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Asymptotic behaviour of non-isotropic random walks with heavy tails

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Abstract A random flight on a plane with non-isotropic displacements at the moments of direction changes is considered. In the case of exponentially distributed flight lengths a Gaussian limit theorem is proved for the position of a particle in the scheme of series when jump lengths and non-isotropic displacements tend to zero. If the flight lengths have a folded Cauchy distribution the limiting distribution of the particle position is a convolution of the circular bivariate Cauchy distribution with a Gaussian law.

Keywords Random flights, non-Gaussian limit theorem, Bessel functions

1 Introduction

We consider the problem of random flights in Euclidean spaces defined by a series of displacements, $\bar{r}_j$, the magnitude and direction of each one being independent of all the previous ones. This model was introduced by Karl Pearson in 1905 and has a long and interesting history, both as a purely mathematical problem in probability theory and as a model for various physical and chemical processes [2]. The majority of papers investigate the problem of random flights with the orientation of movements uniformly distributed over a sphere, and deviations separated by exponentially distributed or Dirichlet distributed time lapses (cf. discussions in [5, 6, 10]). For the

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most recent developments in the studying of the random walks in a random environment we refer to [3] and the papers cited therein. In this short note we introduce a novel feature in the form of non-isotropic displacements at the moments of the direction changes. As a model of this non-isotropic perturbation we consider Hadamard's (or componentwise) product of a fixed deterministic vector (Δ_1, Δ_2) with the unit vector $\bar{e} = (\cos\theta_j, \sin\theta_j)$ in the direction of the previous movement. In the following analysis these perturbations are assumed to be small and the direction changes are frequent enough. In this note we are mainly interested in the case where the distribution of the i.i.d. flight lengths has a heavy tail, say it follows a folded Cauchy distribution.

More precisely, we consider a planar, non-isotropic random walk performed by a particle taking steps (X_j, Y_j), $j \in \mathbf{N}$. We assume that

$$X_j = (R_j + \Delta_1)\cos\theta_j, \qquad Y_j = (R_j + \Delta_2)\sin\theta_j, \tag{1}$$

where θ_j and $R_j = |\bar{r}_j|$ are independent positive random variables (hereafter r.v.'s), and $\Delta_1 \neq \Delta_2$ are deterministic positive real numbers. We assume that θ_j are uniformly distributed in $[0, 2\pi)$ and positive r.v.'s R_j are identically distributed with density $f(r)$, $r > 0$. Clearly, after n steps the position reached by the moving particle is given by

$$\hat{X}_n = \sum_{j=1}^{n} X_j, \qquad \hat{Y}_n = \sum_{j=1}^{n} Y_j. \tag{2}$$

A possible sample path of the random walk (2) is depicted in Figure 1 and can

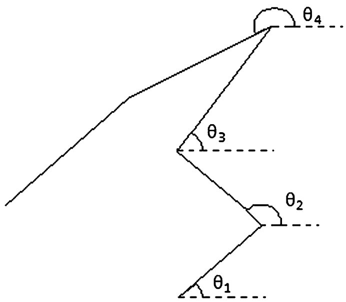

Fig. 1. A sample path of the random walk

be interpreted as the position of a particle taking jumps at integer-valued times, with arbitrary orientation. Since $\Delta_1 \neq \Delta_2$, the distribution of the random walk $(\hat{X}_n, \hat{Y}_n)$, $n \geq 1$, as well as that of its asymptotic limiting process, is not rotation invariant. If the angles θ_j are non-uniformly distributed on $[0, 2\pi)$ the resulting random motion is anisotropic as well, this case will be studied elsewhere. Here two qualitatively different examples are considered: 1) the exponential distribution of the i.i.d. flight lengths, and 2) the folded Cauchy distribution when all the moments m_r, $r \geq 1$ are infinite. Naturally, in the former case under a suitable scaling one obtains the Gaussian limit with independent components having different variances. In the latter case the limiting law is a convolution of a circular bivariate Cauchy distribution with a Gaussian law.

2 Main results

In case 1) we are working under the following assumptions

- (i) The jump lengths $R_j = |\bar{r}_j|, j = 1, \ldots, n$ are exponentially distributed with parameter
$$\mu^{(n)} = \frac{\mu}{t} n^{-1/2}. \tag{3}$$
- (ii) The asymmetry conditions: for some $C_1, C_2 > 0$ (we are interested in the case $C_1 \neq C_2$)
$$\Delta_i^{(n)} = C_i n^{-1/2}, \quad i = 1, 2, \tag{4}$$

i.e. the displacement vectors decrease with n and are the same for all $1 \leq j \leq n$. Condition (i) means that for fixed values of n the step lengths R_j are i.i.d. with exponential distribution whose parameter $\mu^{(n)}$ is adjusted continuously. One can easily see that $\mathbf{E}X_j = \mathbf{E}Y_j = 0$. Next,

$$\mathbf{E}X_1^2 = \left(\mathbf{E}R_1^2 + 2\Delta_1 \mathbf{E}R_1 + \Delta_1^2\right)\mathbf{E}\left[(\cos\theta)^2\right] \approx n^{-1}\left(\frac{t^2}{\mu^2} + \frac{C_1 t}{\mu} + \frac{C_1^2}{2}\right), \tag{5}$$

in view of the equality $\mathbf{E}[(\cos\theta)^2] = \frac{1}{2}$. In fact, (5) is an identity. Here and in what follows the symbol $\approx$ is used to indicate that LHS is an expansion of RHS up to $O(n^{-k})$ for some $k \in \mathbf{Z}$ but the difference RHS–LHS is $o(n^{-k})$. Similarly,

$$\mathbf{E}Y_1^2 \approx n^{-1}\left(\frac{t^2}{\mu^2} + \frac{C_2 t}{\mu} + \frac{C_2^2}{2}\right).$$

Moreover, X_j and Y_j are dependent but not correlated r.v.'s. These facts suggest that a joint limiting distribution is Gaussian and is represented in the form of two independent diffusions. The proof may be provided by the standard methods via the checking Lindeberg's conditions. However, we prefer to use a direct computation to pave the way for further results for r.v.'s with the heavy tails.

Theorem 1. *Under the assumptions (3) and (4) the sequence $(\hat{X}_n, \hat{Y}_n)$ defined in (2) weakly converges to the zero-mean Gaussian vector (X, Y) where X, Y are independent and possess the variances*

$$\mathrm{Var}(X) = \frac{t^2}{\mu^2} + \frac{C_1 t}{\mu} + \frac{C_1^2}{2}, \qquad \mathrm{Var}(Y) = \frac{t^2}{\mu^2} + \frac{C_2 t}{\mu} + \frac{C_2^2}{2}. \tag{6}$$

Our main result in Theorem 2 below establishes the limiting law for the (folded) Cauchy flights:

$$f(r) = \frac{2}{\pi}\frac{a}{r^2 + a^2}, \quad a > 0, r > 0. \tag{7}$$

Let us remind that the standard circular bivariate Cauchy distribution has the joint PDF (see [8], Ch.II, formula (1.19) or [9])

$$f(x_1, x_2) = \frac{1}{2\pi}\frac{1}{(1 + x_1^2 + x_2^2)^{3/2}}. \tag{8}$$

Theorem 2. *Assume the condition (4) and fix $b > 0$. Let the parameter of the folded Cauchy distribution (7) for the jump lengths be scaled as $a_n = \frac{\pi b}{2n}$. Then the distribution of random vector $(\hat{X}_n, \hat{Y}_n)$ weakly converges as $n \to \infty$ to the convolution of the cumulative distribution functions $F_{X,Y} \circ F_{V,W}$ where (X, Y) is a zero mean Gaussian vector with independent components,*

$$\mathrm{Var}(X) = \frac{C_1^2}{2}, \qquad \mathrm{Var}(Y) = \frac{C_2^2}{2}, \tag{9}$$

and the vector (V, W) has a circular bivariate Cauchy distribution with the shape parameter b, i.e.

$$f_{V,W}(x_1, x_2) = \frac{1}{2\pi} \frac{b}{(b^2 + x_1^2 + x_2^2)^{3/2}}. \tag{10}$$

Remark 1. Let us represent the position of the moving particle as a result of n random flights and n non-isotropic displacements $\hat{X}_n = U_n + T_n$, $\hat{Y}_n = V_n + S_n$. Here $U_n = \sum_{j=1}^n R_j \cos\theta_j$, $T_n = \Delta_1 \sum_{j=1}^n \cos\theta_j$, $V_n = \sum_{j=1}^n R_j \sin\theta_j$, $S_n = \Delta_2 \sum_{j=1}^n \sin\theta_j$. Then

$$\mathbf{E} e^{i\alpha \hat{X}_n + i\beta \hat{Y}_n} = \mathbf{E}\big[e^{i(\alpha U_n + \beta V_n)} e^{i(\alpha T_n + \beta S_n)}\big]. \tag{11}$$

The expectations in the RHS of (11) may be split as $n \to \infty$, cf. (38) below. Moreover T_n and S_n are asymptotically independent. However, the pair (U_n, V_n) asymptotically follows a circular bivariate Cauchy law.

3 Proofs

The initial steps are the same for both Theorems 1 and 2 and valid for any PDF $f(r)$ of flight lengths. They are based on the properties of Bessel functions $J_\nu(x)$ which are solutions of ODEs

$$Ly = x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x \frac{\mathrm{d}y}{\mathrm{d}x} + \big(x^2 - \nu^2\big) y = 0, \tag{12}$$

and admit the expansion [1, 11]

$$J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + k + 1)} \left(\frac{x}{2}\right)^{2m+\nu}. \tag{13}$$

Let us fix a small open neighbourhood U of $(0, 0)$. For $(\alpha, \beta) \in U$ the characteristic function of the steps (X_j, Y_j) reads

$$\begin{aligned}
\varphi(\alpha, \beta) &= \mathbf{E}\big[\exp\big(i\alpha[R + \Delta_1]\cos\theta + i\beta[R + \Delta_2]\sin\theta\big)\big] \\
&= \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\theta \int_0^{\infty} e^{i\alpha(r+\Delta_1)\cos\theta + i\beta(r+\Delta_2)\sin\theta} f(r)\,\mathrm{d}r \\
&= \int_0^{\infty} J_0\Big(\sqrt{\big(\alpha^2 + \beta^2\big) r^2 + 2r\big(\alpha^2 \Delta_1 + \beta^2 \Delta_2\big) + \big(\alpha^2 \Delta_1^2 + \beta^2 \Delta_2^2\big)}\Big) f(r)\,\mathrm{d}r.
\end{aligned} \tag{14}$$

Due to the addition formula of Bessel functions ([4], formula 8.531, page 979)

$$J_0\big(\sqrt{\tilde{r}^2+\rho^2-2\tilde{r}\rho\cos\phi}\big) = J_0(\tilde{r})J_0(\rho) + 2\sum_{k=1}^{\infty} J_k(\tilde{r})J_k(\rho)\cos(k\phi), \tag{15}$$

where, in our case,

$$\tilde{r}^2 = \big(\alpha^2+\beta^2\big)r^2, \qquad \rho^2 = \big(\alpha^2\Delta_1^2+\beta^2\Delta_2^2\big),$$
$$\cos\phi = -\frac{\alpha^2\Delta_1+\beta^2\Delta_2}{\sqrt{\alpha^2+\beta^2}\sqrt{\alpha^2\Delta_1^2+\beta^2\Delta_2^2}}. \tag{16}$$

4 Proof of Theorem 1

For the sake of brevity we omit the upper index $\mu^{(n)}$, low index $\varphi_n(\alpha,\beta)$, etc., whenever it is possible. We can calculate explicitly the characteristic function $\varphi(\alpha,\beta)$ by means of integration term by term. Further, we must keep into account the additional result

$$\int_0^\infty e^{-\alpha x}J_\nu(\beta x)\,\mathrm{d}x = \frac{[\sqrt{\alpha^2+\beta^2}-\alpha]^\nu}{\beta^\nu\sqrt{\alpha^2+\beta^2}}, \qquad \nu>-1, \alpha>0 \tag{17}$$

([4], formula 6.611, page 707).

In view of all these formulas we have that for $f(r)=\mu e^{-\mu r}$

$$\begin{aligned}
&\varphi(\alpha,\beta)\\
&\quad = J_0\Big(\sqrt{\alpha^2\Delta_1^2+\beta^2\Delta_2^2}\Big)\int_0^\infty \mu e^{-\mu r}J_0\big(r\sqrt{\alpha^2+\beta^2}\big)\,\mathrm{d}r\\
&\qquad + 2\sum_{k=1}^\infty J_k\Big(\sqrt{\alpha^2\Delta_1^2+\beta^2\Delta_2^2}\Big)\cos\bigg(k\ \mathrm{arcos}\bigg[-\frac{\alpha^2\Delta_1+\beta^2\Delta_2}{\sqrt{\alpha^2+\beta^2}\sqrt{\alpha^2\Delta_1^2+\beta^2\Delta_2^2}}\bigg]\bigg)\\
&\qquad \times \int_0^\infty J_k\big(r\sqrt{\alpha^2+\beta^2}\big)\mu e^{-\mu r}\,\mathrm{d}r\\
&\quad = J_0\Big(\sqrt{\alpha^2\Delta_1^2+\beta^2\Delta_2^2}\Big)\frac{\mu}{\sqrt{\mu^2+\alpha^2+\beta^2}}\\
&\qquad + 2\mu\sum_{k=1}^\infty J_k\Big(\sqrt{\alpha^2\Delta_1^2+\beta^2\Delta_2^2}\Big)\cos\bigg(k\ \mathrm{arcos}\bigg[-\frac{\Delta_1\alpha^2+\Delta_2\beta^2}{\sqrt{\alpha^2+\beta^2}\sqrt{\alpha^2\Delta_1^2+\beta^2\Delta_2^2}}\bigg]\bigg)\\
&\qquad \times \frac{(\alpha^2+\beta^2)^{-k/2}}{\sqrt{\alpha^2+\beta^2+\mu^2}}\big(\sqrt{\mu^2+\alpha^2+\beta^2}-\mu\big)^k.
\end{aligned} \tag{18}$$

A crucial point is now to preserve only the relevant terms of the expansion of $\varphi_n(\alpha,\beta)$ in view of the evaluation of the limit for the characteristic function $\lim_{n\to\infty}[\varphi_n(\alpha,\beta)]^n$, taking into account that $\mu=\mu^{(n)}$, $\Delta_i=\Delta_i^{(n)}$, $i=1,2$. Since

$$\frac{(\sqrt{\mu^2+\alpha^2+\beta^2}-\mu)^k}{\sqrt{\mu^2+\alpha^2+\beta^2}} = \big(\sqrt{\mu^2+\alpha^2+\beta^2}\big)^{k-1}\bigg(1-\frac{\mu}{\sqrt{\mu^2+\alpha^2+\beta^2}}\bigg)^k, \tag{19}$$

we can cut the terms of the expansion for $k \geq 2$. To justify this fact let us use the expansion (13). All the terms with $k \geq 2$ in (18) contain the factors $(\alpha^2\Delta_1^2+\beta^2\Delta_2^2)^{k/2}$ and in view of assumptions (3) and (4) it is easy to check that the sum of these terms is $o(n^{-1})$ uniformly over $(\alpha,\beta) \in U$. Hence,

$$\begin{aligned}
\left[\varphi_n(\alpha,\beta)\right]^n \approx \Bigg[& \frac{\mu}{\sqrt{\alpha^2+\beta^2+\mu^2}} J_0\Big(\sqrt{\alpha^2\Delta_1^2+\beta^2\Delta_2^2}\Big) \\
& - 2J_1\Big(\sqrt{\alpha^2\Delta_1^2+\beta^2\Delta_2^2}\Big) \frac{\alpha^2\Delta_1+\beta^2\Delta_2}{\sqrt{\alpha^2+\beta^2}\sqrt{\alpha^2\Delta_1^2+\beta^2\Delta_2^2}} \\
& \times \frac{\mu}{\sqrt{\alpha^2+\beta^2}} \frac{1}{\sqrt{\alpha^2+\beta^2+\mu^2}} \big(\sqrt{\alpha^2+\beta^2+\mu^2}-\mu\big)\Bigg]^n \\
= \Bigg[& \frac{J_0(\sqrt{\alpha^2\Delta_1^2+\beta^2\Delta_2^2})}{\sqrt{1+\frac{\alpha^2+\beta^2}{\mu^2}}} \\
& - 2J_1\Big(\sqrt{\alpha^2\Delta_1^2+\beta^2\Delta_2^2}\Big) \frac{(\alpha^2\Delta_1+\beta^2\Delta_2)}{(\alpha^2+\beta^2)\sqrt{\alpha^2\Delta_1^2+\beta^2\Delta_2^2}} \\
& \times \mu\left(1-\frac{\mu}{\sqrt{\alpha^2+\beta^2+\mu^2}}\right)\Bigg]^n.
\end{aligned} \tag{20}$$

For $n \to \infty$, $\Delta_i^{(n)} \to 0, i = 1, 2$, and since $J_0(x) \approx 1 - (\frac{x}{2})^2$ as $x \to 0$ and $J_1(x) \approx \frac{x}{2}$ for small values of x we can obtain the following relationship

$$\begin{aligned}
\left[\varphi_n(\alpha,\beta)\right]^n \approx \Bigg[& \left(1-\frac{\alpha^2\Delta_1^2+\beta^2\Delta_2^2}{4}\right)\left(1-\frac{\alpha^2+\beta^2}{2\mu^2}\right) \\
& - \frac{\alpha^2\Delta_1+\beta^2\Delta_2}{\alpha^2+\beta^2}\mu\left(1-\frac{1}{\sqrt{1+\frac{\alpha^2+\beta^2}{\mu^2}}}\right)\Bigg]^n.
\end{aligned} \tag{21}$$

We now take the equalities

$$\frac{1}{\sqrt{1+x}} = \sum_{k=0}^{\infty}\binom{-1/2}{k}x^k = \sum_{k=0}^{\infty}(-1)^k\binom{2k}{k}\frac{x^k}{2^{2k}}, \tag{22}$$

and thus for small values of x we have that $(1+x)^{-1/2} \approx 1-\frac{x}{2}$. In conclusion, by writing explicitly $\mu^{(n)}$, $\Delta_i^{(n)}$, $i = 1, 2$, as in the assumptions (i) and (ii) we have that

$$\begin{aligned}
\left[\varphi_n(\alpha,\beta)\right]^n \approx \Bigg[& \left(1-\frac{\alpha^2\Delta_1^2+\beta^2\Delta_2^2}{4}\right)\left(1-\frac{\alpha^2+\beta^2}{2(\mu^{(n)})^2}\right) \\
& - \frac{\alpha^2\Delta_1+\beta^2\Delta_2}{\alpha^2+\beta^2}\mu^{(n)}\left(\frac{\alpha^2+\beta^2}{2(\mu^{(n)})^2}\right)\Bigg]^n
\end{aligned}$$

$$= \left[\left(1 - \frac{\alpha^2 \Delta_1^2 + \beta^2 \Delta_2^2}{4} \right) \left(1 - \frac{\alpha^2 + \beta^2}{2(\mu^{(n)})^2} \right) - \frac{\alpha^2 \Delta_1 + \beta^2 \Delta_2}{2\mu^{(n)}} \right]^n \quad (23)$$

(by assumptions (i) and (ii))

$$\approx \left[1 - \frac{\alpha^2 C_1^2 + \beta^2 C_2^2}{4n} - \frac{(\alpha^2 + \beta^2) t^2}{2n\mu^2} - \left(\alpha^2 \frac{C_1}{\sqrt{n}} + \beta^2 \frac{C_2}{\sqrt{n}} \right) \frac{t}{2\sqrt{n}\mu} \right]^n$$
$$= \left(1 - \frac{\alpha^2 C_1^2 + \beta^2 C_2^2}{4n} - \frac{(\alpha^2 + \beta^2) t^2}{2n\mu^2} - \frac{\alpha^2 C_1 + \beta^2 C_2}{2} \frac{t}{\mu} \frac{1}{n} \right)^n \to e^{-\frac{\alpha^2 \mathrm{Var} X + \beta^2 \mathrm{Var} Y}{2}}. \quad (24)$$

This concludes the proof of the Theorem 1.

Remark 2. An interesting question is what are the implications of the assumption that Δ_1^2 and Δ_2^2 are negligible with respect to Δ_1 and Δ_2, Let us apply the following formula ([4], formula 6.616, page 710)

$$\int_0^\infty e^{-\alpha x} J_0\big(\beta \sqrt{x^2 + 2\gamma x}\big)\, \mathrm{d}x = \frac{1}{\sqrt{\alpha^2 + \beta^2}} e^{\gamma(\alpha - \sqrt{\alpha^2 + \beta^2})}. \quad (25)$$

By using (25), the characteristic function $\varphi(\alpha, \beta)$ can be written as

$$\varphi(\alpha, \beta) \approx \frac{\mu}{\sqrt{\mu^2 + \alpha^2 + \beta^2}} e^{\frac{\alpha^2 \Delta_1 + \beta^2 \Delta_2}{\alpha^2 + \beta^2}(\mu - \sqrt{\mu^2 + \alpha^2 + \beta^2})}. \quad (26)$$

Therefore, the characteristic function of $(\hat{X}_n, \hat{Y}_n)$, in view of the assumptions on $\mu^{(n)}$ and $\Delta_i^{(n)}$, $i = 1, 2$, becomes

$$[\varphi_n(\alpha, \beta)]^n \approx \left[1 - \frac{\alpha^2 + \beta^2}{2(\mu^{(n)})^2} + \frac{\alpha^2 \Delta_1 + \beta^2 \Delta_2}{\alpha^2 + \beta^2} \left(\mu^{(n)} - \mu^{(n)} \sqrt{1 + \frac{\alpha^2 + \beta^2}{(\mu^{(n)})^2}} \right) \right]^n$$
$$\approx \left[1 - \frac{\alpha^2 + \beta^2}{2(\mu^{(n)})^2} - \frac{\alpha^2 \Delta_1 + \beta^2 \Delta_2}{\alpha^2 + \beta^2} \frac{\alpha^2 + \beta^2}{2\mu^{(n)}} \right]^n$$
$$\to e^{-\frac{\alpha^2}{2}(\frac{t^2}{\mu^2} + \frac{C_1 t}{\mu}) - \frac{\beta^2}{2}(\frac{t^2}{\mu^2} + \frac{C_2 t}{\mu})}. \quad (27)$$

In the steps above we made use of $\frac{\mu^{(n)}}{\sqrt{(\mu^{(n)})^2 + \alpha^2 + \beta^2}} \approx 1 - \frac{\alpha^2 + \beta^2}{2(\mu^{(n)})^2}$ as $n \to \infty$ in view of (22), and used assumption (i) afterwards. In contrast to (6) the terms $\frac{C_i^2}{2}$ are missing from the limiting expression (27). We conclude that the approximation considered above leads to the linearization of the limiting variances with respect to C_1 and C_2.

5 Proof of Theorem 2

In the case of jump lengths with a folded Cauchy distribution (7) the CLT is not applicable. Again, our goal is to calculate the limiting characteristic function keeping the relevant terms in the asymptotic expansion. We omit the lower index in a_n, φ_n whenever it is possible. The equalities (14) and (15) imply

$$\varphi(\alpha,\beta) = J_0(\sqrt{\alpha^2\Delta_1^2+\beta^2\Delta_2^2})\frac{2a}{\pi}\int_0^\infty \frac{J_0(r\sqrt{\alpha^2+\beta^2})}{r^2+a^2}\,\mathrm{d}r - 2J_1(\sqrt{\alpha^2\Delta_1^2+\beta^2\Delta_2^2})\frac{\alpha^2\Delta_1+\beta^2\Delta_2}{\sqrt{\alpha^2+\beta^2}\sqrt{\alpha^2\Delta_1^2+\beta^2\Delta_2^2}} \times \frac{2a}{\pi}\int_0^\infty \frac{J_1(r\sqrt{\alpha^2+\beta^2})}{r^2+a^2}\mathrm{d}r + o(n^{-1}). \tag{28}$$

As in Theorem 1 all the terms of the asymptotic expansion (28) with $k \geq 2$ contain the multiplyer of $(\alpha^2\Delta_1^2+\beta^2\Delta_2^2)^{k/2}$ and the remaining sum is $o(n^{-1})$ uniformly over $(\alpha,\beta)\in U$. Indeed, in view of (13) for $k\geq 2$ we have

$$\left|J_k\Big(\sqrt{\alpha^2\Delta_1^2+\beta^2\Delta_2^2}\Big)\right| < Cn^{-k/2}.$$

The module of the term $\cos(k\phi)$ in (15) is estimated by 1. Let $c=\sqrt{\alpha^2+\beta^2}$ and δ be the Kronecker delta-function. Note that for any $\kappa>0$

$$\frac{2a}{\pi}\int_0^\infty \frac{J_k(cr)}{r^2+a^2}\,\mathrm{d}r < C\Big(n\int_0^{n^{-2/k-\kappa}} r^{k/2}\,\mathrm{d}r + n^{-1}\int_{n^{-2/k-\kappa}}^\infty \frac{J_k(cr)}{r^2}\,\mathrm{d}r\Big) < C\Big(n^{-\kappa-\kappa k/2}+\frac{1+\kappa}{n}\ln(n)\delta_{k=2}+n^{-1-\kappa k/2}\delta_{k>2}\Big), \tag{29}$$

and a similar bound holds from below. Hence, the series is absolutely convergent. For these reasons it remains to consider the first two terms in the expansion (28).

For this aim note that

$$\int_0^\infty \frac{J_1(cr)}{r^2+a^2}\,\mathrm{d}r = \frac{c}{a^2c^2}\Big[\int_0^\infty J_1(u)\,\mathrm{d}u - \int_0^\infty \frac{J_1(u)u^2}{u^2+(ac)^2}\,\mathrm{d}u\Big], \tag{30}$$

and $\int_0^\infty J_1(u)\,\mathrm{d}u = -\int_0^\infty J_0'(u)\,\mathrm{d}u = 1$. By differentiating w.r.t. the parameter the formula 6.532.4 from [4] one gets

$$\int_0^\infty \frac{J_1(u)u^2}{u^2+(ac)^2}\,\mathrm{d}u = acK_1(ac), \tag{31}$$

where K_1 stands for the Macdonald function [1, 11]. So, we obtain that

$$I_1 = \int_0^\infty \frac{J_1(r\sqrt{\alpha^2+\beta^2})}{r^2+a^2}\mathrm{d}r = \frac{1}{a^2(\sqrt{\alpha^2+\beta^2})}\big(1-a\sqrt{\alpha^2+\beta^2}K_1\big(a\sqrt{\alpha^2+\beta^2}\big)\big). \tag{32}$$

If $a=a_n=\frac{\pi b}{2n}$ then for large n we have $K_1(ac)=\frac{1}{a_nc}+\frac{a_nc}{4}(2\gamma-1)+o(n^{-1})$ where $\gamma=0.57721566$ stands for the Euler–Mascheroni constant and

$$I_1 = \frac{1}{a_n^2c}\Big[1-a_nc(\frac{1}{a_nc}+\frac{a_nc}{4}(2\gamma-1)+o\big(n^{-1}\big)\Big] = -\frac{2\gamma-1}{4}c+O\big(n^{-1}\big).$$

As a result we obtain that the second term in (28) is $O(n^{-3/2})$ and does not contribute asymptotically.

Alternatively, according to [4], formula 6.532.1 for non-integer ν

$$I_\nu(a) = \int_0^\infty \frac{J_\nu(x)}{x^2+a^2}\,\mathrm{d}x = \frac{\pi}{a\sin(\pi\nu)}\big(\bar{J}_\nu(a) - J_\nu(a)\big), \tag{33}$$

where $\bar{J}_\nu(a)$ stands for the Anger function which is a solution of the inhomogeneous Bessel equation $Ly = (x-\nu)\frac{\sin\pi x}{\pi}$ [1]. By definition the Anger functions always coincide with the Bessel functions for the integer values of ν. The following identity is well-known [7]

$$\bar{J}_\nu(x) = \frac{\sin\pi\nu}{\pi}\sum_{l=-\infty}^{\infty}(-1)^l\frac{J_l(x)}{\nu-l}. \tag{34}$$

So, we use l'Hôpital's rule to evaluate the integral $I_1(a) = \lim_{\nu\to1} I_\nu(a)$ in (33) and obtain

$$\lim_{n\to\infty}\lim_{\nu\to1}\frac{\pi}{a_n\sin(\pi\nu)}\big(\bar{J}_\nu(a_n) - J_\nu(a_n)\big) = 0.$$

Anyway, we need to evaluate the first term in expansion (28). According to [4], formula 6.532.6

$$\int_0^\infty \frac{J_0(bx)}{x^2+a^2}\,\mathrm{d}x = \frac{\pi}{2a}\big(I_0(ab) - L_0(ab)\big),$$

where I_0 stands for the modified Bessel function and L_0 is the modified Struve function. Let us remind that the modified Struve function satisfies the inhomogeneous Bessel equation [1, 11]

$$Ly = x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - \big(x^2+\nu^2\big)y = \frac{4(x/2)^{\nu+1}}{\sqrt{\pi}\,\Gamma(\nu+1/2)}. \tag{35}$$

By using expansions of the modified Struve functions in a neighbourhood of 0 (see [1, 11])

$$L_0(x) = \left(\frac{x}{2}\right)\sum_{k=0}^{\infty}\frac{1}{(\Gamma(3/2+k))^2}\left(\frac{x}{2}\right)^{2k}, \tag{36}$$

we write two terms of the asymptotic expansion

$$\frac{2a}{\pi}\int_0^\infty \frac{J_0(r\sqrt{\alpha^2+\beta^2})}{r^2+a^2}\mathrm{d}r \approx 1 + \frac{a^2(\alpha^2+\beta^2)}{4} - \frac{a\sqrt{\alpha^2+\beta^2}}{2\Gamma(3/2)^2} + \cdots$$

Finally, in view of the formula 6.565.3 of [4]

$$\int_0^\infty \frac{x^{\nu+1}}{(x^2+a^2)^{\nu+3/2}}J_\nu(bx)\mathrm{d}x = \frac{b^\nu\sqrt{\pi}}{2^\nu a\Gamma(\nu+3/2)}e^{-ab}. \tag{37}$$

Applying (37) for $\nu = 0$ we obtain that the characteristic function of the circular bivariate Cauchy law has the form $e^{-b\sqrt{\alpha^2+\beta^2}}$. So, in the limit we obtain the product of the characteristic functions of the Gaussian law and the circular bivariate Cauchy distribution

$$\lim_{n\to\infty}\left[\varphi_n(\alpha,\beta)\right]^n = \exp\left(-\frac{C_1^2}{2}\alpha^2 - \frac{C_2^2}{2}\beta^2\right)\exp\left(-b\sqrt{\alpha^2+\beta^2}\right), \tag{38}$$

and Theorem 2 is proved.

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6 Appendix section

It is interesting that for odd $2n+1$ the integral $\int_0^\infty \frac{J_{2n+1}(cr)}{r^2+a^2}\,\mathrm{d}r$ can be expressed in terms of the Macdonald function $K_{2n+1}(ac)$ and for even $2n$ the integral $\int_0^\infty \frac{J_{2n}(cr)}{r^2+a^2}\,\mathrm{d}r$ is a linear combination of the modified Bessel function $I_{2n}(ac)$ and the modified Struve functions $L_k(ac)$, $k \le 2n$. Let us remind that the Macdonald function $K_\nu(x)$ is a positive solution of the equation

$$Ly = x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - \left(x^2+\nu^2\right)y = 0 \tag{39}$$

vanishing when $x \to \infty$ and the modified Struve function is defined in (35).

For odd order, Bessel functions $\int_0^\infty \frac{J_{2n+1}(cr)}{r^2+a^2}\,\mathrm{d}r$, $a, c > 0$ take the form

$$2n+1 = 1 : \frac{1}{a^2c} - \frac{K_1(ac)}{a},$$

$$2n+1 = 3 : \frac{K_3(ac)}{a} + \frac{1}{a^2c} - \frac{8}{a^4c^3},$$

$$2n+1 = 5 : -\frac{K_5(ac)}{a} + \frac{1}{a^2c} - \frac{24}{a^4c^3} + \frac{384}{a^6c^5},$$

$$2n+1 = 7 : \frac{K_7(ac)}{a} + \frac{1}{a^2c} - \frac{48}{a^4c^3} + \frac{1920}{a^6c^5} - \frac{46080}{a^8c^7},$$

$$2n+1 = 9 : -\frac{K_9(ac)}{a} + \frac{1}{a^2c} - \frac{80}{a^4c^3} + \frac{5760}{a^6c^5} - \frac{322560}{a^8c^7} + \frac{10321920}{a^{10}c^9},$$

$$2n+1 = 11 : \frac{K_{11}(ac)}{a} + \frac{1}{a^2c} - \frac{120}{a^4c^3} + \frac{13440}{a^6c^5} - \frac{1290240}{a^8c^7} + \frac{92897280}{a^{10}c^9} - \frac{3715891200}{a^{12}c^{11}}.$$

For even order, Bessel functions $\int_0^\infty \frac{J_{2n}(cr)}{r^2+a^2}\,\mathrm{d}r$, $a, c > 0$ take the form

$$2n = 0 : \frac{\pi I_0(ac)}{2a} - \frac{\pi L_0(ac)}{2a},$$

$$2n = 2 : \frac{c}{3} - \frac{\pi I_2(ac)}{2a} + \frac{\pi L_2(ac)}{2a},$$

$$2n = 4 : \frac{c}{15} + \frac{\pi I_4(ac)}{2a} - \frac{\pi L_2(ac)}{2a} + \frac{3\pi L_3(ac)}{a^2 c},$$

$$2n = 6 : \frac{a^2 c^3}{105} + \frac{c}{35} - \frac{\pi I_6(ac)}{2a} + \frac{\pi L_4(ac)}{2a} - \frac{5\pi L_3(ac)}{a^2 c} + \frac{40\pi L_4(ac)}{a^3 c^2},$$

$$2n = 8 : \frac{a^2 c^3}{63} + \frac{c}{63} + \frac{\pi I_8(ac)}{2a} - \frac{\pi L_4(ac)}{2a} + \frac{12\pi L_5(ac)}{a^2 c} - \frac{84\pi L_4(ac)}{a^3 c^2} + \frac{840\pi L_5(ac)}{a^4 c^3}.$$

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