{\prime}}$ ) in Eq. (1.30) produces a CS equation of the same form, with $R^{\prime}$ replacing $R$ and

$$
\begin{align*}
& \beta_{R^{\prime}}\left(g_{R^{\prime}}\right)=\beta_{R}\left\{d g_{R^{\prime}} / d g_{R}\right\} /\left\{1+\beta_{R} d / d g_{R} \ln \left(m_{R^{\prime}} / m_{R}\right)\right\}  \tag{1.38}\\
& \gamma_{R^{\prime}}\left(g_{R^{\prime}}\right)=\left\{\gamma_{R}+\xi_{3}^{-1} \beta_{R} d g_{R^{\prime}} / d g_{R}\right\} /\left\{1+\beta_{R} d / d g_{R} \ln \left(m_{R^{\prime}} / m_{R}\right)\right\}
\end{align*}
$$

Observe that $\beta, \gamma$ are $R$-independent in the 1 -loop approximation.
The main complication for $R=R(\lambda)$ is that $m_{R} \partial / \partial m_{R}$ in Eq. (1.30) has to be replaced by $\left\{m_{R} \partial / \partial m_{R}+\lambda \cdot \partial / \partial \lambda\right\}$ so that we still get the infinitesimal scale transformation $\eta \partial / \partial \eta$ in (1.36). In general, the $\beta$ and $\gamma$ functions depend on dimensionless ratios such as $m_{R}^{2} / \lambda^{2}, \lambda_{1}^{2} / \lambda_{2}^{2}, \ldots$, as well as on $g_{R}$.

The generalization to amplitudes $G_{A B C}^{(b, f)}$ involving several composite operators A, B, C, ..., can be readily carried through. The analogue of Eq. (1.7) is

$$
\begin{aligned}
\mathscr{L}(\chi)= & \mathscr{L}(0)+\chi_{A}(A+\Delta A)+\chi_{B}(B+\Delta B)+\ldots \\
& +\chi_{A}^{2} \Delta(A A)+\chi_{A} X_{B} \Delta(A B)+\chi_{A} X_{C} \Delta(A C)+\ldots+\chi_{B}^{2} \Delta(B B) \\
& +\ldots \\
& +\ldots
\end{aligned}
$$

where $\Delta(A B C . .$.$) is a local vertex with cut-off-dependent normalization. Con-$ wider the terms linear in $X$. For general $A$, there may be many counterterms $\triangle A$ with dimensionality less than or equal to that of $A$ [see Eq. (1.8)]. In other words, the Z -factors have to be treated as matrices. However, by adding suitable linear combinations of the counterterm vertices $\Delta A, \Delta B, \ldots$, we can always redefine $A, B, C, \ldots$, such that they are multiplicatively renormalized:

$$
A+\Delta A=Z_{A}(\Lambda) A \quad, \quad B+\Delta B=Z_{B}(\Lambda) B, \ldots(1.40)
$$

Equation (1.39) also contains counterterms $O\left(\chi^{p}\right)(p>1)$ produced by superficially divergent PI subdiagrams with $p$ composite-operator insertions. However, these terms are very easy to isolate. For example, if $A(x)$ and $B(y)(x, y=$ coordinates $)$ couple to the same 1PI diagram, the induced counterterm must be proportional to $P\left(\partial_{x}\right) \delta^{4}(x-y)$, where $P$ is a polynomial; this is so because a 1PI diagram (made convergent internally) shrinks to a point when the divergent part is taken (Fig. 1). In momentum space, we have $P\left(-i q_{A B}\right)$, where $q_{A B}$ is the momentum transfired from $A$ to $B$. So the mode of renormalization is given by

$$
\begin{align*}
\left\{G_{A B C}^{(b, f)}\right\}_{R}= & z_{A}^{-1} z_{B}^{-1} z_{C}^{-1} \ldots \ldots z_{3}^{-b / 2} z_{2}^{-f / 2}\left\{G_{A B C}^{(b, f)}\right\}_{\text {Bare }} \\
& +\sum_{i} P_{i}\left(q_{\text {subset } i}\right) \mathcal{A}_{i} \tag{1.41}
\end{align*}
$$

where $P_{i}$ is a polynomial in momentum-transfer variables $q_{\text {subset }} i$ for the $i^{\text {th }}$ subset of $\{A, B, C, \ldots\}$ and $t_{i}$ is a nontrivial amplitude involving the remaining operators plus the counterterm operator $\Delta\left(i^{\text {th }}\right.$ subset). There is a factor $Z_{Q}^{-1}$ for each insertion $Q$, in addition to the usual factors $Z_{3}^{-1 / 2}, Z_{2}^{-1 / 2}$ for each external boson and fermion 1 ind in a complete or connected Green's function. The subtractive renormalization $\sum_{i} P_{i}, t_{i}$ lack absorptive parts in channels orespodding to the polynomial dependence $P_{i}\left(q_{\text {subset }}\right)$.

By repeating the previous analysis, we end up with a CS equation for $G_{A B C}^{(b, f)} \ldots$,

$$
\begin{aligned}
{\left[m_{R} \frac{\partial}{\partial m_{R}}\right.} & \left.+\beta_{R}\left(g_{R}\right) \frac{\partial}{\partial g_{R}}+b \gamma_{R}\left(g_{R}\right)+f \bar{\gamma}_{R}\left(g_{R}\right)+\gamma_{A B C \ldots}^{(R)}\left(g_{R}\right)\right]\left\{G_{A B C \ldots}^{(b, \ldots)}\left(q ; g_{R}, m_{R}\right)\right\}_{R} \\
& \left.+\sum_{i} \mathcal{P}_{i}\left(q_{\text {subset } i}\right) \mathcal{G}_{i} \text { (other } q ; g_{R}, m_{R}\right) \\
& =-i \delta_{R}\left(g_{R}\right)\left\{G_{A B C \ldots}^{(b, f)}\left(\Delta ; q ; g_{R}, m_{R}\right)\right\}_{R}
\end{aligned}
$$

where all factors are cut-off independent, $T_{i}$ is a polynomial, and

$$
\begin{equation*}
\bar{\gamma}_{R}\left(g_{R}\right)=\Lambda \frac{\partial}{\partial \Lambda} \ln \left\{Z_{2}\left(\Lambda / m_{R}, g_{B}\right)\right\}^{-1 / 2} \tag{1.43}
\end{equation*}
$$

is the fermionic analogue of $\gamma_{R}\left(g_{R}\right)$. The important thing to notice is that there is an addition rule satisfied by

$$
\begin{equation*}
\gamma_{A B C \ldots}^{(R)}\left(g_{R}\right)=-\Lambda \frac{\partial}{\partial \Lambda} \ln \left[Z_{A} Z_{B} Z_{C} \cdots\right] \tag{1.44}
\end{equation*}
$$

the $\gamma$-function describing the combined effects of the product $A B C . .$. on wavefunction renormalization. For each operator $Q$, there is a characteristic $\gamma$-function

$$
\begin{equation*}
\gamma_{Q}^{(R)}\left(g_{R}\right)=-\Lambda \frac{\partial}{\partial \Lambda} \ln Z_{Q}\left(\Lambda / m_{R}, g_{B}\right) \tag{1.45}
\end{equation*}
$$

which satisfies the formula

$$
\begin{equation*}
\gamma_{A B C}^{(R)} \ldots\left(g_{R}\right)=\gamma_{A}^{(R)}\left(g_{R}\right)+\gamma_{B}^{(R)}\left(g_{R}\right)+\gamma_{C}^{(R)}\left(g_{R}\right)+\ldots \tag{1.46}
\end{equation*}
$$

Conserved currents (e.g. $\theta_{\mu \nu}$ and the electromagnetic current $J_{\mu}$ ) obey adentitles such as

$$
\begin{equation*}
i\left(p^{\prime}-p\right)_{\mu} G_{J_{\mu}}^{(0,2)}\left(p^{\prime}, p\right)=G^{(0,2)}\left(p^{\prime}\right)-G^{(0,2)}(p) \tag{1.47}
\end{equation*}
$$

irrespective of which rules ( $B$ or any $R$ ) are chosen. Hence there is no wavefunction renormalization factor for a conserved current

$$
\begin{equation*}
Z_{J_{\mu}}\left(R^{\prime}, R\right)=Z_{J_{\mu}}=1 \tag{1.48}
\end{equation*}
$$

and the corresponding $\gamma$-function (1.45) vanishes identically:

$$
\begin{equation*}
\gamma_{J_{\mu}}=0 \tag{1.49}
\end{equation*}
$$

Of course, there can still be subtractive renormalization induced by timeordering products of conserved currents. For example, the renormalization of $T\langle 0| J_{\mu}(x) J_{\nu}(0)|0\rangle$ involves a subtraction proportional to $\left(\partial_{\mu} \partial_{\nu}-g_{\mu \nu} \partial^{2}\right) \delta^{4}(x)$. Partially conserved currents such as the chiral $\operatorname{SU}(3) \operatorname{XSU}(3)$ currents ( $\pi_{\mu}^{a}, \pi_{5 \mu}^{a}$ ) of current algebra become conserved at short distances, so they also satisfy (1.48) and (1.49).

## 2. ASYMPTOTIC SOLUTIONS

This chapter concerns situations in which the mass-inserted amplitude $G(\Delta)$ can be neglected in the CS equation. In perturbation theory, an appropriate asymptotic 1 imit is specified by Weinberg's theorem ${ }^{11}$ ).

### 2.1 Logarithms in perturbation theory

Weinberg's theorem extends the power-counting method to the problem of estimating the asymptotic behaviour of a Feynman amplitude as some of its external momenta become large. Actually, it is a theorem in real-variable analysis which describes the asymptotic behaviour of infinite multiple integrals, so its application to Feynman integrals is necessarily restricted to the Euclidean region. However, Pohlmeyer ${ }^{24}$ ) has proven a corresponding theorem for Minkowski space.

Let $\mathcal{A}$ be a Feynman amplitude which depends on external momenta $q_{1}, \ldots, \mathrm{q}_{\mathrm{K}}$ and $r_{1}, \ldots, r_{N}$, where the $q$-momenta are large:

$$
\begin{align*}
A & =A\left(q_{1} \ldots q_{k} ; r_{1} \ldots . r_{N}\right) \\
q_{i}=\eta \ell_{i} & +c_{i}, \quad\left(\eta \rightarrow \infty ; \ell_{i}, c_{i}, r_{j} \text { all fixed }\right)  \tag{2.1}\\
i & =1 \ldots N
\end{align*}
$$

All momenta are understood to be incoming, so momentum conservation requires

$$
\begin{align*}
\sum_{j=1}^{N} r_{j}+\sum_{i=1}^{K} q_{i} & =0  \tag{2.2a}\\
\sum_{i=1}^{K} l_{i} & =0 \tag{2.2b}
\end{align*}
$$

Imagine the large $O(n)$ momenta (indicated by heavy lines in Fig. 2) percolating through all vertices and propagators of a subgraph $\mathcal{G}^{\prime}$ of a typical graph 9. Naturally, we arrange that the external lines of $\mathscr{G}^{\prime}$ include all external lines of $\mathcal{G}$ which carry the $O(n)$ momenta $q_{1}, \ldots, q_{K}$, and that it is kinematically possible for all internal lines of $G^{\prime}$ to carry $O(\eta)$ momenta. Each subgraph can
contribute to the asymptotic behaviour of 9 . The idea is to count the powers produced by propagators, vertices and loop integrations in $~_{9}^{\prime}$ ', keeping the intermediate momenta $k_{1}, \ldots, k_{M}$ fixed. The effect of loop integration is illustrated by the l-loop boson self-energy amplitude (Fig. 3) for a scalar Yukawa theory (interaction $g \bar{\psi} \psi \phi)^{*}$ ):

$$
\begin{align*}
\Gamma_{R}^{(2,0)}\left(q^{2}\right)_{1-\text { loop }}=- & g_{R}^{2} \int \frac{d^{4} p}{(2 \pi)^{4}} \operatorname{Tr}\left\{(\not p-m)^{-1}(\not p+q-m-m)^{-1}\right. \\
& - \text { subtraction }\} \tag{2,3}
\end{align*}
$$

This example shows that it is necessary to count 4 powers of $\eta$ for each loop integral $\int d^{4} p$ inside $\left.{ }^{* *}\right) \mathscr{G}^{\prime}$. Consequently, each $\mathscr{G}^{\prime}$ contributes a characteristic power $O\left(\eta^{\prime \prime}\right)\left(\xi^{\prime}\right)$ ) (with logarithmic corrections), where the dimensionality $\left.\xi^{\prime}\right)\left(\rho_{\prime}^{\prime}\right)$ of $\varsigma^{\prime}$ is given by

$$
\begin{align*}
D\left(\xi^{\prime}\right)= & \sum \operatorname{dim}\left\{\text { propagators and vertices of } B_{j}^{\prime}\right\} \\
& +4\left\{\text { number of independent loops in } G^{\prime}\right\} \tag{2.4}
\end{align*}
$$

So far, the only restriction placed on the vectors $l_{i}$ is the momentumconservation equation (2.2b). Another obvious requirement is that none of the $\ell_{i}$ should vanish -- otherwise, some of the $q$-momenta would not be $O(\eta)$. However, the power-counting rules introduced above do not necessarily work unless additional conditions are imposed. For example, the 4 boson and 3 fermion propagators of the subgraph $\varrho^{\prime}$ of Fig. 4 (Yukawa theory) normally contribute $O\left(\eta^{-11}\right)$. However, at the "exceptional" momentum point ${ }^{25)} \ell_{1}=-\ell_{2}$, the fermion propagator in the middle of $G^{\prime}$ no longer carries momentum $O(n)$ :

$$
A\left(l_{1}^{\prime} ; l_{i}^{2} \neq 0 ; l_{1}=-l_{2}\right)=O\left(\eta^{-10}\right)
$$

Additional exceptional points exist in Minkowski space:

$$
\mathcal{A}\left(l^{\prime} ; l_{i}^{2} \neq 0 ; \ell_{1} \neq-l_{2} ;\left(l_{1}+l_{2}\right)^{2}=0\right)=O\left(\eta^{-9}\right) \text { or } 0\left(\eta^{-10}\right)
$$

[depending on whether $\left(l_{1}+\ell_{2}\right) \cdot\left(c_{1}+c_{2}+k_{1}\right)$ vanishes or not]. The general conclusion is that tree graphs can be more singular than $O\left(n^{(\sqrt{\prime})\left(\mathcal{G}^{\prime}\right)}\right.$ ) if some of the partial sums $I^{\prime} \ell$ happen to be light-1ike.

Not surprisingly, loop integrals also misbehave at these exceptional points. Consider the l-loop amplitude (Fig. 5)

[^3]\[

$$
\begin{align*}
& I\left(q_{1} \ldots q_{k}\right)=i^{k} \int \frac{d^{4} p}{(2 \pi)^{4}} \prod_{i=1}^{k}\left[\left(p+Q_{i}\right)^{2}-m^{2}\right]^{-1}, \quad(K \geqslant 3)  \tag{2.5a}\\
& \left(Q_{i}=\sum_{j=1}^{i} q_{j}, q_{i}=\eta l_{i}+c_{i}, \quad \sum_{i=1}^{k} l_{i}=\sum_{i=1}^{k} c_{i}=0\right)
\end{align*}
$$
\]

which can be simply represented as a Feynman-parameter integral ${ }^{26)}$ :

$$
\begin{equation*}
I=\frac{i^{K+1}(K-3)!}{16 \pi^{2}} \int_{0}^{1} \prod_{i=1}^{K} d \alpha_{i} \delta\left(1-\sum \alpha\right) /\left[\sum_{i<j} \alpha_{i} \alpha_{j}\left(Q_{i}-Q_{j}\right)^{2}-m^{2}\right]^{K-2} \tag{2.5b}
\end{equation*}
$$

The 1 imit $n \rightarrow \infty$ is equivalent to setting $m$ and $c_{i}$ to zero in (2.5a) or (2.5b). If the massless amplitude $I\left(m=0=c_{i}\right)$ exists, the result is

$$
\begin{equation*}
I(\eta) \sim \eta^{4-2 k} I\left(m=0=c_{i} ; \eta=1\right) \quad,(\eta \rightarrow \infty) \tag{2.6}
\end{equation*}
$$

which agrees with naive power-counting:
$D\left(\xi^{\prime}\right)=4-2 k$ (with $\xi^{\prime}=$ complete graph $\mathcal{H}$ ).
However, $I\left(m=0=c_{i}\right)$ may not exist because of infrared singularities; in that case, $\eta^{2 K-4} I(\eta)$ blows up as $\eta$ tends to infinity. For example, if all partial sums $\sum^{\prime} \ell$ are 1 ight-1ike but non-zero, there is a singularity at $p=0$ :

$$
I(\eta) \sim i^{\kappa} \eta^{4-2 k} \int \frac{d^{4} p}{(2 \pi)^{4}} \prod_{i=1}^{k}\left[p^{2}+2 p \cdot \sum_{j=1}^{i} l_{j}+O\left(\eta^{-1}\right)\right]^{-1}
$$

The true asymptotic behaviour can be found by substituting

$$
\left(Q_{i}-Q_{j}\right)^{2}=\eta r_{i j}+O(1)
$$

into Eq. (2.5b), with

$$
r_{i j}=2\left(\sum_{k=i}^{j} l_{k}\right) \cdot\left(\sum_{k=i}^{j} c_{k}\right)
$$

thus, if $r_{i j}$ all have the same sign, the answer is

$$
\begin{align*}
I(\eta) \sim & \eta^{2-k}\left(\frac{i^{k+1}(k-3)!}{16 \pi^{2}}\right) \int_{0}^{1} \prod_{i=1}^{k} d \alpha_{i} \delta(1-\Sigma \alpha) /\left[\sum_{i<j} \alpha_{i} \alpha_{j} r_{i j}\right]^{k-2} \\
& \left(\left(\Sigma^{\prime} l\right)^{2}=0, \quad \Sigma^{\prime} l \neq 0, \eta \rightarrow \infty ; k \geqslant 3\right)
\end{align*}
$$

because the resulting $\alpha$-integral is manifestly convergent.
In general, loop integrals become infrared singular at thresholds for the production of intermediate zero-mass particles. These thresholds occur when at least one of the momenta ${ }^{\prime}$ ' $\ell$ becomes light -like. This is precisely the condition for exceptional asymptotic behaviour of tree graph amplitudes, so we conclude that the general requirement for momenta to be non-exceptional ${ }^{25 \text { ) is }}$

$$
\begin{equation*}
\left(\sum^{\prime} l\right)^{2} \neq 0 \tag{2.7}
\end{equation*}
$$

for all partial sums $\sum^{\prime} \ell$ associated with connected amplitudes.

## Note

i) In coordinate space, the non-exceptional $n \rightarrow \infty$ limit corresponds to the short-distance limit

$$
x_{1} \ldots x_{K} \longrightarrow y \quad \text { (non-exceptional), }
$$

where $x_{1}, \ldots, x_{K}$ are coordinates conjugate to the momenta $q_{1}, \ldots, q_{K}$. If $K^{\prime}$ partial sums $\sum^{\prime} \ell$ become light-like $\left(0<K^{\prime}<K\right)$, the $x$-coordinates are free to tend to $K^{\prime}+1$ distinct $y$-coordinates:

$$
x_{1} \ldots x_{K} \longrightarrow y_{1} \ldots y_{k^{\prime}+1} \quad \text { (exceptional). }
$$

ii) I have glossed over a complication caused by renormalization of ultraviolet divergences. Counterterms which remove logarithmic divergences develop logarithmic infrared singularities
as $m, c_{i}$, and all subtraction points $\lambda$ are effectively scaled to zero by the $\eta \rightarrow \infty$ limit. Consequently, there are logarithmic corrections to naive powercounting at non-exceptional momenta [as in Eq. (2.3)]:

$$
\begin{equation*}
\varphi^{\prime} \text { contribution }=O\left[\eta^{D\left(\xi^{\prime}\right)}\{\text { Polynomial in } \ln \eta\}\right] \tag{2.8}
\end{equation*}
$$

Additional powers of $\eta$ are not generated: the candidate
counterterm $(m=\lambda=0)=\int^{\Lambda} d^{4 \ell} p / p^{4 \ell-d(\Gamma)}, \quad(d(\Gamma)<0)$ is not permitted because it is ultraviolet convergent*).

The main features and limitations of the theorem have now been exposed. It states that the amplitude $\mathcal{A}(\mathcal{S})$ for a graph $\mathcal{G}$ satisfies the asymptotic bound

$$
\begin{equation*}
A(y)=0\left[\eta^{\operatorname{Max} x\left(y^{\prime}\right)} \ln ^{\beta(\xi)} \eta\right] \tag{2.9}
\end{equation*}
$$

for the non-exceptional 1 imit $\eta \rightarrow \infty$ specified by Eqs. (2.1), (2.2), and (2.7). The power Max $!\left(G^{\prime}\right)$ denotes the maximum dimensionality attained in the set of all subgraphs $\mathcal{G}^{\prime}$ of $\mathfrak{G} ;($ see Fig. 2). The logarithmic power $\beta(\mathcal{G})$ (integer $\geq 0$ ) is determined by counting powers of in $\Lambda$ in counterterms which remove the ultraviolet divergences of subgraphs with maximal dimensionality ${ }^{27)}$.

If propagators for external lines of $\mathcal{G}^{\prime}$ are excluded, theories with dimensionless coupling constants obey the rule

[^4]\[

$$
\begin{align*}
D\left(\mathcal{l}^{\prime}\right)=4 & -\left\{\text { number of bosons external to } \mathscr{l}^{\prime}\right\} \\
& -\frac{3}{2}\left\{\text { number of fermions external to } \xi^{\prime}\right\} \tag{2.10}
\end{align*}
$$
\]

Hence the maximum dimensionality is found by minimizing the number $M$ of intermediate lines in Fig. 2. If composite operators (with sources $X$ ) are present, the generalization of (2.10) is

$$
D\left(g^{\prime}\right)=4-\sum \operatorname{dim}\left\{\text { external lines of } \mathcal{l}^{\prime}\right\}
$$

where each source $X$ is represented by a line (external or intermediate in Fig. 2) and given an appropriate dimensionality. For example, the source $X_{\phi^{2}}$ for the composite operator $\phi^{2}(x)$ carries dimensionality 2 , so the complete graph shown in Fig. 5 has dimensionality $4-2 \mathrm{~K}$; [compare Eq. (2.6)].

According to Eq. (2.10) and (2.10'), the value of $9\left(9^{\prime}\right)$ in renormalizable theories is completely specified by the number and nature of the lines carrying momenta $k_{1}, \ldots, k_{M}$ and $q_{1}, \ldots, q_{K}$ in Fig. 2. This means that Fig. 2 can be applied to a collection of Feynman diagrams for the amplitude
t $\left(q_{1}, \ldots, q_{K} ; r_{1}, \ldots, r_{N}\right)$ if all sets of intermediate lines $\left\{k_{1}, \ldots, k_{M}\right\}$ permitted by selection rules are considered.

Now we examine the limit $n \rightarrow \infty$ for the CS equation [Eqs. (1.36), (1.42)]

$$
\begin{align*}
\partial_{G} G(\eta q)+ & \{\text { subtraction terms }\}=-i \delta G(\Delta ; \eta q) \\
\partial_{G} & =\eta \partial / \partial \eta-\beta \partial / \partial g-\gamma_{G} \tag{2.11}
\end{align*}
$$

for connected amplitudes $G(\eta q), G(\Delta ; \eta q)$ at non-exceptional momenta $q$. Graphs contributing to $G(n q)$ do not have external legs with fixed momenta (i.e. $N=0$ in Fig. 2), so for each graph $\mathfrak{G}$, the dominant subgraph is $\mathcal{G}$ itself:

$$
\begin{equation*}
G(\eta q)=O\left[\eta^{2(G)} l_{n}^{\beta} \eta\right] \quad, \quad[D|G|=D(G)] \tag{2.12}
\end{equation*}
$$

The dominant subgraphs*) for $G(\Delta)$ are shown in Fig. 6:

$$
\begin{aligned}
& D\left(\xi^{\prime} ; \text { Fig. ba }\right)= \begin{cases}D(G)-2 & \left(k_{1}, k_{2}=\text { bosons }\right), \\
D(G)-3 & \left(k_{1}, k_{2}=\text { fermions }\right) ;\end{cases} \\
& D\left(\xi^{\prime} ; \text { Fig. bb }\right)=D(G)-\operatorname{dim} \chi_{\Delta}
\end{aligned}
$$

[^5]Here $X_{\Delta}$ is the source for $\bar{\psi} \psi$ or $\phi^{2}$ at zero momentum:

$$
\begin{equation*}
\operatorname{dim} x_{\Delta}=1 \text { or } 2 \tag{2.14}
\end{equation*}
$$

Equations (2.13) and (2.14) imply

$$
\begin{array}{ll}
G(\Delta ; \eta q)=O\left[\eta^{D(G)-1} \ln ^{\beta^{\prime}} \eta\right] & \text { (fermions present), } \\
G(\Delta ; \eta q)=O\left[\eta^{D(G)-2} \ln { }^{\beta^{\prime \prime}} \eta\right] \quad \text { (no fermions) } \tag{2.15b}
\end{array}
$$

so the leading power $G^{\text {as. }}$ of $G$, defined by

$$
\begin{equation*}
G(\eta q)=G^{a s}(\eta q)+O\left[\eta(G)-1 \ln n^{\beta^{\prime \prime \prime}} \eta\right] \tag{2.16}
\end{equation*}
$$

satisfies the equation ${ }^{*}$ )

$$
\begin{equation*}
\partial_{G} G^{\text {as. }}(\eta q)+\{\text { sub. terms }\}^{\text {as. }}=0 \tag{2.17}
\end{equation*}
$$

with

$$
\begin{align*}
\{\text { subtraction terms }\} & =\sum_{i} \rho_{i} \ell_{i} \quad[\text { Eq. }(1.42)] \\
=\{\text { sub. terms }\}^{\text {as. }}+0[\eta(G)-1 & \left.+\ln ^{\beta^{\prime v}} \eta\right] \tag{2.18}
\end{align*}
$$

In other words, $G(\Delta)$ is asymptotically negligible ${ }^{5}$ ).
The restriction to non-exceptional momenta ${ }^{25}$ ) is essential. For example, apart from a few special cases ${ }^{1}$, $G(\Delta)$ is not asymptotically negligible if $G$ is an on-shell amplitude. Also, additional mass insertions usually do not reduce the asymptotic behaviour further: in most cases, the amplitudes $G\left(n \Delta^{\prime} s\right)$ and $G\left(n+1 \Delta^{\prime} s\right)$ in the CS equation**)

$$
\begin{align*}
\partial_{G(n \Delta \prime s)} G\left(n \Delta \Delta^{\prime} s\right) & +\{\text { sub. terms }\}=-i \delta G\left(n+1 \Delta^{\prime} s\right) \\
\left.\partial_{G(n} \Delta^{\prime} s\right) & =\partial_{G}-n \gamma_{\Delta}  \tag{2.19}\\
\gamma_{\Delta} & =-\Lambda \partial / \partial \Lambda \ln Z_{\Delta}\left(\Lambda / m_{R}, g_{B}\right)
\end{align*}
$$

have the same asymptotic power $S(G)$ - 2. The situation for $G(\Delta \Delta)$ is illustrated in Fig. 7:

[^6]\[

$$
\begin{align*}
& D\left(\xi^{\prime} ; \text { Fig. Ta }\right)= \begin{cases}D(G)-2 & \left(k_{1}, k_{2}=\text { bosons }\right), \\
D(G)-3 & \left(k_{1}, k_{2}=\text { fermions }\right) ;\end{cases}  \tag{2.20}\\
& D\left(\xi^{\prime} ; \text { Fig. } 7 b\right)=D(G)-2 \operatorname{dim} x_{\Delta} .
\end{align*}
$$
\]

If $\Delta$ is proportional to $\int d^{4} x \bar{\psi} \psi(x)$, the asymptotic power is reduced from I) $(G)-1$ for $G(\Delta)$ to $\int(G)-2$ for $G(\Delta \Delta)$. Otherwise, there is no reduction; clearly, any number of $\Delta$ insertions can be attached to the lower blob in Fig. Fa without changing the asymptotic power.

In perturbation theory, the general form of $G^{\text {as. consistent with Eq. (2.17) }}$ is

$$
\begin{equation*}
G^{a s}(\eta)=\eta^{D(G)} \sum_{N} g^{N} \sum_{p=0}^{\ell(N)} \ln ^{p} \eta G_{N, p} \tag{2.21}
\end{equation*}
$$

where $\ell(N)$ is the number of loops in $N^{\text {th }}$ order diagrams for the amplitude $G$. The presence of $\partial / \partial g$ in the CS operator $\partial_{G}$ means that (2.17) relates the $N^{\text {th }}$ order coefficient $G_{N, p}$ to lower-order coefficients. The consequences of this are best illustrated with a simple example, the photon propagator in $Q_{E D}{ }^{28}$ ):

$$
\begin{gather*}
D_{F_{\mu \nu}}^{\prime}(q)=-\frac{i}{q^{2}}\left(g_{\mu \nu}-q_{\mu} q_{\nu} / q^{2}\right) d\left(q^{2} / m^{2}, \alpha\right)+\left\{q_{\mu} q_{\nu} / q^{4} \text { term }\right\}  \tag{2.22}\\
{\left[\alpha=(\text { charge })^{2} / 4 \pi\right]}
\end{gather*}
$$

Charge renormalization is simpler than in Eq. (1.16) because gauge invariance implies ${ }^{9}$ ) $Z_{1}=Z_{2}$ :

$$
\begin{equation*}
\alpha_{R}=Z_{3} \alpha_{B} \tag{2.23}
\end{equation*}
$$

It follows that the combination $\alpha d\left(\mathrm{q}^{2} / \mathrm{m}^{2}, \alpha\right)$ has no wave-function renormalization, so its CS equation has no $\gamma$-function or subtraction terms and the leading asymptoxic power $\alpha d^{\text {as. }}\left(\eta^{2} q^{2} / m^{2}, \alpha\right)$ satisfies the equation*)

$$
\begin{equation*}
[\eta \partial / \partial \eta-\alpha \beta(\alpha) \partial / \partial \alpha]\left(\alpha d^{\alpha s .}\right)^{-1}=0 \tag{2.24}
\end{equation*}
$$

with

$$
\begin{equation*}
\beta(\alpha)=-\Lambda \partial / \partial \Lambda \ln Z_{3}\left(\Lambda / m_{R}, \alpha_{B}\right) \tag{2.25}
\end{equation*}
$$

If the perturbative expansions

$$
\begin{align*}
{\left[d^{\alpha s}(\eta, \alpha)\right]^{-1} } & =\sum_{n=0}^{\dddot{ }} \alpha^{n} a_{n}(\eta) \quad, \quad\left(a_{0}(\eta) \equiv 1\right) \\
\beta(\alpha) & =\sum_{n=1}^{\dddot{m}} \alpha^{n} b_{n} \tag{2.26}
\end{align*}
$$

[^7]are substituted in Eq. (2.24), the result is a set of equations
\[

$$
\begin{equation*}
\eta \partial / \partial \eta a_{n+1}(\eta)=\sum_{r=0}^{n} a_{r}(\eta)(r-1) b_{n-r+1} \quad, \quad(n=0,1,2, \ldots) \tag{2.27}
\end{equation*}
$$

\]

which imply that a two-loop calculation ${ }^{29}$ ) of $\beta(\alpha)\left(b_{1}=2 / 3 \pi, b_{2}=1 / 2 \pi^{2}\right)$ is sufficient to determine the leading logarithm in any order:

$$
\begin{align*}
& a_{1}(\eta)=-b_{1} \ln \eta+a_{1}(\eta) \\
& a_{n}(\eta)=-b_{2} b_{1}^{n-2}(\ln \eta)^{n-1} /(n-1)+0\left[(\ln \eta)^{n-2}\right] \quad,(n \geqslant 2) \tag{2.28}
\end{align*}
$$

Examination of the cut-off dependence of photon self-energy diagrams shows that the $n$-loop graphs of Fig. 8 (plus suitable counterterm graphs) are responsible for the $(\ln n)^{n-1}$ dependence of $a_{n}(n)$ for $\left.n \geq 2 *\right)$.

The next problem is to decide how to sum the logarithms in Eq. (2.21). The leading-logarithm approximation is very easy to analyse, but it usually fails. For example, Eq. (2.28) yields

$$
\begin{equation*}
\left(\alpha d^{a s .}\right)^{-1}=\{\text { leading logs }\}+\left\{1^{s t} \text { non-leading }\right\}+\ldots \tag{2.29}
\end{equation*}
$$

with

$$
\begin{align*}
\{\text { leading logs }\} & =\alpha^{-1}\{1-(2 \alpha / 3 \pi) \ln \eta\}  \tag{2.30}\\
\left\{1^{\text {st }} \text { non-leading }\right\} & =a_{1}(1)+(3 / 4 \pi) \ln \{1-(2 \alpha / 3 \pi) \ln \eta\} \tag{2.31}
\end{align*}
$$

Equation (2.30) is unsatisfactory as an asymptotic approximation for ( $\left.\alpha \mathrm{d}^{\text {as. }}\right)^{-1}$ because it implies that $\alpha \mathrm{d}^{\text {as. }}$ has a "ghost" pole ${ }^{30}$ ) (i.e. a pole at a space-1ike momentum point). This is not a genuine difficulty, because the "non-leading" term (2.31) dominates (2.30) in the relevant region ( $\ln \eta \sim 3 \pi / 2 \alpha$ ). In fact, the leading-logarithm approximation works only if $n$ is much smaller:

$$
\begin{equation*}
1-(2 \alpha / 3 \pi) \ln \eta>\alpha \tag{2.32}
\end{equation*}
$$

As $\eta$ increases, more and more non-leading logarithms [i.e. higher orders in the expansion of $\beta(\alpha)]$ become important. When $n$ becomes sufficiently large, the perturbation expansion for $\beta(\alpha)$ cannot be truncated at finite order, and all of the logarithms in (2.21) must be summed.

[^8]Further progress depends on the following assumptions:
I) The theory exists at finite values of the coupling constant $g$, where $\beta(g)$, $\gamma_{G}(g), G(g), G^{\text {as. }}(\mathrm{g}), \ldots$, are differentiable functions of $g$. Perturbation theory is generated by asymptotically expanding these functions about $g=0$.
II) At finite $g$, the leading power of $G$ is $G^{\text {as. }}$, as in perturbation theory. Very little is known about the validity of (I) in non-trivial theories; for example:
i) Does the radius of convergence of the perturbation series vanish ${ }^{31)}$ ?
ii) If so, does unitarity (or something equally respectable) uniquely specify the continuation from $g=0$ to finite $g$ ?
iii) Are inequivalent theories generated by different orders of summation ${ }^{3}$ ) [e.g. of diagrams, or of terms in (2.21)]?

In general, the status of assumption (II) is equally uncertain. However, for asymptotically free theories ${ }^{8}$ ), it is not difficult to show ${ }^{7}$ ) that (II) works, provided that the validity of (I) is assumed.

### 2.2 Characteristics of solutions

Consider Eq. (2.17) when subtraction terms are absent and there is only one coupling constant g :

$$
\begin{equation*}
\left[\eta \partial / \partial \eta-\beta(g) \partial / \partial g-\gamma_{G}(g)\right] G^{a s}(\eta q ; g, m)=0 \tag{2.33}
\end{equation*}
$$

The general solution of (2.33) is most conveniently written ${ }^{1 \theta}$ )

$$
\begin{equation*}
G^{\text {as. }}(\eta q ; g, m)=G^{\text {as. }}(q ; \bar{g}, m) \exp \int_{g}^{\bar{g}} d x \gamma_{G}(x) / \beta(x) \tag{2.34}
\end{equation*}
$$

where the auxiliary function $\bar{g}=\bar{g}(g, \eta)$ (known as the "effective coupling constant") is defined by

$$
\begin{equation*}
\int_{g}^{\bar{g}} d x / \beta(x)=\ln \eta \quad ; \quad \bar{g}(g, 1)=g \tag{2.35}
\end{equation*}
$$

The $\eta$-dependence of $G^{\text {as. }}(\eta q)$ is entirely contained in the $\bar{g}$-dependence of the right-hand side of Eq. (2.34). For consistency, we must suppose that the integral $\int d x / \beta(x)$ diverges at least once to $+\infty$ and once to $-\infty$. Otherwise, Eq. (2.35) does not permit the variable $\eta$ to run freely from 0 (the infrared limit) to $\infty$ (the ultraviolet limit).

If the divergence in $\int d x / \beta(x)$ is caused by the range of integration becoming infinite,

$$
\begin{equation*}
\left|\int_{g}^{ \pm \infty} d x / \beta(x)\right|=\infty \tag{2.36}
\end{equation*}
$$

then Eq. (2.35) implies that $\vec{g}$ diverges as $\ln \eta$ becomes infinite. There is no good argument for supposing that this does not happen in real life. Nevertheless, people usually do not bother with this case because of its lack of predictive power: additional assumptions for the $g \rightarrow \infty$ behaviour of the functions $\mathrm{G}^{\text {as. }}(\mathrm{q} ; \mathrm{g}, \mathrm{m})$ must be introduced in order to determine their asymptotic behaviour from Eq. (2.34).

Alternatively, $\beta(x)$ may possess "eigenvalues" or "fixed points" $g_{\infty}$,

$$
\begin{equation*}
\beta\left(g_{\infty}\right)=0 \tag{2.37}
\end{equation*}
$$

which make $\int d x / \beta(x)$ diverge; (e.g. the origin $x=0$ is a fixed point). According to Eq. (2.35) , $\bar{g}$ approaches the nearest fixed point $g_{\infty}$ as $\ln \eta$ tend s to $+\infty$ (ultraviolet-stable fixed point) or to $-\infty$ (infrared-stable fixed point) ${ }^{33}$ ); see Fig. 9. A series of fixed points $g_{\infty}^{\prime}, g_{\infty}^{\prime \prime}$, ... (as in Fig. 10), produces independent regions I, II, III, ..., in coupling-constant space. The effective coupling constant is restricted to the region in which the "physical" coupling constant $g$ lies*). For example, if $g$ lies within region III in Fig. 10 , the relevant IR- and UV-stable fixed points are $g_{\infty}^{\prime \prime}$ and $g_{\infty}^{\prime \prime \prime}$ :

$$
\begin{equation*}
g_{\infty}^{\prime \prime}=\lim _{\eta \rightarrow 0} \bar{g}<\bar{g}(g, \eta)<g_{\infty}^{\prime \prime \prime}=\lim _{\eta \rightarrow \infty} \bar{g} \tag{2.38}
\end{equation*}
$$

Generally, IR-stable fixed points are less interesting because the correction ( $G$ - $G^{\text {as. }}$ ) from mass insertions is expected to dominate $G^{\text {as. }}$ when $\eta$ becomes too small.

It is convenient to discuss the asymptotic properties of Eq. (2.34) for the case $\bar{g} \rightarrow g_{\infty}$ in terms of the function ${ }^{34},{ }^{35}$ )

$$
\begin{equation*}
\varepsilon(\eta)=(\ln \eta)^{-1} \int_{g}^{\bar{g}} d x\left\{\gamma_{G}(x)-\gamma_{G}\left(g_{\infty}\right)\right\} / \beta(x) \tag{2,39}
\end{equation*}
$$

which is closely related to the exponential factor in (2.34):

$$
\begin{equation*}
\exp \int_{g}^{\bar{g}} d x \gamma_{G}(x) / \beta(x)=\eta^{\gamma_{G}\left(g_{\infty}\right)+\varepsilon(\eta)} \tag{2.40}
\end{equation*}
$$

A change of integration variable $x \rightarrow v$ in Eq. (2.39), with

$$
x=\bar{g}\left(g, \eta^{v}\right) \quad, \quad d x / \beta(x)=\ln \eta d v
$$

yields the formula

$$
\begin{equation*}
\varepsilon(\eta)=\int_{0}^{1} d v\left\{\gamma_{G}\left[\bar{g}\left(g, \eta^{v}\right)\right]-\gamma_{G}\left(g_{\infty}\right)\right\} \tag{2.41}
\end{equation*}
$$

*) If $g$ is exactly equal to $g_{\infty}$, Eq. (2.33) can be solved directly:

$$
G^{\text {as. }}\left(\eta q ; g_{\infty}, m\right)=\eta^{\gamma_{G}\left(g_{0 \infty}\right)} G^{\text {as. }}\left(q ; g_{\infty}, m\right)
$$

From Eq. (2.41), we conclude ${ }^{34}$ )

$$
\begin{equation*}
\varepsilon(\eta) \rightarrow 0 \quad, \quad(|\ln \eta| \rightarrow \infty) \tag{2.42}
\end{equation*}
$$

because $\bar{g}\left(g, \eta^{v}\right)$ tends to $g_{\infty}$ for $v \neq 0$, and $\gamma_{G}(x)$ is bounded and continuous [according to assumption (I)].

The asymptotic behaviour of $G^{\text {as. }}(\mathrm{nq})$ is obtained by expanding $\overline{\mathrm{g}}$ about $\mathrm{g}_{\infty}$ in Eq. (2.34). The leading term

$$
\begin{equation*}
G^{a s .}(\eta q ; g, m) \sim \eta^{\gamma_{G}\left(g_{\infty}\right)+\varepsilon(\eta)} G^{a s}\left(q ; g_{\infty}, m\right) \tag{2.43}
\end{equation*}
$$

is almost entirely determined by properties of the theory at the fixed point ${ }^{3-5},{ }^{33}$ ). All dependence on the region $g \leq x<g_{\infty}$ is contained in the factor $\eta^{\varepsilon(n)}$, which controls the over-all normalization of the leading term. If the region $x \approx g_{\infty}$ generates a singularity of the integral in Eq. (2.39), this factor also modifies the leading power $\eta^{\gamma_{G}\left(g_{\infty}\right)}$ in Eq. (2.43): e.g.,

$$
\begin{equation*}
\eta^{\varepsilon(\eta)} \sim \eta^{c /(\ln \eta)^{p}} \quad \text { or } \quad \eta^{c /(\ln |\ln . \eta|)^{p}} \tag{2.44}
\end{equation*}
$$

The precise form of the modification depends on ${ }^{34}$ ):
i) the strength of the singularity in $\left\{\gamma_{G}(x)-\gamma_{G}\left(g_{\infty}\right)\right\} / \beta(x)$ as $x$ tends to $g_{\infty}$;
ii) the rate at which $\bar{g}$ approaches $g_{\infty}$. This is controlled by the strength of the zero in $\beta(x)$ at $x=g_{\infty}$; [see Eq. (2.35)]. For example ${ }^{32}$ ), if $g_{\infty}$ is an infinite-order zero of $\beta(x), \bar{g}$ approaches $g_{\infty}$ very slowly, e.g.

$$
\bar{g}-g_{\infty}=O\left[(\ln |\ln \eta|)^{-p}\right] \quad, \quad(p>0)
$$

compared with the rate at which $\overline{\mathrm{g}}$ tends to a simple zero:

$$
\begin{equation*}
\bar{g}-g_{\infty}=0\left[\eta^{\beta^{\prime}\left(g_{\infty}\right)}\right] \quad, \quad\left(\beta^{\prime}\left(g_{\infty}\right) \neq 0\right) \tag{2.45}
\end{equation*}
$$

## Note

a) If there are several coupling constants $\vec{g}$ as in Eq. (1.37), the effective coupling constant $\vec{g}$ is a vector in coupling-constant space defined by the equations

$$
\eta \partial / \partial \eta \overrightarrow{\vec{g}}(\vec{g}, \eta)=\vec{\beta}(\vec{g}) \quad ; \quad \overrightarrow{\bar{g}}(\vec{g}, 1)=\vec{g}
$$

The exponent in Eq. (2.34) becomes

$$
(\ln \eta)^{-1} \int_{0}^{1} d v \gamma_{G}\left[\overrightarrow{\vec{g}}\left(\vec{g}, \eta^{v}\right)\right]
$$

and the condition for a fixed point $\vec{g}_{\infty}$ is

$$
\vec{\beta}\left(\vec{g}_{\infty}\right)=0
$$

A fixed point is UV-stable if it attracts $\vec{g}$ from $a Z Z$ directions in $\vec{g}$-space as $\eta$ tends to infinity.
b) The simplest example of a subtractively renormalized amplitude is the twopoint function

$$
\begin{equation*}
\int d^{4} x e^{i q \cdot x} T\langle 0| J_{\mu}(x) J_{v}(0)|0\rangle^{1 P I}=\left(q_{\mu} q_{\nu}-g_{\mu \nu} q^{2}\right) \pi\left(q^{2} / m^{2}, g\right) \tag{2.46}
\end{equation*}
$$

whose absorptive part is measured in the inclusive process $\mathrm{e}^{+} \mathrm{e}^{-} \rightarrow$ hadrons. According to Eq. (1.49), there is no g-dependent $\gamma$-function, so the analogue of (2.33) is

$$
\begin{equation*}
[\eta \partial / \partial \eta-\beta(g) \partial / \partial g] \pi^{\text {as. }}\left(\eta^{2} q^{2} / m^{2}, g\right)+K(q)=0 \tag{2.47}
\end{equation*}
$$

with solution

$$
\begin{equation*}
\pi^{a s}\left(\eta^{2} q^{2} / m^{2}, g\right)=\pi^{a s}\left(q^{2} / m^{2}, \bar{g}\right)-\int_{g}^{\bar{g}} d x K(x) / \beta(x) \tag{2.48}
\end{equation*}
$$

### 2.3 Broken scale invariance

Now we consider the possibility that Green's functions are asymptotically scale-invariant ${ }^{4}$ ). This means that the leading term (2.43) should be proportional to a pure power $\eta^{\gamma}{ }_{G}\left(g_{\infty}\right)$ in the UV limit $\eta \rightarrow \infty$.

To ensure that corrections of the type (2.44) are absent, we have to assume that the function $\varepsilon(\eta)$ tends to zero sufficiently rapidly:

$$
\begin{equation*}
\varepsilon(\eta)=O\left[(\ln \eta)^{-1}\right] \tag{2.49}
\end{equation*}
$$

Referring to Eq. (2.39), we see that the condition for asymptotic scale invariance is ${ }^{34}$ )

$$
\begin{equation*}
I_{G}=\int_{g}^{g_{\infty}} d x\left\{\gamma_{G}(x)-\gamma_{G}\left(g_{\infty}\right)\right\} / \beta(x)=\text { convergent } \tag{2.50}
\end{equation*}
$$

Equation (2.50) is supposed to be valid for all Green's functions $G$, so this looks like a very strong assumption. However, if $g_{\infty}$ is a simple zero of $\beta(x)$,

$$
\begin{equation*}
\beta\left(g_{\infty}\right)=0, \quad \beta^{\prime}\left(g_{\infty}\right)<0, \quad\left(g_{\infty}=\text { UV-stable }\right) \tag{2.51}
\end{equation*}
$$

(2.50) is automatically satisfied because assumption (I) says that $\gamma_{G}(x)$ is diffferentiable*).

[^9]The asymptotic expansion of Eq. (2.34) can now be specified more precisely than in Eqs. $(2.42)$ and $(2.43)^{5,18)}$ :

$$
\begin{align*}
G^{\text {as. }}(\eta q ; g, m)= & \eta^{\gamma_{G}\left(g_{\infty}\right)} \exp I_{G} G^{\text {as. }}\left(q ; g_{\infty}, m\right) \\
& +0\left[\eta^{\left.\left\{\gamma_{G}\left(g_{\infty}\right)-\left|\beta^{\prime}\left(g_{\infty}\right)\right|\right\}\right]} \quad,(\eta \rightarrow \infty)\right. \tag{2.52}
\end{align*}
$$

Equation (2.52) indicates that non-leading terms decrease as a power $0\left[\eta^{-\left|\beta^{\prime}\left(g_{\infty}\right)\right|}\right]$ relative to the leading term. This result is a consequence of Eq. (2.45) and the fact that the most important non-leading contributions arise from $O\left(\bar{g}-g_{\infty}\right)$ terms in the expansions of $G^{\text {as. }}(q ; \bar{g}, m)$ and the exponential factor in Eq. (2.34).

According to assumption (II), contributions to the asymptotic behaviour of $G(n q)$ due to mass insertions also decrease as a power $0\left[\eta^{-p(\Delta)}\right](p(\Delta)>0)$ relfive to the leading term in the expansion of $\mathrm{G}^{\text {as. }}(\mathrm{nq})$. Equation (2.52) and assumption (II) imply the result

$$
\begin{align*}
G(\eta q ; g, m) & =\eta^{\gamma_{G}\left(g_{\infty}\right)}\left\{1+0\left(\eta^{-p}\right)\right\} \exp I_{G} G^{a s}\left(q ; g_{\infty}, m\right) \\
p & =\operatorname{Min}\left\{p(\Delta),\left|\beta^{\prime}\left(g_{\infty}\right)\right|\right\} \tag{2.53}
\end{align*}
$$

Now suppose that $G$ is the complete Green's function $G_{A B C}$.... where A, B, C, ..., are renormalized field operators (simple or composite). Equation (2.53) is not directly applicable because of subtractive renormalization (1.41) induced by the time-ordering operation in

$$
\begin{align*}
& G_{A B C} \quad \delta^{4}\left(q_{1}+q_{2}+\ldots\right) \\
& \quad=\int d^{4} x_{1} d^{4} x_{2} \ldots \exp i\left(q_{1} \cdot x_{1}+q_{2} \cdot x_{2}+\ldots\right) T\langle 0| A\left(x_{1}\right) B\left(x_{2}\right) C\left(x_{3}\right) \ldots|0\rangle \tag{2.54}
\end{align*}
$$

However, this problem can be trivially circumvented by considering the unordered product instead. According to the addition rule (1.46) and assumption (I), the characteristic power $\gamma_{G}\left(g_{\infty}\right)$ in Eq. (2.53) is given by the rule

$$
\begin{equation*}
\gamma_{G}\left(g_{\infty}\right)-\gamma_{G}(0)=\gamma_{A}\left(g_{\infty}\right)+\gamma_{B}\left(g_{\infty}\right)+\ldots \tag{2.55}
\end{equation*}
$$

Thus each operator $Q$ contributes $\left.\eta_{Q} \mathcal{S}_{\infty}\right)$ to the asymptotic behaviour of $G$. In coordinate space, the result is

$$
\begin{align*}
& \langle 0| A\left(\rho x_{1}\right) B\left(\rho x_{2}\right) C\left(\rho x_{3}\right) \ldots|0\rangle \\
& \quad \sim \rho^{-\left(d_{A}+d_{B}+d_{c}+\ldots\right)} f\left(x_{1}, x_{2}, x_{3}, \ldots\right) \quad, \quad\left(\rho=\eta^{-1} \rightarrow 0\right) \tag{2.56}
\end{align*}
$$

where

$$
\begin{equation*}
d_{Q}=\{\text { canonical dimension of } Q\}+\gamma_{Q}\left(g_{\infty}\right) \tag{2.57}
\end{equation*}
$$

is the dynamical dimension ${ }^{4}$, ${ }^{3}$ ) of the operator $Q$. Equation (2.56) says that dynamical dimension is an additively conserved quantity at short distances. This rule, abstracted from the preceding field-theoretic machinery, is the basis for Wilson's theory of broken scale invariance ${ }^{4}$ ).

If $Q$ is a conserved operator, the "anomalous" term $\gamma_{Q}\left(g_{\infty}\right)$ vanishes because of Eq. (1.49): for example,

$$
\begin{equation*}
d_{J_{\mu}}=3, \quad d_{\theta_{\mu \nu}}=4 \tag{2.58}
\end{equation*}
$$

In particular, if $J_{\mu}$ is the electromagnetic current for hadrons, the two-point function $\langle 0| J_{\mu} J_{V}|0\rangle$ obeys the familiar condition

$$
\begin{gather*}
\langle 0| J_{\mu}(x) J_{\nu}(0)|0\rangle \sim\left(R / 12 \pi^{4}\right)\left(g_{\mu \nu} \partial^{2}-\partial_{\mu} \partial_{\nu}\right)\left(x^{2}-i \varepsilon x_{0}\right)^{-2}  \tag{2.59}\\
\left(x_{\alpha} \rightarrow 0\right)
\end{gather*}
$$

so the total cross-section for $\mathrm{e}^{+} \mathrm{e}^{-}+$hadrons is predicted to be asymptotically scale-invariant ${ }^{36}$ ):

$$
\begin{equation*}
\sigma\left(e^{+} e^{-} \rightarrow \text { hadrons }\right) / \sigma^{a s}\left(e^{+} e^{-} \rightarrow \mu^{+} \mu^{-}\right) \longrightarrow R,\left(q^{2} \rightarrow \infty\right) \tag{2.60}
\end{equation*}
$$

Here $q$ is the sum of the $e^{+}$and $e^{-}$momenta, and

$$
\sigma^{a s}\left(e^{+} e^{-}-\mu^{+} \mu^{-}\right)=4 \pi \alpha^{2} / 3 q^{2}
$$

is a convenient normalizing factor. Asymptotic corrections to Eq. (2.60) are $O\left[\left(q^{2}\right)^{-P / 2}\right]$, where $P$ is the power defined in Eq. (2.53). [In fact, Eqs. (2.59) and (2.60) are valid for any UV-stable fixed point $g_{\infty}$; if $g_{\infty}$ is not a simple zero of $\beta(x)$, the corrections to (2.60) decrease logarithmically.]

A practical difficulty of this theory is that we do not know how to compute quantities like $R$, because they are characteristic of the non-trivial interacting theory at $g=g_{\infty}$.

### 2.4 Asymptotic freedom

A theory is asymptotically free ${ }^{8,37)}$ if
i) the fixed point at the origin is UV-stable: for each coupling constant $g_{i}$, the condition

$$
\begin{equation*}
g_{i} \beta_{i}\left(g_{1}, g_{2}, \ldots\right)<0,(i=1,2, \ldots) \tag{2.61}
\end{equation*}
$$

is satisfied in the neighbourhood of the origin $\vec{g}=0$.
ii) the value of $\vec{g}$ is chosen such that, as $\eta$ increases from 1 to $\infty$, $\vec{g}$ describes a path linking $\vec{g}$ with the origin:

$$
\begin{equation*}
\vec{g} \longrightarrow 0 \quad, \quad(\eta \longrightarrow \infty) \tag{2.62}
\end{equation*}
$$

Equation (2.62) is not an automatic consequence of (2.61) because it assumes the absence of barriers such as non-trivial fixed points which can prevent $\overrightarrow{\mathrm{g}}$ from reaching points close to the origin. The important feature of these theories is that most of the terms in Eq. (2.43) can be computed explicitly. In particular, $G^{\text {as. }}\left(q ; g_{\infty}, m\right)$ and $\gamma_{G}\left(g_{\infty}\right)$ are trivially given by free-field theory because the relevant fixed point $g_{\infty}$ vanishes.

Checking Eq. (2.61) is just a matter of computing one-loop*) contributions to $B(g)$ in perturbation theory. Most theories do not obey Eq. (2.61): in QED, the coefficient $b_{1}$ of the one-loop term in (2.26) is positive ( $b_{1}=+2 / 3 \pi$ ), and more generally ${ }^{38}$ ), a four-dimensional renormalizable field theory cannot be asymptotically free unless non-Abelian gauge mesons are present. Furthermore, Eq. (2.61) is satisfied only by special classes of gauge theories. The simplest and most interesting case ${ }^{8},{ }^{39}$ ) involves massless gauge fields $A_{\mu}^{a}$ interacting with themselves and with fermions $\psi^{i}$ belonging to a representation $R$ of the gauge group G. For applications to strong interactions ${ }^{1,8}, 39$ ) the fermions are supposed to be current quarks ${ }^{40}$ ) distinguished by properties of "colour" and "flavour":

$$
\left\{\psi^{i}\right\}=\left(\begin{array}{ccccc}
u_{1} & d_{1} & s_{1} & c_{1} & \ldots . .  \tag{2.63}\\
u_{2} & d_{2} & \ldots & \cdots & \\
\vdots & & & &
\end{array}\right)
$$

The flavours u, d, s, c, ... (= up, down, strange, charmed, ... quarks) are gauge-invariant and transform under observable symmetry groups $G_{\text {obs }}$ such as chiral $\operatorname{SU}(3) \times \operatorname{SU}(3)$. Each flavour carries a colour index K which is transformed by $G$ but not by $G_{o b s}\left(e . g . ~ u \rightarrow u_{K}, K=1,2, \ldots\right.$ ). For example $e^{40}$ ), if $K=1,2,3$ refers to the fundamental representation 3 of $G=S U(3)$, the complete fermionic representation $R$ is a direct sum:

$$
\begin{equation*}
R=\underline{3}_{u}+\underline{3}_{d}+\underline{3}_{s}+\ldots \tag{2.64}
\end{equation*}
$$

The Feynman rules of the theory involve a coupling constant $g$, the structure constants $c^{a b c}$ of the gauge group $G$, and the matrix generators $\tau^{a}$ of $G$ for the representation $R$. If necessary, the fermions can be provided with a gaugeinvariant mass matrix $\mathcal{M}$ in flavour space to ensure that symmetries like chiral $\operatorname{SU}(3) \times \operatorname{SU}(3)$ are softly broken. Propagators and vertices are generated by the Lagrangian**)

$$
\begin{equation*}
\mathscr{L}=-\frac{1}{4} F^{2}+\bar{\psi}\left(i \not \wp_{f}-\mathcal{M}\right) \psi+\mathscr{L}_{g . f .}+\mathscr{L}_{\text {ghost }} \tag{2.65}
\end{equation*}
$$

*) Two-loop, if the one-1oop terms accidentally vanish.
**) For background, consult review articles ${ }^{41-43}$ ) on the quantization and renormalization of gauge theories.
where

$$
\begin{equation*}
F_{\mu \nu}^{a}=\partial_{\mu} A_{\nu}^{a}-\partial_{\nu} A_{\mu}^{a}+g c^{a b c} A_{\mu}^{b} A_{\nu}^{c} \tag{2.66}
\end{equation*}
$$

is the gauge-covariant field-strength tensor, and

$$
\begin{equation*}
\left(D_{f}\right)_{\mu}=\partial_{\mu}-i g A_{\mu}^{a} \tau^{a} \tag{2.67}
\end{equation*}
$$

is the gauge-covariant derivative for the fermionic representation $R$. The terms $F^{2}, \bar{\psi} \rrbracket_{f} \psi$ and $\bar{\psi} \mathcal{M} \psi$ are invariant under infinitesimal gauge transformations

$$
\begin{align*}
\delta A_{\mu}^{a} & =-D_{\mu}^{a b} \delta \omega^{b}+O\left(\delta \omega^{2}\right) \\
\delta \psi & =-i g \delta \omega^{a} \tau^{a} \psi+O\left(\delta \omega^{2}\right) \tag{2.68}
\end{align*}
$$

where

$$
\begin{equation*}
D_{\mu}^{a b}=\partial_{\mu} \delta^{a b}-g c^{a b c} A_{\mu}^{c} \tag{2.69}
\end{equation*}
$$

is the covariant derivative for the adjoint representation of $G$, and $\delta \omega^{a}(x)$ are arbitrary non-singular functions of the coordinate $x$. The term $\mathscr{L}_{\mathrm{g}, \mathrm{f}}$ in Eq. (2.65) specifies a gauge for which the propagator of $A_{\mu}^{a}$ is well defined; e.g. a Fermitype gauge-fixing term

$$
\begin{equation*}
\mathscr{L}_{\text {g.f. }}=-\frac{1}{2 \xi}\left(\partial^{\mu} A_{\mu}^{a}\right)^{2} \tag{2.70}
\end{equation*}
$$

generates the propagator

$$
\begin{equation*}
{\stackrel{\Gamma}{A_{\mu}^{a}} A_{\nu}^{b}}_{b}=-\frac{i \delta^{a b}}{k^{2}}\left\{g_{\mu v}-(1-\xi) k_{\mu} k_{v} / k^{2}\right\} \tag{2.71}
\end{equation*}
$$

Unphysical contributions of $\mathscr{L}_{\text {g.f }}$ to loop integrals must be cancelled by including a ghost Lagrangian $\left.\left.n^{44}, 4\right)^{\frac{g}{5}}\right)^{f} \mathscr{L}_{\text {ghost }}$ in Eq. (2.65). The choice of $\mathscr{L}_{\text {g.f }}$ in Eq. (2.70) produces a ghost Lagrangian

$$
\begin{equation*}
\mathscr{L}_{\text {ghost }}=\left(\partial^{\mu} \phi^{a}\right)^{*} D_{\mu}^{a b}(A) \phi^{b} \tag{2.72}
\end{equation*}
$$

where the ghost field $\phi^{a}$ is a Lorentz scalar with Fermi-Dirac statistics. Equations (2.65) to (2.72) refer to the unrenormalized (B) representation.

One way of computing coupling-constant renormalization is to compare $O\left(A^{2}\right)$ and $O\left(A^{3}\right)$ terms in $\mathscr{P}$. Wave-function renormalization

$$
\begin{equation*}
\left(A_{\mu}^{a}\right)_{R}=Z_{3}^{-1 / 2}\left(A_{\mu}^{a}\right)_{B} \tag{2.73}
\end{equation*}
$$

and a rescaling of the gauge parameter*)

$$
\begin{equation*}
\xi_{R}=Z_{3}^{-1} \xi_{B} \tag{2.74}
\end{equation*}
$$

[^10]account for the $O\left(A^{2}\right)$ terms. Three-meson 1PI divergences produce a $Z_{1}-$ factor [similar to that in Eq. (1.10)]
\[

$$
\begin{equation*}
-\frac{1}{2} g_{B} c^{a b c}\left(\partial^{\mu} A^{\nu}-\partial^{\nu} A^{\mu}\right)_{B}^{a}\left(A_{\mu}^{b} A_{\nu}^{c}\right)_{B}=-\frac{1}{2} g_{R} Z_{1} c^{a b c}\left(\partial^{\mu} A^{\nu}-\partial^{\nu} A^{\mu}\right)_{R}^{a}\left(A_{\mu}^{b} A_{\nu}^{c}\right)_{R} \tag{2.75}
\end{equation*}
$$

\]

so coupling-constant renormalization is given by

$$
\begin{equation*}
g_{R}=z_{3}^{3 / 2} z_{1}^{-1} g_{B} \tag{2.76}
\end{equation*}
$$

Note
Instead of (2.75), we could have considered one of the other g-dependent vertices in $\mathscr{L}: O\left(A^{4}\right), O(\bar{\psi} A \psi)$, or $O\left(\phi^{*} A \phi\right)$. For example, if $\tilde{z}_{1}, \tilde{z}_{3}$ are the renormalization factors for ghost amplitudes,

$$
\begin{align*}
& \phi_{R}=\widetilde{Z}_{3}^{-1 / 2} \phi_{B} \\
& g_{B}^{a b c}\left(\partial^{\mu} \phi^{a}\right)_{B}^{*}\left(A_{\mu}^{b} \phi^{c}\right)_{B}=g_{R} \widetilde{Z}_{1} c^{a b c}\left(\partial^{\mu} \phi^{a}\right)_{R}^{*}\left(A_{\mu}^{b} \phi^{c}\right)_{R} \\
& \text { the result is } \\
& g_{R}=\widetilde{Z}_{1}^{-1} Z_{3}^{1 / 2} \widetilde{Z}_{3} g_{B} \tag{2.78}
\end{align*}
$$

As long as the regularization procedure ${ }^{46}, 47$ ) is chosen such that gauge Ward identities ${ }^{43,48-50)}$ are satisfied, the consistency of equations such as (2.76) and (2.78) is assured ${ }^{48}$ ):

$$
\begin{equation*}
z_{1}\left|\tilde{z}_{1}=z_{3}\right| \tilde{z}_{3} \tag{2.79}
\end{equation*}
$$

A computation of the one-loop contributions to $g_{R} / g_{B}$ yields (after a bit of algebra) the basic result ${ }^{8}, 39$ )

$$
\begin{align*}
\beta(g) & =-b g^{3}+O\left(g^{5}\right) \\
b & =\left\{\frac{11}{3} C_{2}(G)-\frac{4}{3} T(R)\right\} / 16 \pi^{2} \tag{2.80}
\end{align*}
$$

with

$$
\begin{array}{lll}
C_{2}(G) \delta^{a b}=c^{a c d} c^{b c d} \\
T(R) \delta^{a b}= & & \left(C_{2}(G)>0\right)  \tag{2.82}\\
\operatorname{Tr}\left\{\tau^{a} \tau^{b}\right\} \quad, & (T(R) \geqslant 0)
\end{array}
$$

Hence Eq. (2.61) (b>0) is satisfied if there are not too many fermions in the theory. The rate at which $\overline{\mathrm{g}}$ approaches the origin is a direct consequence of Eqs. (2.35) and (2.80):

$$
\begin{equation*}
\bar{g}^{2}=(2 b \ln \eta)^{-1}+0\left[\ln \ln \eta /(\ln \eta)^{2}\right] \quad,(\eta \rightarrow \infty) \tag{2.83}
\end{equation*}
$$

Most Green's functions $G(n q)$ are not asymptotically scale-invariant because the one-loop term $\mathrm{c}_{\mathrm{G}} \mathrm{g}^{2}$ in the perturbative expansion

$$
\begin{equation*}
\gamma_{G}(g)-\gamma_{G}(0)=c_{G} g^{2}+O\left(g^{4}\right) \tag{2.84}
\end{equation*}
$$

causes the integral

$$
\int d x\left(\gamma_{G}(x)-\gamma_{G}(0)\right) / \beta(x)
$$

in Eq. (2.50) to diverge logarithmically at $x=0$. The rate at which the addtional power $\varepsilon(n)$ [Eq. (2.39)] approaches zero is governed by the equation

$$
\begin{align*}
g_{G} & \equiv \int_{g}^{0} d x\left[\left(\gamma_{G}(x)-\gamma_{G}(0)\right) / \beta(x)+c_{G} / b x\right] \\
& =\lim _{\eta \rightarrow \infty}\left[\varepsilon(\eta) \ln \eta+\left(c_{G} / b\right) \ln (\bar{g} / g)\right] \tag{2.85}
\end{align*}
$$

i.e. $\varepsilon(n)$ decreases more slowly than $(\ln n)^{-1}$ :

$$
\varepsilon(\eta) \sim\left(c_{G} / 2 b\right) \ln \ln \eta / \ln \eta
$$

This means that the leading singularity of $G^{\text {as. }}(n q)$ differs from the singularity ${ }_{\eta} \gamma_{G}(0)$ observed in free-field theory; there exist logarithmic modifications of the form $\left.{ }^{8}, 39\right)(1 \mathrm{n} \eta)^{\mathrm{C}_{\mathrm{G}} / 2 \mathrm{~b}}$ :

$$
\begin{equation*}
G^{\text {as. }}(\eta q ; g, m)=\eta^{\gamma_{G}(0)}\left(2 b g^{2} \ln \eta\right)^{c_{G} / 2 b} G^{\text {as. }}(q ; 0, m) \exp \oiint_{G}\left\{1+0\left(\frac{\ln \ln \eta}{\ln \eta}\right)\right\} \tag{2.86}
\end{equation*}
$$

This is a non-perturbative result. It depends on the validity of assumption (I), because the factor $\exp \mathscr{I}_{G}$ depends on the values of $\beta(x)$ and $\gamma_{G}(x)$ in the region $0<x<g$. Also, a $(\ln \eta)^{c_{G} / 2 b}$ singularity cannot be generated by finite-order perturbation theory for $G(n q)$, because in general, the power $c_{G} / 2 b$ is not an integer.

Of course, if there is no wave-function renormalization for $G$ [ie. $\left.\gamma_{G}(x) \equiv \gamma_{G}(0)\right]$, the leading singularity is the same as in free-field theory (parton model). In particular, the ratio

$$
\begin{align*}
R\left(q^{2} / m^{2} ; g\right) & =\sigma\left(e^{+} e^{-} \rightarrow \text { hadrons }\right) / \sigma^{\text {ass }}\left(e^{+} e^{-} \rightarrow \mu^{+} \mu^{-}\right)  \tag{2.87}\\
& =R^{\text {as. }}\left(q^{2} / m^{2} ; q\right)+\{\text { non-leading powers }\}
\end{align*}
$$

tends to the asymptotic value ${ }^{8}, 39$ )

$$
\begin{equation*}
R=\sum_{\substack{\text { flavours } \\ \text { colours }}} Q^{2}, \quad(Q=\text { quark charge }) \tag{2.88}
\end{equation*}
$$

It is not difficult to find the leading asymptotic correction to this result. The two-loop contribution $R^{\text {as. }}\left(\mathrm{q}^{2} / \mathrm{m}^{2} ; g\right)$ is $3 \mathrm{~g}^{2} \mathrm{~T}(\mathrm{R}) / 16 \pi^{2}$ relative to the one-loop contribution $\sum Q^{2}$; [the factor $3 g^{2} / 16 \pi^{2}$ can be extracted from the Jost-Luttinger calculation ${ }^{29}$ ) for QED]. So, if the right-hand side of the equation

$$
\begin{equation*}
R^{a s .}\left(\eta^{2} q^{2} / m^{2} ; g\right)=R^{a s}\left(q^{2} / m^{2} ; \bar{g}\right) \tag{2.89}
\end{equation*}
$$

is expanded in $\bar{g}$ about $\bar{g}=0$, the result is ${ }^{51}$ )

$$
R^{\text {as. }}\left(\eta^{2} q^{2} / m^{2} ; g\right)=\sum Q^{2}\left\{1+3 T(R) \bar{g}^{2} / 16 \pi^{2}+0\left(\tilde{g}^{4}\right)\right\},(\eta \rightarrow \infty) \cdot(2.90)
$$

Thus the first non-leading term is positive and decays logarithmically. This result illustrates the difference between asymptotic freedom and asymptotic freefield behaviour; in free-field theory, the leading correction is negative and approaches zero very rapidly: $O\left(\eta^{-4}\right)$.

Asymptotic freedom allows us to make very precise statements about amplitudes which involve a short-distance limit. Unfortunately the reverse is true elsewhere, especially for the problem of deciding whether the theory possesses a respectable S-matrix or not. Most Yang-Mills theories [e.g. Eq. (2.65)] do not possess a perturbative S-matrix because of infrared singularities at thresholds for the production of massless gauge mesons, and in any case, we do not want massless states to appear in the hadronic spectrum. The perturbative solution to this problem is to break the gauge symmetry spontaneously, i.e. to introduce enough Higgs-Kibble ${ }^{52 \text { ) }}$ scalar fields $\Phi$ to make all of the gauge mesons massive. The presence of $\Phi^{4}$ couplings means that Eq. (2.61) is much more difficult to satisfy ${ }^{39,53 \text { ). }}$ Nevertheless, asymptotically free theories with scalar mesons do exist ${ }^{54}$ ) and in particular, there are models ${ }^{55}$ ) in which all perturbative states are massive. An immediate reason for not pursuing this line further is that it involves an unrealistic perturbative constraint. The spectrum of a summed-up theory is unlikely to bear much resemblance to the perturbative spectrum*), and hadronic states are definitely not perturbative.

Another approach is to assume that there is a dynamical mechanism which breaks gauge invariance spontaneously ${ }^{58}$ ). There are no $\Phi$ fields; instead, the infrared singularities of perturbation theory are supposed to sum to scalar meson poles which simulate the Higgs-Kibble mechanism in a non-perturbative manner. Dynamical generation of mass from Lagrangians which contain no dimensionful parameters has been exhibited ${ }^{59}$ ) for four-dimensional scalar $Q E D$ and some two-dimensional models.

However, the most likely possibility is that the strong interaction gauge symnetry (colour) is exact ${ }^{39}, 60$ ). One would break gauge invariance spontaneously only if it were desirable to produce a hadronic spectrum with lots of non-degenerate coloured states, including states with quark-like quantum numbers. Note that

[^11]isospin and ordinary $\operatorname{SU}(3)$ have nothing to do with breaking a strong gauge group, because they are approximate degeneracy symmetries. Similarly, observable hadronic currents $j_{\mu}$ (electromagnetic, chiral, etc.) are colour-invariant; otherwise, instead of current conservation or partial conservation, we would have
\[

$$
\begin{equation*}
D^{a b} \cdot j^{b}=\text { zero or soft operator } \tag{2.91}
\end{equation*}
$$

\]

where $D^{a b}$ is the appropriate covariant derivative (2.67) with strong coupling constant g .

So we return to the colour-flavour model mentioned at the beginning of this section. All flavours, observable operators and observable states are colour singlets. Perturbative amplitudes become infrared singular either as some of the external momenta are taken on-she11, or as a partial sum of the external momenta becomes light-1ike. We are better off with the singularities because then we can at least contemplate the possibility ${ }^{60}$ ) that these infrared effects, summed to all orders in perturbation theory, confine quarks, gluons and coloured "bound states" (constituent quarks, etc.). The precise criterion for this is poorly understood, but there are some hopeful signs:
i) The Bloch-Nordsieck solution ${ }^{61}$ ) of the infrared problem in QED, which results in the existence of observable photon and fermion states, fails when applied to Yang-Mills theories ${ }^{62,63)}$.
ii) The lattice approximation ${ }^{64}$ ) for the colour-flavour model indicates quark confinement.

It is surprising that perturbative infrared singularities in gauge-invariant channels receive so little attention, because there we do have a precise criterion: observable amplitudes must not be infrared singular. In other words, when summed to all orders, the singularities should conspire to shift all gauge-invariant thresholds away from zero mass. Gauge-invariant singularities occur even if the fermions in Eq. (2.65) are all massive $(\mathcal{H} \neq 0)$. For example, consider the stressenergy tensor

$$
\begin{align*}
& \theta_{\mu \nu}=-F_{\mu \alpha} F_{v}^{\cdot \alpha}+g_{\mu \nu} F^{2} / 4  \tag{2.92}\\
&\left.-g_{\mu \nu} \bar{\psi}\left(i \not D_{f}-\mathcal{M}\right) \psi+\gamma_{\mu} \stackrel{\rightharpoonup}{D}_{v}^{f}+\gamma_{\nu} \stackrel{\leftrightarrow}{D_{\mu}^{f}}\right) \psi \\
&+\{\text { renormalization counterterms }\}
\end{align*}
$$

The diagram shown in Fig. 11a is responsible for the singularity

$$
\begin{array}{r}
i \int d^{4} x e^{i q \cdot x} T\langle 0| \theta_{\alpha \beta}|x| \theta_{\gamma \delta}(0)|0\rangle \sim-\left(d(G) / 60 \pi^{2}\right) q_{\alpha} q_{\beta} q_{\gamma} q_{\delta} \ln q^{2}  \tag{2.93}\\
+0\left(q^{2}\right) \quad,\left(q^{2} \rightarrow 0\right)
\end{array}
$$

where $d(G)$ is the number of generators of the gauge group G. More generally, amplitudes $\mathrm{T}\langle 0| \mathrm{O}_{1} \mathrm{O}_{2} \mathrm{O}_{3} \ldots|0\rangle$ involving gauge-invariant operators $\mathrm{O}_{\mathrm{i}}$ can be
infrared singular whenever the diagrams of Fig. 11b (intermediate gluon or ghostantighost pair) are permitted by selection rules. Of course, additional singularities appear if some of the fermions are massless.

Note that observable amplitudes must be regular (not merely non-singular) at light-like momentum transfers. For example, if $Q$ is the gauge-invariant operator

$$
\begin{equation*}
Q(x)=\varepsilon_{\alpha \beta \gamma \delta} F^{\alpha \beta} F^{\gamma \delta}+\{\text { renormalization counterterms }\} \tag{2.94}
\end{equation*}
$$

there is a non-singular branch cut at $q^{2}=0$ in lowest-order perturbation theory for the two-point function

$$
\begin{align*}
i \int d^{4} x e^{i q \cdot x} T\langle 0| Q(x) Q(0)|0\rangle \sim & -\left(2 d(G) / \pi^{2}\right) q^{4} \ln q^{2}  \tag{2.95}\\
& +0\left(g^{2}\right),\left(q^{2} \rightarrow 0\right)
\end{align*}
$$

The theory makes sense only if branch cuts like $q^{4} \ln q^{2}$ disappear when the sum to all orders in perturbation theory is performed. This means that a complete analysis of the infrared problem involves non-singular graphs with more than two intermediate gluons or ghosts (Fig. 11c), as well as the diagrams of Fig. 11b.

The non-perturbative nature of the problem can be readily appreciated by considering the extreme case $\mathcal{M}=0$ in (2.65), as advocated by Gross and Neveu ${ }^{59}$ ) [but without breaking the colour symmetry*)]. The only dimensional parameter in the theory is the renormalization subtraction point $\lambda$. Amplitudes obey equations such as (2.33) and (2.48) exactly (with $\lambda$ replacing $m$ ), so the infrared behaviour for the limit in which all external momenta approach zero can be discussed in terms of a non-trivial infrared-stable fixed point $g_{\infty}$ (possibly at infinity). For example, consider Eq. (2.48), (which is also valid for form factors of $\left.T\left\langle\theta_{\alpha \beta} \theta_{\gamma \delta}\right\rangle\right)$. The criterion is that

$$
\begin{equation*}
\lim _{\eta \rightarrow 0} \pi\left(\eta^{2} q^{2} / \lambda^{2} ; g\right)_{\mathcal{M}=0}=\pi\left(q^{2} / \lambda^{2} ; g_{\infty}\right)_{\mathcal{M}=0}-\int_{g}^{g_{\infty}} d x \frac{K(x)}{\beta(x)} \tag{2.96}
\end{equation*}
$$

should be finite with zero absorptive part, so $K(x)$ must approach zero sufficiently strongly as $x \rightarrow g_{\infty}$ to make $\int^{g \infty} d x K(x) / B(x)$ converge.

The case $\mathcal{M} \neq 0$ is more complicated. Apart from the lattice approximation, the main lines of analysis will probably involve:
i) theorems ${ }^{63,65 \text { ) that infrared singularities of amplitudes are given by the }}$ theory with massive fields omitted (except as external lines), up to finite renormalizations;
ii) the improved CS equation ${ }^{7)}$, to be considered in the next chapter.

[^12]3. OTHER METHODS
3.1 Gel1-Mann -- Low analysis ${ }^{3}$ )

Let us return to the subject of coupling-constant renormalization in quantum electrodynamics.

The coupling constant $\alpha$ used in practical calculations ( $\alpha^{-1}=137.036, \ldots$ ) refers to the amplitude for an electron to emit a zero-frequency photon. According to Eq. (2.23), charge renormalization is controlled by the wave-function renormalization of the photon propagator

$$
\begin{equation*}
D_{F_{\mu \nu}}^{\prime}(q)=-\left(\frac{i}{q^{2}}\right)\left(g_{\mu \nu}-q_{\mu} q_{\nu} / q^{2}\right) d\left(q^{2} / m^{2}, \alpha\right)-i \xi q_{\mu} q_{\nu} / q^{4} \tag{3.1}
\end{equation*}
$$

so the choice of $\alpha$ as coupling constant corresponds to the normalization condition

$$
\begin{equation*}
d(0, \alpha)=1 \tag{3.2}
\end{equation*}
$$

for $D_{F \mu \nu}^{\prime}(q)$ at $q^{2}=0$.
Consider another renormalization procedure in which the coupling constant $\alpha_{\lambda}$ is given by the amplitude for the emission of a virtual photon with space-like momentum $\lambda$. Now the photon propagator $\left(D_{F \mu \nu}^{\prime}\right) \lambda_{\lambda}$ depends on $q, \lambda, m$, and $\alpha_{\lambda}$,

$$
\begin{equation*}
\alpha_{\lambda}\left(D_{F_{\mu \nu}}^{\prime}\right)_{\lambda}=-\left(\frac{i}{q^{2}}\right)\left(g_{\mu \nu}-q_{\mu} q_{\nu} / q^{2}\right) D\left(q^{2} / \lambda^{2}, m^{2} / \lambda^{2}, \alpha_{\lambda}\right)-i \alpha_{\lambda} \xi_{\lambda} q_{\mu} q_{\nu} / q^{4} \tag{3.3}
\end{equation*}
$$

and is normalized at $q^{2}=\lambda^{2}$ :

$$
\begin{equation*}
D\left(1, m^{2} / \lambda^{2}, \alpha_{\lambda}\right)=\alpha_{\lambda} \tag{3.4}
\end{equation*}
$$

When the renormalization prescription is altered by changing $\lambda$, the normalization of $\left(D_{F \mu \nu}^{\prime}\right) \lambda$ changes by a factor $3_{3}^{-1}$ [as in Eq. (1.21)], but the combination $\alpha_{\lambda}\left(D_{F \mu \nu}^{\prime}\right) \lambda$ is invariant:

$$
\begin{align*}
D\left(q^{2} / \lambda_{1}^{2}, m^{2} / \lambda_{1}^{2}, \alpha_{\lambda_{1}}\right) & =D\left(q^{2} / \lambda_{2}^{2}, m^{2} / \lambda_{2}^{2}, \alpha_{\lambda_{2}}\right)  \tag{3.5}\\
& =\alpha d\left(q^{2} / m^{2}, \alpha\right) \tag{3.6}
\end{align*}
$$

For $q^{2}=\lambda_{2}^{2}$, Eqs. (3.5) and (3.6) become

$$
\begin{align*}
\alpha_{\lambda \sqrt{\rho}} & =D\left(\rho, m^{2} / \lambda^{2}, \alpha_{\lambda}\right)  \tag{3.7}\\
& =\propto d\left(\rho \lambda^{2} / m^{2}, \alpha\right) \tag{3.8}
\end{align*}
$$

where $\rho$ is a convenient scale parameter:

$$
\begin{equation*}
\lambda_{2}=\lambda_{1} \sqrt{\rho} \quad, \quad \lambda_{1}=\lambda \tag{3.9}
\end{equation*}
$$

The result of substituting (3.7) into (3.5) is a functional equation for the renormalization-group transformation*) $\lambda \rightarrow \lambda \sqrt{\rho}$ :

$$
\begin{equation*}
D\left(q^{2} / \lambda^{2}, m^{2} / \lambda^{2}, \alpha_{\lambda}\right)=D\left(q^{2} / \rho \lambda^{2}, m^{2} / \rho \lambda^{2}, D\left(\rho, m^{2} / \lambda^{2}, \alpha_{\lambda}\right)\right) \tag{3.10}
\end{equation*}
$$

Application of $\partial / \partial \rho$ to Eq. (3.10) at $\rho=1$ produces the renormalization-group equation

$$
\begin{equation*}
\left[X \partial / \partial X+M \partial / \partial M-\psi\left(\alpha_{\lambda}, M\right) \partial / \partial \alpha_{\lambda}\right] D\left(X, M, \alpha_{\lambda}\right)=0 \tag{3.11}
\end{equation*}
$$

with $X=q^{2} / \lambda^{2}, M=m^{2} / \lambda^{2}$, and

$$
\begin{equation*}
\psi\left(\alpha_{\lambda}, M\right)=\left[\partial D\left(\rho, M, \alpha_{\lambda}\right) / \partial \rho\right]_{\rho=1} \tag{3.12}
\end{equation*}
$$

In general, Eq. (3.11) cannot be integrated because $\psi$ depends on two variables. However, GML ${ }^{3}$ ) observed that all renormalized Feynman integrals remain convergent when $M$ is set equal to zero:

$$
\begin{equation*}
m=0, \quad q^{2} \neq 0 \quad \lambda^{2} \neq 0 \tag{3.13}
\end{equation*}
$$

For example, the subtracted integral

$$
\begin{equation*}
I(q, \lambda, m)=\int d^{4} p\left(p^{2}-m^{2}\right)^{-1}\left\{\left[(p+q)^{2}-m^{2}\right]^{-1}-\left[(p+\lambda)^{2}-m^{2}\right]^{-1}\right\} \tag{3.14}
\end{equation*}
$$

remains both infrared and ultraviolet convergent when the condition (3.13) is imposed. For the general case, ultraviolet power counting is not affected by setting $m$ to zero, but infrared convergence has to be checked, either by inspection ${ }^{3}$ ) or by appealing to Kinoshita's theorem ${ }^{1}, 67,68$ ). The result is an integrable equation of the same form as Eq. (2.33),

$$
\begin{equation*}
\left[x \partial / \partial X-\psi\left(\alpha_{\lambda}\right) \partial / \partial \alpha_{\lambda}\right] D\left(X, 0, \alpha_{\lambda}\right)=0 \tag{3.15}
\end{equation*}
$$

where

$$
\begin{equation*}
\psi(z)=\psi(z, 0) \tag{3.16}
\end{equation*}
$$

is known as the Gell-Mann/Low function.
Instead of analysing $D\left(x, 0, \alpha_{\lambda}\right)$ further, let us make use of the connection (3.6)-(3.8) between $D\left(X, M, \alpha_{\lambda}\right)$ and the conventionally renormalized theory. Thus Eq. (3.12) can be rewritten

$$
\begin{align*}
\psi\left(\alpha d\left(\lambda^{2} / m^{2}, \alpha\right)\right. & \left., m^{2} / \lambda^{2}\right)=\left[(\partial / \partial \rho) \alpha d\left(\rho \lambda^{2} / m^{2}, \alpha\right)\right]_{\rho=1}  \tag{3.17}\\
= & \partial / \partial \ln \left(-\lambda^{2}\right)\left[\alpha d\left(\lambda^{2} / m^{2}, \alpha\right)\right]
\end{align*}
$$

Order-by-order in perturbation theory, the leading power in $\mathrm{m}^{2}$ at $\mathrm{m}=0$ can be isolated by substituting the expansions

[^13]\[

$$
\begin{align*}
& \psi\left(z, m^{2} / \lambda^{2}\right)=\Psi(z)+O\left[m^{2}\{\operatorname{logs} \text { of } m\}\right]  \tag{3.18}\\
& \alpha d\left(\lambda^{2} / m^{2}, \alpha\right)=\alpha d^{a s}\left(\lambda^{2} / m^{2}, \alpha\right)+O\left[m^{2}\{\log s \text { of } m\}\right] \tag{3.19}
\end{align*}
$$
\]

Equation (3.18) depends on the existence of the $M \rightarrow 0$ limit (3.13), while (3.19) is just a special case of (2.16); [i.e, $\alpha \mathrm{d}^{\text {as }}$ is the amplitude which appears in Eqs. (2.24) and (2.26)]. The result is an integrable equation

$$
\begin{equation*}
\psi\left(\alpha d^{a s .}\left(\lambda^{2} / m^{2}, \alpha\right)\right)=\partial / \partial \ln \left(-\lambda^{2}\right)\left[\alpha d^{a s .}\left(\lambda^{2} / m^{2}, \alpha\right)\right] \tag{3.20}
\end{equation*}
$$

with solution

$$
\begin{equation*}
\ln \left(-\lambda^{2} / m^{2}\right)=\int_{\alpha d^{a s .}(-1, \alpha)}^{\alpha d^{a s .}\left(\lambda^{2} / m^{2}, \alpha\right)} \frac{d z}{\psi(z)} \tag{3.21}
\end{equation*}
$$

The analysis of the GML equation (3.21) is very similar to that of Eq. (2.35). The sign of the first term in the perturbative expansion ${ }^{3}$,69)

$$
\psi(z)=z^{2} / 3 \pi+z^{3} / 4 \pi^{2}+z^{4}\left(\zeta(3)-\frac{101}{96}\right) / 3 \pi^{3}+O\left(z^{5}\right)(3.22)
$$

implies that $z=0$ is an infrared-stable fixed point. If there is an ultravioletstable fixed point $\alpha_{0}$,

$$
\begin{equation*}
\psi\left(\alpha_{0}\right)=0 \quad, \quad\left(\alpha<\alpha_{0}\right) \tag{3.23}
\end{equation*}
$$

it follows from (3.21) that $\alpha_{0}$ is the asymptotic charge of the photon propagator at short distances:

$$
\begin{equation*}
\alpha d^{a s}\left(\lambda^{2} / m^{2}, \alpha\right) \longrightarrow \alpha_{0} \quad,\left(-\lambda^{2} \rightarrow \infty\right) \tag{3.24}
\end{equation*}
$$

If $\psi(z)$ does not vanish for $z>0$, we must suppose that $\int d z / \psi(z)$ diverges at $z=\infty$ :

$$
\begin{equation*}
\alpha d^{\text {as. }}\left(\lambda^{2} / m^{2}, \alpha\right) \longrightarrow \infty \quad,\left(-\lambda^{2} \rightarrow \infty\right) \tag{3.25}
\end{equation*}
$$

Otherwise the theory is inconsistent because of the presence of a tachyonic singrarity in $\alpha d^{\text {as. }}\left(\lambda^{2} / \mathrm{m}^{2}, \alpha\right)$ at

$$
\begin{equation*}
\left(\lambda^{2}\right)_{\text {tach. }}=-m^{2} \exp \int_{q(\alpha)}^{\infty} d z / \psi(z) \tag{3.26}
\end{equation*}
$$

[where $\mathrm{q}(\alpha)$ is shorthand for $\alpha \mathrm{d}^{\text {as. }}(-1, \alpha)$ ]. It is now obvious why the "ghost" (tachyon) problem ${ }^{30}$ ) discussed previously [(Eq. (2.30)] is not a genuine diffficulty: it is not reasonable to assume that the small-z behaviour $z^{2} / 3 \pi$ is a good approximation for $\psi(z)$ at large $z$.

The connection between $\psi(z)$ and the $\beta$-function (2.25) is found by applying the CS differential operator

$$
\left[2 \partial / \partial \ln \left(-\lambda^{2}\right)-\alpha \beta(\alpha) \partial / \partial \alpha\right]
$$

to Eq. $\left.(3.21)^{32}\right)$ :

$$
\begin{align*}
\psi(q(\alpha)) & =\frac{1}{2} \alpha \beta(\alpha) d q(\alpha) / d \alpha \\
q(\alpha) & =\alpha d^{a s .}(-1, \alpha) \tag{3.27}
\end{align*}
$$

For conventional on-shell renormalization, the perturbative expansions of $q(\alpha)$ and $B(\alpha)$ are $\left.^{3}, 29\right)$

$$
\begin{align*}
& q(\alpha)=\alpha-5 \alpha^{2} / 9 \pi+\left(\zeta(3)+\frac{65}{648}\right) \alpha^{3} / \pi^{2}+O\left(\alpha^{4}\right) \\
& \beta(\alpha)=2 \alpha / 3 \pi+\alpha^{2} / 2 \pi^{2}-121 \alpha^{3} / 144 \pi^{3}+O\left(\alpha^{4}\right) \tag{3.28}
\end{align*}
$$

A feature of the GML formulation is that $\psi(z)$ is a universal function. Unlike $\beta(z)$, it does not depend on the renormalization prescription $\lambda$. Adler's proposal ${ }^{32)}$ for computing the fine-structure constant

$$
\begin{equation*}
\beta\left(\alpha_{\infty}\right)=0 \quad, \quad\left(\alpha_{\infty}^{-1}=137.036 \ldots ? ?\right) \tag{3.29}
\end{equation*}
$$

illustrates this point. Equation (3.29) involves the $\beta$-function for on-shell renormalization. This makes sense because the fine-structure constant is the onshell coupling constant. A different renormalization procedure would produce a different $\beta$-function $\left[0\left(\alpha^{3}\right)\right.$ terms and higher] with an eigenvalue numerically different from $\alpha_{\infty}$. On the other hand, the corresponding asymptotic charge

$$
\begin{equation*}
\alpha_{0}=q\left(\alpha_{\infty}\right) \tag{3.30}
\end{equation*}
$$

is renormalization-group invariant. It does not refer to a particular point in $\lambda$-space, so apart from the constraint $\alpha<\alpha_{0}$, its value has nothing to do with the value of the fine-structure constant.

To obtain renormalization-group equations for multiplicatively renormalized amplitudes $G$, one can use the fact that the ratio

$$
G\left(p^{2} / \lambda^{2}, m^{2} / \lambda^{2}, \alpha_{\lambda}\right) / G\left(q^{2} / \lambda^{2}, m^{2} / \lambda^{2}, \alpha_{\lambda}\right)
$$

is $\lambda$-independent for arbitrary sets of external momenta $p$ and $q$. For example, the analogue of Eq. (3.11) is

$$
\begin{gather*}
{\left[x \partial / \partial x+M \partial / \partial M-\psi\left(\alpha_{\lambda}, M\right) \partial / \partial \alpha_{\lambda}-x\left(\alpha_{\lambda}, M\right)\right] G\left(x, M, \alpha_{\lambda}\right)=0} \\
x\left(\alpha_{\lambda}, M\right)=-\left[(\partial / \partial \rho) \ln G\left(\rho^{-1}, M / \rho, D\left(\rho, M, \alpha_{\lambda}\right)\right)\right]_{\rho=1} \tag{3.32}
\end{gather*}
$$

The analysis of the $M=0$ limit is the same as that of Eq. (2.33), so $X\left(\alpha_{0}, 0\right)$ is the power of $X$ which characterizes the large-X behaviour of $G$ :

$$
\begin{equation*}
\text { anomalous dimension }=\gamma_{G}\left(g_{\infty}\right)-\gamma_{G}(0)=\frac{1}{2}\left[X\left(\alpha_{0}, 0\right)-X(0,0)\right] \tag{3.33}
\end{equation*}
$$

Of course, we expect anomalous dimension to be an invariant of the renormalization group. This can be checked from Eq. (1.38) and by considering transformations on Eq. (3.32).

More complicated cases of coupling-constant renormalization can be handled by finding the appropriate generalization of the invariant function $D$ in Eq. (3.3). For example, consider the invariant

$$
\begin{equation*}
\bar{D}=\left[G^{(2)}\left(p_{1}\right) G^{(2)}\left(p_{2}\right) G^{(2)}\left(p_{3}\right) G^{(2)}\left(p_{4}\right)\right]^{1 / 2} \Gamma^{(4)}\left(p_{1}, p_{2}, p_{3}, p_{4}\right) \tag{3.34}
\end{equation*}
$$

with $G^{(2)}=$ boson propagator, $\Gamma^{(4)}=4-1$ eg lPI vertex, $p_{i}=$ external momenta, $\sum p=0$. In analogy with (3.4), the coupling constant $g_{\lambda}$ for the $\phi^{4}$ interaction is given by $\vec{D}$ at the subtraction point $p_{i}=\lambda_{i}$. The $M \rightarrow 0$ limit can be taken if $p_{i}$ and $\lambda_{i}$ are not exceptional. The resulting $\psi(g)$ function is universal with respect to transformations which scale all of the $\lambda_{i}$ together. In gauge theories, one vertex (e.g. 3-gluon) can be used to define $\bar{D}$ and $g_{\lambda}$, and then gauge Ward identities fix the normalization of the other vertices (4-gluon, ghost-gluon, etc.).

The GML approach involves the same assumptions [(I) and (II) in the discussion after (2.32)] as the CS method, so the choice of CS or GML method (or variation thereof) is mostly a matter of convenience; [e.g. the GML method is ideal for two-dimensional $Q E D^{70}$ )].

### 3.2 Improved CS equation

According to assumption (II), non-leading powers in the asymptotic expansion of a perturbative amplitude do not compete with the leading power when summed to all orders. These non-leading powers reflect the presence of masses (or other dimensionful coupling constants) in the Lagrangian $\mathscr{L}$. The rule ${ }^{4}, 7,33,71$ ) for estimating these terms involves the maximal dynamical dimension $d_{\Delta}$ of the mass operators of $\mathscr{L}$; e.g.

$$
d_{\Delta}=\left\{\begin{array}{lll}
3+\gamma_{\Delta}\left(g_{\infty}\right) & , & (\Delta=\bar{\psi} \psi)  \tag{3.35}\\
2+\gamma_{\Delta}\left(g_{\infty}\right) & , & \left(\Delta=\phi^{2} / 2\right)
\end{array}\right.
$$

where $\gamma_{\Delta}$ is the $\gamma$-function ( 1.46 ) for the composite operator $\Delta$. It states that the $N^{\text {th }}$ set of non-leading powers of a perturbative amplitude $G(n q)$ sums to an amplitude which is*)

$$
\begin{equation*}
O\left[\eta^{-N\left(4-d_{\Delta}\right)}\{\text { logs of } \eta\}\right], \quad(N=1,2, \ldots . \text { not too large }) \tag{3.36}
\end{equation*}
$$

relative to the summed-up leading power $G^{\text {as. }}(n q)$. Thus we must assume ${ }^{4}$ )

$$
\begin{equation*}
d_{\Delta}<4 \tag{3.37}
\end{equation*}
$$

in order to ensure the validity of assumption (II). Equation (3.37) is automatically satisfied ${ }^{7}$ ) if the theory is asymptotically free because the relevant anomalous dimension $\gamma_{\Delta}(0)$ vanishes.

Wilson introduced Eq. (3.36) as one of the rules of his theory of broken scale invariance ${ }^{4}$ ) and later justified it ${ }^{* *)}$ within the context of the renormalization group by applying a variant of the GML analysis due to Eriksson ${ }^{72)}$. The

[^14]conventional GML procedure is to renormalize the mass on-shell; the renormalized mass is taken to be the position of the pole in the propagator and is regarded as being independent of the subtraction point $\lambda$. [This accounts for the absence of a nontrivial coefficient for the operator $M \partial / \partial M$ in Eq. (3.11) and (3.31).] Ericsson's method involves a mass parameter $m_{\lambda}$ defined in terms of the propagator at the subtraction point $\lambda$. In bare outline, what happens is that a change of $\lambda$ induces a mass renormalization
\[

$$
\begin{equation*}
m_{\lambda^{\prime}}=3_{m}\left(\lambda^{\prime}, \lambda\right) m_{\lambda} \tag{3.38}
\end{equation*}
$$

\]

as well as coupling-constant and wave-function renormalization. Because of the factor $3_{m}$, there is a function $\left(m_{\lambda}^{2} / \lambda^{2}\right) \chi_{m}\left(\alpha_{\lambda}, m_{\lambda}^{2} / \lambda^{2}\right)$ associated with $\lambda \partial / \partial \lambda\left(m_{\lambda}^{2} / \lambda^{2}\right)$ in the same way that $\psi\left(\alpha_{\lambda}, m_{\lambda}^{2} / \lambda^{2}\right)$ is related to $\lambda \partial / \partial \lambda\left(\alpha_{\lambda}\right)$ [as in Eqs. (3.7) and (3.12), except that the change in renormalization prescription produces a diffferent two-variable $\psi$-function]. The variable $\mathrm{m}_{\lambda}^{2} / \lambda^{2}$ acts like a coupling constank in the renormalization-group equations. Thus the anomalous dimension of $\Delta$, $\chi_{\mathrm{m}}\left(\alpha_{0}, 0\right) / 2\left[\right.$ or $\chi_{\mathrm{m}}\left(\alpha_{0}, 0\right)$ for scalars $]$, appears when the theory is expanded about the fixed point $\left(\alpha_{0}, 0\right)$ in ( $\alpha_{\lambda}, m_{\lambda}^{2} / \lambda^{2}$ ) space. Equation (3.37) is the condition for this fixed point to be UV-stable in the $m_{\lambda}^{2} / \lambda^{2}$ direction.

The result (3.36) was rediscovered by Weinberg ${ }^{7}$ ) as a direct consequence of his improved CS equation


Equation (3.39) is characteristic of "mass-independent" renormalization prescriptions in which there is a renormalization parameter $\mu$ but the $\beta$ and $\gamma$ functions do not depend on $m_{R} / \mu$. Instead of a mass-insertion term, there is an ordinary derivative $m_{R} \partial / \partial m_{R}$ multiplied by a $\gamma$ function $\gamma_{m}\left(g_{R}\right)$ associated with mass renormalization. This means that Eq. (3.39) can be directly integrated without taking a zero mass limit.

The main step in the derivation of (3.39) is to find a suitable renormalization prescription. One possibility ${ }^{7}$ ) is to multiply $g_{B}, m_{B}$, and $G_{B}$ by $Z$-factors given by setting $m_{B}=0$ in the corresponding factors $Z\left(g_{B}, \Lambda / \mu, m_{B} / \mu\right)$ which relate bare and renormalized quantities in the GML prescription. Here, $\mu$ is the GML reference momentum $\lambda$ and $Z_{m}=m_{R} / m_{B}$ is the $Z$-factor for the composite operator $\bar{\psi} \psi$. Assuming that cutoff independent renormalized amplitudes are produce d ${ }^{7}{ }^{73}$ ), this prescription is obviously mass-independent:

$$
\begin{align*}
\beta\left(g_{R}\right) & =\lim _{\Lambda \rightarrow \infty}(\mu \partial / \partial \mu) g_{R}\left(g_{B}, \Lambda / \mu\right) \\
\gamma_{m}\left(g_{R}\right) & =\lim _{\Lambda \rightarrow \infty}(\mu \partial / \partial \mu) \ln Z_{m}\left(g_{B}, \Lambda / \mu\right)  \tag{3.40}\\
\gamma\left(g_{R}\right) & =\lim _{\Lambda \rightarrow \infty}(\mu \partial / \partial \mu) \ln Z_{G}\left(g_{B}, \Lambda / \mu\right)
\end{align*}
$$

Equation (3.39) is an immediate consequence of combining the formulas

$$
\begin{gather*}
G_{R}=Z_{G} G_{B} \\
(\mu \partial / \partial \mu) G_{B}\left(\text { fixed } q, g_{B}, m_{B}\right)=0 \tag{3.41}
\end{gather*}
$$

Weinberg also observed that all of the properties needed in the derivation of Eq. (3.39) are contained in 't Hoof's analysis ${ }^{74}$ ) of the renormalization group for dimensionally renormalized ${ }^{46}, 75-77$ ) amplitudes. This line was subsequently studied in detail by many authors ${ }^{78}$ ). Briefly, the analysis depends on the following theorem ${ }^{74,79)}$ : in the dimensional renormalization scheme, counterterm vertices are polynomials in the renormalized mass parameter $m_{R}$ as well as in the external momenta $q^{*}$ ). Thus $Z$-factors depend only on the unity of mass ${ }^{74}$ ) $\mu$ (which accompanies each loop integral $\mu^{4-n} \int d^{n} p$ ), the complex parameter $n$, and $g_{B}$, which has dimensionality $(4-n) k$ (with $k=1 / 2$ for gauge theories, $k=1$ for $\phi^{4}$ coupling):

$$
\begin{equation*}
Z=Z\left(g_{B} \mu^{(n-4) \kappa}, n\right) \tag{3.42}
\end{equation*}
$$

Once again, Eq. (3.39) follows directly from (3.41) with ${ }^{78}$ )

$$
\begin{align*}
& \beta\left(g_{R}\right)=\lim _{n \rightarrow 4}(\mu \partial / \partial \mu) g_{R}\left(g_{B} \mu^{(n-4) k}, n\right) \\
& \gamma_{m}\left(g_{R}\right)=\lim _{n \rightarrow 4}(\mu \partial / \partial \mu) \ln Z_{m}\left(g_{B} \mu^{(n-4) k}, n\right)  \tag{3.43}\\
& \gamma\left(g_{R}\right)=\lim _{n \rightarrow 4}(\mu \partial / \partial \mu) \ln Z_{G}\left(g_{B} \mu^{(n-4) k}, n\right)
\end{align*}
$$

The analogue of Eq. $(1.35)$ is

$$
\begin{equation*}
\left[\eta \partial / \partial \eta+m_{R} \partial / \partial m_{R}+\mu \partial / \partial \mu-d_{G}\right] G_{R}\left(\eta q ; g_{R}, m_{R}, \mu\right)=0 \tag{3.44}
\end{equation*}
$$

where the integer $d_{G}$ is the mass dimensionality of $G$, so if $d_{G}$ is absorbed into the definition of the $\gamma$ function

$$
\begin{equation*}
\gamma_{G}\left(g_{R}\right)=d_{G}+\gamma\left(g_{R}\right) \tag{3.45}
\end{equation*}
$$

Eq. (3.39) can be written

$$
\begin{equation*}
\left[\eta \partial / \partial \eta-\beta\left(g_{R}\right) \partial \partial \partial g_{R}+\left\{1-\gamma_{m}\left(g_{R}\right)\right\} m_{R} \partial / \partial m_{R}-\gamma_{G}\left(g_{R}\right)\right] G_{R}\left(\eta q ; g_{R}, m_{R}, \mu\right)=0 \tag{3.46}
\end{equation*}
$$ The general solution of Eq. (3.46) is ${ }^{7}$ )

$$
\begin{equation*}
G_{R}\left(\eta q ; g_{R}, m_{R}, \mu\right)=G_{R}(q ; \bar{g}, \bar{m}, \mu) \exp \int_{g_{R}}^{\bar{g}} d x \gamma_{G}(x) / \beta(x) \tag{3.47}
\end{equation*}
$$

where, in addition to the effective coupling constant $\bar{g}=\bar{g}\left(g_{R}, \eta\right)$, there is an "effective mass"

$$
\begin{equation*}
\bar{m}=m_{R} \eta^{-1} \exp \int_{g_{R}}^{\bar{g}} d x \gamma_{m}(x) / \beta(x) \tag{3.48}
\end{equation*}
$$

[^15]The $\eta \rightarrow \infty$ limit of Eq. (3.47) is controlled by the asymptotic behaviour of both $\overline{\mathrm{g}}$ and $\overline{\mathrm{m}}$ *):

$$
\begin{align*}
\bar{m} & =m_{R} \eta_{m}^{\gamma_{m}\left(g_{\infty}\right)-1} \exp \int_{g_{R}}^{\bar{g}} d x\left\{\gamma_{m}(x)-\gamma_{m}\left(g_{\infty}\right)\right\} / \beta(x)  \tag{3.49}\\
& \sim m_{R} \eta_{m}^{\gamma_{m}\left(g_{\infty}\right)-1}\left\{\log _{5} \text { of } \eta\right\} \quad, \quad\left(\bar{g} \rightarrow g_{\infty}\right)
\end{align*}
$$

Now, apart from finite renormalization, the massless amplitude $G_{R}\left(q ; g_{R}, 0, \mu\right)$ may be identified with the leading-power amplitudes $G^{\text {as. }}(q ; g, m)$ and $G\left(X, 0, g_{\lambda}\right)$ of the CS and GML analyses. Therefore the condition ${ }^{7}$ )

$$
\begin{equation*}
\bar{m} \longrightarrow 0 \quad, \quad \gamma_{m}\left(g_{\infty}\right)<1 \tag{3.50}
\end{equation*}
$$

is essential if assumption (II) is to be satisfied. Making the identification

$$
d_{\Delta}=\left\{\begin{array}{lll}
3+\gamma_{m}\left(g_{\infty}\right), & (\Delta=\bar{\psi} \psi)  \tag{3.51}\\
2+2 \gamma_{m}\left(g_{\infty}\right), & \left(\Delta=\phi^{2} / 2\right)
\end{array}\right.
$$

we see that Eq. (3.50) is equivalent to Eq. (3.37).
Evidently, the terms $N=1,2, \ldots$, of Eq. (3.36) correspond to increasing powers of $\overline{\mathrm{m}}$ in the Taylor expansion of Eq. (3.47) about $\overline{\mathrm{m}}=0$ :

$$
\begin{equation*}
G_{R}\left(\eta q ; g_{R}, m_{R}, \mu\right) \sim\left\{\exp \int_{g_{R}}^{\vec{g}} d x \gamma_{G}(x) / \beta(x)\right\} \sum_{r=0}^{\cdots} \frac{\bar{m}^{r}}{r!}\left(\frac{\partial}{\partial m}\right)^{r} G_{R}\left(q ; g_{R}, m, \mu\right)_{m=0} \tag{3.52}
\end{equation*}
$$

However, it is necessary to insert the warning "not too large" in (3.36) because in general, the limit

$$
\begin{equation*}
\lim _{m \rightarrow 0}\left(\frac{\partial}{\partial m}\right)^{r} G_{R}\left(q ; g_{R}, m, \mu\right) \quad, \quad(\text { non-exceptional } q) \tag{3.53}
\end{equation*}
$$

does not exist if $r$ is too large ${ }^{7}$ ). If $m_{R}$ is a fermionic mass parameter, trouble first appears when $(\partial / \partial m)^{3}$ acts on a single internal propagator, producing a logerithmic infrared singularity in the limit $m \rightarrow 0$ :

$$
\begin{equation*}
\bar{m}^{3}\left(\frac{\partial}{\partial \bar{m}}\right)^{3} \int^{1} d^{4} p /(\not p-\bar{m})=0\left(\bar{m}^{3} \ln \bar{m}\right) \tag{3.54}
\end{equation*}
$$

Higher derivatives $(r>3)$ do not reduce the power further:

$$
\begin{equation*}
\text { e.g. } \quad \bar{m}^{5}(\partial / \partial \bar{m})^{5} \int^{1} d^{4} p /(\not p-\bar{m})=O\left(\bar{m}^{3}\right) \tag{3.55}
\end{equation*}
$$

For scalar masses $m_{R}$, the $r=2$ term (3.53) is infrared singular.
The restriction of (3.36) to small values of $N$ means that the non-leading terms are not completely under control. In particular, the logarithmic powers $\beta$ in $0\left(\bar{m}^{3} 1 n^{\beta} \bar{m}\right)$ (for fermions) should be further investigated; for example, are there models in which $\beta$ is bounded as the order of perturbation grows? I think

[^16]that $\beta$ is larger than 1 in general, because the coefficient of the term (3.54) in the expansion of $\bar{m}^{3}(\partial / \partial \bar{m})^{3} G_{R}$ is a zero-mass amplitude evaluated at exceptional momenta:
\[

$$
\begin{equation*}
\text { coefficient }=\lim _{m \rightarrow 0} G_{R}(p=0,-p=0 ; q ; \bar{g}, m, \mu) \tag{3.56}
\end{equation*}
$$

\]

Here ( $p,-p$ ) are external momenta for the fermion-antifermion pair obtained by cutting the internal propagator $(p-m)^{-1}$ in (3.54). In particular, the zeromass limit can produce logarithms in the fermionic self-energy which modify the denominator

$$
\begin{equation*}
(\not p-\bar{m})^{-1} \longrightarrow\left(\not p \ln ^{\alpha}\left(p^{2} / \mu^{2}\right)-\bar{m}\right)^{-1} \tag{3.57}
\end{equation*}
$$

and generate extra logarithms of $\overline{\mathrm{m}}$ in Eq. (3.54).
Equation (3.46) is valid for all momenta $q$, so there is no need to restrict its application to ultraviolet limits. Thus the discussion of infrared singlarities at the end of Chapter II can be extended in an obvious way to include the case $\mathcal{H} \neq 0$. For example, the analogue of Eq. (2.96) is
$\lim _{\eta \rightarrow 0} \pi_{R}\left(\eta^{2} q^{2} ; g_{R}, M_{R}, \mu\right)=\pi_{R}\left(q^{2} ; g_{\infty}, M_{\infty}, \mu\right)-\int_{g_{R}}^{g_{\infty}} d x \frac{K(x)}{\beta(x)}$
Unfortunately, the asymptotic value $\mathcal{M}_{\infty}$ of $\overline{\mathcal{M}}(n \approx 0)$ is not known because the IR-stable fixed point $g_{\infty}$ is not at the origin.
4. OPERATOR-PRODUCT EXPANSIONS

In order to apply the renormalization group to topics such as current algebra and deep inelastic electroproduction, it is necessary to consider ultraviolet limits for subsets of the external lines of amplitudes. The appropriate tool is Wilson's expansion for operator products at short distances ${ }^{4}$ ).

### 4.1 Short-distance limit

The short-distance expansion ${ }^{4}, 15,80-83$ )

$$
\begin{equation*}
\prod_{i=1}^{k} A_{i}\left(x+\rho \xi_{i}\right) \sim \sum_{n} \mathscr{G}_{n}\left(\rho \xi_{j}-\rho \xi_{k}\right) O_{n}(x) \quad,\left(\rho \rightarrow 0 \text {, fixed } x, \xi_{i}\right) \tag{4.1}
\end{equation*}
$$

is an asymptotic expansion of the operator product $\Pi A_{i}$ into local operators $O_{n}(x)$ (including the identity operator $I$ ), with $c$-number coefficient functions $\mathcal{C}_{n} d^{-}$ pending on the various coordinate differences $\rho\left(\xi_{j}-\xi_{k}\right)$. The product $\Pi A_{i}$ may be unordered (e.g. multiple commutators), or ordered (T-product, anti-T-product, ...), or some combination thereof. The terms $C_{n} O_{n}$ in the sum $\sum_{n}$ are ordered according to the strength of the singularity of $\mathcal{C}_{n}$ in the scaling variable $\rho^{*)}$ :

$$
\begin{equation*}
\lim _{\rho \rightarrow 0}\left[\left\{\pi A_{i}-\sum_{n=0}^{N} \ddots_{n}(\rho) O_{n}(0)\right\} / \ddots_{N}(\rho)\right]=0,(N=0,1,2, \ldots) \tag{4.2}
\end{equation*}
$$

[^17]The effect of expanding about $x^{\prime}=x-\rho \xi$ instead of $x$ is to change the contributions of derivative operators:

$$
\begin{equation*}
O_{n}\left(x^{\prime}+\rho \xi\right) \sim \sum_{r=0} \frac{\rho^{r}}{r!}\left(\xi \cdot \frac{\partial}{\partial x^{\prime}}\right)^{r} O_{n}\left(x^{\prime}\right) \tag{4.3}
\end{equation*}
$$

In free-field theory, the expression (4.1) is trivially obtained by Taylorexpanding the Wick expansion of $\Pi A_{i}$. For example, consider the Wick expansion

$$
\begin{align*}
& T\left\{\mathcal{F}_{\mu}^{i}(\xi) \mathcal{F}_{\nu}^{j}(0)\right\}=\left\{\bar{\psi}(\xi) \gamma_{\mu}\left(\lambda^{i} / 2\right) \psi(\xi) \bar{\psi}(0) \gamma_{\nu}\left(\lambda^{j} / 2\right) \psi(0)\right\} I \\
& +: \bar{\psi}(\xi) \gamma_{\mu}\left(\lambda^{i} / 2\right) \sqrt[\psi(\xi) \bar{\psi}(0) \gamma_{\nu}\left(\lambda^{j} / 2\right) \psi(0): \quad+: \bar{\psi}(0) \gamma_{\nu}(\lambda \dot{j} / 2) \bar{\psi}(0) \bar{\psi}(\xi) \gamma_{\mu}\left(\lambda^{i} / 2\right) \psi(\xi): ~]{2}  \tag{4.4}\\
& +: \bar{\psi}(\xi) \gamma_{\mu}\left(\lambda^{i} / 2\right) \psi(\xi) \bar{\psi}(0) \gamma_{\nu}\left(\lambda^{j} / 2\right) \psi(0):
\end{align*}
$$

for $\operatorname{SU}(3)$ currents

$$
\begin{equation*}
\mathcal{F}_{\mu}^{i}(x)=: \bar{\psi}(x) \gamma_{\mu}\left(\lambda^{i} / 2\right) \psi(x): \quad, \quad\left(\lambda^{i}=\text { matrices for ordinary } \operatorname{SU}(3)\right) \tag{4.5}
\end{equation*}
$$

with

$$
\begin{align*}
\dot{\psi}(0) \vec{\psi}(\xi) & =i \int \frac{d^{4} q}{(2 \pi)^{4}} e^{i q \cdot \xi}(\not q-m+i \varepsilon)^{-1} \\
& \sim-\frac{i}{2 \pi^{2}} \psi /\left(\xi^{2}-i \varepsilon\right)^{2} \tag{4.6}
\end{align*}
$$

When expanded about $\xi=0$, each term on the right-hand side of (4.4) contributes to the Wilson expansion:

$$
\begin{align*}
& O_{n}=I \quad(\text { first term of }(4.4)) \text {; } \\
& O_{n}=\text { two-fermion operators }=: \bar{\psi}_{(x)}\{\lambda, \gamma \text { matrices }\}\left\{\partial \partial \partial_{\partial} \ldots\right\} \psi(x): \text {, } \\
& \text { (second and third terms of (4.4)); }  \tag{4.7}\\
& O_{n}=\text { four-fermion operators } \\
& =: \bar{\psi}_{(x)}\{\lambda, \gamma\}\left\{\underset{\partial \partial \ldots \partial\}}{\longleftrightarrow} \psi_{(x)} \bar{\psi}_{(x)}\{\lambda, \gamma\}\left\{\partial \partial_{\partial}^{\longleftrightarrow} \partial\right\} \psi(x):,\right. \\
& \text { (last term of (4.4)). }
\end{align*}
$$

The expansion of $\overline{\psi(0)} \bar{\psi}(\xi)$ produces terms of the form $\xi^{p}$ or $\xi^{\mathrm{p}} \ln \left(\xi^{2} \mathrm{~m}^{2}\right)$, so we have ( $p=$ integer)

$$
\begin{equation*}
\wp_{n}(\xi)=\xi_{\alpha} \xi_{\beta} \ldots \xi_{\omega}\left(\xi^{2}\right)^{p}\left\{\text { constant , or constant. } \ln \left(m^{2} \xi^{2}\right)\right\} \tag{4.8}
\end{equation*}
$$

The logarithm is at least $0\left(\mathrm{~m}^{3} \mathrm{ln} \mathrm{m}\right)$ in Eq. (4.6) [compare (3.54)], so it contributes to unimportant non-leading terms in the Wilson expansion.

The expansion (4.1) remains valid in renormalized perturbation theory ${ }^{71}, 81,83,86$ ). The operators $A_{i}, O_{n}$ are provided with counterterms so that they yield finite matrix elements. In general, there are more operators $O_{n}$ than in free-field theory. For example, the expansion of $T\left\{\mathbb{F}_{\mu} T_{\nu}\right\}$ involves six-fermion, eightfermion, ..., operators as well as renormalized versions of the operators in
(4.7). The operators $O_{n}$ can also depend on other fields appearing in the Lagrangian (egg. gluon and ghost fields in gauge theories). The properties ${ }^{71}, 87$ ) of the coefficient functions $C_{n}$ are very similar to those of the asymptotic parts $G^{\text {as. }}$ of renormalized Green's functions; they become coupling-constant dependent and exhibit extra logarithmic singularities compared with Eq. (4.8),

$$
\begin{equation*}
\varrho_{n}(\xi) \sim \xi_{\alpha} \xi_{\beta} \ldots \xi_{\omega}\left(\xi^{2}\right)^{p} \text { Polynomial }\left(\ln \xi^{2}, g_{R}\right) \tag{4.9}
\end{equation*}
$$

or more generally [in analogy with Eq. (2.21)]

$$
\begin{equation*}
\mathscr{C}_{n}\left(\rho \xi_{j}-\rho \xi_{k}\right) \sim \rho^{p} \rho_{o l y n o m i a l}\left(\ln \rho, g_{R}\right) \quad,(p=\text { integer }) \tag{4.10}
\end{equation*}
$$

Since the most interesting applications of Eq. (4.1) are non-perturbative, a proof based on the general principles of axiomatic field theory would be very desirable. The main difficulty to be overcome is that there is no general argomont which forbids the appearance of infinite oscillations in matrix elements of $\Pi A_{i}$ as $\rho$ tends to zero ${ }^{\theta 2}$ ). A rather complete analysis is possible ${ }^{82}$ ) if oscillations are assumed to be absent.

Most practical applications ${ }^{4}, 8^{8}$ ) involve products $\Pi A_{i}$ of observable hadronic currents: the electromagnetic current $J_{\mu}$, the energy-momentum tensor $\theta_{\mu \nu}$, the currents $\mathbb{T}_{\mu}^{i}, \pi_{S \mu}^{i}$ associated with chiral $\operatorname{SU}(3) \times \operatorname{SU}(3)$, and so on. It is a consequence of (4.1) that the corresponding operators $O_{n}$ are also observable. Naturally ${ }^{4}$ ), the expansion respects the conservation and covariance properties of $\Pi A_{i}$ for exact symmetries (Poincaré invariance, isospin, charge conjugation, etc.), so it is often convenient to isolate a sector ( $J, I, \ldots$ ) of the expansion containing operators $O_{n}$ with spin $J$, isospin $I$, etc. As a rule ${ }^{4}$, the leading power of $\rho$ in a given sector also conserves symmetries which are only approximate [e.g. ordinary $S U(3)$, chiral $S U(2) \times S U(2)]$. For example, the leading contribution proportional to $\theta_{\mu \nu}$ in the expansion

$$
\begin{equation*}
\mathcal{F}_{5 \mu}^{i}(x+\xi) \mathcal{F}_{5 \nu}^{j}(x-\xi) \sim \ldots+\oint_{\mu \nu \alpha \beta}^{i j}(\xi) \theta^{\alpha \beta}(x)+\ldots,(\xi \rightarrow 0) \tag{4.11}
\end{equation*}
$$

satisfies the constraints

$$
\begin{gather*}
\mathscr{C}_{\mu \nu \alpha \beta}^{i j}=\delta^{i j} \varphi_{\mu \nu \alpha \beta} \\
\partial^{\mu} \varphi_{\mu \nu \alpha \beta}(\xi)=0=\partial^{\nu} \varphi_{\mu \nu \alpha \beta}(\xi) \tag{4.12}
\end{gather*}
$$

As in the discussion of the asymptotic properties of Green's functions, it is desirable to classify contributions to a given sector by powers of the scaling variable $\rho$, not by logarithms*):

$$
\begin{equation*}
\left[\varphi_{n+1}(\rho) / \varphi_{n}(\rho)\right]_{(J, I, \ldots)}=0\left[\rho^{-p(n)}\right],(p(n)>0) \tag{4.13}
\end{equation*}
$$

[^18]The most important point is that the expansion (4.1) should not depend on the matrix element to which it is applied. The general condition for this to be true is

$$
\begin{equation*}
\langle F| \prod_{i j}\left\{A_{i}\left(x+\rho \xi_{i}\right) B_{j}\left(y_{j}\right)\right\}|I\rangle \sim \sum_{n} \varphi_{n}(\rho)\langle F| \pi_{j}\left\{O_{n}(x) B_{j}\left(y_{j}\right)\right\}|I\rangle \tag{4.14}
\end{equation*}
$$

where $|I\rangle,|F\rangle$ are vacuum or on-shell states, the matrix element is complete (i.e. includes disconnected pieces), $\Pi_{i, j}$ and $\Pi_{j}$ refer to unordered or ordered products*), and the limit $\rho \rightarrow 0$ is taken with
i) all momenta and other parameters of $|I\rangle,|F\rangle$ fixed,
ii) the coordinates $x, \xi, y_{j}$ fixed $\left(y_{j} \neq x\right)$.

Note
a) When ordered products (such as $T$-products) are involved, there can be ambiguities proportional to products of $\delta\left(\xi_{i}-\xi_{j}\right)$ and $\delta\left(y_{i}-y_{k}\right)$ and derivatives thereof. For example, the renormalization of the left-hand side of Eq. (4.14) may involve counterterms of the form

$$
X_{A_{1}} X_{A_{2}} \ldots \quad X_{B_{1}} X_{B_{2}} \ldots \quad \Delta\left(A_{1} A_{2} \ldots B_{1} B_{2} \ldots\right)
$$

in Eq. (1.39). The $\rho \rightarrow 0$ limit of these $\delta$-function terms need not match the $\delta$-functions associated with counterterms

$$
X_{O_{n}} X_{B_{1}} X_{B_{2}} \ldots \Delta\left(O_{n} B_{1} B_{2} \ldots .\right.
$$

on the right-hand side.
b) Even in free-field theory, the state vector

$$
\begin{gather*}
|\varphi(\rho)\rangle=\int d^{4} x \prod_{i} d^{4} \xi_{i} f\left(x, \xi_{1}, \xi_{2}, \ldots\right)\left[\left\{\pi A_{i}-\sum_{n=0}^{N}{\left.\left.\mathcal{O}_{n} O_{n}\right\} / \mathcal{G}_{N}\right]|I\rangle}_{(f=\text { suitable smearing function })}\right.\right.
\end{gather*}
$$

does not vanish:

$$
\begin{align*}
& \langle\varphi(\rho) \mid \varphi(\rho)\rangle \rightarrow 0  \tag{4.16}\\
& \langle F \mid \varphi(\rho)\rangle \rightarrow 0, \quad(\rho \rightarrow 0)
\end{align*}
$$

i.e. the limit $\rho \rightarrow 0$ is "weak".

For most operators $B_{j}\left(y_{j}\right)$, it is absolutely mandatory that the coordinates $y_{j}$ (and not the corresponding momenta) be held fixed as $\rho$ tends to zero. This point will be illustrated by considering the connected amplitude of Fig. 12a in the tree approximation, with

$$
\begin{equation*}
B(y)=: \bar{\psi}(y)\left(\partial^{2}\right)^{25} \psi(y): \tag{4.17}
\end{equation*}
$$

[^19]The most singular q-number contribution generated by the Wick expansion (4.4) is

$$
\begin{equation*}
T\left\{\mathcal{F}_{\mu}^{i}(\xi) \mathcal{F}_{\nu}^{j}(0)\right\} \sim f^{i j k} \mathscr{G}_{\mu \nu}(\xi) \mathcal{F}_{\lambda}^{k}(0) \quad, \quad(\xi \longrightarrow 0) \tag{4.18}
\end{equation*}
$$

where $\mathcal{C}_{\mu \nu \lambda}(\xi)$ scales as $\xi^{-3}$ and $f^{i j k}$ are the structure constants for ordinary $\operatorname{SU}(3)$. [This singularity generates the equal-time commutator

$$
\left[\mathcal{F}_{0}^{i}, \mathcal{F}_{0}^{j}\right]_{E . T .}=i f^{i j k} \delta^{3}(\vec{\xi}) \mathcal{F}_{0}^{k}
$$

of current algebra.] Thus the momentum-space singularity is

$$
\begin{equation*}
\int d^{4} \xi e^{i q \cdot \xi} \varphi_{\mu \nu \lambda}(\xi)=O(q) \quad,(q \rightarrow \infty) \tag{4.19}
\end{equation*}
$$

The graphs responsible for this contribution are shown in Fig. 12b.
We must also consider the graphs of Fig. 12c. If the coordinates $y, z_{1}, z_{2}$ are held fixed as $\xi \rightarrow 0$, the leading term contributed by these graphs is independent of $\xi\left[\right.$ i.e. $O\left(\xi^{+3}\right)$ relative to (4.18)] and is given by the graph of Fig. 12d, with

$$
\begin{equation*}
Q_{\mu \nu}(0)=: \bar{\psi}(0) \gamma_{\mu}\left(\lambda^{i} / 2\right) \psi(0) \bar{\psi}(0) \gamma_{\nu}\left(\lambda^{j} / 2\right) \psi(0): \tag{4.20}
\end{equation*}
$$

This agrees with the result of applying the limit $\xi \rightarrow 0$ to the last term of Eq. (4.4). Thus the contribution $Q_{\mu \nu}$ is always present in the expansion of $\mathrm{T}\left\{\mathbb{F}_{\mu} \widetilde{F}_{\nu}\right\}$ and does not depend on whether $B(y)$ is present or not.

On the other hand, if we transform to momentum space and let the momenta of the currents $\pi_{\mu}, \pi_{V}$ become large ( $q=\eta \ell+c ; \eta \rightarrow \infty, \ell^{2} \neq 0$ ) with the momenta of $\psi, \bar{\psi}$, and B held fixed, the result

$$
\begin{equation*}
\{\text { Fig. } 12(\mathrm{c}) \text { graphs }\}=O\left(q^{48}\right) \tag{4.21}
\end{equation*}
$$

dominates the "expected" term (4.19). Thus naive substitution of the operatorproduct expansion for $T\left\{\pi_{\mu} \pi_{\nu}\right\}$ gives the wrong answer.

However, it is very easy to explain the result (4.21) in the language of operator-product expansions. For simplicity, let us set the momentum of the B operator equal to zero and consider the limit $\xi \rightarrow 0$ of the amplitude

$$
\begin{equation*}
A\left(z_{1}, z_{2}, \xi\right)=\int d^{4} y T\langle 0| \bar{\psi}\left(z_{1}\right) \psi\left(z_{2}\right) B(y) \mathcal{F}_{\alpha}^{i}(\xi) \mathcal{F}_{\beta}^{j}(0)|0\rangle \tag{4.22}
\end{equation*}
$$

The point is that, in addition to the expansion of $T\left\{\pi_{\mu} \pi_{\nu}\right\}$, it is necessary to consider the expansion

$$
\begin{align*}
T\left\{B(y) \mathcal{F}_{\alpha}^{i}(\xi) \mathcal{F}_{\beta}^{j}(0)\right\} & \sim \varphi_{\alpha \beta}^{i j}(\xi, y): \bar{\psi} \psi:+\ldots,(\xi, y \rightarrow 0)  \tag{4.23}\\
\bigodot_{\alpha \beta}^{i j}(\rho \xi, \rho y) & =O\left(\rho^{-56}\right) \tag{4.24}
\end{align*}
$$

Substituting into (4.22), we find

$$
A\left(z_{1}, z_{2}, \xi\right) \sim \int d^{4} y b_{\alpha \beta}^{i j}(\xi, y) T\langle 0| \bar{\psi}\left(z_{1}\right) \psi\left(z_{2}\right): \bar{\psi}(0) \psi(0):|0\rangle,(\xi \rightarrow 0)_{(4.25)}
$$

Counting powers in coordinate space, we have four powers of $\rho$ for the integral $\int d^{4} y$ to be added to the power -56 in (4.24):

$$
\begin{equation*}
\int d^{4} y \mathscr{b}_{\alpha \beta}^{i j}(\xi, y)=O\left(\xi^{-52}\right) \quad, \quad(\xi \longrightarrow 0) \tag{4.26}
\end{equation*}
$$

Fourier transforming ( 4.26 ) yields the result

$$
\begin{equation*}
\int d^{4} \xi e^{i q, \xi} \int d^{4} y \zeta_{\alpha \beta}^{i j}(\xi, y)=O\left(q^{48}\right) \quad, \quad(q \rightarrow \infty) \tag{4.27}
\end{equation*}
$$

in agreement with Eq. (4.21). In other words, it is not sufficient to substitute (4.18) into (4.22) because, for any non-zero value of $\xi$ (no matter how small), the integral $\int d^{4} y$ always contains the region $y=O(\xi)$ in which the contribution (4.23) dominates.

It can be seen from the literature ${ }^{15,71,81,86,90)}$ that a complete derivation of the Wilson expansion in perturbation theory is necessarily non-trivial. Indeed, explicit derivations of (4.14) have been given only for the following special cases

$$
\begin{equation*}
\prod_{i=1}^{L} A_{i}\left(x+\rho \xi_{i}\right)=\prod_{i=1}^{L}\left\{\phi\left(x+\rho \xi_{i}\right) \text { or } \partial_{\alpha} \partial_{\beta} \ldots \partial_{\omega} \phi\left(x+\rho \xi_{i}\right)\right\} \tag{4.28}
\end{equation*}
$$

$$
\prod_{j=1}^{N} B_{j}\left(y_{j}\right)=\prod_{j=1}^{N} \phi\left(y_{j}\right)
$$

in $\phi^{4}$ theory; (for L arbitrary, see Ref. 90). However, the main idea of the derivation is the same for all cases (i.e. simple or composite operators $A_{i}, B_{j}$ in any renormalizable theory) and can be summarized fairly simply.

Consider the complete time-ordered amplitude (Fig. 13)

$$
\begin{equation*}
A=T\langle 0| \prod_{i=1}^{L} \prod_{j=1}^{N} A\left(\rho \xi_{i}\right) B_{j}\left(y_{j}\right)|0\rangle \tag{4.29}
\end{equation*}
$$

which corresponds to the choice $x=0,|I\rangle=|F\rangle=|0\rangle$ in (4.14). In Fig. 13, wavy lines correspond to sources $X$ for the operators $A_{i}, B_{j}[$ as in Eq. (1.7) $]$ and nonwavy lines indicate propagators of fields (gluon, ghost, fermion, etc.) from which the Lagrangian and $A_{i}, B_{j}$ are constructed. Some of the source functions $X$ can be chosen to generate the field operators themselves; e.g. see the lines labelled $A_{1}$ and $B_{2}$ in Fig. 13.

The amplitude $\mathcal{A}$ includes disconnected contributions, so the first step is to extract all factors of the form

$$
\begin{equation*}
G_{s}=T\langle 0| \prod_{i \in S} A_{i}\left(\rho \xi_{i}\right)|0\rangle \tag{4.30}
\end{equation*}
$$

where $S$ is any non-empty subset of $\{1, \ldots, L\}$. We already know how to deal with the $\rho \rightarrow 0$ limit of $G_{S}$, because in momentum space, it becomes the $\eta \rightarrow \infty$ limit of
$\mathrm{G}_{\mathrm{S}}(\mathrm{nq}+\mathrm{c})$ at non-exceptional q . Summing over all possible sets S (including the complete set $\{1, \ldots, L\}$ ), we have ${ }^{15}, 81$ )

$$
\begin{equation*}
T \prod_{i=1}^{L} A_{i}\left(\rho \xi_{i}\right)=: \prod_{i=1}^{L} A_{i}\left(\rho \xi_{i}\right):+\sum_{S} T\langle 0| \prod_{i \in S} A_{i}\left(\rho \xi_{i}\right)|0\rangle: \prod_{i \leqslant s} A_{i}\left(\rho \xi_{i}\right): \tag{4.31}
\end{equation*}
$$

where the generalized Wick products*)

$$
\left.: \pi_{i}^{\prime} A_{i}\left(\rho \xi_{i}\right): \quad, \quad \pi^{\prime}=\pi_{i \notin s} \text { or } \prod_{i=1}^{L}\right)
$$

correspond to time-ordered amplitudes

$$
\begin{equation*}
A^{\prime}=T\langle 0| \prod_{j=1}^{N}: \pi_{i}^{\prime} A_{i}\left(\rho \xi_{i}\right): B_{j}\left(y_{j}\right)|0\rangle \tag{4.32}
\end{equation*}
$$

constructed from graphs in which every $A_{i}$ operator in the product $\Pi^{\prime} A$ is connected to $a B_{j}$ operator. The problem is now reduced to that of obtaining expansions for the $\mathfrak{t}^{\prime}$ amplitudes. The expansion for the original amplitude $t$ can then be reconstructed from Eq. (4.31), with the amplitudes $G_{S}$ appearing as factors in various coefficient functions. [Observe that we chose to extract only those factors (4.30) which do not depend on the $B_{j}$ operators.]

Let us denote the Fourier transform of (4.32) by

$$
\begin{equation*}
\text { F.T. }\left[\mathcal{A}^{\prime}\right]=\tilde{A}^{\prime}\left(q_{1} \ldots q_{k} ; r_{1} \ldots r_{N}\right) \delta\left(\sum_{i=1}^{k} q_{i}+\sum_{j=1}^{N} r_{j}\right) \tag{4.33}
\end{equation*}
$$

where $K$ is the number of $A_{i}$ operators in (4.32) and the momenta $\left\{q_{1}, \ldots, q_{K}\right\}$ and $\left\{r_{1}, \ldots, r_{N}\right\}$ are conjugate to the coordinates $\left\{\rho \xi_{i}\right\}$ and $\left\{y_{1}, \ldots, y_{N}\right\}$, respectively. The notation is chosen such that Eqs. (2.1) and (2.2) and Fig. 2 can be applied directly to the amplitude $\tilde{x}^{\prime}$. Figure 2 involves decomposing $\tilde{x}^{\prime}$ into an "upper blob" $\mathscr{U}\left(\mathcal{G}^{\prime}\right.$ in Fig. 2) and a "lower blob" $\mathscr{L}$ :

$$
\begin{equation*}
\tilde{A}^{\prime}=\int \prod_{i=1}^{M} d^{4} k_{i} U\left(q_{1} \ldots q_{N} ; k_{1} \ldots k_{M}\right) \mathscr{L}\left(k_{1} \ldots k_{M} ; r_{1} \ldots r_{N} ; \sum r=-\sum q\right) \tag{4.34}
\end{equation*}
$$

The $\mathcal{U}$ amplitude is defined in terms of graphs $\mathcal{G}^{\prime}$ which become lPI when the coordinates $\xi_{i}$ are set equal: i.e. $\xi_{i}=\xi$, for all $i$ in $\Pi_{i}^{\prime}$ (as shown in Fig. 14). The rationale for this definition is that it isolates all graphs $\mathcal{G}^{\prime}$ which are able to carry $O(\eta)$ momenta simultaneously along all internal lines; i.e. the decomposition (4.34) is designed to facilitate power-counting for the limit $\mathrm{q}_{1}, \ldots, \mathrm{q}_{\mathrm{K}} \rightarrow \infty$.
Note
i) One of the consequences of the definition is that propagators for the intermediate lines $k_{1}, \ldots, k_{M}$ (including self-energy corrections) are incorporated in the amplitude $\mathscr{L}$.

[^20]ii) The expression for $\mathscr{U}$ includes a momentum-conservation $\delta$-function
\[

$$
\begin{equation*}
\delta\left(\sum q+\sum k\right) \tag{4.35a}
\end{equation*}
$$

\]

so $U$ can be considered as a function of ( $K+M$ ) independent variables $\left(q_{1}, \ldots, q_{K}, k_{1}, \ldots, k_{M}\right)$. Also, some of the graphs $G^{\prime}$ may be disconnected; each connected component gives rise to a factor

$$
\begin{equation*}
\delta\left(\Sigma^{\prime} q+\Sigma^{\prime \prime} k\right) \tag{4.35b}
\end{equation*}
$$

in the expression for $\mathcal{U}$, where $\sum^{\prime}$ and $\sum^{\prime \prime}$ denote non-empty partial sums.
iii) In Fig. 2 (as applied to $\tilde{E}^{\prime}$ ), all external lines should be understood to be wavy source lines. Previously (Figs. 6b and 7b), we allowed some of the source lines $r_{1}, \ldots, r_{N}$ to be intermediate as well. However, for reasons which will shortly become evident, we now require that none of the intermediate lines $k_{1}, \ldots, k_{M}$ be wavy (as in Fig. 14).
The amplitude $U$ can be expanded in $k=\left(k_{1}, \ldots, k_{M}\right)$ about the point $\left.{ }^{*}\right)$

$$
\begin{equation*}
k=(0,0, \ldots, 0) \tag{4.36}
\end{equation*}
$$

with all momenta $q_{p-1}=\left(q_{1}, \ldots, q_{K}\right)$ held fixed:

$$
\begin{array}{r}
U(q, k)=\left.\sum_{n=0}^{p-1} \frac{1}{n!}\left(k \cdot \frac{\partial}{\partial \lambda}\right)^{n} U(q, \lambda)\right|_{\lambda=0}+R_{p}(q, k) \\
\left(p \geqslant 1 ; \text { notation: } k \cdot \partial / \partial \lambda=\sum_{i=1}^{M} k_{i}^{\mu} \partial / \partial \lambda_{i}^{\mu}\right) \tag{4.37}
\end{array}
$$

Consider the dimensionality of the amplitudes appearing in (4.37). The definition of dimensionality given by Eq. (2.4) does not take account of the extra $\delta$-functions (4.35) which we have chosen to incorporate in the definition of $U$. Since each $\delta$-function contributes a power -4 , we have

$$
\begin{equation*}
D(U)=D\left(\xi^{\prime}\right)-4\left\{\text { number of connected components of } \xi^{\prime}\right\} \tag{4.38}
\end{equation*}
$$

Each derivative $\partial / \partial \lambda$ acting on $\mathcal{U}$ lowers its dimensionality by 1 :

$$
\begin{align*}
D\left\{(\partial / \partial \lambda)^{n} U(q, \lambda)\right\} & =D(U)-n \\
D\left\{R_{p}\right\} & =D(U)-p \tag{4.39}
\end{align*}
$$

We see that Eq. (4.37), when substituted into (4.34), looks like a Wilson expansion with coefficient functions $(\partial / \partial \lambda)^{n} \Psi(q, \lambda=0)$ multiplying operators $O_{n}$ given by momentum-space vertices

$$
\begin{equation*}
(k)^{n}=\left(k_{i_{1}}\right)_{\mu_{1}}\left(k_{i_{2}}\right)_{\mu_{2}} \cdots\left(k_{i_{n}}\right)_{\mu_{n}} \tag{4.40}
\end{equation*}
$$

[^21]However, the prescription (4.37) is not generally satisfactory: the remainder amplitude $\Re_{p}$ need not be $O\left(\eta^{-p} 1 n^{\beta} \eta\right)$ relative to $U$ if $\mathcal{G}^{\prime}$ contains subgraphs with sufficiently large dimensionality. It is necessary to find a prescription for constructing a remainder amplitude in which the dimensionalities of all subgraphs are simultaneously lowered.

The answer is obvious if the complete graph $\mathcal{G}$ for the amplitude $\tilde{\mathcal{t}}^{\prime}$ can be uniquely decomposed into a nested set of subgraphs (Fig. 15)

$$
\begin{equation*}
g_{1}^{\prime} \subset g_{2}^{\prime} \subset \ldots \subset \xi_{h}^{\prime} \tag{4.41}
\end{equation*}
$$

Let $U_{1}, U_{2}, \ldots, U_{h}$ be the corresponding "upper blob" amplitudes, with

$$
\begin{align*}
U_{2}\left(\{k\}_{2}, q\right)= & \int\{d k\}_{1} U_{21}\left(\{k\}_{2},\{k\}_{1}\right) U_{1}\left(\{k\}_{1}, q\right),  \tag{4.42}\\
U_{(m+1)} & =\int\{d k\}_{m} U_{(m+1) m} U_{m}, \text { etc. }
\end{align*}
$$

so that the complete amplitude can be written

$$
\begin{align*}
\tilde{A}^{\prime} & =\int\{d k\}_{h} \mathscr{L}_{h}\left(r,\{k\}_{h}\right) \mathcal{U}_{h}\left(\{k\}_{h}, q\right)  \tag{4.43}\\
& =\int \prod_{m=1}^{h}\{d k\}_{m} \mathscr{L}_{h} U_{h(h-1)} U_{(h-1)(h-2)} \ldots U_{21} U_{1}
\end{align*}
$$

We begin by subtracting the smallest blob $\mathcal{U}_{1}$ as in (4.37),

$$
\begin{equation*}
U_{1} \rightarrow\left(1-t_{1}\right) U_{1} \tag{4.44}
\end{equation*}
$$

where $t_{m}$ is shorthand for the Taylor operator

$$
\begin{equation*}
t_{m}=\left.\sum_{r=0}^{p(m)-1} \frac{1}{r!}\left(k \cdot \frac{\vec{\partial}}{\partial \lambda}\right)_{m}^{r}\right|_{\lambda=0} \quad, \quad(m=1,2,3, \ldots, h) . \tag{4.45}
\end{equation*}
$$

For the next graph $\mathscr{G}_{2}^{\prime}$, we have

$$
\begin{equation*}
u_{2} \longrightarrow\left(1-t_{2}\right) \int\{d k\}_{1} u_{21}\left(1-t_{1}\right) u_{1} \tag{4.46}
\end{equation*}
$$

and so on, until we arrive at a completely subtracted amplitude

$$
\begin{equation*}
\tilde{y}_{p}^{\prime}=\int \prod_{m=1}^{h}\{d k\}_{m} \mathscr{L}_{h}\left(1-t_{h}\right) u_{h(h-1)}\left(1-t_{h-1}\right) \ldots\left(1-t_{2}\right) u_{21}\left(1-t_{1}\right) u_{1} \tag{4.47}
\end{equation*}
$$

in which the dimensionalities of the subgraphs $\mathscr{Y}_{1}^{\prime}, \ldots, \mathscr{S}_{h}^{\prime}$ are reduced by $p(1), \ldots, p(h)$, respectively. The difference between $\tilde{\mathscr{A}}^{\prime}$ and $\tilde{S}_{p}^{\prime}$ is a finite series in the form of an operator-product expansion:

$$
\begin{equation*}
\tilde{A}^{\prime}=\sum_{n=0}^{n(p)} \tilde{\mathscr{\varphi}}_{n}^{\prime}(q) \mathscr{A}_{n}(r)+\tilde{\varphi}_{p}^{\prime} \tag{4.48}
\end{equation*}
$$

The prescription for constructing $\tilde{S}_{p}^{\prime}$ must be generalized to include cases in which there are overlapping subgraphs. Recall (Fig. 14) that the subgraphs $\mathscr{G}^{\prime}$ are defined such that, when the coordinates $\xi_{i}$ are identified, the result is a 1PI graph $\Gamma^{\prime}$ associated with the class of operators

$$
\begin{equation*}
Q(\xi)=: \pi_{i}^{\prime} A_{i}(\xi):, \cdots \quad \pi^{\prime}\left\{\partial \cdots \partial A_{i}(\xi)\right\}:, \ldots \tag{4.49}
\end{equation*}
$$

The superficial degree of divergence of $\Gamma^{\prime}$ is trivially related to the dimensionality of the amplitude $\mathcal{U}$ associated with $\mathcal{G}^{\prime}$ :

$$
\begin{equation*}
D(u)=d\left(\Gamma^{\prime}\right)-4 K \tag{4.50}
\end{equation*}
$$

Furthermore, the subtractions $t_{1}, t_{2}, \ldots, t_{h}$ in (4.47) become renormalization counterterms $\Delta Q$ for the operators $Q$ when the coordinates $\xi_{i}$ are set equal to $\xi$. So for the general case, the prescription is that subtractions used to construct $\tilde{S}_{p}^{\prime}$ from $\tilde{\mathscr{A}}^{\prime}\left(\xi_{i} \neq \xi\right.$ ) should correspond to renormalization counterterms $\Delta Q$ (possibly over subtracted) for the amplitude $\tilde{\mathscr{t}}^{\prime}\left(\xi_{i}=\xi\right)$. This ensures that all subdimensionalities of $\tilde{S}_{p}^{\prime}$ are reduced relative to those of $\tilde{\mathcal{t}}^{\prime}\left(\xi_{i} \neq \xi\right)$, because the renormalization theorem ${ }^{12}$ ) says that all of the $d\left(\Gamma^{\prime}\right)$ can be lowered to any desired value by including sufficient counterterms $\Delta Q$ in Eq. (1.7). Thus Eq. (4.48) remains valid for the general case.

An example of overlapping subgraphs is given in Fig. 16. There are two nested sets

$$
\xi_{1}^{\prime} \subset \xi_{2 a}^{\prime} \subset \xi_{3}^{\prime}, \xi_{1}^{\prime} \subset \xi_{2 b}^{\prime} \subset \xi_{3}^{\prime}
$$

with $\mathscr{\Im}_{2 \mathrm{a}}^{\prime}$ and $\mathscr{G}_{2 \mathrm{~b}}^{\prime}$ overlapping. The remainder amplitude is given by the formula

$$
\begin{align*}
\tilde{y}_{p}^{\prime}= & \iint\{d k\}_{3}\{d k\}_{1} \mathscr{L}_{3}\left(1-t_{3}\right) \\
& \cdot\left[\int\{d k\}_{2 a} u_{3(2 a)}\left(1-t_{2 a}\right) U_{(2 a) 1}-\int\{d k\}_{2 b} u_{3(2 b)} t_{2 b} u_{(2 b) 1}\right]\left(1-t_{1}\right) U_{1} \tag{4.51}
\end{align*}
$$

No account has been taken of the need for renormalization counterterms for the operator $O_{n}$ (for example), so this discussion applies only to unrenormalized (B) amplitudes*). In coordinate space, Eqs. (4.48) and (4.31) imply

$$
\begin{gather*}
A_{B}=\sum_{n=0}^{n(p)}\left(\varphi_{n}\right)_{B}\left(A_{n}\right)_{B}+\varphi_{p}\left(p \xi_{i}, y_{j}\right)_{B} \\
\left(A_{n}\right)_{B}=T\langle 0| \prod_{j=1}^{N} O_{n}(0) B_{j}\left(y_{j}\right)|0\rangle_{B} \tag{4.52}
\end{gather*}
$$

where $S_{p}$ is constructed from Eq. (4.31) and the Fourier transforms of $\tilde{S}_{p}^{\prime}$ amplitubes.

According to the rule (1.41), $\mathcal{A}$ and $\mathcal{I}_{\mathrm{n}}$ are renormalized as follows:

$$
\begin{align*}
& A_{R}=\prod_{i j} Z_{A_{i}}^{-1} Z_{B_{j}}^{-1} A_{B}+\delta\left[\xi_{i}-\xi_{j}, \rho \xi_{i}-y_{j}, y_{i}-y_{j}\right] \\
& \left(A_{n}\right)_{R}=Z_{n}^{-1} \prod_{j} Z_{B_{j}}^{-1}\left(A_{n}\right)_{B}+\delta\left[y_{i}, y_{i}-y_{j}\right] \tag{4.53}
\end{align*}
$$

Some of the $Z$-factors (e.g. $Z_{n}$ ) may be matrices. The terms $\delta\left[x_{i}, y_{j}, \ldots\right]$, which involve $\delta$-functions of the coordinate differences $x_{i}, y_{j}, \ldots$, are subtractions

[^22]generated by counterterms $\Delta\left(A_{1} A_{2} \ldots B_{1} B_{2} \ldots\right)$ and $\Delta\left(0_{n} B_{1} B_{2} \ldots\right)$ in Eq. (1.39); [see note (a) following Eq. (4.14)]. In general, the terms $\sum_{n} \mathcal{C}_{n} \mathcal{A}_{\mathrm{n}}$ and $S_{\mathrm{p}}$ mix under renormalization. However, in renormalizable theories, renormalization does not increase the subdimensionalities of an amplitude. Therefore, the term $O\left(\delta_{p}\right)$ in the equation
\[

$$
\begin{equation*}
\left(\varphi_{n}\right)_{R}=Z_{n} \prod_{i} Z_{A_{i}}^{-1}\left(\varphi_{n}\right)_{B}+\delta\left[\xi_{i}-\xi_{j}\right]+O\left(\varphi_{p}\right) \tag{4.54}
\end{equation*}
$$

\]

can be absorbed into the definition of the renormalized amplitude $\left(S_{p}\right){ }_{R}$ without changing its maximum subdimensionality. So the result is

$$
\begin{equation*}
A_{R}=\sum_{n=0}^{n(p)}\left(G_{n}\right)_{R}\left(A_{n}\right)_{R}+\{\delta \text {-functions }\}+\left(f_{p}\right)_{R} \tag{4.55}
\end{equation*}
$$

where the remainder amplitude obeys the bound

$$
\begin{equation*}
\left(f_{p} \mid A\right)_{R}=O\left(\rho^{p} \ln ^{\beta} \rho\right), \quad(\underset{\rho \rightarrow 0}{ }=\text { positive integer ; } ; \tag{4.56}
\end{equation*}
$$

Note
i) The terms $C_{n} 0_{n}$ do not depend on the operators $B_{j}$ because the sets of intermediate lines $\left\{\mathrm{k}_{1}, \ldots, \mathrm{k}_{\mathrm{M}(\mathrm{m})}\right\}_{\mathrm{m}}$ do not involve sources. Because of this restriction on intermediate lines, the coordinates $\xi_{i}, y_{j}$ must be held fixed as $\rho$ tends to zero in (4.56).
ii) Strictly speaking, the term \{ $\delta$-functions\} [absent for the special case (4.28)] violates Eq. (4.1) because it need not be negligible relative to $\mathrm{C}_{\mathrm{n}}$ and may depend on the $\mathrm{B}_{\mathrm{j}}$ operators. However, the violation is of the same form as the $\delta$-function ambiguities of ordered products, so it is of little practical importance.
iii) It is permissible to fix the momenta of simple (i.e. non-composite) operators $B_{j}$ in Eqs. (4.55) and (4.56). For example, consider the operator $B_{2}$ in Fig. 13; even if its momentum is fixed, its source line cannot be intermediate because the corresponding subgraph would not be 1 PI for $\xi_{i}=\xi$.
iv) The on-shell states $|I|,|F\rangle$ of Eq. (4.14) can be recovered by choosing some of the $B_{j}$ in (4.55) to be suitable interpolating operators. If simple operators are used, remark (iii) is relevant. More generally, the on-shell limit isolates the set of graphs for which the interpolating operator (simple or composite) communicates with other external lines via a single internal line. For this subset, the source of the interpolating field cannot be an intermediate line, so the time-ordered case of (4.14) is obtained.

The set of allowed discontinuities of $T\langle F| \Pi_{i j} A_{i} B_{j}|I\rangle$ is restricted by the locality of the operator $O_{n}$ on the right-hand side of Eq. (4.14). If the ordering of the $A_{i}$ operators with respect to each other (but not to the $B_{j}$ operators) is
partially or completely removed, $T\langle F| \Pi_{j} O_{n} B_{j}|I\rangle$ remains completely time-ordered, while the iع-prescription for coefficient functions is adjusted accordingly; egg.

$$
\begin{align*}
T\left\{A_{1}(\xi) A_{2}(0)\right\} & \sim\left(\xi^{2}-i \varepsilon\right)^{\alpha} O_{1}(0) \\
A_{1}(\xi) A_{2}(0) & \sim\left(\xi^{2}-i \varepsilon \xi_{0}\right)^{\alpha} O_{1}(0) \quad, \quad(\xi \longrightarrow 0) \tag{4.57}
\end{align*}
$$

It is also possible to unorder part of $\Pi_{j}{ }^{B} j$ on each side of (4.14), retaining the time-ordering between $A_{i}\left(\rho \xi_{i}\right)$ (or $\left.O_{n}\right)$ and $B_{j}\left(y_{i}\right)$ for all $i, j$. Special precautions are necessary when unordering a $A_{i} B_{j}$ pair. In general, the relevant $B_{j}$ operator should not appear in the middle of the partially or completely unordered product $\Pi_{i} A_{i}: ~ e . g$.

$$
\begin{align*}
& \langle 0| A_{1}(\xi) A_{2}(0) B(y)|0\rangle \sim\left(\xi^{2}-i \varepsilon \xi_{0}\right)^{\alpha}\langle 0| O_{1}(0) B(y)|0\rangle \\
& \langle 0| A_{1}(\xi) B(y) A_{2}(0)|0\rangle \sim ? ?, \quad(\xi \rightarrow 0, y \text { fixed }) \tag{4.58}
\end{align*}
$$

These remarks are important ${ }^{91}$ ) for the process

$$
\begin{equation*}
e^{+}+e^{-} \longrightarrow h+X \tag{4.59}
\end{equation*}
$$

where $h$ is an observed hadron (momentum $p_{h}$ ) and $X$ denotes other final-state hadrons with combined momentum $\mathrm{p}_{\mathrm{x}}$. The one-particle inclusive cross-section is propertional to

$$
\begin{equation*}
\bar{W}_{\mu \nu}=\sum_{x}\langle 0| J_{\mu}(0)|X h\rangle\langle X h| J_{\nu}(0)|0\rangle(2 \pi)^{4} \delta^{4}\left(p_{x}+p_{h}-q\right) \tag{4.60}
\end{equation*}
$$

where $q$ is the momentum of the virtual photon. If $\Phi_{h}$ is an interpolating operator for $h$, the corresponding off-shell amplitude in coordinate space is

$$
\begin{equation*}
W_{\mu \nu}=\langle 0| T\left\{J_{\mu}(\xi) \Phi_{h}\left(y_{1}\right)\right\} \widetilde{T}\left\{J_{v}(0) \Phi_{h}\left(y_{2}\right)\right\}|0\rangle \tag{4.61}
\end{equation*}
$$

where the coordinate $\xi$ is conjugate to $q$ and $\bar{T}$ denotes the anti-T-product. It is not possible to substitute an operator-product expansion for $J_{\mu} J_{V}$ because local operators $O_{n}$ cannot be produced by the combination $T\left(A_{1} B_{1}\right) \bar{T}\left(A_{2} B_{2}\right)$ [in the notation of (4.14)]. Mueller ${ }^{91}$ ) has suggested an expansion of $w_{\mu \nu}$ in which there are coefficient functions (different from the Wilson coefficient functions) which do not multiply local operators $O_{n}$. Derivations have been given for some non-gauge theories ${ }^{91,92 \text { ). }}$

### 4.2 Renormalization-group properties

A particular sector ( $J, I, \ldots$ ) of the operator-product expansion (4.1) is obtained by projecting out intermediate sets $\left\{k_{1}, \ldots, k_{M}\right\}$ with total quantum numbers ( $J, I, \ldots$ ), or equivalently, by a judicious choice of operators $B_{j}$ in Eq. (4.29) :

$$
A \longrightarrow A_{(J, 1, \ldots)}
$$

The corresponding operators $\left\{0_{n}\right\}$ have quantum numbers ( $J, I, \ldots$ ). For simplicity, let us assume that there is a single operator $O_{n}$ (matrix element $f_{n}$ ) which contribute the first $p$ powers of the scaling parameter $\rho$ to the asymptotic expansion of . $\ell$, order-by-order in perturbation theory:

$$
\begin{align*}
\left.A_{(J, I}, \ldots\right) / A_{n}= & F_{n}\left(\rho \xi_{i}, y_{j} ; g, m, \mu\right) \\
= & G_{n}\left(\rho \xi_{i} ; g, m, \mu\right)\left\{1+O\left(\rho^{p} \ln { }^{\beta} \rho\right)\right\}  \tag{4.62}\\
& (\rho \rightarrow 0, p=\text { positive integer })
\end{align*}
$$

As indicated by the notation, we choose to carry out the analysis using a massindependent renormalization prescription ${ }^{*}$ ).

The improved CS equation for the ratio $\mathrm{F}_{\mathrm{n}}$ is (neglecting $\delta$-function subfractions)

$$
\begin{equation*}
\left[\mu \partial / \partial \mu+\beta(g) \partial / \partial g+\gamma_{m}(g) m \partial / \partial m+\gamma_{( }(g)-\gamma_{n}(g)\right] \tag{4.63}
\end{equation*}
$$

$$
F_{n}\left(\rho \xi_{i}, y_{j} ; g, m, \mu\right)=0
$$

where $\gamma(g)$ and $\gamma_{n}(g)$ are the $\gamma$-functions for the amplitudes $t$ and $t_{n}$ [with $\left.\gamma(0)=0=\gamma_{n}(0)\right]$. The addition rule (1.46) implies

$$
\begin{aligned}
& \gamma(g)=\sum_{i} \gamma_{A_{i}}(g)+\sum_{j} \gamma_{B_{j}}(g) \\
& \gamma_{n}(g)=\gamma_{0_{n}}(g)+\sum_{j} \gamma_{B_{j}}(g)
\end{aligned}
$$

so all dependence on $\gamma$-functions for the $B_{j}$ operators drops out in the difference

$$
\begin{equation*}
\gamma(g)-\gamma_{n}(g)=\sum_{i} \gamma_{A_{i}}(g)-\gamma_{O_{n}}(g) \tag{4.64}
\end{equation*}
$$

Equations (4.62)-(4.64) imply

$$
\begin{align*}
{\left[\mu \partial / \partial \mu+\beta \partial / \partial g+\gamma_{m} m \partial / \partial m+\sum_{i} \gamma_{A_{i}}\right.} & \left.-\gamma_{O_{n}}\right] \mathscr{U}_{n}  \tag{4.65}\\
& =0\left[\rho^{p} \ln ^{\beta} \rho \mathscr{b}_{n}(\rho)\right]
\end{align*}
$$

Since $\mathcal{C}_{\mathrm{n}}$ does not depend on $y_{j}, \mu \partial / \partial \mu$ can be eliminated in terms of $\mathrm{m} \partial / \partial \mathrm{m}$ and either $\rho \partial / \partial \rho$ or $\eta \partial / \partial \eta\left(\eta=\rho^{-1}\right)$ using a homogeneity equation of the form (3.44): egg.

$$
\begin{equation*}
\left[\mu \partial / \partial \mu+m \partial / \partial m+\eta \partial / \partial \eta-d_{\ell}\right] \tilde{\varphi}_{n}\left(\eta q_{i} ; g, m, \mu\right)=0 \tag{4.66}
\end{equation*}
$$

where $\tilde{C}_{n}(q)$ is the Fourier transform of $C_{n}(\xi)$ and $d_{C}$ is the mass dimensionality of $\tilde{\mathrm{C}}_{\mathrm{n}}$. Thus the analogue of Eq. (3.46) is

$$
\begin{align*}
{\left[\eta \partial / \partial \eta-\beta \partial / \partial g+\left(1-\gamma_{m}\right) m n \partial / \partial m\right.} & \left.-\left(d_{\varphi}+\sum_{i} \gamma_{A_{i}}-\gamma_{0_{n}}\right)\right] \tilde{\varphi}_{n}  \tag{4.67}\\
& =O\left[\eta^{-p} \ln \beta^{\beta^{\prime}} \tilde{\varphi}_{n}\right]
\end{align*}
$$

*) See Ref. 7. The corresponding analysis for the ordinary CS equation ${ }^{18}, 71,86,87$ ) involves expanding $T\left\langle\Delta \Pi_{i j} A_{i} B_{j}\right\rangle$, where $\Delta$ is the zero-momentum mass-insertion operator $\left[\mathrm{e} . \mathrm{g}, \Delta=\int \mathrm{d}^{4} \mathrm{x}: \bar{\psi}(\mathrm{x}) \psi(\mathrm{x}):\right]$. Straightforward substitution of (4.1) works for the leading power in a given sector, but for non-leading powers, the expansion of $: \psi \psi: \Pi_{i} A_{i}$ also contributes. Compare Eqs. (4.17)-(4.27).

If $m$ is a fermionic mass, the amplitudes

$$
\begin{equation*}
\tilde{\mathscr{b}}_{n}^{(r)}\left(\eta q_{i} ; g, \mu\right)=\left.\left(\frac{\partial}{\partial m}\right)^{r} \tilde{\mathscr{b}}_{n}\left(\eta q_{i} ; g, m, \mu\right)\right|_{m=0} \tag{4.68}
\end{equation*}
$$

exist in perturbation theory for $r=0,1,2$, but not for $r=3,4, \ldots$, so Eq. (4.67) becomes

$$
\begin{gather*}
{\left[\eta \partial / \partial \eta-\beta \partial / \partial g-\left\{d_{\varphi}+\sum_{i} \gamma_{A_{i}}-\gamma_{0_{n}}-r\left(1-\gamma_{m}\right)\right\}\right] \tilde{\mathcal{U}}_{n}^{(r)}=0}  \tag{4.69}\\
0 \leqslant r \leqslant \operatorname{Min}(2, p-1)
\end{gather*}
$$

The consequences of (4.69) are analogous to those of Eq. (2.33):
i) Broken scale invariance:

$$
\begin{align*}
b_{n}^{(r)} & \sim \text { const. } \rho^{-P(r, n)}, \\
P(r, n) & =\sum_{i} d_{A_{i}}-d_{O_{n}}-r\left(4-d_{\Delta}\right) \tag{4.70}
\end{align*}
$$

For $r=0$, Eq. (4.70) extends to coefficient functions ${ }^{4}$ ) the rule (2.56) that dynamical dimensions are conserved at short distances. Non-leading terms $r=1,2, \ldots$, are damped by a power $r\left(4-d_{\Delta}\right)$ of $\rho$, as in Eq. (3.36), if the condition (3.37) is satisfied. For the special case

$$
\begin{equation*}
A_{i}=\text { (partially) conserved currents, } O_{n}=\text { mass operator } \tag{4.71}
\end{equation*}
$$

there is a cancellation ${ }^{4}, 7$ ) of anomalous dimensions for the leading term ( $\mathrm{r}=1$ )

$$
\begin{equation*}
P(r=1, n=\Delta)=\sum_{i} d_{A_{i}}-4=\text { integer } \tag{4.72}
\end{equation*}
$$

which is important for the short-distance analysis of $\eta \rightarrow 3 \pi$ decay and other weak and electromagnetic corrections to strong interactions.
ii) Asymptotic freedom ${ }^{7}, 93$ ):

$$
\begin{aligned}
& \varphi_{n}^{(r)} \sim \text { const. } \rho^{-p(r, n)}(\ln \rho)^{\lambda(r, n)}\{1+O(\ln |\ln \rho| / \ln \rho)\} \\
& p(r, n)=\sum_{i} \operatorname{dim} A_{i}-\operatorname{dim} O_{n}-r \\
& \lambda(r, n)=\left\{\sum_{i} c_{A_{i}}-c_{o_{n}}+r c_{m}\right\} / 2 b
\end{aligned}
$$

Here, $\operatorname{dim} Q$ denotes the canonical dimension of $Q, b$ is defined in Eq. (2.80), and $c_{Q}$ is the one-loop coefficient in the expansion

$$
\begin{equation*}
\gamma_{Q}(g)=c_{Q} g^{2}+O\left(g^{4}\right) \tag{4.74}
\end{equation*}
$$

The result (4.73) generalizes Eq. (2.86). If $A_{i}$ and $O_{n}$ are conserved or partially conserved currents for $r=0$, or if Eq. (4.71) is applicable ${ }^{7}$ ) with $r=1$, the logarithmic power $\lambda(r, n)$ vanishes and the leading contribution to $\mathrm{C}_{\mathrm{n}}^{(\mathrm{r})}$ (including the constant of proportionality) is given by the free-field result. [The last statement assumes that the complete operator $O_{n}$ is generated in the expansion of $\Pi A_{i}$ in free-field theory. This need not be the
case for $O_{n}=\theta_{\mu \nu} ;$ egg. if the operators $A_{i}$ are chiral currents, only the fermionic part of $\theta_{\mu \nu}$ is generated. The result ${ }^{93)}$ is that the constant of proportionality is computable and differs from the constant which normalizes the term $\theta_{\mu \nu}$ (fermionic) in free-field theory.]
If we want to check these predictions for short-distance behaviour by measuring the amplitudes $\tilde{C}_{n}\left(q_{i}\right)$, it is essential that the momenta $\left\{q_{i}\right\}$ be nonexceptional. For example, consider the connected matrix element

$$
\begin{aligned}
W_{\mu \nu}= & (2 \pi)^{-1} \int d^{4} x e^{i q \cdot x}\langle p|\left[J_{\mu}(x), J_{\nu}(0)\right]|p\rangle \\
= & -\left(g_{\mu \nu} / M\right) F_{1}\left(\xi, q^{2}\right)+\left(p_{\mu} p_{\nu} / p \cdot q\right) F_{2}\left(\xi, q^{2}\right) \\
& +\left(i \varepsilon_{\mu v \lambda \eta} p^{2} q^{\eta} / 2 p . q\right) F_{3}\left(\xi, q^{2}\right) \\
& + \text { other terms if } J_{\mu} \text { not completely conserved } ; \\
-1 \leqslant \xi= & -q^{2} / 2 p \cdot q=\omega^{-1} \leqslant 1 ; F_{i}\left(-\xi, q^{2}\right)= \pm F_{i}\left(\xi, q^{2}\right) ; \\
& |p\rangle=\text { nucleon, mass } M, \text { momentum } p
\end{aligned}
$$

For $0 \leq \xi \leq 1, W_{\mu \nu}$ is proportional to the total cross-section for deep inelastic lepton-nucleon scattering, where $q$ is the momentum transferred by the current and $\mathrm{F}_{\mathrm{i}}$ are the usual structure functions. As is well known ${ }^{94}, 95$ ), the Bjorken 1 imit $-q^{2} \rightarrow \infty$ at fixed $\xi$ corresponds to $x^{2} \rightarrow 0$, not $x_{\mu} \rightarrow 0$. In the language of Eq. (2.1) and (2.7), this is because the 1 imit involves exceptional values of q :

$$
q=\eta l+c ; \eta \rightarrow \infty, \quad l^{2}=0, \quad p . l \neq 0, \quad c . l \neq 0 .(4.76)
$$

Also well known ${ }^{96}$ ) is the fact that the leading Wilson coefficient function in the spin-J sector is asymptotically proportional to the moment

$$
M_{n}^{i}\left(q^{2}\right)=\int_{0}^{1} d \xi \xi^{n} F_{i}\left(\xi, q^{2}\right) \quad,(n=J-1, J-2, J-1 \ldots \text { for } i=1,2,3 . .)(4.77)
$$

ie. the $x_{\mu} \rightarrow 0$ imit corresponds to the 1 imit $-q^{2} \rightarrow \infty$ of $M_{n}$. It does not matter that the non-exceptional 1 imit ( $\ell^{2} \neq 0$ ) is not kinematically consistent with the bound $|\xi| \leq 1$. This bound is relevant for the commutator amplitude (4.75), whereas the lack of exceptionality of the $n \rightarrow \infty$ limit should be checked for the timeordered analogue of (4.75):

$$
\begin{align*}
& T=T\left(\zeta, q^{2}\right), \quad \zeta=2 p \cdot q / \sqrt{-q^{2}} \\
& -q^{2} \rightarrow \infty \text { at fixed } \zeta \tag{4.78}
\end{align*}
$$

(For simplicity, the indices $\mu, \nu$ have been dropped.) Given a suitably normalized spin-J projection*) of $T$,

$$
\begin{equation*}
T_{n}\left(q^{2}\right) \propto\left(q^{2}\right)^{\text {power }} \int_{-1}^{1} d v\left(1-v^{2}\right)^{1 / 2} C_{n}(v) T\left(-i v, q^{2}\right) \tag{4.79}
\end{equation*}
$$

[^23]the $-q^{2} \rightarrow \infty$ limit of $T_{n}$ is non-exceptional and is determined by $\tilde{\mathrm{C}}_{n}(q)$, and the commutator amplitude is recovered by taking the absorptive part:
\[

$$
\begin{gather*}
\mu_{n}\left(q^{2}\right)=A b s T_{n}\left(q^{2}\right)  \tag{4.80}\\
\mu_{n} / M_{n} \longrightarrow 1 \quad, \quad\left(-q^{2} \rightarrow \infty\right)
\end{gather*}
$$
\]

The difference between $\mu_{n}$ and $M_{n}$ is that only spin-J operators contribute to $\mu_{n}$, whereas non-leading powers in the expansion of $M_{n}$ involve a mixture of spins.

It would be desirable to directly measure the $q-$ dependence of the moments in order to distinguish the alternatives

$$
M_{n}\left(q^{2}\right) \sim \begin{cases}K_{n} & \text { (Bj. scaling) } \\ K_{n}^{\prime}\left(\ln q^{2}\right)^{-\lambda(n)} & (\text { As. Freedom, Eq. (4.73)) } \\ K_{n}^{\prime \prime}\left(-q^{2}\right)^{-a(n) / 2} & \text { (Br. scale invariance, Eq. (4.70)) }\end{cases}
$$

where $K_{n}, K_{n}^{\prime}$, and $K_{n}^{\prime \prime}$ are $q$-independent and the powers $\lambda(n), a(n)$ are given by

$$
\begin{equation*}
\lambda(n)=c_{o_{n}} / 2 b \quad, \quad a(n)=\gamma_{o_{n}}\left(g_{\infty}\right) \tag{4.82}
\end{equation*}
$$

However, it has been observed ${ }^{9 \theta}$ ) that $M_{n}\left(q^{2}\right)$ and $\mu_{n}\left(q^{2}\right)$ (small $n$ ) differ significantly for $1 \leq-q^{2} \leq 10(\mathrm{GeV})^{2}$, so there is an ambiguity caused by non-leading terms not being negligible. Thus it is not surprising that all three possibilities*) (4.81) continue to be discussed in the literature. For example, Parisi ${ }^{100}$ ) and Nachtmann ${ }^{98}$ ) present the case for large anomalous dimensions $a(n)(\approx 1 / 2$ or 1 ). Parisi's method [in which the moment equations (4.77) are inverted] has since been applied to asymptotically free theories ${ }^{101,102)}$.

The most important qualitative feature of (4.81) is that, if Bjorken scaling is not valid, $\mathrm{F}_{\mathrm{i}}\left(\xi, \mathrm{q}^{2}\right)$ should rise at small $\xi$ and fall near $\xi=1$ as $-\mathrm{q}^{2}$ increases. This is because the smallest value of $n(n=0)$ corresponds to a conserved operator such as $\theta_{\mu \nu}(\lambda(0)=0=a(0))$ and positivity requirements ${ }^{97}$ ) force $a(n)$ and $\lambda(n)$ to increase with $n(\lambda(n), a(n)>0, n>0)$, so all moments decrease except for the asymptotically constant $n=0$ moment. Eventually, $\mathrm{F}_{\mathrm{i}}$ collapses to a $\delta$-function:

$$
\begin{equation*}
F_{i}\left(\xi, q^{2}\right) \longrightarrow 2\left(K_{0}^{\prime} \text { or } K_{0}^{\prime \prime}\right) \delta(\xi) \tag{4.83}
\end{equation*}
$$

Recently, the FNAL muon-scattering group ${ }^{103}$ ) reported indications that the structure functions exhibit this behaviour.

[^24]
### 4.3 Complications for Yang-Mills theories

In many cases, the operator $\mathrm{O}_{\mathrm{m}}$ is not multiplicatively renormalized. Instead, there is a set of operators $\overrightarrow{0}_{m}$ which generate each other as renormalization counterterms

$$
\begin{equation*}
\overrightarrow{\mathrm{O}}_{m} \longrightarrow \overrightarrow{\mathrm{O}}_{m} \overleftrightarrow{\mathrm{Z}}_{m}^{-1} \tag{4.84}
\end{equation*}
$$

where the $Z$-factor and its $\gamma$-function

$$
\begin{equation*}
\stackrel{\gamma}{m}=\lim _{n \rightarrow 4}\left(\mu \partial / \partial \mu \stackrel{Z}{Z}_{m}\right) \stackrel{\square}{Z}_{m}^{-1} \tag{4.85}
\end{equation*}
$$

are matrices. [I am using dimensional regularization, as in (3.43), so $0_{m}$ has been substituted for $0_{n}$ in order to avoid confusion with the complex parameter $n$ ].

According to Eq. (1.8), the counterterms $\Delta Q$ of a vertex $Q$ have canonical dimension $\operatorname{dim}(\Delta Q)$ less than or equal to $\operatorname{dim}(Q)$. Counterterms with $\operatorname{dim}(\Delta Q)<\operatorname{dim}(Q)$ are not generated unless masses or other dimensionful coupling constants are present; e.g. $\bar{\psi} \not \partial \psi$ generates $m \bar{\psi} \psi$. So, if we restrict ourselves to the leading singularities $G^{\text {as. }}$ and $\mathcal{C}_{n}(r=0)$, it is sufficient to compute all counterterms of the zero-mass theory:

$$
\begin{equation*}
\operatorname{dim}(\Delta Q)=\operatorname{dim}(Q) \tag{4.86}
\end{equation*}
$$

In particular, the asymptotic behaviour of the moments $M_{m}\left(q^{2}\right)$ in (4.81) is controlled by a term $\vec{C}_{m} \cdot \overrightarrow{0}_{m}$ in the operator-product expansion of $J_{\mu} J_{\nu}$, where all operators $\left(O_{m}\right)_{i}$ in the set $\vec{O}_{m}$ have the same canonical dimension*)

$$
\begin{equation*}
\operatorname{dim} O_{m}=J+2 \quad, \quad\left(J=\text { spin of } O_{m}\right) \tag{4.87}
\end{equation*}
$$

and where the coefficient functions $\vec{C}_{m}(r=0)$ satisfy the renormalization-group equation

$$
\begin{gathered}
{\left[\eta \partial / \partial \eta-\beta(g) \partial / \partial g+{\stackrel{\leftrightarrow}{\gamma_{m}}}_{m}(g)-d(m)\right]{\overrightarrow{\varphi_{m}}}_{m}(\eta q ; g, \mu)=0} \\
\left(d(m)=\text { mass dimensionality of } \mathscr{b}_{m}\right)
\end{gathered}
$$

The new feature of the solution of Eq. (4.88) [compared with the solution (2.34) of Eq. (2.33)] is that matrices occurring in the exponential must be ordered:

$$
\begin{gather*}
\vec{\varphi}_{m}(\eta q ; q, \mu)=\eta^{-d(m)}\left\{\exp -\int_{g}^{\bar{g}} d x \overleftrightarrow{\gamma}_{m}(x) / \beta(x)\right\}_{\text {ord. }}{\overrightarrow{\rho_{m}}}_{m}(q ; \bar{g}, \mu)  \tag{4.89}\\
\{\exp \ldots\}_{\text {ord }}=\sum_{r=0}^{\infty}(-1)^{r} \int_{g}^{\bar{g}} d x_{1} \int_{x_{1}}^{\bar{g}} d x_{2} \ldots \int_{x_{r-1}}^{\bar{g}} d x_{r} \\
\cdot \overleftrightarrow{\gamma}_{m}\left(x_{1}\right) \overleftrightarrow{\gamma}_{m}\left(x_{2}\right) \ldots \overleftrightarrow{\gamma}_{m}\left(x_{r}\right) / \prod_{i=1}^{r} \beta\left(x_{i}\right)
\end{gather*}
$$

[^25]The limit $\eta \rightarrow \infty$ applied to (4.89) produces the following results:
i) Broken scale invariance: anomalous dimensions are given by the eigenvalues of $\overleftrightarrow{\gamma}_{m}\left(g_{\infty}\right)$.
ii) Asymptotic freedom ${ }^{93}$ ) : the constant $c_{0_{m}}$ in Eq. (4.73) should be replaced by an eigenvalue of the matrix $\stackrel{\leftrightarrow}{c}_{m}$ given by

$$
\begin{equation*}
\overleftrightarrow{\gamma}_{m}(g)=g^{2} \overleftrightarrow{c}_{m}+O\left(g^{4}\right) \tag{4.90}
\end{equation*}
$$

In each case, the dominant contribution in Eq. (4.81) is obtained by substituting the smallest eigenvalue in Eq. (4.82).

The application of these rules to the Yang-Mills Lagrangian (2.65) is not entirely straightforward. The trouble is that, whatever the gauge, the mixing matrix $\overleftrightarrow{Z}$ for $\operatorname{SU}(3)$-singlet twist-two operators is enormous. It would be an onerous task to compute all of the matrix elements of $\stackrel{\leftrightarrow}{c}_{m}$, extract the eigenvalues, and (by considering a sufficiently large class of gauges) determine which eigenvalues are physically relevant. Indeed, the original calculations ${ }^{93}$ ) are based on a simplified prescription which is assumed to be equivalent to the above rules. So the problem is to verify the correctness of the results for twist-two operators and more generally, to find labour-saving prescriptions for arbitrary-twist operators.

Let us begin with the tree approximation for the upper blob graphs $\mathcal{G}^{\prime}$ in Figs. 2 and 14 (i.e, the zero-loop approximation for coefficient functions). Even for this case, the complete set of allowed operators is complicated. The intermediate sets $\left\{\mathrm{k}_{1}, \ldots, \mathrm{k}_{\mathrm{M}}\right\}$ may include Fadde'ev-Popov ghosts as well as fermions and gauge mesons, so some of the tree-approximation operators are ghostdependent and hence not manifestly gauge-invariant. However, if we restrict ourselves to twist-two operators, it is easy to see that intermediate ghost lines cannot appear in the tree approximation for $\mathcal{G}^{\prime}$. An analysis of Abelian gauge theories by Gross and Treiman ${ }^{104}$ ) can be readily generalized to show that the allowed operators are given by symmetrizing the gauge-invariant combinations

$$
\begin{array}{r}
0^{\mu_{1} \ldots \mu_{s}=} \bar{\psi} \gamma^{\mu_{1}} D_{f}^{\mu_{2}} \ldots D_{f}^{\mu_{J}}\left\{1 \text { or } \lambda^{i} / 2\right\} \psi \\
\left(\lambda^{i}=\text { ordinary }-\operatorname{SU}(3) \text { matrices }\right) \tag{4.91}
\end{array}
$$

in the Lorentz indices $\mu_{1}, \ldots, \mu_{J}$ and removing traces. The symmetric-traceless part can be projected out by multiplying the operators in (4.91) by the source ${ }^{93}$ )

$$
\begin{equation*}
x_{\mu_{1}} x_{\mu_{2}} \ldots \quad x_{\mu_{J}}, \quad\left(x^{2}=0\right) \tag{4.92}
\end{equation*}
$$

The result is a pair of spin-J operators, one an $S U(3)$ singlet, the other an octet:

$$
\begin{align*}
Q_{1} & =\bar{\psi} \chi \cdot \gamma\left(i \chi \cdot D_{f}\right)^{J-1} \psi \\
Q(8) & =\bar{\psi} \chi \cdot \gamma\left(i \chi \cdot D_{f}\right)^{J-1}\left(\lambda^{i} / 2\right) \psi \tag{4.93}
\end{align*}
$$

The next step is to consider the renormalization of these operators so that the one-loop contributions $\stackrel{\leftrightarrow}{c}$ to the mixing matrix can be isolated. In this context, the one -100p approximation means that we are considering one -loop counterterms $\Delta_{1} Q$ of $Q$, plus one-loop counterterms $\Delta_{1} \Delta_{1} Q$ of $\Delta_{1} Q$, plus $\Delta_{1} \Delta_{1} \Delta_{1} Q, \ldots$, and so on, until no new vertices are produced. All of these counterterms necessarily appear as operators in the Wilson expansion ${ }^{*}$ ); otherwise, the expansion would not transform consistently under renormalization-group transformations.

Renormalization produces complications because the Lagrangian (2.65) which generates the Feynman rules is not manifestly gauge-invariant. Instead, it is invariant under the following set of transformations, introduced by Becchi, Roust and Stor ${ }^{43}, 50$ ):

$$
\begin{align*}
& \delta A_{\mu}^{a}=-D_{\mu}^{a b}(A) \phi^{b} \delta \lambda, \delta \psi=-i g \phi^{a} \delta \lambda \tau^{a} \psi  \tag{4.94}\\
& \delta \phi^{a}=C^{a}(A) \delta \lambda, \delta \phi^{a}=\frac{1}{2} c^{a b c} \phi^{b} \phi^{c} \delta \lambda
\end{align*}
$$

Observe that $A_{\mu}^{a}$ and $\psi$ undergo a gauge transformation (2.68) with gauge function $\delta \omega^{a}$ given by $\phi^{a} \delta \lambda$. Since the ghost field obeys Fermi-Dirac statistics, the parameter $\delta \lambda$ should be treated as an anticommuting number ${ }^{106)}$. The function $C^{a}(A)$ refers to the choice of gauge; according to the quantization rules ${ }^{45}, 49$ ), the gauge-fixing term can be written in the form

$$
\begin{equation*}
\mathscr{L}_{g \cdot f .}=-\frac{1}{2}\left\{C^{a}(A)\right\}^{2} \tag{4.95}
\end{equation*}
$$

with corresponding ghost Lagrangian given by

$$
\begin{equation*}
\mathscr{L}_{\text {ghost }}=\phi^{a *} \delta C^{a}(A) / \delta \lambda \tag{4.96}
\end{equation*}
$$

As in Eqs. (2.65)-(2.72), Eq. (4.94) involves unrenormalized quantities.
The rules for counterterms $\Delta Q$ of a gauge-invariant operator $Q$ are as follows:
i) If a given order of perturbation theory produces no ghost-dependent terms in $\Delta Q, \Delta Q$ is manifestly gauge-invariant. Roughly speaking, this is because $\phi \delta \lambda$ acts as a c-number for the set of Feynman diagrams being considered.
ii) The appearance of ghost-dependent counter terms in $\Delta Q$ implies a lack of explicit gauge invariance for the accompanying meson-dependent counterterms. A nontrivial analysis ${ }^{107-109)}$ based on the Ward identities associated with (4.94) shows that $\Delta Q$ is symmetric under a source-dependent generalization of (4.94).

[^26]Important special cases of rule (i) are:
a) The operator $Q(8)$ is multiplicatively renormalized ${ }^{93}$ ) in Lorentz-invariant gauges, because octet twist-two operators cannot depend on ghosts and $Q(8)$ is the only vertex which is both Lorentz- and gauge-invariant.
b) All counterterms $\Delta Q$ in the axial gauge ${ }^{110 \text { ) }}$

$$
\begin{equation*}
N^{\mu} A_{\mu}=0, \quad\left(N^{\mu}=\text { Fixed 4-vector }\right) \tag{4.97}
\end{equation*}
$$

are manifestly gauge-invariant because the relevant gauge-fixing term

$$
\begin{equation*}
\dot{L}_{g \cdot f .}=-\lim _{\alpha \rightarrow 0} \frac{1}{2 \alpha}(N, A)^{2} \tag{4.98}
\end{equation*}
$$

produces a vanishing interaction term $\mathrm{g} \phi^{*} \cdot(\mathrm{~N} . \mathrm{A} \times \phi)$ in (4.96).
Since the renormalization of $Q(8)$ presents no difficulties, let us concentrate on the mixing matrix generated by the singlet operator $Q_{1}$ in Eq. (4.93). The one-loop counterterms $\Delta_{1} Q_{1}$ include $Q_{1}$ itself plus a mesonic operator generated by the set of IPI diagrams displayed in Fig. 17. Only fermion propagators are involved, so the new counterterm is proportional to the gauge-invariant combination

$$
\begin{equation*}
Q_{2}=x^{\mu} F_{\mu \alpha}(i X . D)^{J-2} F^{\alpha \nu} x_{\nu} \tag{4.99}
\end{equation*}
$$

The next set of counterterms $\Delta_{1} Q_{2}$ includes $Q_{1}$ and vertices given by the divergent parts of the diagrams in Fig. 18. As noted by Gross and Wilczek ${ }^{93}$ ), the ghost contributions from Fig. 18a do not vanish. This means that the accompanying mesonic terms (Fig. 18b) are necessarily not gauge-invariant ${ }^{111}$ ).

The prescription adopted in Ref. 93 is to construct a $2 \times 2$ submatrix $\overleftrightarrow{Z}(2)$ of $\stackrel{\leftrightarrow}{Z}$ corresponding to the gauge-invariant combinations $Q_{1}, Q_{2}$. Three of the elements of $\overleftrightarrow{Z}(2)\left(Q_{1} \rightarrow Q_{1}, Q_{1} \rightarrow Q_{2}, Q_{2} \rightarrow Q_{1}\right)$ are obtained from $\Delta_{1} Q_{1}$ and $\Delta_{1} Q_{2}$ without ambiguity. The diagonal element $Q_{2} \rightarrow Q_{2}$ is defined to be the contribution from 1PI diagrams in which $Q_{2}$ is coupled to only two external lines, both of which are put on-shell. Thus the rows and columns of $\underset{Z}{Z}$ are chosen to refer to a basis vector with elements $\left\{Q_{1}, Q_{2}, O\left(A^{2}\right)\right.$ operators which vanish on-shell, other operators, i.e. $\left.O\left(A^{3}\right), O\left(A^{4}\right), \ldots, O\left(\phi^{*} \phi\right), O\left(\phi^{*} A \phi\right), \ldots\right\}$, and $\underset{Z}{Z}(2)$ is taken to be the $2 \times 2$ submatrix in the upper left-hand corner of $\stackrel{\leftrightarrow}{Z}$. The answer is supposed to be given by the eigenvalues of $\stackrel{\leftrightarrow}{Z}(2)$. Gross and Wilczek ${ }^{93}$ ) checked their answer by repeating their on-shell prescription for the axial gauge (4.97) in the lightlike case $\mathrm{N}^{2}=0$.

This prescription is not obviously correct because in general, the eigenvalues of a submatrix do not coincide with any of the eigenvalues of the complete matrix. The fact that Gross and Wilczek obtain the same answer in two sets of gauges is not conclusive; it merely suggests that the prescription is gaugeinvariant for twist-two operators. No matter how rigorous the proof of gauge invariance may be, the answer is not correct unless it is an eigenvalue of the complete mixing matrix $\stackrel{\leftrightarrow}{Z}$.

Indeed, Kluberg-Stern and Zuber ${ }^{112}$ ) have observed that the prescription fails for the twist-4, spin-0 operator $\left(F_{\mu \nu}\right)^{2}$ in the usual Fermi gauges (2.70). Subsequent analysis ${ }^{107,109)}$ has shown this to be the rule rather than the exception. However, the exceptional case turns out to be that of twist-two operators. For reasons which can be understood only by consulting the detailed analysis ${ }^{107 \text { ), the }}$ eigenvalues of $\overleftrightarrow{Z}(2)$ are indeed eigenvalues of $\overleftrightarrow{Z}$. Instead of continuing in gauges with ghosts, I shall explain what happens for axial gauges ${ }^{105}$ ).

All Green's functions are homogeneous in the fixed vector $N_{\mu}$, because the only source of $N_{\mu}$-dependence is Eq. (4.98) which generates the free propagator for gauge mesons:

$$
\begin{equation*}
\Delta_{\mu \nu}(k, N)=-\frac{i}{k^{2}}\left\{g_{\mu \nu}-\frac{k_{\mu} N_{\nu}+k_{\nu} N_{\mu}}{k \cdot N}+\frac{N^{2} k_{\mu} k_{\nu}}{(k \cdot N)^{2}}\right\} \tag{4.100}
\end{equation*}
$$

In order to carry out the renormalization program, it is essential that the lightlike case $N^{2}=0$ chosen by Gross and Wilczek be avoided. The chief characteristic of integrals involving denominators $(\mathrm{k} \cdot \mathrm{N})^{-\ell}$ for $\mathrm{N}^{2}=0$ is that it is impossible to maintain power-counting ${ }^{113}$ ); there are insufficient scalar products available. For example, a momentum-dependent scalar counterterm involving $N_{\mu}$ necessarily contains a factor

$$
\begin{equation*}
V(N, p, q)=\int_{i=1}^{r} N \cdot p_{i} / N \cdot q_{i} \quad,\left(p, q=\text { momerita }, N^{2}=0\right) \tag{4.101}
\end{equation*}
$$

because it must be homogeneous in $N_{\mu}$. Since $V$ is not a polynomial in $p$ and $q$, power-counting does not work. In principle, renormalization is still permitted for the non-local vertex $V$, because the denominators $\left(N \cdot q_{i}\right)^{-1},(N \cdot k)^{-\ell}$ are given principal-value singularities and hence have no absorptive part. However, explicit calculations ${ }^{105}$ ) of meson and fermion self-energies demonstrate the existence of divergent parts proportional to $\ln q^{2}$, where $q$ is the momentum of the dressed propagator. The absorptive part does not vanish, so the $\mathrm{N}^{2}=0$ gauge is not renormalizable. For $N^{2} \neq 0$, these difficulties disappear ${ }^{105,114 \text { ) ; the } N_{\mu}-1 . ~-~-~}$ dependence of counterterms always takes the form

$$
\begin{gather*}
\text { counterterm }=\left\{N_{\mu_{1}} \ldots N_{\mu_{2 r}} /\left(N^{2}\right)^{r}\right\}\left\{N_{\mu} \text {-independent }\right\}  \tag{4.102}\\
(r=\text { integer })
\end{gather*}
$$

or some linear combination thereof, and all counterterms are polynomials in momentum space.

Consider the renormalization mixing matrix $Z$ generated by the operator $Q_{1}$ in axial gauges $\left(N^{2} \neq 0\right)$. In view of the trouble caused by ghosts in Lorentzinvariant gauges, it is tempting to assume that the absence of ghosts in axial gauges means that no spurious mixing occurs. The simplest way to disprove this idea is to compute the divergent part of the diagram shown in Fig. 19 for the case $\operatorname{spin} J=2^{105 \text { ), }}$

$$
\begin{array}{r}
P . P . \Gamma_{i j}(J=2)=\left\{g^{2} \delta_{i j} C_{2}(R) / 4 \pi^{2}(n-4)\right\}\left\{-\frac{4}{3} x x \cdot q+N X . N q \cdot X / N^{2}\right. \\
\left.-2 \phi(N . X)^{2} / N^{2}-X X \cdot N q . N / N^{2}\right\} \tag{4.103}
\end{array}
$$

where P.P. denotes the pole part of the PI amplitude $\Gamma_{i j}$ and $C_{2}(R)$ is the value of the quadratic Casimir operator for the fermionic representation $R$ of the gauge group:

$$
\begin{equation*}
\left(\tau^{a} \tau^{a}\right)_{i j}=\delta_{i j} C_{2}(R) \tag{4.104}
\end{equation*}
$$

In addition to the matrix element $Q_{2} \rightarrow Q_{1}$ given by the term proportional to $\mathrm{XX} \cdot \mathrm{q}$, three $N_{\mu}$-dependent vertices are generated. They appear in the following list of independent gauge-invariant operators $P_{r}$ evaluated at zero momentum [ie. $P_{r} \rightarrow$ zeroth-order 1PI amplitude $\left.\left\langle\psi_{j}(-q) P_{r} \bar{\psi}_{i}(q)\right\rangle\right]$ :

$$
\begin{array}{ll}
P_{1}=i \bar{\psi} N X \cdot D_{f} \psi \chi \cdot N / N^{2} & \longrightarrow \\
P_{2}=i \bar{\psi} \not D_{f} \psi(X \cdot N)^{2} / N^{2} & \delta_{i j} N X \cdot q X \cdot N / N^{2}  \tag{4.105}\\
P_{3}=i \bar{\psi} \chi N \cdot D_{f} \psi X \cdot N / N^{2} & \longrightarrow \delta_{i j} \phi(X \cdot N)^{2} / N^{2} \\
P_{4}=i \bar{\psi} \nsim N \cdot D_{f} \psi(X \cdot N)^{2} /\left(N^{2}\right)^{2} \longrightarrow \delta_{i j} \chi N \cdot q X \cdot N / N^{2} \\
& \delta_{i j} N N \cdot q(X \cdot N)^{2} /\left(N^{2}\right)^{2}
\end{array}
$$

(Here, and in the rules stated previously, the characterization "gauge-invariant" refers only to the operator-dependent parts of counterterms. The rules do not forbid dependence on gauge-dependent $c$-numbers such as $N_{\mu}$.) Since we are considering the case $J=2$, the 1 list (4.105) is restricted to operators which are quadratic in $\chi$. Equations (4.103) and (4.105) become very complicated for rbitracy spins J. Similar results can be obtained for the diagrams in Fig. 20; except for the case $J=2$, P.P.Г(J) is $N_{\mu}$-dependent.

Evidently, the mixing matrix $\overleftrightarrow{z}$ is very large. However, some simple features emerge if we define its rows and columns in terms of a well-chosen basis vector. Let us introduce operator classes $C_{p}$, where an operator $Q$ belongs to $C_{p}$ if it is of the form

$$
\begin{equation*}
Q=(X . N)^{p}\{X . N \text { - independent terms }\} \quad,(p=0,1,2, \ldots, J) \tag{4.106}
\end{equation*}
$$

The basis vector is partitioned in the following way: $\left\{\left(C_{0}\right.\right.$ operators $)$, ( $C_{1}$ operators), ( $C_{2}$ operators), $\ldots,\left(C_{J}\right.$ operators $\left.)\right\}$. The result is the partitioned matrix $\overleftrightarrow{Z}$ shown in Fig. 21. There is a series of submatrices $M_{0}, M_{1}, M_{2}, \ldots, M_{J}$ which occupy the main diagonal, where $M_{p}$ describes the mixing of operators within the class $C_{p}$.

A11 counterterms $\Delta Q$ of the operator $Q$ in (4.106) contain the factor $(X \cdot N)^{P}$, because $(X \cdot N)^{\mathrm{P}}$ commutes with loop integrals. Consequently, operators in the class $C_{p}$ do not generate counterterms with smaller values of $p$ :

$$
\begin{equation*}
C_{p} \longrightarrow C_{p^{\prime}}, \quad\left(p^{\prime} \geqslant p\right) . \tag{4.107}
\end{equation*}
$$

Equation (4.107) implies the identity

$$
\begin{equation*}
\operatorname{det}(\stackrel{Z}{Z}-\lambda \overleftrightarrow{I})=\prod_{p=0}^{J} \operatorname{det}\left(M_{p}-\lambda I_{p}\right) \tag{4.108}
\end{equation*}
$$

because all submatrices below the diagonal set ( $M_{0}, M_{1}, M_{2}, \ldots, M_{J}$ ) vanish*). Hence the eigenvalues of $\overleftrightarrow{Z}$ are given by the eigenvalues of $M_{0}, M_{1}, M_{2}, \ldots, M_{J}$.

The only $X \cdot N$-independent gauge-invariant combinations are $Q_{1}, Q_{2}$, and

$$
\begin{equation*}
Q_{3}=\left(N^{\alpha} x^{\beta} F_{\alpha \beta}\right)(i X . D)^{J-2}\left(N^{\gamma} \chi^{\delta} F_{\gamma \delta}\right) / N^{2} \tag{4.109}
\end{equation*}
$$

so the matrix $M_{0}$ is $3 \times 3$. The diagrams in Fig. 17 are $N_{\mu}$-independent, so the matrix element $Q_{1} \rightarrow Q_{3}$ vanishes. Also, an explicit calculation of the diagrams in Fig. 20 yields the result ${ }^{105}$ )

$$
\begin{align*}
\text { P.P. } \Gamma^{a b}(J)= & \frac{g^{2} C_{2}(G)}{2 \pi^{2}(n-4)}\left\langle A_{\alpha}^{a}(q) Q_{2} A_{\beta}^{b}(-q)\right\rangle_{\text {1PI , bare }}  \tag{4.110}\\
& \cdot\left\{1-1 / J(J-1)-1 /(J+1)(J+2)+\sum_{j=2}^{J} j^{-1}\right\}+O(X \cdot N)
\end{align*}
$$

There are nonzero terms proportional to the bare vertices of $Q_{2}$ and (for $J \geq 4$ ) $X \cdot N$-dependent operators, but there is no term proportional to

$$
\begin{gather*}
\left\langle A_{\alpha}^{a}(q) Q_{3} A_{\beta}^{b}(-q)\right\rangle_{1 P I, \text { bare }}=2 \delta^{a b}\left(N \cdot q X_{\alpha}-X_{\cdot q} N_{\alpha}\right)\left(N \cdot q X_{\beta}-X_{\cdot q} N_{\beta}\right)(X \cdot q)^{J-2} / N_{(4.111)}^{2},  \tag{4.111}\\
(J=2,4,6, \ldots)
\end{gather*}
$$

Hence the matrix element $Q_{2} \rightarrow Q_{3}$ also vanishes. Furthermore, explicit calculations (with $N^{2} \neq 0$ ) show that the $2 \times 2$ submatrix of $M_{0}$ generated by $Q_{1}$ and $Q_{2}$ is exactly the same as the submatrix $\stackrel{\leftrightarrow}{Z}(2)$ obtained in Ref. 93; for example, the $Q_{2} \rightarrow Q_{2}$ matrix element can be checked by combining Eq. (4.110) with the one -loop result ${ }^{116}$ )

$$
Z_{3}(\text { axial gauge })=1-\left(11 C_{2}(G)-4 T(R)\right) g^{2} / 24 \pi^{2}(n-4)
$$

$$
\begin{equation*}
+O\left(g^{4}\right), \quad\left(N^{2} \neq 0\right) \tag{4.112}
\end{equation*}
$$

for the $\overleftrightarrow{Z}$-factor of the gauge-meson field $A_{\mu}^{a}$.
The results for $M_{0}$ are summarized in Fig. 22. Clearly, the eigenvalues of $\overleftrightarrow{Z}(2)$ are also eigenvalues of both $M_{0}$ and $\overleftrightarrow{Z}$. Therefore the results in Ref. 93 are correct, provided that the other eigenvalues of $\overleftrightarrow{Z}$ are unphysical. [A separate discussion is necessary in order to verify this last point ${ }^{105)}$.]

The same procedure works for operators with arbitrary twist. Further simplification can be achieved by applying the following theorems ${ }^{105}$ ):
i) The presence of a factor $D^{\mu} F_{\mu \nu},{ }_{\neq} \psi$, or $\bar{\psi} \not \varnothing_{f}$ in the gauge-invariant combinatron $Q$ means that $\Delta Q$ also contains one of these factors.

[^27]ii) All $N_{\mu}$-dependent counterterms contain either $D^{\mu_{F}}{ }_{\mu \nu},{ }_{f} \psi$, or $\bar{\psi}_{f}$ as a factor. Theorem (i) is the analogue of a theorem of Kluberg-Stern and Zuber ${ }^{107 \text { ) for gauges }}$ with ghosts. Some consequences of theorem (ii) are evident in the twist-two case:
a) The result $Q_{2}+Q_{3}$, obtained in the one-loop approximation in Eq. (4.110), is true to all orders of perturbation theory.
b) The operator $P_{4}$, defined in Eq. (4.105), cannot contribute to Eq. (4.103), and the operators $P_{1}$ and $P_{3}$ necessarily appear in the combination $\left(P_{1}-P_{3}\right)$.

## $4.4 n \rightarrow 3 \pi$ decay

Finally, let us consider a problem in current algebra ${ }^{117 \text { ), the soft-meson }}$ theorems for $\eta \rightarrow 3 \pi$ decay. We shall see that it is essential that the analysis be carried out in terms of short-distance expansions. Only then does it become clear that there is an extra term ${ }^{118}$ ) which must be added to the conventional predictions ${ }^{119-121}$ ) for the decay amplitudes.

We begin by recalling the standard assumption ${ }^{122 \text { ) for chiral symmetry- }-1.0 \mid}$ breaking terms in the energy density for strong interactions:

$$
\begin{equation*}
\theta_{00}=\bar{\theta}_{00}+u_{0}+c u_{8} \quad, \quad(c \simeq-1.25) \tag{4.113}
\end{equation*}
$$

The operator $\bar{\theta}_{00}$ is $S U(3) \times S U(3)$ invariant. The scalar-density operators $u_{0}, u_{\theta}$ belong to the $(3, \overline{3}) \oplus(\overline{3}, 3)$ representation formed by the set $\left\{u_{i}, v_{j}\right.$; $i, j=0,1,2, \ldots, 8 ; u_{i}=$ scalar density, $v_{j}=$ pseudoscalar density\}. In gauge theories with explicit chiral symmetry breaking $[\mathcal{M} \neq 0$ in Eq. (2.65)], ( $u_{0}+\mathrm{cu}_{8}$ ) is given by the renormalized version of a mass term in $\bar{\psi} \cdot \mathcal{M} \psi$.

It is natural to assume that the strong-interaction energy density induced by second-order electromagnetic effects is*)

$$
\begin{equation*}
H_{e m}(\text { pure })=-e^{2} \int d^{4} x D^{\mu \nu}(x) T\left\{J_{\mu}(x) J_{v}(0)\right\} \tag{4.114}
\end{equation*}
$$

where $D^{\mu \nu}(x)$ is the free photon propagator in coordinate space and $J_{\mu}$ is the electromagnetic current for hadrons; (see Fig. 23). However, the consequences of (4.114) disagree with experiment:
a) Sutherland ${ }^{123}$ ) showed that the decay amplitudes $\mathrm{A}\left(\eta \rightarrow \pi^{+} \pi^{-} \pi^{0}\right)$ and $\mathrm{A}\left(\eta \rightarrow 3 \pi^{0}\right)$ vanish in the $\operatorname{SU}(2) \times S U(2)$ approximation in which one of the pions (charged or uncharged) is soft.
b) Dashen ${ }^{124}$ ) obtained the rule

$$
\begin{align*}
\left\langle K^{+}\right| \mathcal{H}_{\text {em }}(\text { pure })\left|K^{+}\right\rangle & -\left\langle K^{0}\right| \mathcal{H}_{e m}(\text { pure })\left|K^{0}\right\rangle  \tag{4.115}\\
\simeq & \simeq \pi^{+} \mid \mathcal{H}_{e m}(\text { pure })\left|\pi^{+}\right\rangle-\left\langle\pi^{0}\right| \mathcal{H}_{e m}(\text { pure })\left|\pi^{0}\right\rangle
\end{align*}
$$

*) Added note: To normalize $\mathcal{H e}_{e m}$ correctly, set $D_{\mu \nu}(x)$ equal to $\left(i g_{\mu \nu} / 8 \pi^{2}\right)\left(x^{2}-i \varepsilon\right)^{-1}$.
which is not consistent with the observed $\mathrm{K}^{+}-\mathrm{K}^{0}$ and $\pi^{+}-\pi^{0}$ mass differences.

The standard remedy ${ }^{119}, 121$ ) is to include a $u_{3}$ tadpole ${ }^{125}$ ) in the expression for the energy density:

$$
\begin{aligned}
& \text { total energy density }=\theta_{00}+\mathcal{H}_{\text {em }} \text { (tadpole) } \\
& \mathcal{H}_{e m}(\text { tadpole })=\mathcal{H}_{e m}(\text { pure })+c_{3} u_{3}, \quad\left(c_{3}=O\left(e^{2}\right)\right) .(4.117)
\end{aligned}
$$

In this scheme, Eq. (4.115) is retained and the tadpole is supposed to be responsible for the deviation

$$
\begin{equation*}
\delta m^{2}=m^{2}\left(K^{+}\right)-m^{2}\left(K^{0}\right)-m^{2}\left(\pi^{+}\right)+m^{2}\left(\pi^{0}\right) \tag{4.118}
\end{equation*}
$$

from Dashen's rule; e.g. the $S U(3)$ approximation implies

$$
\begin{align*}
&\langle\pi| u_{3}|\pi\rangle \simeq 0 \\
&\left\langle K^{+}\right| u_{3}\left|K^{+}\right\rangle  \tag{4.119}\\
& \delta m^{2} \simeq 2 c_{3}\left\langle K^{+}\right| u_{3}\left|K^{+}\right\rangle \tag{4.120}
\end{align*}
$$

Similarly, Sutherland's null result for $\mathcal{H e}_{\text {em }}$ (pure) is retained and the decay amplitudes

$$
\begin{equation*}
\mathrm{T}=-\langle 3 \pi| \mathcal{H}_{\mathrm{em}}(\text { tadpole })|\eta\rangle \tag{4.121}
\end{equation*}
$$

are supposed to be given by

$$
\begin{equation*}
T \simeq-c_{3}\langle 3 \pi| u_{3}|\eta\rangle \tag{4.122}
\end{equation*}
$$

in the $\operatorname{SU}(2) \times \operatorname{SU}(2)$ approximation. The amplitude $T\left(n \rightarrow \pi^{+} \pi^{-} \pi^{0}\right)$ still vanishes when a charged pion becomes soft, but the soft $-\pi^{0}$ limit produces a new equal-time commutator

$$
\begin{align*}
& i\left[F_{5}^{3}, u_{3}\right]=\left(\sqrt{2} v_{0}+v_{8}\right) / \sqrt{3}=v  \tag{4.123}\\
& T\left(\eta \rightarrow \pi^{0}(\text { soft }) 2 \pi\right)=\left(c_{3} / F_{\pi}\right)\langle 2 \pi| v|\eta\rangle \tag{4.124}
\end{align*}
$$

where $F^{i}, F_{5}^{j}(i, j=1, \ldots, 8)$ are the charges which generate chiral $\operatorname{SU}(3) \times \operatorname{SU}(3)^{122)}$,

$$
\begin{equation*}
F^{i}=\int d^{3} x \mathcal{F}_{0}^{i}(x) \quad, \quad F_{5}^{i}=\int d^{3} x \mathcal{F}_{50}^{i}(x) \tag{4.125}
\end{equation*}
$$

the pion decay constant*) $\mathrm{F}_{\pi} \simeq 94 \mathrm{MeV}$ is given by

$$
\begin{equation*}
\left\langle\pi^{0}(q)\right| \mathcal{F}_{5 \mu}^{3}(0)|0\rangle=-i q_{\mu} F_{\pi} \tag{4.126}
\end{equation*}
$$

and the matrix element $\langle 2 \pi| v|n\rangle\left(=\left\langle 2 \pi^{0}\right| v|n\rangle=\left\langle\pi^{+} \pi^{-}\right| v|n\rangle\right)$ is evaluated at zero momentum transfer. If $\langle 2 \pi| v|n\rangle$ is not negligibly small, the energy dependence of the decay spectra is given by the formulas ${ }^{120,126)}$
*) Note the extra factor 2 in the definitions of chiral currents and $F_{\pi}$ in
Ref. 119: e.g. $F_{\pi}($ Ref. 119$)=2 F_{\pi}(4.126)$. Ref. 119: e.g. $F_{\pi}($ Ref. 119 $)=2 F_{\pi}(4.126)$.

$$
\begin{aligned}
& \mathrm{T}\left(\eta \rightarrow 3 \pi^{0}\right) \simeq \text { constant }=\left(c_{3} / F_{\pi}\right)\langle 2 \pi| v|\eta\rangle, \\
& \mathrm{T}\left(\eta \rightarrow \pi^{+} \pi^{-} \pi^{0}\right) \simeq\left(1-2 E_{0} \mid m_{\eta}\right)\left(c_{3} \mid F_{\pi}\right)\langle 2 \pi| v|\eta\rangle, \\
& \quad\left(E_{0}=\pi^{0} \text { energy with } \eta \text { at rest }\right),
\end{aligned}
$$

in agreement with experiment ${ }^{127}$ ). Contracting another pion yields the equal-time commutator

$$
\begin{align*}
i\left[F_{5}^{3}, v\right] & =-u_{3}  \tag{4.128}\\
\langle 2 \pi| v|\eta\rangle & \simeq F_{\pi}^{-1}\left\langle\pi^{0}\right| u_{3}|\eta\rangle \tag{4.129}
\end{align*}
$$

Equations (4.120), (4.127), and (4.129) yield the result $\left.{ }^{119}, 120\right)$

$$
\begin{align*}
& T\left(\eta \rightarrow 3 \pi^{0}\right) \simeq \delta m^{2} / \sqrt{3} F_{\pi}^{2}  \tag{4.130}\\
& T\left(\eta \rightarrow \pi^{+} \pi^{-} \pi^{0}\right) \simeq\left(1-2 E_{0} / m_{\eta}\right) \delta m^{2} / \sqrt{3} F_{\pi}^{2}
\end{align*}
$$

This corresponds to a partial decay width ${ }^{119} \simeq 65 \mathrm{eV}$ for $\eta \rightarrow \pi^{+} \pi^{-} \pi^{0}$ which is considerably smaller than the experimental value ${ }^{128}$ ), $204 \pm 29 \mathrm{eV}$. It can be argued that the prediction ( 4.130 ) is not expected to be as accurate as (4.127) because it involves approximate $\operatorname{SU}(3) \times \operatorname{SU}(3)$, not $\mathrm{SU}(2) \times \operatorname{SU}(2)$. Even so, the agreement with experiment is not impressive.

Now we consider the short-distance approach to $\eta \rightarrow 3 \pi$ decay. Wilson ${ }^{4}$ ) observed that the Sutherland analysis and its generalization to include tadpoles assume that the expression (4.114) for $\mathscr{H}_{\mathrm{em}}$ (pure) converges. For the theory of broken scale invariance or asymptotically free theories, this is not true ${ }^{4}, 7$ ). When the operator-product expansion

$$
\begin{equation*}
T\left\{\mathrm{~J}_{\mu}(\mathrm{x}) \mathrm{J}_{\nu}(0)\right\} \sim \sum_{n} \mathscr{E}_{n \mu \nu}(\mathrm{x}) \mathrm{O}_{n}(0) \quad, \quad\left(\mathrm{x}_{\alpha} \rightarrow 0\right) \tag{4.131}
\end{equation*}
$$

is substituted in Eq. (4.114) to test the ultraviolet convergence of the integral $\int d^{4} x$, Lorentz invariance implies

$$
\begin{equation*}
\int d^{4} x \mathscr{C}_{n \mu \nu}(x) D^{\mu \nu}(x)=0 \tag{4.132}
\end{equation*}
$$

for all coefficient functions except those which multiply scalar operators. In the isospin-1 sector*), we have

$$
\begin{equation*}
\mathrm{T}\left\{J_{\mu}(x) J_{\nu}(0)\right\} \sim \mathscr{G}_{\mu \nu}(x) u_{3}(0),\left(x_{\alpha} \rightarrow 0,(J, I)=\left(0^{+}, 1\right) \text { sector }\right) \tag{4.133}
\end{equation*}
$$

This is an example of the special case (4.71), so the formula

$$
\begin{equation*}
\mathscr{G}_{\mu \nu}(x) \sim(\text { constant })\left(g_{\mu \nu} x^{2}+2 x_{\mu} x_{\nu}\right) /\left(x^{2}\right)^{2}, \quad\left(x_{\alpha} \rightarrow 0\right) \tag{4.134}
\end{equation*}
$$

is valid for both broken scale invariance ${ }^{4}$ ) (imespective of the anomalous dimension $d$ of $u_{3}, 1 \leq d<4$ ) and asymptotic freedom ${ }^{7}$ ). Since $D^{\mu \nu}(x)$ goes as $x^{-2}$, we see that the integral (4.114) diverges logarithmically.

[^28]The integral must be properly subtracted so that the result is a convergent expression to which current-algebraic techniques can be applied. The divergence is logarithmic, so a single subtraction suffices ${ }^{4}$ ):

$$
\begin{equation*}
\mathcal{H}_{e m}=-e^{2} \int d^{4} x D^{\mu \nu}(x)\left[T\left\{J_{\mu}(x) J_{\nu}(0)\right\}-f_{\mu \nu}(x) u_{3}(0)\right]+f u_{3}(0) \tag{4.135}
\end{equation*}
$$

The finite constant $f$ is introduced to adjust the finite part of the $u_{3}$ contribution to fit electromagnetic mass differences.

Previously, we saw that the tadpole term was introduced as an extra parameter in response to the difficulties associated with Eq. (4.114). Here, the $u_{3}{ }^{-}$ dependence of the electromagnetic interaction arises naturally. Indeed, the distinction between an electromagnetic term $\mathcal{H e}_{\mathrm{em}}$ (pure) and a strong isospin-breaking term $c_{3} u_{3}$, which can be made for Eqs. (4.116) and (4.117), no longer exists.

Wilson ${ }^{4}, 129$ ) concludes that the interaction (4.135) yields the same results for $\eta \rightarrow 3 \pi$ decay as the tadpole interaction (4.117). He identifies the $f u_{3}$ term as the Coleman-Glashow tadpole $\left.{ }^{125}\right) c_{3} u_{3}$, and argues that the finiteness of the other term means that Sutherland's argument applies to it, in the same way that the Sutherland argument applies to $\mathcal{H}_{\mathrm{em}}$ (pure) if $\int \mathrm{d}^{4} \mathrm{x}$ in Eq. (4.114) happens to converge. However, for reasons which will now be explained, I believe that this conclusion has to be modified.

First, we note that the quantities $C_{\mu \nu}(x), f$ in Eq. (4.135) are not absolute; they can be specified only with respect to a particular subtraction condition. The choice of a subtraction condition from the infinite set of possibilities is purely a matter of convenience. A label i will be added to a symbol to indicate its dependence on a given subtraction condition $i$ :

$$
\begin{align*}
& f, \mathscr{C}_{\mu \nu}(x) \rightarrow f_{i}, \mathscr{C}_{\mu \nu}^{i}(x) \\
& I_{i}= \int d^{4} x T\left\{J_{\mu}(x) J_{\nu}(0)-\mathscr{C}_{\mu \nu}^{i}(x) u_{3}(0)\right\} D^{\mu \nu}(x) \tag{4.136}
\end{align*}
$$

Of course, the total electromagnetic energy density $\mathcal{J}_{\text {em }}$ is observable, so it does not depend on $i$ :

$$
\begin{equation*}
\mathcal{H}_{e m}=-e^{2} I_{i}+f_{i} u_{3}(0) \tag{4.137}
\end{equation*}
$$

For example, we may decide to define the subtraction condition $i=7$ by imposing the constraint

$$
\begin{equation*}
\langle p| I_{7}|p\rangle=\langle n| I_{7}|n\rangle \tag{4.138}
\end{equation*}
$$

in which case $f_{7}$ is proportional to the proton-neutron mass difference:

$$
\begin{equation*}
f_{7}\left\{\langle p| u_{3}|p\rangle-\langle n| u_{3}|n\rangle\right\}=m_{p}-m_{n} \tag{4.139}
\end{equation*}
$$

i) As this example shows, the principles of renormalization are not confined to perturbation theory.
ii) It is not possible to substitute the leading $\mathrm{x}^{-2}$ singularity in Eq. (4.134) for $\mathcal{C}_{\mu \nu}(x)$, because that would introduce an infrared singularity in $\int d^{4} x$ at $x=\infty$.
iii) As a rule (e.g. in perturbation theory), $\mathcal{C}_{\mu \nu}^{i}(x)$ should include the complete leading power of $T\left\{J_{\mu}(x) J_{\nu}(0)\right\}$ as $x \rightarrow 0$. In particular, if the theory is asymptotically free, terms in $T\left\{J_{\mu}(x) J_{V}(0)\right\}$ which are only logarithmically less singular than $\mathrm{x}^{-2}$ can also produce logarithmic infinities in $\int \mathrm{d}^{4} \mathrm{x}$, so they must be included in the subtraction.
iv) For a given prescription $i, C_{\mu \nu}^{i}(x)$ remains unspecified up to functions $\Delta_{\mu \nu}$ which satisfy

$$
\begin{equation*}
\int d^{4} x D^{\mu \nu}(x) \Delta_{\mu \nu}(x)=0 ; \varphi_{\mu \nu}^{i} \longrightarrow \varphi_{\mu \nu}^{i}+\Delta_{\mu \nu} \tag{4.140}
\end{equation*}
$$

Now consider the amplitudes for $\eta \rightarrow 3 \pi$ decay:

$$
\begin{equation*}
A=-\langle 3 \pi| \mathcal{H}_{e m}|\eta\rangle \tag{4.141}
\end{equation*}
$$

\{The symbol A is used instead of T [Eq. (4.121)] to avoid confusing the results implied by the interactions (4.117) and (4.135).\} The $u_{3}$ subtraction does not affect the soft $-\pi^{\ddagger}$ limit because it commutes with the corresponding axial charges. Therefore we recover the desirable result [ $\mathrm{SU}(2) \times \operatorname{SU}(2)$ approximation]

$$
\begin{align*}
A\left(\eta \rightarrow 3 \pi^{0}\right) & \simeq A_{0} \\
A\left(\eta \rightarrow \pi^{+} \pi^{-} \pi^{0}\right) & \simeq\left(1-2 E_{0} / m_{\eta}\right) A_{0} \tag{4.142}
\end{align*}
$$

where the constant $A_{0}$ is given by

$$
\begin{align*}
A_{0} & =A\left(\eta \longrightarrow \pi^{0}(\text { soft }) 2 \pi\right) \\
& =e^{2}\left\langle 2 \pi, \pi^{0}(\text { soft })\right| I_{i}|\eta\rangle+\left(f_{i} \mid F_{\pi}\right)\langle 2 \pi| v|\eta\rangle \tag{4.143}
\end{align*}
$$

and is assumed not to be small.
It is evident that the question of whether Sutherland's argument is applicable or not has nothing to do with the convergence of the integral in (4.135). All of the integrals $I_{i}$ are finite. If we consider a pair $I_{1}, I_{2}$ corresponding to inequivalent subtraction conditions $\left(f_{1} \neq f_{2}\right)$, Eq. (4.137) implies

$$
\begin{align*}
e^{2}\left(I_{1}-I_{2}\right) & =\left(F_{1}-f_{2}\right) u_{3}  \tag{4.144}\\
e^{2}\left\langle 2 \pi, \pi^{0}(\text { soft })\right|\left(I_{1}-I_{2}\right)|\eta\rangle & =-\left(f_{1}-f_{2}\right)\langle 2 \pi| v|\eta\rangle / F_{\pi} \tag{4.145}
\end{align*}
$$

so Sutherland's argument cannot be valid for both of the convergent integrals $I_{1}$ and $I_{2}$. In fact, we can define a subtraction prescription $i=S$ by requiring the validity of the "Sutherland condition"

$$
\begin{equation*}
\left\langle 2 \pi, \pi^{0}(\text { soft })\right| I_{s}|\eta\rangle=0 \tag{4.146}
\end{equation*}
$$

or equivalently,

$$
\begin{equation*}
A_{0}=\left(f_{S} \mid F_{\pi}\right)\langle 2 \pi| v|\eta\rangle \tag{4.147}
\end{equation*}
$$

The same reasoning can be applied to Dashen's rule, Eq. (4.115). Obviously, its derivation assumes the convergence of $\int d^{4} x$ in Eq. (4.114). For the energy density (4.135), the problem is that the $u_{3}$ term in Eq. (4.144) does not satisfy the rule; according to Eq. (4.119), it contributes to the total mass difference $\delta m^{2}$ defined in Eq. (4.118). So there is no analogue of Dashen's rule for the subtracted integrals $I_{i}$ except for a special subtraction prescription $i=D$ in which the "Dashen condition"

$$
\begin{align*}
\left\langle K^{+}\right| I_{D}\left|K^{+}\right\rangle & -\left\langle K^{0}\right| I_{D}\left|K^{0}\right\rangle \\
& =\left\langle\pi^{+}\right| I_{D}\left|\pi^{+}\right\rangle-\left\langle\pi^{0}\right| I_{D}\left|\pi^{0}\right\rangle \tag{4.148}
\end{align*}
$$

is true by definition. The formula for the corresponding constant $f_{D}$ is

$$
\begin{align*}
& \delta m^{2} \\
& =f_{D}\left[\left\langle K^{+}\right| u_{3}\left|K^{+}\right\rangle-\left\langle K^{0}\right| u_{3}\left|K^{0}\right\rangle-\left\langle\pi^{+}\right| u_{3}\left|\pi^{+}\right\rangle+\left\langle\pi^{0}\right| u_{3}\left|\pi^{0}\right\rangle\right] \tag{4.149}
\end{align*}
$$

The conclusion is that the interaction (4.135) produces the same results for $\eta \rightarrow 3 \pi$ decay as the tadpole Hamiltonian (4.117) only if the subtraction prescriptions S and D happen to coincide [either exactly, or within the $\operatorname{SU}(3) \times \operatorname{SU}(3)$ approximation]. There is no reason to suppose that this is the case: in general, there will be a renormalization mismatch:

$$
\begin{equation*}
f_{S} \neq f_{D} \tag{4.150}
\end{equation*}
$$

The analogue of the previous $S U(3) \times \operatorname{SU}(3)$ analysis can be carried out by observing that the $\mathrm{SU}(3)$ approximation to Eq. (4.149) looks like Eq. (4.120):

$$
\begin{equation*}
\delta m^{2} \simeq 2 F_{D}\left\langle K^{+}\right| u_{3}\left|K^{+}\right\rangle \simeq F_{D} \sqrt{3}\left\langle\pi^{0}\right| u_{3}|\eta\rangle \tag{4.151}
\end{equation*}
$$

Combining Eqs. (4.129), (4.147), and (4.151), we obtain the result

$$
\begin{equation*}
A_{0} \simeq\left(f_{S} / f_{D}\right) \delta m^{2} / \sqrt{3} F_{\pi}^{2} \tag{4.152}
\end{equation*}
$$

Apart from the extra factor $\mathrm{f}_{\mathrm{S}} / \mathrm{f}_{\mathrm{D}}$, the answer is the same as Eq. (4.130). The obvious conclusion is that this factor is responsible for the discrepancy between the conventional answer (4.130) and experiment, with

$$
\begin{equation*}
\left|F_{S} / F_{D}\right|_{\operatorname{expt} .} \simeq 1.8 \tag{4.153}
\end{equation*}
$$

At present, an independent theoretical estimate of $f_{S} / f_{D}$ is not available, but it is easy to see that a connection with operator-product expansions exists. A combination of Eqs. $(4.143)$ and (4.147) yields the result

$$
\begin{align*}
& \quad \begin{aligned}
F_{\pi}\left\langle 2 \pi, \pi^{0}(\text { soft })\right| I_{D}|\eta\rangle & =e^{-2}\left(F_{S}-F_{D}\right)\langle 2 \pi| v|\eta\rangle \\
& =\lim _{q \rightarrow 0} i \int d^{4} x e^{i q \cdot y} T\langle 2 \pi| \partial^{\alpha} F_{5 \alpha}^{3}(y) I_{D}|\eta\rangle
\end{aligned}
\end{align*}
$$

$$
\begin{align*}
\partial_{y}^{\alpha} T\left\{\mathcal{F}_{5 \alpha}^{3}(y) I_{D}\right\}= & T\left\{\partial^{\alpha} \mathcal{F}_{5 \alpha}(y) I_{D}\right\}+i e^{-2}\left(f_{S}-f_{D}\right) \delta^{4}(y) v(0)  \tag{4.155}\\
& +Q_{\alpha} \partial^{\alpha} \delta^{4}(y)+Q_{\alpha \beta} \partial^{\alpha} \partial^{\beta} \delta^{4}(y)+\ldots
\end{align*}
$$

where the series of derivative terms $Q_{\alpha} \alpha^{\alpha} \delta^{4}(y), Q_{\alpha \beta} \partial^{\alpha}{ }^{\alpha}{ }^{\beta} \delta^{4}(y), \ldots$, is finite, and does not contribute to the $q \rightarrow 0$ limit. The term proportional to $v(0) \delta^{4}(y)$ is not an equal-time commutator because the operator $I_{D}$ [defined by Eq. (4.136)] is bilocal, not local. Indeed, Sutherland's method, which works for the bilocal $\mathbb{H}_{e m}$ (pure) in (4.114) when $\int d^{4} x$ converges, cannot be applied to $I_{D}$. If we suppose that the derivative $\partial_{y}^{\alpha}$ can be commuted with $\int d^{4} x$ in (4.136), the left-hand side of (4.155) becomes

$$
\begin{align*}
\int d^{4} \times D^{\mu \nu}(x) & (\partial / \partial y)^{\alpha} T\left[\mathcal{F}_{5 \alpha}^{3}(y)\left\{J_{\mu}(x) J_{\nu}(0)-\mathcal{G}_{\mu \nu}^{D}(x) u_{3}(0)\right\}\right] \\
& =T\left\{\partial^{\alpha} \mathcal{F}_{5 \alpha}^{3}(y) I_{D}\right\}+i \delta^{4}(y) v(0) \int d^{4} x D^{\mu \nu}(x) \mathcal{B}_{\mu \nu}^{D}(x) \tag{4.156}
\end{align*}
$$

because the equal-time commutators with $J_{\mu}, J_{\nu}$ vanish, and the commutator with $u_{3}$ is given by Eq. (4.123). The factor $\int d^{4} x D^{\mu \nu} C_{\mu \nu}^{D}$ is infinite, so the method fails.

The simplest way to understand why the $\delta^{4}(y) v(0)$ term appears is to note that it must correspond to a singularity $O\left(y^{-3}\right)$ in the trilocal $T\left\{\mathbb{F}_{5 \alpha} I_{D}\right\}$. Equation (4.155) and the identity

$$
\begin{equation*}
\partial^{\alpha}\left[y_{\alpha} /\left(y^{2}-i \varepsilon\right)^{2}\right]=-2 \pi^{2} i \delta^{4}(y) \tag{4.157}
\end{equation*}
$$

imply

$$
\begin{align*}
& \int d^{4} x D^{\mu \nu}(x) T\left[\mathcal{F}_{5 \alpha}^{3}(y)\left\{J_{\mu}(x) J_{\nu}(0)-\mathscr{f}_{\mu \nu}^{D}(x) u_{3}(0)\right\}\right] \\
& \sim-\left(2 \pi^{2} e^{2}\right)^{-1}\left(f_{S}-f_{D}\right)\left\{y_{\alpha} /\left(y^{2}-i \varepsilon\right)^{2}\right\} v(0) \quad, \quad\left(y_{\alpha} \rightarrow 0\right) \tag{4.158}
\end{align*}
$$

for the pseudoscalar isoscalar sector of the expansion of the trilocal. The lefthand side of (4.158) can be analysed in the same manner as the example discussed in Eqs. (4.17)-(4.27) and Fig. 12. The following regions must be considered:
i) The limit $y \rightarrow 0$ for the product of two operators associated with the subtraction term:

$$
\begin{equation*}
T\left\{\mathcal{F}_{5 \alpha}^{3}(y) u_{3}(0)\right\} \sim\left\{y_{\alpha} / 2 \pi^{2}\left(y^{2}-i \varepsilon\right)^{2}\right\} v(0) \tag{4.159}
\end{equation*}
$$

The normalization is fixed by Eqs. (4.123) and (4.157).
ii) The limit $x, y \rightarrow 0$ for the product of three operators

$$
\begin{equation*}
T\left\{F_{5 \alpha}^{3}(y) J_{\mu}(x) J_{v}(0)\right\} \quad \sim \quad \mathcal{E}_{\alpha \mu v}(x, y) \vee(0) \tag{4.160}
\end{equation*}
$$

Equation (4.160) is yet another example of the special case (4.71), so the equation

$$
\begin{equation*}
\mathscr{C}_{\alpha \mu \nu}(\rho x, \rho y)=O\left(\rho^{-5}\right) \quad, \quad(\rho \rightarrow 0) \tag{4.161}
\end{equation*}
$$

is valid for either broken scale invariance or asymptotic freedom. In general, equal-time commutators from the regions $x-y \ll x, y$ and $y \ll x$ also have to be considered, but for the present case, there is no contribution because $F_{5}^{3}$ commutes with $J_{\mu}$ and $J_{V}$ at equal times.

Substituting Eqs. (4.159) and (4.160) into Eq. (4.158), we arrive at the desired result:

$$
\begin{align*}
& \int d^{4} x D^{\mu \nu}(x) {\left[f_{\alpha \mu \nu}(x, y)-b_{\mu \nu}^{D}(x) y_{\alpha} / 2 \pi^{2}\left(y^{2}-i \varepsilon\right)^{2}\right] } \\
& \sim-\left(2 \pi^{2} e^{2}\right)^{-1}\left(f_{S}-f_{D}\right) y_{\alpha} /\left(y^{2}-i \varepsilon\right)^{2} \quad(y \rightarrow 0) \tag{4.162}
\end{align*}
$$

In contrast with the situation in Eq. (4.156), the integral $\int d^{4} x$ converges at $x=0$, because the $x^{-2}$ singularity of $C_{\alpha \mu \nu}(x, y)$ in the 1 imit $x \ll y$ is precisely cancelled off by the subtraction term. Observe that if we count powers in the integral ( -5 for $\mathcal{C}_{\alpha \mu \nu},-2$ for $D^{\mu \nu}$, and +4 for $\int d^{4} x$ ), the result $-5-2+4=-3$ agrees with the power $O\left(y^{-3}\right)$ which appears on the right-hand side of Eq. (4.162).

It must be emphasized that the only similarity between this analysis and the short-distance analysis ${ }^{4}, 89$ ) of $\pi^{0} \rightarrow 2 \gamma$ decay is that both involve coefficient functions for products of three operator $\mathrm{s}^{130}$ ). The anomalous constant ${ }^{131}$ ) S for $\pi^{0} \rightarrow 2 \gamma$ decay can be produced by the function $C_{\alpha \beta \gamma}(x, y)$ in

$$
\begin{equation*}
T\left\{J_{\alpha}(x) J_{\beta}(0) \mathcal{F}_{5 \gamma}^{3}(y)\right\} \sim C_{\alpha \beta \gamma}(x, y) I \quad, \quad(x, y \rightarrow 0) \tag{4.163}
\end{equation*}
$$

if it is sufficiently singular ${ }^{4}$ ):

$$
\begin{equation*}
\lim _{\rho \rightarrow 0} \rho^{q} C_{\alpha \beta \gamma}(\rho x, \rho y) \neq 0 \quad, \quad(x, y \neq 0) \tag{4.164}
\end{equation*}
$$

Thus, when the operator $(\partial / \partial y)_{\gamma}$ is applied to the left-hand side of (4.163), the leading power $C_{\alpha \beta \gamma}$ can generate a contact term which scales as $\rho^{-10}$ *) :

$$
\begin{equation*}
\partial_{y}^{\gamma} C_{\alpha \beta \gamma}(x, y)=-\frac{S}{2 \pi^{2}} \varepsilon_{\alpha \beta \mu \nu} \partial_{x}^{\mu} \partial_{y}^{\nu} \delta^{4}(x) \delta^{4}(y) \tag{4.165}
\end{equation*}
$$

[compare with Eq. (4.157)]. On the other hand, the analysis of $\eta \rightarrow 3 \pi$ decay does not involve $\delta^{4}(x) \delta^{4}(y)$ or its derivatives explicitly, and should not be understood
*) Conserved $\delta$-function ambiguities in $C_{\alpha \beta \gamma}\left[\right.$ e.g. $\left.\varepsilon_{\alpha \beta} \beta \nu\left(\partial_{x}\right) \gamma^{\partial} x^{\mu} \partial_{y}^{\nu} \delta^{4}(x) \delta^{4}(y)\right]$ are irrelevant: they cannot contribute to the $\partial_{x} \partial y^{\delta} \delta^{4}(x) \delta^{4}(y)$ term. In the literature, beware of incorrect statements that the anomaly is controlled by shortdistance singularities of $\mathrm{T}\left\{\mathrm{J}_{\alpha} \mathrm{J}_{\beta} \partial^{\gamma_{T_{5 \gamma}}^{3}}\right\}$.
purely in terms of the failure of a naive manipulation [ $\partial_{y}$ not commuting with $\left.\int d^{4} x D^{\mu \nu}(x)\right]$. The important point for $\eta \rightarrow 3 \pi$ decay (irrelevant for $\pi^{0} \rightarrow 2 \gamma$ decay) is that one must keep track of the initial subtraction prescription.

Unfortunately, there is no analogue of the fact that the answer for $\pi^{0} \rightarrow 2 \gamma$ decay can be computed directly from the leading-power coefficient function $C_{\alpha \beta \gamma}(x, y)$. Whatever function is chosen for the leading power*) $\mathcal{C}_{\alpha \mu \nu}(x, y)$ in the integrand of (4.162), there is no way of sidestepping the difficulty that we do not know the correct expression for $C_{\mu \nu}^{D}$. Since this function refers explicitly to the subtraction prescription $D$, there is no hope of calculating it in a reasonably model-independent fashion: it depends on complicated non-asymptotic details of the strong interactions. Instead, one should look for measurable amplitudes involving integrals which can be related to the integral in Eq. (4.162).

So far, the $U(1)$ problem ${ }^{132}$ ) of quark models has been ignored. For a long time, it has been known that free-quark models (quark-parton model and its abstractions) are not consistent with approximate chiral symmetry. Such models contain an observable axial baryonic current

$$
\begin{equation*}
\mathcal{F}_{5 \alpha}^{0}=\bar{\psi}_{i} \gamma_{\alpha} \gamma_{5} \psi \tag{4.166}
\end{equation*}
$$

[e.g. generated by $U(6) \times U(6)$ algebra $\left.{ }^{133}\right)$ ] which is partially conserved. The consequences of this are disastrous*):
i) Glashow ${ }^{134}$ ) obtained the result

$$
\begin{equation*}
m_{\eta}^{2} \simeq m_{\pi}^{2} \tag{4.167}
\end{equation*}
$$

which corresponds to the fact that a linear combination $K_{5 \alpha}$ of $\mathbb{T}_{5 \alpha}^{0}$ and $\mathbb{T}_{5 \alpha}^{8}$ becomes conserved in the $\operatorname{SU}(2) \times \operatorname{SU}(2)$ 1imit $(\mathrm{c} \rightarrow-\sqrt{2})$ :

$$
\begin{equation*}
\partial^{\alpha} K_{5 \alpha}=-(\sqrt{2}+c) v / \sqrt{3} \tag{4.168}
\end{equation*}
$$

More generally, there must be an observable pseudoscalar isoscalar particle $L$ with mass $m_{L}$ given by ${ }^{119,132 \text { ) }}$

$$
\begin{equation*}
m_{L}^{2} \lesssim 3 m_{\pi}^{2} \tag{4.169}
\end{equation*}
$$

ii) Brandt and Preparata ${ }^{135}$ ) pointed out that Eq. (4.168) implies

$$
\begin{equation*}
\langle 2 \pi| v|\eta\rangle=0 \tag{4.170}
\end{equation*}
$$

[^29]because the matrix element is evaluated at zero-momentum transfer. Their solution is to abandon approximate chiral symmetry, but I think that too many good chiral-symmetric predictions [e.g. for $\mathrm{K}_{\ell 3}$ decay $\left.{ }^{136}\right)$ ] have to be dismissed as accidents for this approach to be believable.

For gauge theories, the divergence of $\tilde{J}_{5 \alpha}^{0}$ is anomalous, but there is a gauge non-invariant symmetry current

$$
\begin{equation*}
\mathcal{F}_{5 \alpha}^{S}=\mathcal{F}_{5 \alpha}^{0}-\left(g^{2} S / 4 \pi^{2}\right) \varepsilon_{\alpha \beta \mu \nu} A_{a}^{\beta}\left(\partial^{\mu} A_{a}^{\nu}+\frac{1}{3} g c^{a b c} A_{b}^{\mu} A_{c}^{\nu}\right) \tag{4.171}
\end{equation*}
$$

which is partially conserved, so $v$ is still equal to a total divergence:

$$
\begin{equation*}
v \quad \propto \partial^{\alpha} K_{5 \alpha}^{S} \tag{4.172}
\end{equation*}
$$

In order to obtain a non-vanishing result for $\langle 2 \pi| v|n\rangle$, Kogut and Susskind ${ }^{129}$ ) proposed the formula

$$
\begin{equation*}
\langle 2 \pi| F_{5 \alpha}^{s}(0)|\eta\rangle \sim(\text { constant }) q_{\alpha} / q^{2}, \quad(q \rightarrow 0) \tag{4.173}
\end{equation*}
$$

where $q$ is the momentum carried by $\mathbb{T}_{5 \alpha}^{S}$. To ensure that the zero-mass pole is unobservable experimentally, they suppose that it is a linear combination of positive and negative metric propagators, $i / q^{2}$ and $-i / q^{2}$, which cancel for observable operators but not for unobservable operators such as*) $\mathbb{F}_{5 \alpha}^{S}$. Weinberg ${ }^{119 \text { ) }}$ has shown that the Kogut-Susskind mechanism also invalidates the bad result (4.169).

Remarks at the end of Chapters 2 and 3 about the infrared behaviour of current amplitudes can be extended to amplitudes involving $\pi_{5 \mu}^{S}$. Since $\pi_{5 \mu}^{S}$ is partially conserved, it is not multiplicatively renormalized, so the form factor $\pi_{1}\left(q^{2}\right)$ for the amplitude

$$
\begin{equation*}
i \int d^{4} x e^{i q \cdot x} T\langle 0| \mathcal{F}_{5 \alpha}^{s}(x) \mathcal{F}_{5 \beta}^{s}(0)|0\rangle=\left(q_{\alpha} q_{\beta}-g_{\alpha \beta} q^{2}\right) \pi_{1}\left(q^{2}\right)+g_{\alpha \beta} \pi_{2}\left(q^{2}\right) \tag{4.174}
\end{equation*}
$$

satisfies the same renormalization-group equations as $\pi\left(q^{2}\right)$ in Eq. (2.46) [but with a different subtraction function $\left.K(x) \rightarrow K_{1}(x)\right]$. To obtain a pole in $\pi_{1}\left(q^{2}\right)$ at $\mathrm{q}^{2}=0$,

$$
\begin{equation*}
\lim _{\eta \rightarrow 0} \eta^{2} \pi_{1}\left(q^{2} ; \bar{g}, \bar{M}, \mu\right)=\text { finite } \neq 0 \tag{4.175}
\end{equation*}
$$

it is necessary to assume $\overline{\mathrm{g}} \rightarrow \infty$, or $\overline{\mathcal{M}} \rightarrow \infty$, or both; otherwise the basic assumptions of the renormalization-group method are violated. If we assume the infraredsingularity theorems ${ }^{63}, 65$ ) mentioned at the end of Chapter 2 to be applicable here, the effective coupling constant $\bar{g}^{\prime}$ obtained by omitting massive fields should also diverge as $\eta$ tends to zero.
*) The gauge-invariant charge $\int d^{3} x \mathscr{F}_{50}^{S}(x)$ may be observable.

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## REFERENCES

1) See the lectures given by T. Appelquist at this School.
2) E.C.G. Stueckelberg and A. Petermann, Helv. Phys. Acta 26, 499 (1953).
3) M. Gell-Mann and F.E. Low, Phys. Rev. 95, 1300 (1954).
4) K.G. Wilson, Phys. Rev. 179, 1499 (1969).
5) C.G. Callan, Jr., Phys. Rev. D 2, 1541 (1970);
K. Symanzik, Commun. Math. Phys. 18, 227 (1970).
6) G. 't Hooft, Nuclear Phys. B61, 455 (1973).
7) S. Weinberg, Phys. Rev. D 8, 3497 (1973).
8) H.D. Politzer, Phys. Rev. Letters 30, 1346 (1973);
D.J. Gross and F. Wilczek, Phys. Rev. Letters 30, 1343 (1973);
G. 't Hooft, unpublished remarks at the Marseilie meeting on Yang-Mills' theories (June, 1972).
9) The standard textbooks are: N.N. Bogoliubov and D.V. Shirkov, Introduction to the theory of quantized fields, Interscience, New York (1959); S.S. Schweber, Relativistic quantum field theory, Harper and Row, New York (1961); J.D. Bjorken and S.D. Drell, Relativistic quantum fields, McGraw-Hill, New York (1965). See also lectures by S. Coleman, Renormalization and symmetry, Proc. 1971 Int. Summer School of Physics "Ettore Majorana", Editrice Compositori, Bologna (1973).
10) F.J. Dyson, Phys. Rev. 75, 1736 (1949);
A. Salam, Phys. Rev. 82, 217 (1951); ibid. 84, 426 (1951).
11) S. Weinberg, Phys. Rev. 118, 838 (1960).
12) N.N. Bogoliubov and O.S. Parasiuk, Doklady Akad. Nauk (USSR) 100, 25, 429 (1955); O.S. Parasiuk, ibid. 100, 643 (1955); N.N. Bogoliubov and O.S. Parasiuk, Acta Math. 97, $2 \overline{27}$ (1957); 0.S. Parasiuk, Ukr. Math. J. 12, 287 (1960); K. Hepp, Commun. Math. Phys. 2, 301 (1966).
13) The procedure can be formulated without using regulators: W. Zimmermann, Commun. Math. Phys, 6, 161 (1967), and 15, 208 (1969); Lectures on Elementary Particles and Quantum Field Theory, 1970 Brandeis Sunmer Institute in Theoretical Physics, ed. S. Deser, M. Grisaru and H. Pendleton, MIT Press, Cambridge, Mass. (1971), Vol. I, p. 395.
14) However, the situation is not completely hopeless. Interesting suggestions have been made recently by: G. Parisi, Recent progress in Lagrangian field theory and applications, Marseille, June 24-28 (1974), ed. C.P. Korthals-Altes, E. de Rafael and R. Stora, p. 21; Nuclear Phys. B100, 368 (1975); D.I. Blokhintsev, A.V. Efremov and D.V. Shirkov, JINR preprint E2-8027 (1974); K. Symanzik, DESY preprints 75/12 and 75/24 (1975); F. Jegerlehner, Nuclear Phys. B100, 21 (1975).
15) W. Zimmermann, Ann. Phys. (N.Y.) 77, 536, 570 (1973).
16) Cf. Sect. 19.9 of the book by Bjorken and Drell, Ref. 9.
17) M. Astaud and B. Jouvet, Compt. Rend. 264, 1433 (1967); Nuovo Cimento 53A, 841 (1968) and 63A, 5 (1969); M. Astaud, Nuovo Cimento 66A, 111 (1970).
18) S. Co1eman, Dilatations, Proc. 1971 Int. Summer School of Physics "Ettore Majorana", Editrice Compositori, Bologna (1973).
19) See the review by P. Carruthers, Broken scale invariance in particle physics, Phys. Reports 1C, No. 1 (1971).
20) The use of $B$ rules and a regulator $\Lambda$ is not essential. For example, J.H. Lowenstein, Commun. Math. Phys. 24, 1 (1971) uses Zimmermann's normal-product algorithm (Refs. 13, 15).
21) This method for deriving the CS equation was suggested by S. Coleman, Ref. 18; S. Coleman and R. Jackiw, Ann. Phys. (N.Y.) 67, 552 (1971).
22) Similarly, the Ward identities of conformal transformations involve an additional vertex -i $\beta_{R} \int d^{4} x x_{\mu} \phi^{4}(x)$ in $\int d^{4} x x_{\mu} \theta V_{\nu}^{V}$ : B. Schroer, Nuovo Cimento Letters 2, 627 (1971); G. Parisi, Phys. Letters 39B, 643 (1972); C.G. Callan, Jr. and D.J. Gross, Broken conformal invariance, Princeton report (1972); N.K. Nielsen, Nuclear Phys. B65, 413 (1973); S. Sarkar, Phys. Letters 50B, 499 (1974) and Nuclear Phys. B83, 108 (1974); N.K. Nielsen, Nuclear Phys. B97, 527 (1975).
23) S.L. Adler and W.A. Bardeen, Phys. Rev. D 4, 3045 (1971); D 6, 734 (E) (1972); S.L. Adler, Phys. Rev. D 5, 3021 (1972) ; D ㄱ, 1948 (E) (1973).
24) K. Pohlmeyer, DESY report 74/36 (1974).
25) This terminology was invented by K. Symanzik, Springer Tracts in Modern Physics 57, 222 (1971). There is some question as to whether Eq. (2.7) is adequate when some of the momenta $\ell_{i}$ are time-like: K. Symanzik, Comm. Math. Phys. 34, 7 (1973) and Springer Lecture Notes in Physics, Vol. 32. The problem is formulated differently in Ref. 24.
26) For the basic formulas, see J.D. Bjorken and S.D. Drell, Relativistic quantum mechanics, McGraw-Hill, New York (1964), pp. 170-171, and the textbooks mentioned in Ref. 9. The properties of Feynman-parametric integrals are discussed in: R.J. Eden, P.V. Landshoff, D.I. Olive and J.C. Polkinghorne, The analytic S-matrix, Cambridge University Press (1966).
27) For a precise mathematical analysis of the logarithmic power $\beta(\mathcal{G})$, consult J.P. Fink, J. Math. Phys. 9, 1389 (1968).
28) Compare Chapter VIII of the book by Bogoliubov and Shirkov, Ref. 9.
29) Two-loop calculation of $Z_{3}: R$. Jost and J.M. Luttinger, Helv. Phys. Acta 23, 201 (1950). Three-loop calculation: E. de Rafael and J. Rosner, Ann. Phys. (N.Y.) 82, 369 (1974).
30) L.D. Landau, A.A. Abrikosov and I.M. Khalatnikov, Dokl. Acad. Nauk 95, 497, 773,1177 (1954) and 96, 261 (1954); L.D. Landau and I. Pomeranchuk, Dok1. Acad. Nauk 102, 489 (1955); L.D. Landau, in Niels Bohr and the development of physics, Pergamon Press, London (1955), p. 52.
31) F.J. Dyson, Phys. Rev. 85, 631 (1952); B. Simon, Fundamental interactions in physics and astrophysics, 1972 Coral Gables Conference, Plenum Press, New York (1973).
32) S.L. Adler, Phys. Rev. D 5, 3021 (1972).
33) K.G. Wilson, Phys. Rev. D 3, 1818 (1971).
34) R.J. Crewther, S.-S. Shei and T.-M. Yan, Phys. Rev. D 8, 3396 (1973), Appendix $B$,
35) R.J. Crewther, Proc. 2nd Int. Conf. on Elementary Particles (Aix-en-Provence), Journal de Physique, Tome 34, Colloque C-1 (1973), p. 111.
36) 0. Nachtmann, Phys. Letters 51B, 469 (1974) discusses asymptotic corrections to this result for the special case $g=g_{\infty}$.
1) For an extensive review, see H.D. Politzer, Asymptotic freedom: an approach to strong interactions, Phys. Reports 14C, 129 (1974). Reference 35 summarizes some early contributions to the subject.
2) S. Coleman and D.J. Gross, Phys. Rev. Letters 31, 851 (1973).
3) D.J. Gross and F. Wilczek, Phys. Rev. D 8, 3633 (1973).
4) M. Gell-Mann, Acta Phys. Austriaca Supp1. IX, 733 (1972).
5) E.S. Abers and B.W. Lee, Phys. Reports 9C, 1 (1973); M. Veltman, Proc. 1973 Bonn Symposium on Electron and Photon Interactions at High Energies (North-Holland), H. Rollnick, W. Pfeil (ed.); S. Coleman, in Laws of hadronic matter, 1973 International Summer School of Physics "Ettore Majorana", Erice, ed. A. Zichichi (Academic Press, New York and London, 1975), p. 138.
6) G. 't Hooft and M. Veltman, Diagrammar, CERN Report 73/9 (1973).
7) J. Zinn-Justin, Lecture Notes in Physics, Vol. 37 (Springer Verlag, Berlin, 1975); C. Becchi, Lectures at this School.
8) R.P. Feynman, Acta Phys. Polon. 24, 697 (1963); B. de Witt, Phys. Rev. Letters 12, 742 (1964) and Phys. Rev. 162, 1195, 1239 (1967); L.D. Fadde' ev and V.N. Popov, Phys. Letters 25B, 29 (1967); L.D. Fadde'ev, Kiev ITP Report 67-36 [Trans. FNAL-THY-57 (1973)]; S. Mandelstam, Phys. Rev. 175, 1580 (1968); E.S. Fradkin and I.V. Tyutin, Phys. Rev. D 2 , 2841 (1970).
9) G. 't Hooft, Nuclear Phys. B33, 173 (1971).
10) Dimensional regularization: G. 't Hooft and M. Veltman, Nuclear Phys. B44, 189 (1972) and Ref. 42; C.G. Bollini and J.J. Giambiagi, Phys. Letters 40B, 566 (1972).
11) Covariant regularization: A.A. Slavnov, Kiev preprint ITP 71-83 E (1971); B.W. Lee and J. Zinn-Justin, Phys. Rev. D 5, 3121 (1972).
12) A.A. Slavnov, Ref. 47 and Teor. Mat. Fiz. 10, 153 (1972) [Translation: Theor. Math. Phys. 10, 99 (1972)]; J.C. Taylor, Nuclear Phys. B33, 436 (1971).
13) G. 't Hooft and M. Veltman, Nuclear Phys. B50, 318 (1972); B.W. Lee and J. Zinn-Justin, Phys. Rev. D 5, 3121, 3137, 3155 (1972) and D 고, 1049 (1973).
14) C. Becchi, A. Rouet and R. Stora, Comm. Math. Phys. 42, 127 (1975) and Recent progress in Lagrangian field theory and applications, Marseille, June 24-28 (1974), ed. C.P. Korthals-Altes, E. de Rafael and R. Stora, p. 6 .
15) T. Appelquist and H. Georgi, Phys. Rev. D 8, 4000 (1973); A. Zee, Phys. Rev. D 8, 4038 (1973).
16) P.W. Higgs, Phys. Letters 12, 132 (1964); Phys. Rev. Letters 13, 508 (1964); Phys. Rev. 145, 1156 (1966); F. Englert and R. Brout, Phys. Rev. Letters 13, 321 (1964); G.S. Guralnik, C.R. Hagen and T.W.B. Kibble, Phys. Rev. Letters 13, 585 (1964); T.W.B. Kibble, Phys. Rev. 155, 1554 (1967). Review article: J. Bernstein, Rev. Mod. Phys. 46, 7 (1974); 46, $855(\mathrm{E})$ (1974).
17) T.-P. Cheng, E. Eichten and L.-F. Li, Phys. Rev. D 9, 2259 (1974).
18) S. Ferrara and B. Zumino, Nuclear Phys. B79, 413 (1974).
19) N.-P. Chang, Phys. Rev. D 10, 2706 (1974); E. Ma, Phys. Rev. D 11, 322 (1975).
20) H.A. Bethe and E.E. Salpeter, Phys. Rev. 82, 309 (1951); M. Gell-Mann and F.E. Low, Phys. Rev. 84, 350 (1951); E.E. Salpeter and H.A. Bethe, Phys. Rev. 84, 1232 (1951); D. Lurié, Particles and fields, Interscience, New York (1968), Ch. 9.
21) R. Dashen, B. Hasslacher and A. Neveu, Phys. Rev. D 10, 4114, 4130, 4138 (1974): D 11, 3424 (1975); R. Jackiw and J. Goldstone, Phys. Rev. D 11, 1486 (1975); R. Jackiw, Collective phenomena in quantum field theory, Acta Phys. Polon. B (to be published); L.D. Fadde'ev, Institute for Advanced Study (Princeton) preprint (1975).
22) F. Englert and R. Brout, Phys. Rev, Letters 13, 321 (1964); H. Pagels, Phys. Rev. D 7, 3689 (1973); R. Jackiw and K. Johnson, Phys. Rev. D 8, 2386 (1973); J. Cornwall and R. Norton, Phys. Rev. D 8, 3338 (1974) S. Sarkar, Nuclear Phys. B56, 493 (1973); J. Cornwall, Phys. Rev. D 10, 500 (1974); E. Eichten and F. Feinberg, Phys. Rev. D 10, 3254 (1974); E.C. Poggio, E. Tomboulis and S.-H.H. Tye, Phys. Rev. D 11, 2839 (1975).
23) S. Coleman and E. Weinberg, Phys, Rev. D ㄱ, 1888 (1973); D.J. Gross and A. Neveu, Phys. Rev. D 10, 3235 (1974).
24) S. Weinberg, Phys. Rev. Letters 31, 494 (1973); Phys. Rev. D 8, 4482 (1973); H. Fritzsch, M. Gell-Mann and H. Leutwyler, Phys. Letters 47B, 365 (1973); H. Fritzsch and M. Gell-Mann, Proc. XVI International Conference on HighEnergy Physics (1972), Vol. 2, p. 135.
25) F. Bloch and A. Nordsieck, Phys. Rev. 52, 54 (1937); D.R. Yennie, S.C. Frautschi and H. Suura, Ann. Phys. (N.Y.) 13, 379 (1961); K.E. Eriksson, Nuovo Cimento 19, 1010 (1961); $\overline{\text { D. Zwanziger, Phys. Rev. }}$ D 11, 3481, 3504 (1975).
26) S. Weinberg, Phys. Rev. 140, B516 (1965).
27) T. Appelquist and J. Carazzone, Phys. Rev. D 11, 2856 (1975).
28) K.G. Wilson, Phys. Rev. D 10, 2445 (1974) and Recent progress in Lagrangian field theory and applications, Marseille, June 24-28 (1974), ed. C.P. Korthals-Altes, E. de Rafael and R. Stora, P. 125; J. Kogut and L. Susskind, Phys. Rev. D 11, 395 (1975); V. Baluni and J.F. Willemsen, MIT preprint CTP-468 (1975).
29) K. Symanzik, Commun. Math. Phys. 34, 7 (1973).
30) N.N. Bogoliubov and D.V. Shirkov, JETP (USSR) 30, 77 (1956) [Translation: Soviet Physics JETP 3, 57 (1956)].
31) T. Kinoshita, J. Math. Phys. 3, 650 (1962).
32) A. Sirlin, Phys. Rev. D $\underline{5}, 2132$ (1972).
33) M. Baker and K. Johnson, Phys. Rev. 183, 1292 (1969).
34) R.J. Crewther, S.-S. Shei and T.-M. Yan, Phys. Rev. D 8, 1730 (1973).
35) K. Symanzik, Commun. Math. Phys. 23, 49 (1971).
36) K.E. Eriksson, Nuovo Cimento 30, 1423 (1963).
37) C.G. Callan, Jr., unpublished Princeton report (1973), modified the prescription to make it applicable to theories containing scalar particles.
38) G. 't Hooft, Nuclear Phys. B61, 455 (1973).
39) J.F. Ashmore, Lett. Nuovo Cimento 4, 289 (1972); Commun. Math. Phys. 29, 177 (1973); G.M. Cicuta and E. Montaldi, Lett. Nuovo Cimento 4, $32 \overline{9}$ (1972); E.R. Speer, J. Math. Phys. 15, 1 (1974); G. Leibbrañ̄t, Rev. Mod. Phys. 47, 849 (1975).
40) K.G. Wilson, Phys. Rev. D 7, 2911 (1973), Appendix.
41) J.C. Collins, Nuclear Phys. B92, 477 (1975); P. Breitenlohner and D. Maison, Max-Planck Institute reports, Munich, MPI-PAE/PTh 25/74 and 15/75.
42) M.J. Holwerda, W.L. van Neerven and R.P. van Royen, Nuclear Phys. B75, 302 (1974); J.C. Collins and A.J. Macfarlane, Phys. Rev. D 10, 1201 (1974); J.C. Collins, Phys. Rev. D 10, 1213 (1974); S. -Y. Lee, Phys. Rev. D 10, 1103 (1974); L.-P. Yu, Nuc1ear Phys. B81, 458 (1974).
43) G. 't Hooft and M. Veltman, Ref. 46; J.C. Collins, Nuclear Phys. B80, 341 (1974); unpublished proofs of G. 't Hooft and K. Symanzik (quoted by Collins).
44) K.G. Wilson, On products of quantum field operators at short distances, unpublished Cornell report (1964).
45) W. Zimmermann, Lectures on elementary particles and quantum field theory, 1970 Brandeis Summer Institute in Theoretical Physics, ed. S. Deser, M. Grisaru and H. Pendleton, MIT Press, Cambridge, Mass. (1971), Vol. I, p. 395.
46) K.G. Wilson and W. Zimmermann, Commun. Math. Phys. 24,87 (1972); P. Otterson and W. Zimmermann, Commun. Math. Phys. 24, 107 (1972).
47) R. Brandt, Ann. Phys. 44, 221 (1967); 52, 122 (1969); Fortschritte der Physik 18, 249 (1970).
48) Y. Frishman, Phys. Rev. Letters 25, 966 (1970); G. Altarelli, R.A. Brandt and G. Preparata, Phys. Rev. Letters 26, 42 (1971); R.A. Brandt and G. Preparata, Phys. Rev. Letters 25, 1530 (1970); Nuclear Phys. B27, 541 (1971).
49) H. Fritzsch, M. Gell-Mann, Proceedings of the Coral Gables Conference on Fundamental Interactions at High Energies, January 1971, in Scale invariance and the light cone, Gordon and Breach (1971); J.M. Cornwall and R. Jackiw, Phys. Rev. D 4, 367 (1971).
50) C.G. Callan, Jr., Phys. Rev. D 5, 3202 (1972) and 1973 Cargèse lectures (unpublished).
51) N. Christ, B. Hasslacher and A.H. Mueller, Phys. Rev. D 6, 3543 (1972).
52) K. Wilson, Proceedings of the Coral Gables Conference on Fundamental Interactions at High Energies, January 1971, in Vol. II, Scale invariance and the light cone, Gordon and Breach (1971); Proceedings of the Symposium on Electron and Photon Interactions at High Energies, 1971, p. 116.
53) R.J. Crewther, Phys. Rev. Letters 28, 1421 (1972).
54) T.E. Clark, Nuclear Phys. 81, 263 (1974).
55) A.H. Mueller, Phys. Rev. D 9, 963 (1974).
56) C.G. Callan, Jr. and M.L. Goldberger, Phys. Rev. D 11, 1542, 1553 (1975); N. Coote, Phys. Rev. D 11, 1611 (1975).
57) D.J. Gross and F. Wilczek, Phys. Rev. D 9, 980 (1974); H. Georgi and H.D. Politzer, Phys. Rev. D 9, 416 (19̄74).
58) B.L. Ioffe, Zh. Eksp. Teor. Fiz. Pis'ma Red. 9, 163 (1969) [JETP Lett. 9, 97 (1969)]; Phys. Lett. 30B, 123 (1969); L.S. Brown, in High energy collisions of elementary particles, Lectures in Theoretical Physics, Vol. XII-B (1969), ed. K.T. Mahanthappa and W.E. Brittin (Gordon and Breach, New York, 1971), p. 201; R.A. Brandt, Phys. Rev. Lett. 23, 1260 (1969); Phys. Rev. D 1, 2808 (1970).
59) R. Jackiw, R. van Royen and G.B. West, Phys. Rev. D 2, 2473 (1970); H. Leutwyler and J. Stern, Nuclear Phys. B20, 77 (1970).
60) Originally, this connection was obtained in the light-cone-Bjorken-scaling framework (Refs. 84, 85), but it was immediately recognized [e.g. G. Mack, Nuclear Phys. B35, 592 (1971)] that the Bjorken-scaling assumption is irrelevant.
61) O. Nachtmann, Nuclear Phys. B63, 237 (1973).
62) 0. Nachtmann, Nuclear Phys. B78, 455 (1974).
1) G. Parisi, Nuclear Phys. B59, 641 (1973); C.G. Callan, Jr. and D.J. Gross, Phys. Rev. D 8, 4383 (1973).
2) G. Parisi, Phys. Letters 43B, 207 (1973).
3) G. Parisi, Phys. Letters 50B, 367 (1974); D.J. Gross, Phys. Rev. Lett. 32, 1071 (1974); A. de R(jula, Phys. Rev. Lett. 32, 1143 (1974); D.J. Gross and S.B. Treiman, Phys. Rev. Lett. 32, 1145 (1974); A. de Rújula, S.L. Glashow, H.D. Politzer, S.B. Treiman, F. Wilczek and A. Zee, Phys. Rev. D 10, 1649 (1974); A. de Rújula, H. Georgi and H.D. Politzer, Phys. Rev. D $\overline{10}, 2141$ (1974).
4) For reviews of deep inelastic scattering, see the talks by D. Gross, A. de Rújula and F.J. Gi1man, pp. III-65, IV-90, and IV-149 of Proc. XVII International Conference on High-Energy Physics, London 1974 (Science Research Council, Rutherford Laboratory, 1974).
5) C. Chang et al., Phys. Rev. Lett. 35, 901 (1975).
6) D.J. Gross and S.B. Treiman, Phys. Rev. D 4, 1059 (1971).
7) R.J. Crewther, CERN preprint TH.
8) Thus Eq. (4.94) is an example of a supersymmetric transformation: J. Wess and B. Zumino, Nuclear Phys. B70, 39 (1974); talks by B. Zumino and J. Iliopoulos, pp. I-254 and III-89 of Proc. XVII International Conference on High-Energy Physics, London 1974 (Science Research Council, Rutherford Laboratory, 1974).
9) H. Kluberg-Stern and J.B. Zuber, Saclay preprint DPh-T/75/28 (1975).
10) W.S. Deans and J.A. Dixon, Oxford University preprint 20-75 (1975).
11) S.D. Joglekar and B.W. Lee, FERMILAB-Pub-75/50 THY (1975).
12) R. Arnowitt and S.I. Fickler, Phys. Rev. 127, 1821 (1962); S. Coleman, in Laws of hadronic matter, Proc. 1973 International School of Subnuclear Physics, Erice, Sicily (Academic Press, New York and London, 1975), p. 139 .
13) J.A. Dixon and J.C. Taylor, Nuclear Phys. B78, 552 (1974).
14) H. Kluberg-Stern and J.B. Zuber, Phys. Rev. D 12, 467 (1975).
15) Some of these peculiarities have been discussed by J.M. Cornwall, Phys. Rev. D 10, 500 (1974), Appendix.
16) W. Kummer, Acta Phys. Austriaca 41, 315 (1975), Appendix A.
17) J.A. Dixon and J.C. Taylor, Oxford University preprint 74-74 (1974).
18) W. Kainz, W. Kummer and M. Schweda, Nuclear Phys. B79, 484 (1974); R. Delbourgo, A. Salam and J. Strathdee, Nuovo Cimento 23A, 237 (1974).
19) S.L. Adler and R.F. Dashen, Current algebras and applications to particle physics, W.A. Benjamin, Inc. (New York and Amsterdam, 1968); B. Renner, Current algebras and their applications, Pergamon Press Ltd. (London, 1968).
20) As far as I am aware, the existence of this term has not been noted previously. The only hint in the literature seems to be a remark of D.G. Sutherland, Nuclear Phys. B2, 433 (1967), concerning Harari's analysis of electromagnetic mass differences: H. Harari, Phys. Rev. Lett. 17, 1303 (1966).
21) S. Weinberg, Phys. Rev. D 11, 3583 (1975).
22) D.G. Sutherland, Ref. 118; N. Cabibbo and L. Maiani, Phys. Rev. D 1, 707 (1970), and in Evolution of particle physics, ed. M. Conversi, Academic Press (New York, 1970), p. 50; A.J. Cantor, Harvard University Ph.D. thesis, October, 1969 (unpublished), and Phys. Rev. D 3, 3195, 3205 (1971).
23) S.K. Bose and A.M. Zimerman, Nuovo Cimento 43A, 1165 (1966); R. Ramachandran, Nuovo Cimento 47A, 669 (1967); R.H. Graham, L. O'Raifeartaigh and S. Pakvasa, Nuovo Cimento 48A, 830 (1967); Y.T. Chiu, J. Schechter and Y. Ueda, Phys. Rev. 161, $1 \overline{612}$ (1967). These early references do not include the term $\mathcal{H}_{\mathrm{em}}$ (pure) in Eq. (4.117), so the results have to be modified (Ref. 120) to take account of Dashen's rule, Eq. (4.115).
24) M. Gel1-Mann, Phys. Rev. 125, 1067 (1962); S.L. Glashow and S. Weinberg, Phys. Rev. Lett. 20, $2 \overline{24}$ (1968); M. Gell-Mann, R.J. Oakes and B. Renner, Phys. Rev. $175,2 \overline{195}$ (1968). See the review by H. Pagels, Phys. Reports 16C, 219 (1975).
25) D.G. Sutherland, Phys. Lett. 23, 384 (1966).
26) R. Dashen, Phys. Rev. 183, 1245 (1969).
27) S. Coleman and S.L. Glashow, Phys. Rev. 134, B671 (1964).
28) The amplitudes are taken to be linear in the pion energies. For a discussion of this approximation and of theoretical alternatives to the tadpole interaction (4.117), see J.S. Bell and D.G. Sutherland, Nuclear Phys. B4, 315 (1968).
29) C. Baglin et al., Phys. Lett. 29B, 445 (1969); D.W. Carpenter et al., Phys. Rev. D 1, 1303 (1970).
30) See Particle Data Group, Phys. Lett. 50B, 1 (1974) for the branching ratio. The total width is obtained from the recent Cornell result for $\eta \rightarrow 2 \gamma$ : A. Browman et al., Phys. Rev. Lett. 32, 1067 (1974).
31) Wilson's argument is reviewed by J. Kogut and L. Susskind, Phys. Rev. D 11, 3594 (1975).
32) M. Yamada and N. Nakazawa, Miyazaki preprints MM-1 (1974) and MM-2 (1975), claim to derive a three-current effect for the Sutherland interaction (4.114). However, they make assumptions which are not consistent with the assumed convergence of (4.114).
33) J.S. Bell and R. Jackiw, Nuovo Cimento 60A, 47 (1969); S.L. Adler, Phys. Rev. 177, 2426 (1969). For reviews, see: S.L. Adler, Lectures on Elementary Particles and Quantum Field Theory, Brandeis University Summer Institute, MIT Press (Cambridge, Mass., 1970), Vol. I; R. Jackiw, in S.B. Treiman, R. Jackiw and D.J. Gross, Lectures in Current Algebra and its Applications, Princeton University Press (Princeton, N.J., 1972).
34) See the review by $S$. Weinberg, Proc. XVII International Conference on HighEnergy Physics, London, 1974 (Science Research Counci1, Rutherford Laboratory, 1974), p. III-59.
35) R.P. Feynman, M. Gell-Mann and G. Zweig, Phys. Rev. Lett. 13, 678 (1964).
36) S.L. Glashow, in Hadrons and their interactions, Academic Press Inc. (New York, 1968), p: 83; (I thank R. Jackiw for this reference.) See also S.L. Glashow, R. Jackiw and S.-S. Shei, Phys. Rev. 187, 1916 (1969); M. Gell-Mann, in Proc. Third Topical Conference in Particle Physics, Honolulu (1969), ed. W.A. Simmons and S.F. Tuan, Western Periodicals (Los Angeles, 1970), p. 1.
37) R.A. Brandt and G. Preparata, Ann. Phys. (N.Y.) 61, 119 (1970), Section IVB. Also, see K.G. Wilson, Phys. Rev. D 2, 1478 (1970), footnote 14.
38) G. Donaldson et a1., Phys. Rev. Letters 31, 337 (1973).

## Figure captions

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Fig. 1


Fig. 2


Fig. 3


Fig. 4


K $\left(\frac{1}{2} \varphi^{2}\right)$ INSERTIONS
Fig. 5


Fig. 6


Fig. 7



Fig. 8


Fig. 9


Fig. 10

(a)

(b)

(c)

Fig. 11

(a)

(b)



Fig. 12


Fig. 13

$g^{\prime}$


Fig. 14



Fig. 16


Fig. 17

(a)

(b)

Fig. 18


Fig. 19


$\Gamma_{\alpha \beta}^{a b}(\mathrm{~J})$
Fig. 20


Fig. 21


Fig. 22


Fig. 23


[^0]:    *) A 1 PI diagram is a connected graph which cannot be separated into two disconnected pieces by cutting one of its internal lines. The corresponding 1PI amplitude is understood not to include propagators for external lines.

[^1]:    *) More precisely, write $p_{j}^{\prime}=\eta r_{j}$ and let $\eta$ tend to $\infty$, keeping all $r_{j}, p_{i}$ ( $\neq$ any $\left.p_{j}^{\prime}\right)$, and $q_{i}$ fixed and not allowing any partial sum $\sum^{\prime} r$ of the $r j$ to vanish. The latter condition ensures that a $\mathrm{p}^{\prime}$-dependent propagator (which carries momentum $n \sum^{\prime} r+\sum 1 \quad p_{p}^{\prime} p+\sum^{\prime} q$ ) has the expected asymptotic dependence on $n$. The special directions $\sum^{7} r=0$ are covered by looking at appropriate subsets $\mathrm{S}^{\prime}$ of S .
    **) A primitively divergent graph becomes convergent if any internal line is cut.

[^2]:    *) For the general case, the involved combinations of Hepp's proof ${ }^{12)}$ seem to be unavoidable. Authors of textbooks often choose to discuss a specific theory such as quantum electrodynamics, where skeleton expansions and Ward identities permit some simplification.

[^3]:    *) Here, the relevant subgraph $\mathscr{G}^{\prime}$ is the complete graph 9. All other subgraphs involve restricting $\int d^{4} p$ to a finite volume, so they contribute $O\left(q^{-1}\right)$.
    **) The intermediate lines $\mathrm{k}_{1}, \ldots, \mathrm{k}_{\mathrm{M}}$ may be parts of closed loops in $\mathfrak{G}$, but this is not relevant for the subgraph ( $9^{\prime}$. It becomes relevant only when a larger subgraph is considered.

[^4]:    *) In other words, we forbid "oversubtraction": once a 1PI amplitude $\Gamma$ becomes superficially convergent $[d(\Gamma)<0]$, counterterms which further reduce $d(\Gamma)$ are not introduced in Eq. (1.3).

[^5]:    *) Graphs with one intermediate boson line and $\mathfrak{J}\left(\Theta^{\prime}\right)=\mathscr{D}(G)-2$ are possible if a $\phi^{3}$ interaction is added to the Lagrangian (1.10) or (1.12).

[^6]:    *) Unless a selection rule forces it to vanish, $G^{\text {as. }}$ almost invariably receives contributions from the simplest Feynman graphs for $G$. If desired, the subtraction terms can be removed by considering unordered products instead of the $T$-product $\mathrm{G}_{\mathrm{ABC}}^{(\mathrm{b}} \mathrm{f}$ )... in Eq. (1.42).
    **) See Eqs. (1.28) and (1.45), and the addition rule (1.46).

[^7]:    *) Note that ( $1-d^{-1}$ ) is the photon self-energy amplitude (1PI).

[^8]:    *) The sum of the two-loop graphs goes as $-\mathrm{b}_{2} \alpha_{B}^{2} \ln \Lambda$, and there is an additional power of $\ln \Lambda$ for each of the $n-2$ fermion-loop insertions in the internal photon line: $Z_{3}^{-1}($ Fig, 8$)=0\left[(\ln \Lambda)^{n^{-1}}\right]$. Allother subsets of $n-100 p$ graphs (with $Z_{1}=Z_{2}$ for each subset) are $0\left[(\ln \Lambda)^{n-2}\right]$ so, according to the discussion of Eqs. (2.8) and (2.9), they cannot influence the leading logarithmic behaviour of $a_{n}(n)$.

[^9]:    *) Conversely, if $g_{\infty}$ is not a simple zero of $\beta(x)$, it is unlikely that asymptotic scale invariance can be valid: for every Green's function $G$, $\left\{\gamma_{G}(x)-\gamma_{G}\left(g_{\infty}\right)\right\}$ would have to display a sufficiently strong zero at $x=0$ to compensate for the zero of $\beta(x)$ in Eq. (2,50).

[^10]:    *) The scaling factor in (2.74) is $Z_{3}^{-1}$ because Ward identities forbid ( $\left.\partial, A\right)^{2}$ counterterms ${ }^{45}$ : i.e. $\xi_{B}^{-1}(\partial . A)_{B}^{2}=\xi_{R}^{-1}(\partial . A)_{R}^{2}$. Because of (2.74), the CS differential operator $\partial_{G}$ for $\xi$-dependent Green's functions $G$ contains an additional term $-2 \gamma(g, \xi) \xi \partial / \partial \xi$.

[^11]:    *) A variety of classes of graphs can be summed to produce non-perturbative states. The traditional bound-state picture ${ }^{56}$ ) typically involves the ladder approximation for a scattering amplitude. The semi-classical approaches ${ }^{57}$ ) which are currently fashionable treat observable states as coherent superpositions of infinitely many perturbative states.

[^12]:    *) In order to generate ordinary $\operatorname{SU}(3)$ breaking for the $\mathcal{M}=0$ case, it must be supposed that there exist inequivalent methods of summing the theory, and that one of these methods is $\mathrm{SU}(3)$-asymmetric.

[^13]:    *) The connection between the GML analysis and the work of Stueckelberg and Petermann ${ }^{2}$ ) was noted by Bogoliubov and Shirkov ${ }^{28}, 66$ ).

[^14]:    *) So $p(\Delta)$ can be replaced by $4-{ }_{\Delta}$ in Eq. (2.53).
    **) See Section IV of Ref. 33. Similar conclusions were obtained by Symanzik ${ }^{71}$ ) for $\lambda \phi^{4}$ theory in the CS formalism.

[^15]:    *) One proves (egg. by induction in $\ell$ ) that the $\ell-100 p$ amplitude $\left(\partial / \partial m_{R}\right)_{\Gamma}{ }_{\Gamma}$ converges for sufficiently large integers $p$; here, $\Gamma$ is a 1 PI amplitude
    (Fig. 1) with $\ell^{\prime}-100 p$ counterterms ( $\ell^{\prime}<\ell$ ) included in the Lagrangian. Observe that the result is not true for conventionally renormalized theories: $\mathrm{Z}=\mathrm{Z}\left(\Lambda / \mathrm{m}_{\mathrm{R}}, \mathrm{g}_{\mathrm{B}}\right)=$ polynomial in $\mathrm{g}_{\mathrm{B}}$ and $\mathrm{In}\left(\Lambda / \mathrm{m}_{\mathrm{R}}\right)$.

[^16]:    *) Note the analogy between the $\eta$-dependence of ( $\overline{\mathrm{g}}, \bar{m} / m_{R}$ ) and the $\lambda$-dependence of ( $\alpha_{\lambda}, \mathrm{m}^{2} / \lambda^{2}$ ) in Wilson's analysis ${ }^{33)}$. The mass-independence property simplifies the analysis: $\eta \partial / \partial \eta\left(\bar{g}, \ln \bar{m} / m_{R}\right)$ is given by the integrable expression $\left(\beta(\bar{g}), \gamma_{m}(\bar{g})-1\right)$, whereas the equations for $\lambda \partial / \partial \lambda\left(\alpha_{\lambda}, m_{\lambda}^{2} / \lambda^{2}\right)$ are coupled.

[^17]:     passion does not involve performing an infinite summation $\sum_{n=0}^{\infty}$.

[^18]:    *) The consistency conditions used to analyse $\pi^{0} \rightarrow 2 \gamma$ decay ${ }^{89}$ ) involve coefficient functions classified according to (4.13).

[^19]:    *) More precisely, to products which can be constructed from discontinuities of the completely time-ordered product.

[^20]:    *) The name is appropriate because, in free-field theory, generalized Wick products are the same as ordinary Wick products.

[^21]:    *) If zero-mass propagators are present, the expansion must be performed about a non-exceptional point $\lambda=\left(\lambda_{1}, \ldots, \lambda_{M}\right)$ in $k$-space. The subsequent analysis is unchanged, apart from notational complications due to the $\lambda$-dependence of various amplitudes.

[^22]:    *) Unlike Refs. $15,71,81,86,90$, which deal directly with renormalized amplitudes. The present approach is less satisfactory but avoids some complicated algebra.

[^23]:    *) Details of Eqs. (4.79) and (4.80), such as explicit formulas for $\mu_{n}$, are given in Ref. 97. The functions $C_{n}(v)$ are Gegenbauer polynomials.

[^24]:    *) It has been argued ${ }^{38,99)}$ that Bjorken scaling is not consistent with the renormalization group. Note that the possibility $a(n)=0$ (all $n$ ) for gauge theories with $g_{\infty} \neq 0$ has yet to be excluded.

[^25]:    *) In perturbation theory, the "twist" $\tau(Q)$ of an operator ${ }^{104}$ ) is defined as follows: $\tau(Q)=\operatorname{dim}(Q)-\operatorname{spin}(Q)$. Thus Eq. (4.87) refers to twist-2 operators.

[^26]:    *) To simplify the discussion, I shall assume that operators not generated as counterterms of tree-approximation operators do not appear in the Wilson expansion. This problem is circumvented in Ref. 105.

[^27]:    *) That is, $\overleftrightarrow{Z}$ is block-triangular; see Fig. 21. The same property, involving just two operator classes, has been demonstrated for $\overleftrightarrow{Z}$ in Lorentz-covariant gauges ${ }^{107-109,115)}$.

[^28]:    *) Singularities in the $\operatorname{SU}(3)$-singlet sector also have to be removed, but this is irrelevant for $\eta \rightarrow 3 \pi$ decay and electromagnetic mass differences.

[^29]:    *) For asymptotically free theories, the leading singularity of $\mathcal{C}_{\alpha \mu \nu}$ (but not the entire leading power) coincides with the result for free-field theory.
    **) This is the motivation for not assuming $U(6) \times U(6)$ algebra in Ref. 89 .

