Annali di Matematica pura ed applicata
(IV), Vol. CLXXIV (1998), pp. 13-72

Asymptotic Behaviour of Nonlinear Dirichlet Problems in Perforated Domains (*).

GIANNI DAL MASO - IGOR V. SKRYPNIK

Abstract. – *The asymptotic behaviour of the solutions of nonlinear second order elliptic equations with Dirichlet boundary conditions in performated domains is studied under very mild assumptions on the capacity of the holes.*

0. – Introduction.

In this paper we study the asymptotic behaviour of the solutions of nonlinear second order elliptic equations with Dirichlet boundary conditions in perforated domains.

Let Ω be a bounded open set in the n-dimensional Euclidean space $\boldsymbol{R}^n$ and let Ω_s, $s = 1, 2, \ldots$, be an arbitrary sequence of open subsets of Ω. We consider the sequence of boundary value problems

$$\sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(a_j \left(x, u_s(x), \frac{\partial u_s(x)}{\partial x} \right) \right) = a_0 \left(x, u_s(x), \frac{\partial u_s(x)}{\partial x} \right) \quad \text{in } \Omega_s, \tag{0.1}$$

$$u_s(x) = f(x) \quad \text{in } \partial\Omega_s. \tag{0.2}$$

We assume (see conditions A_1, A_2, and A_3 in Section 1) that the functions $a_j(x, u, p)$, $j = 0, 1, \ldots, n$, and $f(x)$ satisfy the usual conditions which ensure that, for every s, problem (0.1), (0.2) has a solution $u_s(x)$ in $W_m^1(\Omega_s)$. If we extend $u_s(x)$ to Ω by setting $u_s(x) = f(x)$ on $\Omega \backslash \Omega_s$, then our assumptions imply that the sequence $u_s(x)$ is bounded in $W_m^1(\Omega)$. For simplicity of exposition we consider only the case $2 \leqslant m < n$.

The aim of this paper is to study the asymptotic behaviour of $u_s(x)$ as $s \to \infty$ under very weak assumptions on the sets Ω_s. Our main hypothesis is the following condition

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(*) Entrata in Redazione il 27 dicembre 1995.

Indirizzo degli AA.: G. DAL MASO: SISSA, Via Beirut 4, 34013 Trieste, Italy; e-mail: dalmaso@sissa.it; I. V. SKRYPNIK: Institute of Applied Mathematics and Mechanics, Academy of Sciences of Ukraine, R. Luxemburg St. 74, 340114 Donetsk, Ukraine; e-mail: skrypnik@iamm.ac.donetsk.ua

B_1, where $K(x, r)$ denotes the closed cube of centre x and side $2r$, and $C_m(F)$ is the m-capacity of F with respect to a given bounded open set Ω_0 containing $\overline{\Omega}$.

CONDITION B_1. – *There exist a non-negative bounded measure $\nu(B)$, defined for every Borel set $B \subset \Omega$, and a sequence $r_s > 0$, tending to zero as $s \to \infty$, such that the inequality*

$$C_m(K(x, r) \setminus \Omega_s) \leq \nu(K(x, r + r_s)) \tag{0.3}$$

holds for every $x \in \Omega$ and for every $r \geq r_s$ with $K(x, r + r_s) \subset \Omega$.

Using the subadditivity of the m-capacity, it is easy to see that condition B_1 is satisfied when the sets Ω_s are obtained from Ω by removing an increasing number of small closed sets with diameters less than $s^{-n/(n-m)}$ and mutual distances larger that s^{-1}. In this case ν is a suitable multiple of the Lebesgue measure.

Another situation where condition B_1 is trivially satisfied is when all closed sets considered in the previous construction have a non-empty intersection with a given compact smooth manifold $\Sigma \subset \Omega$ of dimension $d > n - m$. In this case, if we assume that the diameters of the closed sets removed from Ω are less than $s^{-d/(n-m)}$ and the mutual distances are larger that s^{-1}, then it is easy to see that B_1 is satisfied with ν equal to a suitable multiple of the d dimensional Hausdorff measure on Σ.

Using the estimates obtained in [6] it is possible to prove that condition B_1 is satisfied also when Ω_s is obtained from Ω by removing an increasing number of closed balls of the appropriate size, whose centers are «uniformly distributed» in a self-similar fractal set of dimension larger than $n - m$.

When ν is a multiple of the Lebesgue measure the problem considered in the present paper is studied in [59]. Similar problems under suitable geometric assumptions on the sets Ω_s are considered also in [54]-[58], [60], and [7]. When the equation (0.1) is linear, the problem has been studied in [33]-[35], [38], [39], [52], and [53] by an orthogonal projection method, in [52], [10], [11] by Brownian motion estimates, in [44]-[46] by Green's function estimates, in [12]-[14] by the energy method, in [48] and [27] by the point interaction approximation, in [5] by probabilistic and capacitary methods. The case of partially perforated domains is considered in [30] and [31]. For weakly connected domains we refer to [50]. The case of random sets Ω_s is studied in [32], [51], [47], [49], [26], [9], [3]. For general compactness results with no geometric hypotheses in the linear case we refer to [2], [1], [20], [15], [43], [18], [19].

In the nonlinear case the problem is studied by Γ-convergence techniques in [17] and [36], provided that (0.1) is the Euler equation of a suitable minimum problem. The special case where $a_j(x, u, p) = |p|^{m-2} p_j$, $j = 1, \ldots, n$, is studied in [2], [37], and [42] under suitable geometric assumptions on the sets Ω_s. When the functions $a_j(x, u, p)$, $j = 1, \ldots, n$, do not depend on u and are odd and homogeneous of degree $m - 1$ with respect to p, the asympotic behaviour of the solutions of (0.1), (0.2) is studied in [21] and [22] without geometric hypotheses on the sets Ω_s. The general compactness result in the non-homogeneous case is proved in [8].

Our main result (Theorem 1.5) allows us not only to predict, in a qualitative way, the form of the boundary value problem satisfied by the limit $u_0(x)$ of the sequence $u_s(x)$ of the solutions of (0.1), (0.2), but also to construct the function $C(x, q)$ which appears in the limit problem in terms of suitable nonlinear capacities associated with equation (0.1) (see condition C). Moreover we obtain (Theorem 1.4 and Section 5) a very precise asymptotic expansion of the sequence $u_s(x)$ in terms of the solution $u_0(x)$ of the limit problem and of suitable nonlinear capacitary potentials associated with equation (0.1).

1. – Statement of the results.

We assume that the functions $a_j(x, u, p)$, $j = 0, 1, \ldots, n$, are defined for $x \in \Omega$, $u \in \mathbf{R}^1$, $p \in \mathbf{R}^n$ and satisfy the following conditions.

CONDITION A_1. – *The functions $a_j(x, u, p)$ are continuous in (u, p) for almost all $x \in \Omega$ and measurable in x for all $u \in \mathbf{R}^1$, $p \in \mathbf{R}^n$; moreover*

$$a_j(x, u, 0) = 0 \quad \text{for } j = 1, \ldots, n \tag{1.1}$$

for all $x \in \Omega$, $u \in \mathbf{R}^1$.

CONDITION A_2. – *There exist positive constants α_0, α_1, α_2, m, m_1, with*

$$0 < \alpha_0 < \alpha_1 \leqslant \alpha_2, \qquad 2 \leqslant m < n, \qquad m \leqslant m_1 < \frac{mn}{n-m}, \tag{1.2}$$

and a function $\gamma(x)$ in $L_r(\Omega)$, with $r > n/m$, such that for every $x \in \Omega$, $u, v \in \mathbf{R}^1$, $p, q \in \mathbf{R}^n$ we have

$$a_0(x, u, p)\, u \geqslant -\alpha_0 |p|^m - \gamma(x)(1 + |u|), \tag{1.3}$$

$$\sum_{j=1}^{n} a_j(x, 0, p)\, p_j \geqslant \alpha_1 (1 + |p|)^{m-2} |p|^2, \tag{1.4}$$

$$\sum_{j=1}^{n} (a_j(x, u, p) - a_j(x, u, q))(p_j - q_j) \geqslant \alpha_1 |p - q|^m, \tag{1.5}$$

$$|a_0(x, u, p)| \leqslant \alpha_2 (|u|^{m_1} + |p|^m)^{(m_1 - 1)/m_1} + \gamma(x), \tag{1.6}$$

$$\sum_{j=1}^{n} |a_j(x, u, p) - a_j(x, v, q)| \leqslant \alpha_2 b(u, v, p, q)(|u - v| + |p - q|), \tag{1.7}$$

where $b(u, v, p, q) = (1 + |u|^{m_1} + |v|^{m_1} + |p|^m + |q|^m)^{(m-2)/m}$.

Note that from (1.1) and (1.7) it follows that

$$|a_j(x, u, p)| \leqslant \alpha_2 (1 + |u|^{m_1} + |p|^m)^{(m-2)/m} (|u| + |p|) \tag{1.8}$$

for every $x \in \Omega$, $u \in \mathbf{R}^1$, $p \in \mathbf{R}^n$, $j = 1, \ldots, n$.

Let us fix a bounded open set $\Omega_0 \subset \mathbf{R}^n$ such that $\overline{\Omega} \subset \Omega_0$. We can extend the functions $a_j(x, u, p)$ to $\Omega_0 \times \mathbf{R}^1 \times \mathbf{R}^n$ preserving all properties mentioned above by setting, e.g., $a_j(x, u, p) = (\mathrm{meas}(\Omega))^{-1} \int_\Omega a_j(y, u, p)\, dy$ for $x \in \Omega_0 \backslash \Omega$, $u \in \mathbf{R}^1$, $p \in \mathbf{R}^n$.

The assumption $2 \leqslant m < n$ is introduced only to simplify the exposition of the results. By similar arguments we can obtain analogous statements also in the case $1 < m < 2$ or $m = n$, under slightly modified hypotheses. For $m > n$ the problem is simplified in view of the compactness of the imbedding of $W_m^1(\Omega)$ in $C^0(\Omega)$.

REMARK 1.1. – Conditions A_1 and A_2 are satisfied when $a_0(x, u, p) = g(x)$, with $g(x) \in L_r(\Omega)$, $r > n/m$, and

$$a_j(x, u, p) = a(x)(1 + |p|^2)^{(m-2)/2} p_j \qquad \text{for } j = 1, \ldots, n\,,$$

where $a(x)$ is a function in $L_\infty(\Omega)$ such that $a(x) \geqslant \alpha$ for some constant $\alpha > 0$.

It is possible to replace condition A_2 by a weaker condition, in particular to replace (1.7) by the inequality

$$\sum_{j=1}^{n} |a_j(x, u, p) - a_j(x, u, q)| \leqslant \alpha_2 b(u, p, q)\, |p - q|\,.$$

In this case the boundary value problem (1.15) has to be changed as in [60], and our results can be partially extended, with minor changes, also to the case

$$a_j(x, u, p) = a(x, u)(1 + |p|^2)^{(m-2)/2} p_j\,, \qquad j = 1, \ldots, n\,.$$

REMARK 1.2. – In condition A_2 inequality (1.4) can be replaced by the weaker inequality

$$\sum_{j=1}^{n} a_j(x, 0, p)\, p_j \geqslant \alpha_1 |p|^m\,, \tag{1.9}$$

if $b(u, v, p, q)$ in (1.7) is replaced by $b_0(u, v, p, q) = (|u|^{m_1} + |v|^{m_1} + |p|^m + |q|^m)^{(m-2)/m}$. This allows us to consider also the model case of the m-Laplacian, which corresponds to the choice

$$a_j(x, u, p) = |p|^{m-2} p_j \qquad \text{for } j = 1, \ldots, n\,.$$

Note that in this case inequality (1.4) is not satisfied, while condition (1.9) holds, and (1.7) is satisfied with $b(u, v, p, q)$ replaced by $b_0(u, v, p, q)$.

Given $f(x) \in W_m^1(\Omega)$, a solution of the boundary value problem (0.1), (0.2) is a function $u(x) \in W_m^1(\Omega_s)$, satisfying $u(x) - f(x) \in \overset{\circ}{W}{}_m^1(\Omega_s)$, such that the integral identity

$$\sum_{j=1}^{n} \int_{\Omega_s} a_j\left(x, u(x), \frac{\partial u(x)}{\partial x}\right) \frac{\partial \varphi(x)}{\partial x_j}\, dx + \int_{\Omega_s} a_0\left(x, u(x), \frac{\partial u(x)}{\partial x}\right) \varphi(x)\, dx = 0 \tag{1.10}$$

holds for an arbitrary function $\varphi(x) \in \overset{\circ}{W}{}_m^1(\Omega_s)$.

Using methods of the theory of monotone operators it is easy to prove the existence of a solution of problem (0.1), (0.2) when $f(x) \in W_m^1(\Omega)$. For every s we denote by $u_s(x)$ one of the possible solutions of problem (0.1), (0.2) and we extend $u_s(x)$ to Ω by setting

$u_s(x)=f(x)$ for $x\in\Omega\backslash\Omega_s$. By condition A_2 the estimate

$$\int_\Omega |u_s(x)|^m\,dx+\int_\Omega \left|\frac{\partial u_s(x)}{\partial x}\right|^m dx\leq R \tag{1.11}$$

holds with a constant R independent of s.

We suppose, in addition, that the following condition is satisfied.

CONDITION A_3. – *The function $f(x)$ belongs to $W^1_\sigma(\Omega_0)$ for some $\sigma>n$.*

Then the function $f(x)$ is bounded and Hölder continuous in Ω, i.e., there exists a constant H such that

$$|f(x)|\leq H\,,\qquad |f(x)-f(y)|\leq H|x-y|^\eta\qquad \text{for } x,y\in\Omega\,, \tag{1.12}$$

where $\eta=1-n/\sigma$.

It is easy to prove, by Moser's method, that the sequence $u_s(x)$ is uniformly bounded. More precisely, the following result holds.

THEOREM 1.3. – *Assume that conditions A_1, A_2, and A_3 are satisfied. Let $u_s(x)$ be a sequence of solutions of problem* (0.1), (0.2) *satisfying condition* (1.11). *Then there exists a constant M independent of s, such that the estimate*

$$\operatorname*{ess\,sup}_{x\in\Omega}|u_s(x)|\leq M \tag{1.13}$$

holds for all s.

PROOF. – For the proof of this theorem see, e.g., [56], § 2, Chapter 9. ■

By (1.11) the sequence $u_s(x)$ contains a weakly convergent subsequence, therefore we may assume that $u_s(x)$ converges weakly in $W^1_m(\Omega)$ to a function $u_0(x)$.

Our main assumption on the sequence Ω_s in condition B_1, which was formulated in the introduction in terms of the m-capacity $C_m(F)$. For every compact set F contained in Ω_0 the m-capacity $C_m(F)$ of F with respect to Ω_0 is defined by

$$C_m(F)=\inf\int_{\Omega_0}\left|\frac{\partial\varphi(x)}{\partial x}\right|^m dx\,, \tag{1.14}$$

where the infimum is taken over all functions $\varphi(x)\in C_0^\infty(\Omega_0)$ which satisfy the equality $\varphi(x)=1$ for $x\in F$.

A crucial role in our paper is played by some special auxiliary functions $v(x,F,q)$, which are defined as the solutions of some model boundary value problems in the domains $\Omega_0\backslash F$. Let F be a compact set contained in Ω_0 and let $\zeta(x)$ be a function of class $C_0^\infty(\Omega_0)$ equal to 1 in F. For every real number q we define $v(x,F,q)$ as the unique function belonging to $q\zeta(x) + \overset{\circ}{W}{}^1_m(\Omega_0 \backslash F)$ which satisfies the integral identity

$$(1.15) \qquad \sum_{j=1}^{n} \int_{\Omega_0 \backslash F} a_j\left(x, 0, \frac{\partial}{\partial x} v(x, F, q)\right) \frac{\partial \varphi(x)}{\partial x_j} \, dx = 0$$

for every $\varphi(x) \in \overset{\circ}{W}{}^1_m(\Omega_0 \backslash F)$.

By conditions A_1 and A_2 the existence and uniqueness of $v(x, F, q)$ follow from the theory of monotone operators. We extend $v(x, F, q)$ to Ω_0 by setting $v(x, F, q) = q$ in F.

For every $u \in \boldsymbol{R}^n$ and for every $r > 0$ let $K(x, r) = \{y \in \boldsymbol{R}^n \colon |y_j - x_j| \leq r, \ j = 1, \ldots, n\}$ be the closed cube of side $2r$ and centre $x = (x_1, \ldots, x_n)$. In Section 4 we shall introduce a special decomposition of the domain Ω of the form

$$\Omega = \left(\bigcup_{\alpha \in I_s} K(x_\alpha^{(s)}, \varrho_s \lambda_s) \right) \cup U_s \,,$$

where λ_s and ϱ_s are sequences of positive real numbers such that $\lambda_s \to \infty$, $\varrho_s \to 0$, and $\lambda_s \varrho_s \to 0$ as $s \to \infty$, $x_\alpha^{(s)} = x_0^{(s)} + 2\lambda_s \varrho_s \alpha$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index with integer coordinates, $x_0^{(s)}$ is a suitable point in the cube $K(0, \lambda_s \varrho_s)$, I_s is the set of all multi-indices α such that $K(x_\alpha^{(s)}, 3\varrho_s \lambda_s) \subset \Omega$, and U_s is the complement of $\bigcup_{\alpha \in I_s} K(x_\alpha^{(s)}, \varrho_s \lambda_s)$ with respect to Ω.

We define $v_\alpha^{(s)}(x, q) = v(x, F, q)$ for $F = K(x_\alpha^{(s)}, (\lambda_s - 2)\varrho_s) \backslash \Omega_s$. Next, we introduce a family of cut-off functions $\varphi_\alpha^{(s)}(x)$ equal to 1 for $x \in K(x_\alpha^{(s)}, (\lambda_s - 2)\varrho_s) \backslash \Omega_s$ and equal to 0 outside $K(x_\alpha^{(s)}, \lambda_s \varrho_s)$ (see (4.15) for the precise definition). Then we introduce the averaging function for $u_0(x)$ defined by

$$(1.16) \qquad u_0^{(s)}(x) = \frac{1}{(\lambda_s \varrho_s)^n} \int_{\Omega} K\left(\frac{|x - y|}{\lambda_s \varrho_s} \right) u_0(y) \, dy \,,$$

where $K(t)$ is an averaging kernel, with $K(t) = 0$ for $|t| \geq 1$, and $u_0(x)$ is the weak limit of $u_s(x)$ in Ω. Finally, by $f_\alpha^{(s)}$ and $u_\alpha^{(s)}$ we denote the mean values of the functions $f(x)$ and $u_0^{(s)}(x)$ in the cube $K(x_\alpha^{(s)}, \lambda_s \varrho_s)$.

In Section 5 we construct the following asymptotic expansion, which is fundamental in our analysis:

$$(1.17) \qquad u_s(x) = u_0^{(s)}(x) + \sum_{\alpha \in I_s} v_\alpha^{(s)}(x, f_\alpha^{(s)} - u_\alpha^{(s)}) \, \varphi_\alpha^{(s)}(x) + R_s(x) \,.$$

To study the asymptotic behaviour of the remainder $R_s(x)$ we need the following assumption.

CONDITION B_2. – *There exists an increasing continuous function $\omega(\varrho)$, satisfying*

$$(1.18) \qquad \int_0^1 \left(\frac{\omega(\varrho)}{\varrho^{n-m}} \right)^{1/(m-1)} \frac{d\varrho}{\varrho} < +\infty \,,$$

such that

$$\nu(K(x, \varrho) \cap \Omega) \leq \omega(\varrho) \tag{1.19}$$

for every cube $K(x, \varrho)$.

From (1.18) it follows that (Lemma 3.1)

$$\lim_{\varrho \to 0} \frac{\omega(\varrho)}{\varrho^{n-m}} = 0 . \tag{1.20}$$

In Section 5 we shall prove the following result.

THEOREM 1.4. – *Assume that conditions* A_1, A_2, A_3, B_1, B_2 *are satisfied, and let* $R_s(x)$ *be the remainder in the asymptotic expansion* (1.17). *Then for every function* $g(x)$ *in* $C_0^\infty(\Omega)$ *the sequence* $g(x)\, R_s(x)$ *converges to zero strongly in* $W_m^1(\Omega)$ *as* $s \to \infty$.

In order to formulate a result about the boundary value problem for the function $u_0(x)$ we introduce a capacity connected with the differential equation (0.1), defined for every compact set $F \subset \Omega_0$ and for every real number q by

$$C_A(F, q) = \sum_{j=1}^{n} \int_{\Omega_0 \setminus F} a_j\left(x, 0, \frac{\partial}{\partial x} v(x, F, q)\right) \frac{\partial}{\partial x_j} v(x, F, q)\, dx , \tag{1.21}$$

where $v(x, F, q)$ is the solution of (1.15). For the main properties of this capacity we refer to [23].

We assume that the following condition is satisfied.

CONDITION C. – *There exists a Borel function* $C(x, q)$, *continuous in* $q \in \boldsymbol{R}^1$, *such that*

$$\lim_{r \to 0}\left(\liminf_{s \to \infty} \frac{C_A(K(x, r) \setminus \Omega_s, q)}{q\nu(K(x, r))}\right) = \lim_{r \to 0}\left(\limsup_{s \to \infty} \frac{C_A(K(x, r) \setminus \Omega_s, q)}{q\nu(K(x, r))}\right) = C(x, q) \tag{1.22}$$

for ν*-almost every* $x \in \Omega$ *and for every* $q \neq 0$.

Condition C is very weak. We shall prove that every sequence Ω_s which satisfies condition B_1 has a subsequence which satisfies condition C (see (6.7), (6.22), and (6.23)). Moreover we shall prove that

$$C(x, 0) = 0 \quad \text{and} \quad |C(x, q)| \leq K(1 + |q|^{m-1})$$

for ν-almost every $x \in \Omega$ and for every $q \in \boldsymbol{R}^1$ (see (6.25) and (6.26)).

Every function $u(x)$ in $W_m^1(\Omega)$ will be identified with its C_m-quasi continuous representative, which is defined for all $x \in \Omega$, except for a set of m-capacity zero. For the definition and properties of C_m-quasi continuous representatives of Sobolev functions we refer to [25], Section 4.8, [29], Section 4, [40], Section 7.2.4, and [61], Chaper 3. By condition B_2 the measure ν belongs to the dual of the Sobolev space $\overset{\circ}{W}{}^1_m(\Omega)$ (see [61], Theorem 4.7.5). Consequently for every $x \in \Omega$ and for every compact set $F \subset \Omega$ the equality $C_m(F) = 0$ implies $\nu(F) = 0$. Therefore the pointwise values of each function $u(x)$ in $W^1_m(\Omega)$ are defined almost everywhere with respect to the measure ν.

The main result of the paper, proved in Section 7, is the following theorem.

THEOREM 1.5. – *Assume that conditions* $A_1, A_2, A_3, B_1, B_2, C$ *are satisfied. Let* $u_s(x)$ *be a sequence of solution of problems* (0.1), (0.2) *which converges weakly in* $W^1_m(\Omega)$ *to a function* $u_0(x)$. *Then* $u_0(x)$ *belongs to* $f(x) + \overset{\circ}{W}{}^1_m(\Omega)$ *and satisfies the integral identity*

$$\sum_{j=1}^{n} \int_{\Omega} a_j\left(x, u_0(x), \frac{\partial u_0(x)}{\partial x}\right) \frac{\partial \varphi(x)}{\partial x_j}\, dx + \int_{\Omega} a_0\left(x, u_0(x), \frac{\partial u_0(x)}{\partial x}\right) \varphi(x)\, dx = \int_{\Omega} C(x, f(x) - u_0(x))\, \varphi(x)\, d\nu(x) \tag{1.23}$$

for every $\varphi(x) \in \overset{\circ}{W}{}^1_m(\Omega) \cap L_\infty(\Omega)$, *where* $C(x, q)$ *is the function defined by* (1.22). *Moreover the sequence* $u_s(x)$ *converges to* $u_0(x)$ *strongly in* $W^1_p(\Omega)$ *for every* $p < m$.

We shall say that a function $u_0(x)$ which satisfies the integral identity (1.23) is a (weak) solution of the equation

$$\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(a_j\left(x, u(x), \frac{\partial u(x)}{\partial x}\right)\right) + C(x, f(x) - u(x))\, \nu = a_0\left(x, u(x), \frac{\partial u(x)}{\partial x}\right) \tag{1.24}$$

in the domain Ω.

2. – Pointwise estimates for solutions and for averaging functions.

In this section we establish some results on integral and pointwise estimates for the auxiliary functions $v(x, F, q)$ introduced in Section 1 as solutions of problem (1.15). We will also obtain some estimates for the averaging functions of the form (1.16).

Throughout the paper, in the proof of the estimates, we shall use the notation c_j, $j = 1, 2, \ldots,$ to indicate a constant which depends only on n, m, α_1, α_2, R, H, M, and $\nu(\Omega)$ (see (1.2), (1.11), (1.12), and (1.13)).

Let us fix a compact set F contained in Ω and let $v(x, q) = v(x, F, q)$. For $0 < \mu \leq |q|$ we introduce the set $E_\mu = \{x \in \Omega_0 : |v(x, q)| \leq \mu\}$.

LEMMA 2.1. – *Assume that conditions* A_1 *and* A_2 *are satisfied, and that* $\operatorname{diam}(F) \leqslant r$. *Then there exists a constant* k_1, *depending only on* α_1, α_2, n, m, *such that*

$$\int_{E_\mu} \left(1 + \left| \frac{\partial v(x, q)}{\partial x} \right|\right)^{m-2} \left| \frac{\partial v(x, q)}{\partial x} \right|^2 dx \leqslant k_1 \mu |q| (r + |q|)^{m-2} C_m(F) \tag{2.1}$$

for every $q \in \boldsymbol{R}^1$ *and for every* μ *with* $0 < \mu \leqslant |q|$.

REMARK 2.2. – It is easy to see that the inequality $0 \leqslant (1/q)\, v(x, q) \leqslant 1$ holds for every $q \neq 0$ and for a.e. $x \in \Omega_0$ (see [59], Lemma 2.1). So we obtain an estimate of the norm of the function $v(x, q)$ in $W_m^1(\Omega_0)$ if we put $\mu = |q|$ in (2.1).

PROOF OF LEMMA 2.1. – First we prove the estimate for $\mu = |q|$. Let y be a point such that $F \subset K(y, r/2)$. Since $m < n$, there exists a constant $c_1 > 0$, depending only on n and m, such that

$$\inf \left\{ \int_{H(y, r)} \left| \frac{\partial \varphi(x)}{\partial x} \right|^m dx \colon \varphi \in C_0^\infty(H(y, r)),\ \varphi(x) = 1\ \forall x \in F \right\} \leqslant c_1 C_m(F),$$

where $H(y, r) = \overset{\circ}{K}(y, r) \cap \Omega_0$ (see [56], Chapter 8, Lemma 2.1). Therefore for every $\varepsilon > 0$ there exists a function $\varphi(x)$ in $C_0^\infty(H(y, r))$ such that $\varphi(x) = 1$ for $x \in F$ and

$$\int_{H(y, r)} \left| \frac{\partial \varphi(x)}{\partial x} \right|^m dx \leqslant c_1 (C_m(F) + \varepsilon). \tag{2.2}$$

Let $z(x) = (2\varphi(x) - 1)_+$, where we use the notation $a_+ = \max\{a, 0\}$ for an arbitrary real number a, and let $G = \{x \in H(y, r) \colon z(x) > 0\} = \{x \in H(y, r) \colon \varphi(x) > 1/2\}$. Using (2.2) and Poincaré's inequality we obtain

$$\operatorname{meas}(G) \leqslant 2^m \int_{H(y, r)} |\varphi(x)|^m dx \leqslant \leqslant 4^m r^m \int_{H(y, r)} \left| \frac{\partial \varphi(x)}{\partial x} \right|^m dx \leqslant 4^m r^m c_1 (C_m(F) + \varepsilon). \tag{2.3}$$

If we use the test function $v(x, q) - qz(x)$ in the integral identity (1.15), from (1.4), (1.8), and Young's inequality we obtain

$$\int_{\Omega_0} \left(1 + \left| \frac{\partial v(x, q)}{\partial x} \right|\right)^{m-2} \left| \frac{\partial v(x, q)}{\partial x} \right|^2 dx \leqslant \leqslant c_2 \int_G \left(|q|^2 \left| \frac{\partial z(x)}{\partial x} \right|^2 + |q|^m \left| \frac{\partial z(x)}{\partial x} \right|^m \right) dx, \tag{2.4}$$

where c_2 depends only on α_1, α_2, n, m. Estimating the right hand side of (2.4) by Hölder's inequality and using (2.2) and (2.3) we get

$$\int_{\Omega_0} \left(1 + \left| \frac{\partial v(x, q)}{\partial x} \right|\right)^{m-2} \left| \frac{\partial v(x, q)}{\partial x} \right|^2 dx \leq k_1 |q|^2 (r + |q|)^{m-2} (C_m(F) + \varepsilon).$$

As $\varepsilon \to 0$ we obtain (2.1) for $\mu = |q|$.

In order to prove inequality (2.1) for $0 < \mu < |q|$, we use test function $\min\{|v(x, q)|, \mu\} - (\mu/q)|v(x, q)|$ in the integral identity (1.15). Then estimate (2.1) for $0 < \mu < |q|$ can be obtained by a standard computation, using the estimate already proved for $\mu = |q|$. ■

We base our study of the behaviour of the sequence $u_s(x)$ on the pointwise estimates of the function $v(x, q)$ given by the following lemma.

LEMMA 2.3. – *Assume that conditions* A_1 *and* A_2 *are satisfied, and that F is contained in a cube $K(y, r)$. For every $x \in \boldsymbol{R}^n$ let $\varrho(x, K(y, r))$ be the distance from x to $K(y, r)$. Then there exists a constant k_2, depending only on α_1, α_2, n, and m, such that*

$$|v(x, q) \leq |q| k_2 \left(\frac{r}{\varrho(x, K(y, r))}\right)^{n-1} \left(\frac{C_m(F)}{r^{n-m}}\right)^{1/(m-1)}, \tag{2.5}$$

for every $x \in \Omega_0$ such that $\varrho(x, K(y, r)) \leq r$.

PROOF. – See [59], Theorem 2.5. ■

In order to obtain the limit boundary value problem we need also some integral estimates of the auxiliary functions $v(x, q)$.

LEMMA 2.4. – *Assume that conditions* A_1 *and* A_2 *are satisfied, and let N be a positive real number. Then there exist two constants k_3 and k_4, depending only on α_1, α_2, n, m, and N, such that*

$$\int_{\Omega_0} \left| \frac{\partial v(x, q')}{\partial x} - \frac{\partial v(x, q'')}{\partial x} \right|^m dx \leq k_3 |q' - q''|^{m/(m-1)} C_m(F), \tag{2.6}$$

$$\left| \frac{1}{q'} C_A(F, q') - \frac{1}{q''} C_A(F, q'') \right| \leq k_4 |q' - q''|^{1/(m-1)} C_m(F), \tag{2.7}$$

$$\left| \frac{1}{q'} C_A(F, q') \right| \leq k_4 |q'|^{1/(m-1)} C_m(F) \tag{2.8}$$

for every compact set $F \subset \Omega$ and for every pair of real numbers q' and q'' such that $0 < |q'|, |q''| \leq N$.

PROOF. – Inequality (2.6) can be proved by using the integral identities corresponding to $v(x, q')$ and $v(x, q'')$, with test function $\varphi(x) = v(x, q') - v(x, q'') - (q' - q'')\, z(x)$, where $z(x)$ is the function introduced in the proof of Lemma 2.1. Subtracting one of the resulting inequalities from the other one and estimating by means of condition A_2 we obtain

$$\int_{\Omega_0} \left| \frac{\partial v'(x)}{\partial x} - \frac{\partial v''(x)}{\partial x} \right|^m dx \leq \tag{2.9}$$

$$\leq c_3 |q' - q''| \int_{\Omega_0} \left(1 + \left| \frac{\partial v'(x)}{\partial x} \right| + \left| \frac{\partial v''(x)}{\partial x} \right|\right)^{m-2} \left| \frac{\partial v'(x)}{\partial x} - \frac{\partial v''(x)}{\partial x} \right| \left| \frac{\partial z(x)}{\partial x} \right| dx\,,$$

where $v'(x) = v(x, q')$ and $v''(x) = v(x, q'')$. In the proof of this lemma the constants $c_3, \ldots, c_6$ depend only on $\alpha_1, \alpha_2, n, m, N$.

From (2.9) we obtain

$$\int_{\Omega_0} \left| \frac{\partial v'(x)}{\partial x} - \frac{\partial v''(x)}{\partial x} \right|^m dx \leq c_4 |q' - q''|^{m/(m-1)} \left(\int_{\Omega_0} \left| \frac{\partial z(x)}{\partial x} \right|^m dx \right)^{1/(m-1)} \cdot$$

$$\cdot \left(\operatorname{meas}(G) + \int_{\Omega_0} \left(\left| \frac{\partial v'(x)}{\partial x} \right| + \left| \frac{\partial v''(x)}{\partial x} \right| \right)^m dx \right)^{(m-2)/(m-1)},$$

and inequality (2.6) follows from (2.1)-(2.3) and from the choice of $z(x)$.

In order to prove (2.7), in the integral identities for $v(x, q')$ and $v(x, q'')$ we use the test functions $(1/q')\, v(x, q') - z(x)$ and $(1/q'')\, v(x, q'') - z(x)$ respectively, with the same function $z(x)$ used in the first part of the proof. Subtracting one of the resulting equalities from the other one we obtain

$$\sum_{j=1}^{n} \int_{\Omega_0} \left(\frac{1}{q'}\, a_j\left(x, \frac{\partial v'(x)}{\partial x}\right) \frac{\partial v'(x)}{\partial x_j} - \frac{1}{q''}\, a_j\left(x, \frac{\partial v''(x)}{\partial x}\right) \frac{\partial v''(x)}{\partial x_j} \right) dx = I(z)\,, \tag{2.10}$$

where

$$I(z) = \sum_{j=1}^{n} \int_{\Omega_0} \left(a_j\left(x, \frac{\partial v'(x)}{\partial x}\right) - a_j\left(x, \frac{\partial v''(x)}{\partial x}\right) \right) \frac{\partial z(x)}{\partial x_j}\, dx\,.$$

We estimate $I(z)$ by using condition (1.7) and we obtain

$$|I(z)| \leqslant c_5 \int_{\Omega_0} \left(1 + \left|\frac{\partial v'(x)}{\partial x}\right| + \left|\frac{\partial v''(x)}{\partial x}\right|\right)^{m-2} \left|\frac{\partial v'(x)}{\partial x} - \frac{\partial v''(x)}{\partial x}\right| \left|\frac{\partial z(x)}{\partial x}\right| dx \leqslant$$

$$\leqslant c_6 \left(\int_{\Omega_0} \left|\frac{\partial v'(x)}{\partial x} - \frac{\partial v''(x)}{\partial x}\right|^m dx\right)^{1/m} \left(\int_{\Omega_0} \left|\frac{\partial z(x)}{\partial x}\right|^m dx\right)^{1/m} \cdot$$

$$\cdot \left(\operatorname{meas}(G) + \int_{\Omega_0} \left(\left|\frac{\partial v'(x)}{\partial x}\right| + \left|\frac{\partial v''(x)}{\partial x}\right|^m\right)\right)^{(m-2)/m},$$

and inequality (2.7) follows from (2.1)-(2.3), (2.6), and from the choice of $z(x)$. Since $(1/q'')\, C_A(F, q'')$ tends to zero as $q'' \to 0$ by Theorem 6.10 of [23], inequality follows from (2.7). ■

We shall now study some properties of the averaging function $u_h(x)$ defined by

$$u_h(x) = \frac{1}{h^n} \int_{\Omega} K\left(\frac{|x-y|}{h}\right) u(y)\, dy\,, \tag{2.11}$$

where $K(t)$ is an infinitely differentiable function on $\boldsymbol{R}^1$, equal to zero for $|t| \geqslant 1$, such that

$$\int_{\boldsymbol{R}^n} K(|x|)\, dx = 1$$

and $0 \leqslant K(t) \leqslant c(n)$ for a suitable constant $c(n)$ depending only on n.

LEMMA 2.5. – *Let $u(x)$ be a function in $W_m^1(\Omega)$. Then there exists a constant k_5, depending only on n and m, such that the inequality*

$$\left|\frac{\partial u_h(x)}{\partial x}\right|^m \leqslant k_5 \frac{1}{h^n} \int_{B(x,h)} \left|\frac{\partial u(y)}{\partial y}\right|^m dy \tag{2.12}$$

holds for every point $x \in \Omega$ and for every $h > 0$ such that the open ball $B(x, h)$ of radius h and centre x is contained in Ω.

PROOF. – See [59], Lemma 3.1. ■

LEMMA 2.6. – *Let $u(x)$ be a function in $W_m^1(\Omega)$. Then there exists a constant k_6, depending only on n and m, such that the inequality*

$$\int_{K(y,r)} |u_h(x) - u(x)|^m dx \leqslant k_6 h^m \int_{K(y,r+h)} \left|\frac{\partial u(x)}{\partial x}\right|^m dx \tag{2.13}$$

holds for every point $y \in \Omega$ and for every pair of positive numbers r and h such that $K(y, r+h) \subset \Omega$.

PROOF. – See [59], Lemma 3.2. ■

Given $x_0 \in \boldsymbol{R}^n$ and $r > 0$, let us consider the family of points $x_\alpha = x_0 + 2r\alpha$ in $\boldsymbol{R}^n$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index with integer coordinates. Let $I(r, h)$ be the set of multi-indices α such that $K(x_\alpha, 2r+h) \subset \Omega$ and, for every integrable function $u(x)$, let

$$u_h(\alpha, r) = \frac{1}{(2r)^n} \int_{K(x_\alpha, r)} u_h(x)\, dx$$

be the mean value of $u_h(x)$ with respect to the cube $K(x_\alpha, r)$, where $u_h(x)$ is defined by (2.11).

LEMMA 2.7. – *Let $u(x)$ be a function in $W_m^1(\Omega)$, let $g_\alpha(x)$, $\alpha \in I(r, h)$, be a family of functions in $L_m(\Omega, \lambda)$, where λ is a positive Borel measure on Ω, and let q be a constant with $1 \leqslant q \leqslant 2$. Assume that, for some positive constant Q, the inequalities*

$$\int_{K(x_\alpha, qr)} |g_\alpha(x)|^m\, d\lambda(x) \leqslant Q \qquad \forall \alpha \in I(r, h) \tag{2.14}$$

are satisfied. Then there exists a constant k_7, depending only on n and m, such that the estimate

$$\sum_{\alpha \in I(r, h)} \int_{K(x_\alpha, qr)} |u_h(x) - u_h(\alpha, qr)|^m\, |g_\alpha(x)|^m\, d\lambda(x) \leqslant \leqslant k_7 \frac{Q}{r^{n-m}} \int_\Omega \left| \frac{\partial u(x)}{\partial x} \right|^m dx \tag{2.15}$$

holds whenever $0 < r \leqslant h$.

PROOF. – Using (2.12) and (2.14) we obtain

$$\sum_{\alpha \in I(r, h)} \int_{K(x_\alpha, qr)} |u_h(x) - u_h(\alpha, qr)|^m\, |g_\alpha(x)|^m\, d\lambda(x) \leqslant$$

$$\leqslant (2\sqrt{n}\, qr)^m \sum_{\alpha \in I(r, h)} \int_{K(x_\alpha, qr)} \left(\int_0^1 \left| \frac{\partial u_h(tx + (1-t)\,\xi_h(\alpha, r))}{\partial x} \right|^m dt \right) |g_\alpha(x)|^m\, d\lambda(x) \leqslant$$

$$\leqslant c_7 \frac{r^m}{h^n} \sum_{\alpha \in I(r, h)} \int_{K(x_\alpha, qr+h)} \left| \frac{\partial u(x)}{\partial x} \right|^m dx \int_{K(x_\alpha, qr)} |g_\alpha(x)|^m\, d\lambda(x) \leqslant$$

$$\leqslant c_7 Q \frac{r^m}{h^n} \sum_{\alpha \in I(r, h)} \int_{K(x_\alpha, qr+h)} \left| \frac{\partial u(x)}{\partial x} \right|^m dx\,.$$

Here $\xi_h(\alpha, r)$ is a suitable point belonging to the cube $K(x_\alpha, qr)$ and c_7 is a constant depending only on n and m. Inequality (2.15) follows from these estimates and from the fact that

$$\sum_{\alpha \in I(r,h)} \chi^{(\alpha)}_{qr+h}(x) \leqslant c_8 \frac{h^n}{r^n} \qquad \text{for } 0 < r \leqslant h\,,$$

where $\chi^{(\alpha)}_{qr+h}(x)$ is the characteristic function of the set $K(x_\alpha, qr+h)$ and c_8 is a constant depending only on n. ■

3. - A Poincaré-Wirtinger inequality.

In this section we shall prove a Poincaré-Wirtinger inequality for measures satisfying condition B_2. We begin with two lemmas concerning the function $\omega(\varrho)$.

LEMMA 3.1. - *Assume that condition* (1.18) *is satisfied. Then*

$$(3.1)\qquad \frac{m-1}{n-m}\left(\frac{\omega(r)}{r^{n-m}}\right)^{1/(m-1)} \leqslant \int_r^{\sqrt{r}} \left(\frac{\omega(\varrho)}{\varrho^{n-m}}\right)^{1/(m-1)} \frac{d\varrho}{\varrho} + \frac{r^{(n-m)/(2(m-1))}}{1-2^{(m-n)/(m-1)}} \int_0^1 \left(\frac{\omega(\varrho)}{\varrho^{n-m}}\right)^{1/(m-1)} \frac{d\varrho}{\varrho}$$

for every $r \leqslant 1/2$. In particular we have

$$(3.2)\qquad \lim_{r\to 0} \frac{\omega(r)}{r^{n-m}} = 0\,.$$

PROOF. - For every $r \leqslant 1/2$ we have

$$\int_r^{\sqrt{r}} \left(\frac{\omega(\varrho)}{\varrho^{n-m}}\right)^{1/(m-1)} \frac{d\varrho}{\varrho} \geqslant (\omega(r))^{1/(m-1)} \int_r^{\sqrt{r}} \frac{1}{\varrho^{(n-m)/(m-1)}} \frac{d\varrho}{\varrho} = \frac{m-1}{n-m}(\omega(r))^{1/(m-1)}\left(\frac{1}{r^{(n-m)/(m-1)}} - \frac{1}{r^{(n-m)/(2(m-1))}}\right).$$

This implies

$$(3.3)\qquad \frac{m-1}{n-m}\left(\frac{\omega(r)}{r^{n-m}}\right)^{1/(m-1)} \leqslant \int_r^{\sqrt{r}} \left(\frac{\omega(\varrho)}{\varrho^{n-m}}\right)^{1/(m-1)} \frac{d\varrho}{\varrho} + \frac{m-1}{n-m}\,\frac{(\omega(r))^{1/(m-1)}}{r^{(n-m)/(2(m-1))}}\,.$$

On the other hand we have

$$\int_r^1 \left(\frac{\omega(\varrho)}{\varrho^{n-m}}\right)^{1/(m-1)} \frac{d\varrho}{\varrho} \geqslant (\omega(r))^{1/(m-1)} \int_r^1 \frac{1}{\varrho^{(n-m)/(m-1)}} \frac{d\varrho}{\varrho} =$$

$$= \frac{m-1}{n-m} (\omega(r))^{1/(m-1)} \left(\frac{1}{r^{(n-m)/(m-1)}} - 1\right) \geqslant \frac{m-1}{n-m} (\omega(r))^{1/(m-1)} \frac{1-2^{(m-n)/(m-1)}}{r^{(n-m)/(m-1)}},$$

where, in the last inequality, we use the fact that $r \leqslant 1/2$. Therefore we obtain

$$\frac{m-1}{n-m} (\omega(r))^{1/(m-1)} \leqslant \frac{r^{(n-m)/(m-1)}}{1-2^{(m-n)/(m-1)}} \int_0^1 \left(\frac{\omega(\varrho)}{\varrho^{n-m}}\right)^{1/(m-1)} \frac{d\varrho}{\varrho}. \tag{3.4}$$

Inequality (3.1) follows now from (3.3) and (3.4), while (3.2) is a consequence of (3.1) and (1.18). ■

Let $\tau(r)$ be the non-decreasing function defined for every $r > 0$ by

$$\tau(r) = \int_0^{\sqrt{r}} \left(\frac{\omega(\varrho)}{\varrho^{n-m}}\right)^{1/(m-1)} \frac{d\varrho}{\varrho} + \frac{r^{(n-m)/(2(m-1))}}{1-2^{(m-n)/(m-1)}} \int_0^1 \left(\frac{\omega(\varrho)}{\varrho^{n-m}}\right)^{1/(m-1)} \frac{d\varrho}{\varrho}. \tag{3.5}$$

By (1.2) and (1.18) we have

$$\lim_{r\to 0} \tau(r) = 0. \tag{3.6}$$

For every pair a, b of real numbers we set $a \wedge b = \min\{a, b\}$.

LEMMA 3.2. – *Assume that condition* (1.18) *is satisfied. Then*

$$\int_0^{+\infty} \left(\frac{\omega(\varrho \wedge r)}{\varrho^{n-m}}\right)^{1/(m-1)} \frac{d\varrho}{\varrho} \leqslant \tau(r) \tag{3.7}$$

for every $r \leqslant 1/2$.

PROOF. – For every $r \leqslant 1/2$ we have

$$\int_0^{+\infty} \left(\frac{\omega(\varrho \wedge r)}{\varrho^{n-m}}\right)^{1/(m-1)} \frac{d\varrho}{\varrho} = \int_0^{r} \left(\frac{\omega(\varrho)}{\varrho^{n-m}}\right)^{1/(m-1)} \frac{d\varrho}{\varrho} + \frac{m-1}{n-m} \left(\frac{\omega(r)}{r^{n-m}}\right)^{1/(m-1)}.$$

The conclusion follows now from Lemma 3.1. ■

We prove now a Poincaré inequality for measures satisfying condition B_2.

PROPOSITION 3.3. – *Assume that condition* B_2 *is satisfied and let* $\tau(r)$ *be the function defined in* (3.5). *Then there exists a constant* k_8, *depending only on* m *and* n, *such* *that*

$$\int_{K(x_0,\, r)\cap\Omega} |u(x)|^m \, d\nu(x) \leqslant k_8(\tau(r))^{m-1} \int_{\mathbf{R}^n} \left| \frac{\partial u(x)}{\partial x} \right|^m dx \tag{3.8}$$

for every cube $K(x_0, r)$ *and for every function* $u(x)$ *in* $W_m^1(\mathbf{R}^n)$ *with compact support.*

PROOF. – Let us fix a cube $K(x_0, r)$ and a function $u(x)$ in $W_m^1(\mathbf{R}^n)$ with compact support. It is well known that

$$|u(x)| \leqslant c_9 \int_{\mathbf{R}^n} \left| \frac{\partial u(y)}{\partial y} \right| |y-x|^{1-n} \, dy \tag{3.9}$$

for C_m-almost every $x \in \mathbf{R}^n$ (see, e.g., [28], Lemma 7.16). By condition B_2 the measure ν belongs to the dual of the Sobolev space $\overset{\circ}{W}{}_m^1(\Omega)$ (see, e.g., [61], Theorem 4.7.5). Therefore inequality (3.9) holds for ν-almost very $x \in K(x_0, r) \cap \Omega$. Thus we have

$$\begin{aligned} &\int_{K(x_0,\, r)\cap\Omega} |u(x)|^m \, d\nu(x) \leqslant \\ &\leqslant c_9 \int_{K(x_0,\, r)\cap\Omega} |u(x)|^{m-1} \left(\int_{\mathbf{R}^n} \left| \frac{\partial u(y)}{\partial y} \right| |y-x|^{1-n} \, dy \right) d\nu(x) = \\ &= \int_{\mathbf{R}^n} \left| \frac{\partial u(y)}{\partial y} \right| \left(\int_{K(x_0,\, r)\cap\Omega} \frac{|u(x)|^{m-1}}{|y-x|^{n-1}} \, d\nu(x) \right) dy \leqslant \\ &\leqslant \left(\int_{\mathbf{R}^n} \left| \frac{\partial u(y)}{\partial y} \right|^m dy \right)^{1/m} \left(\int_{\mathbf{R}^n} \left(\int_{K(x_0,\, r)\cap\Omega} \frac{|u(x)|^{m-1}}{|y-x|^{n-1}} \, d\nu(x) \right)^{m/(m-1)} dy \right)^{(m-1)/m} . \end{aligned} \tag{3.10}$$

By using Hölder's inequality we obtain

$$\begin{aligned} &\int_{\mathbf{R}^n} \left(\int_{K(x_0,\, r)\cap\Omega} \frac{|u(x)|^{m-1}}{|y-x|^{n-1}} \, d\nu(x) \right)^{m/(m-1)} dy \leqslant \\ &\leqslant \int_{\mathbf{R}^n} \left(\int_{K(x_0,\, r)\cap\Omega} \frac{|u(x)|^{m}}{|y-x|^{n-1}} \, d\nu(x) \right) \left(\int_{K(x_0,\, r)\cap\Omega} \frac{1}{|y-z|^{n-1}} \, d\nu(z) \right)^{1/(m-1)} dy \leqslant \\ &\leqslant \int_{K(x_0,\, r)\cap\Omega} |u(x)|^m \left(\int_{\mathbf{R}^n} \frac{1}{|y-x|^{n-1}} \left(\int_{K(x_0,\, r)\cap\Omega} \frac{1}{|y-z|^{n-1}} \, d\nu(z) \right)^{1/(m-1)} dy \right) d\nu(x) . \end{aligned} \tag{3.11}$$

The estimate proved in Theorem 6.1 of [41] gives

$$(3.12)\qquad \int_{R^n} \frac{1}{|y-x|^{n-1}} \left(\int_{K(x_0,\, r)\cap\Omega} \frac{1}{|y-z|^{n-1}}\, d\nu(z) \right)^{1/(m-1)} dy \leqslant$$

$$\leqslant c_{10} \int_0^{+\infty} \left(\frac{\omega(r,\varrho)}{\varrho^{n-m}} \right)^{1/(m-1)} \frac{d\varrho}{\varrho},$$

where $\omega(r,\varrho) = \sup_{x\in R^n} \nu(K(x_0, r)\cap\Omega\cap B(x,\varrho))$. Since $\omega(r,\varrho)\leqslant\omega(r\wedge\varrho)$, from (3.7), (3.11), and (3.12) we obtain

$$\int_{R^n} \left(\int_{K(x_0,\, r)\cap\Omega} \frac{|u(x)|^{m-1}}{|y-z|^{n-1}}\, d\nu(x) \right)^{m/(m-1)} dy \leqslant c_{10}\,\tau(r) \int_{K(x_0,\, r)\cap\Omega} |u(x)|^m\, d\nu(x),$$

which, together with (3.10), implies (3.8). ■

We shall use the following Poincaré-Wirtinger inequality, where

$$u_{y,\,r} = \frac{1}{(2r)^n} \int_{K(y,\, r)} u(x)\, dx .$$

PROPOSITION 3.4. – *Assume that condition* B_2 *is satisfied and let* $\tau(r)$ *be the function defined in* (3.5). *Then there exists a constant* k_9, *depending only on* m *and* n, *such that*

$$(3.13)\qquad \int_{K(y,\, r)} |u(x)-u_{y,\,r}|^m\, d\nu(x) \leqslant k_9\,(\tau(r))^{m-1} \int_{K(y,\, r)} \left| \frac{\partial u(x)}{\partial x} \right|^m dx$$

for every cube $K(y, r)$ *contained in* Ω *and for every function* $u(x)$ *in* $W^1_m(\Omega)$.

PROOF. – For simplicity we assume $y = 0$ and we set $K(r) = K(0, r)$ and $u_r = u_{0,\,r}$ for every $r > 0$. Let us fix a bounded extension operator $T\colon W^1_m(\mathring{K}(1)) \to \mathring{W}^1_m(\mathring{K}(2))$, and for every $r>0$ let us define the extension operator $T_r\colon W^1_m(\mathring{K}(r))\to \mathring{W}^1_m(\mathring{K}(2r))$ by $(T_r u)(x) = (Tu_r)(x/r)$, where $u_r(x) = u(rx)$. It is easy to see that the boundedness of T implies that there exists a constant c_{11}, depending only on m and n, such that

$$(3.14)\qquad \int_{K(2r)} \left| \frac{\partial (T_r u)(x)}{\partial x} \right|^m dx \leqslant c_{11} \left(\int_{K(r)} \left| \frac{\partial u(x)}{\partial x} \right|^m dx + \frac{1}{r^m} \int_{K(r)} |u(x)|^m\, dx \right)$$

for every function $u(x)$ in $W^1_m(\mathring{K}(r))$.

Assume now that $u(x)$ belongs to $W^1_m(\Omega)$ and that $K(r)$ is contained in Ω. Note that $u(x) - u_r = (T_r(u-u_r))(x)$ C_m-almost everywhere (hence ν-almost everywhere) in $K(r)$, since both functions are C_m-quasi continuous and coincide C_m-almost everywhere in $\overset{\circ}{K}(r)$. From (3.8) and (3.14) we obtain

$$\int_{K(r)} |u(x) - u_r|^m \, d\nu(x) = \int_{K(r)} |(T_r(u - u_r))(x)|^m \, d\nu(x) \leq$$

$$\leq k_8 (\tau(r))^{m-1} \int_{K(2r)} \left| \frac{\partial (T_r(u - u_r))(x)}{\partial x} \right|^m dx \leq$$

$$\leq c_{12} (\tau(r))^{m-1} \left(\int_{K(r)} \left| \frac{\partial u(x)}{\partial x} \right|^m dx + \frac{1}{r^m} \int_{K(r)} |u(x) - u_r|^m \, dx \right).$$

The conclusion follows now from this inequality and from the classical version of the Poincaré-Wirtinger inequality

$$\frac{1}{r^m} \int_{K(r)} |u(x) - u_r|^m \, dx \leq c_{13} \int_{K(r)} \left| \frac{\partial u(x)}{\partial x} \right|^m dx$$

(see, e.g., [28], formula (7.45)). ■

4. - Decomposition of the domain and construction of cut-off functions.

In the rest of the paper $u_s(x)$ is a sequence of solutions of problem (0.1), (0.2), which satisfies estimates (1.11) and (1.13) and converges weakly in $W^1_m(\Omega)$ to a function $u_0(x)$ in $f(x) + \overset{\circ}{W}{}^1_m(\Omega)$. We shall always assume that conditions A_1, A_2, A_3, B_1, B_2 are satisfied.

In this section we consider a decomposition of the domain Ω and a family of cut-off functions depending on three sequences ϱ_s, μ_s, λ_s.

Choice of ϱ_s. Let ϱ_s be a sequence of real numbers such that

$$\varrho_s \geq r_s + \left(\int_\Omega |u_s(x) - u_0(x)|^m \, dx \right)^{1/m}, \tag{4.1}$$

$$\lim_{s \to \infty} \varrho_s = 0, \tag{4.2}$$

where r_s is the sequence which appears in condition B_1.

Choice of λ_s. Let $\omega(\varrho)$ be the function which appears in condition B_2, and let t_s be the solution of the equation

$$t_s^{n+1} \left(\frac{\omega(t_s)}{t_s^{n-m}} \right)^{1/(m-1)} = \varrho_s^{n+1}; \tag{4.3}$$

we define λ_s to be the odd integer number which satisfies

$$\lambda_s \leq \frac{t_s}{\varrho_s} < \lambda_s + 2 . \tag{4.4}$$

Choice of μ_s. We define μ_s by

$$\mu_s = \max\left\{ \lambda_s^n \left(\frac{\omega(\lambda_s \varrho_s)}{(\lambda_s \varrho_s)^{n-m}} \right)^{1/(m-1)}, (\lambda_s \varrho_s)^\eta, \sup_{0 < t \leq \lambda_s \varrho_s} \left(\frac{\omega(t)}{t^{n-m}} \right)^{1/m}, \left(\frac{1}{\lambda_s} \right)^{1/2m} \right\}, \tag{4.5}$$

where $\eta = 1 - n/\sigma$ is the exponent in condition (1.12).

LEMMA 4.1. – *Assume that conditions* A_1, A_2, A_3, B_1, B_2 *are satisfied. Then the sequences* ϱ_s, λ_s, μ_s *satisfy the following properties*:

$$\lim_{s \to \infty} \lambda_s = +\infty , \qquad \lim_{s \to \infty} \lambda_s \varrho_s = 0 , \qquad \lim_{s \to \infty} \mu_s = 0 . \tag{4.6}$$

PROOF. – By (4.2) the sequence t_s defined by (4.3) tends to zero as $s \to \infty$. So from the first inequality in (4.4) we obtain the second equality in (4.6).

From (3.2) and (4.3) it follows that

$$\left(\frac{t_s}{\varrho_s} \right)^{n+1} = \left(\frac{t_s^{n-m}}{\omega(t_s)} \right)^{1/(m-1)}$$

tends to infinity and consequently from the second inequality in (4.4) we obtain the first equality in (4.6).

The last equality in (4.6) follows from the other equalities in (4.6), from (3.2), and from the estimate

$$\lambda_s^n \left(\frac{\omega(\lambda_s \varrho_s)}{(\lambda_s \varrho_s)^{n-m}} \right)^{1/(m-1)} = \frac{1}{\lambda_s \varrho_s^{n+1}} (\lambda_s \varrho_s)^{n+1} \left(\frac{\omega(\lambda_s \varrho_s)}{(\lambda_s \varrho_s)^{n-m}} \right)^{1/(m-1)} \leq$$

$$\leq \frac{1}{\lambda_s \varrho_s^{n+1}} t_s^{n+1} \left(\frac{\omega(t_s)}{t_s^{n-m}} \right)^{1/(m-1)} \left(\frac{\lambda_s + 2}{\lambda_s} \right)^{(n-m)/(m-1)} = \frac{1}{\lambda_s} \left(\frac{\lambda_s + 2}{\lambda_s} \right)^{(n-m)/(m-1)},$$

which is a consequence of (4.3) and (4.4). ∎

We introduce now a subdivision of the domain Ω that will be useful for our estimates. Given a point $x_0^{(s)}$ in $K(0, \lambda_s \varrho_s)$, we shall consider the cubic lattice composed of the points $x_\alpha^{(s)} = x_0^{(s)} + 2\lambda_s \varrho_s \alpha$, where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index with integer coordinates, and the set

$$F_s = \bigcup_\alpha \left(K(x_\alpha^{(s)}, \lambda_s \varrho_s) \setminus K(x_\alpha^{(s)}, (\lambda_s - 6) \varrho_s) \right), \tag{4.7}$$

where the union is over all possible multi-indices α with integer coordinates.

LEMMA 4.2. – *There exists a point $x_0^{(s)}$ in $K(0, \lambda_s \varrho_s)$ such that*

$$\nu(F_s \cap \Omega) \leq \frac{7n}{\lambda_s} \nu(\Omega). \tag{4.8}$$

PROOF. – We introduce the strips

$$\Pi_{ki}^{(j)} = \{(x_1, \ldots, x_n) \in \mathbf{R}^n : 2\varrho_s(k-3) \leq x_j - 2i\lambda_s \varrho_s \leq 2\varrho_s(k+3)\}$$

for $j = 1, \ldots, n$, $k = 1, \ldots, \lambda_s$, $i = 0, \pm 1, \pm 2, \ldots$. It is easy to see that for every x in $\mathbf{R}^n$ and for every j we have

$$\sum_{k=1}^{\lambda_s} \sum_{i=-\infty}^{+\infty} \chi_{ki}^{(j)}(x) \leq 7,$$

where $\chi_{ki}^{(j)}$ is the characteristic function of the strip $\Pi_{ki}^{(j)}$. It follows that for every j there exists an integer number k_j, with $1 \leq k_j \leq \lambda_s$, such that

$$\sum_{i=-\infty}^{+\infty} \nu(\Pi_{k_j i}^{(j)} \cap \Omega) \leq \frac{7}{\lambda_s} \nu(\Omega).$$

Define the point $x_0^{(s)}$ by $x_0^{(j)} = ((2k_1 - \lambda_s)\varrho_s, (2k_2 - \lambda_s)\varrho_s, \ldots, (2k_n - \lambda_s)\varrho_s)$. Inequality (4.8) is now an easy consequence of the inclusion

$$F_s \subset \bigcup_{j=1}^{n} \bigcup_{i=-\infty}^{+\infty} \Pi_{k_j i}^{(j)},$$

which follows from the definition of $\Pi_{ki}^{(j)}$ and from the choice of $x_0^{(s)}$. ■

The domain Ω will be decomposed as

$$\Omega = \left(\bigcup_{\alpha \in I_s} K(x_\alpha^{(s)}, \lambda_s \varrho_s) \right) \cup U_s, \tag{4.9}$$

where $x_\alpha^{(s)} = x_0^{(s)} + 2\lambda_s \varrho_s \alpha$ and $x_0^{(s)}$ is defined in Lemma 4.2. In (4.9) I_s is the set of all multi-indices α such that $K(x_\alpha^{(s)}, 3\lambda_s \varrho_s) \subset \Omega$, and U_s is the complement in Ω of the set $\bigcup_{\alpha \in I_s} K(x_\alpha^{(s)}, \lambda_s \varrho_s)$.

Moreover we introduce the notation

$$\begin{cases} K_s(\alpha) = K(x_\alpha^{(s)}, \lambda_s \varrho_s), \\ K_s'(\alpha) = K(x_\alpha^{(s)}, (\lambda_s - 2)\varrho_s), \\ K_s''(\alpha) = K(x_\alpha^{(s)}, (\lambda_s - 1)\varrho_s). \end{cases} \tag{4.10}$$

Let us define the function

$$v_\alpha^{(s)}(x, q) = v(x, K_s'(\alpha) \backslash \Omega_s, q), \tag{4.11}$$

where $v(x, F, q)$ is the function which satisfies the integral identity (1.15). In particular we have $v_\alpha^{(s)}(x, q) = q$ for $x \in K_s'(\alpha)\backslash\Omega_s$. By (4.6) we can assume that the inequalities

$$\lambda_s > 4\,, \qquad 3\lambda_s\varrho_s < 1\,, \qquad \mu_s < 1 \tag{4.12}$$

are satisfied for all s.

Let $u_0^{(s)}(x)$ be the averaging of the function $u_0(x)$ defined by (1.16) and let $f_\alpha^{(s)}$ and $u_\alpha^{(s)}$ be the values of the functions $f(x)$ and $u_0^{(s)}(x)$ in the sube $K_s(\alpha)$.

Let I_s' and I_s'' be the sets of multi-indices defined by

$$\begin{cases} I_s' = \{\alpha \in I_s\colon \, |f_\alpha^{(s)} - u_\alpha^{(s)}\,| > 2\mu_s\}\,, \\ I_s'' = \{\alpha \in I_s\colon \, |f_\alpha^{(s)} - u_\alpha^{(s)}\,| \leqslant 2\mu_s\}\,. \end{cases} \tag{4.13}$$

Let $w_\alpha^{(s)}(x)$ be the function defined by the equalities

$$\begin{cases} w_\alpha^{(s)}(x) = v_\alpha^{(s)}(x, f_\alpha^{(s)} - u_\alpha^{(s)}) & \text{for } \alpha \in I_s'\,, \\ w_\alpha^{(s)}(x) = v_\alpha^{(s)}(x, 2\mu_s) & \text{for } \alpha \in I_s''\,. \end{cases} \tag{4.14}$$

For every function $g(x)$ we denote its positive part by $(g(x))_+ = \max\{g(x), 0\}$. Let us define the cut-off function $\varphi_\alpha^{(s)}(x)$ by

$$\varphi_\alpha^{(s)}(x) = \frac{2}{\mu_s}\min\left\{\left(|w_\alpha^{(s)}(x)\,| - \frac{\mu_s}{2}\right)_+, \frac{\mu_s}{2}\right\},$$

and let $G_\alpha^{(s)}$ be the set where the function $\varphi_\alpha^{(s)}(x)$ is different from zero. Note that $\varphi_\alpha^{(s)}(x) = 1$ for $x \in K_s'(\alpha)\backslash\Omega_s$.

Some properties of the functions $\varphi_\alpha^{(s)}(x)$ will play an important role in the sequel.

LEMMA 4.3. – *Assume that conditions* A_1, A_2, A_3, B_1, B_2 *are satisfied. Then there exists an integer* s_1 *such that*

$$G_\alpha^{(s)} \cap G_\gamma^{(s)} = \emptyset \tag{4.16}$$

for every $s \geqslant s_1$ *and for every* $\alpha, \gamma \in I_s$ *with* $\alpha \neq \gamma$.

PROOF. – It is sufficient to verify that the inclusion

$$G_\alpha^{(s)} \subset K_s''(\alpha) \tag{4.17}$$

holds for s large enough. Usisng the pointwise estimate (2.5) and conditions B_1 and B_2 we obtain the inequality

$$|w_\alpha^{(s)}(x)\,| \leqslant c_{14}\lambda_s^{n-1}\left(\frac{C_m(K_s'(\alpha)\backslash\Omega_s)}{(\lambda_s\varrho_s)^{n-m}}\right)^{1/(m-1)} \leqslant c_{14}\lambda_s^{n-1}\left(\frac{\omega(\lambda_s\varrho_s)}{(\lambda_s\varrho_s)^{n-m}}\right)^{1/(m-1)} \tag{4.18}$$

if $x \in \partial K_s''(\alpha)$. By the maximum principle the same inequality holds for every $x \notin K_s''(\alpha)$.

From (4.5) and (4.6) we have the inequality

$$c_{14}\lambda_s^{n-1}\left(\frac{\omega(\lambda_s\varrho_s)}{(\lambda_s\varrho_s)^{n-m}}\right)^{1/(m-1)} \leq \frac{c_{14}}{\lambda_s}\mu_s < \frac{\mu_s}{2}$$

for sufficiently large s. Consequently from (4.18) we obtain

$$\left(|w_\alpha^{(s)}(x)| - \frac{\mu_s}{2}\right)_+ = 0 \quad \text{for } x \notin K_s''(\alpha),$$

which implies (4.17). ∎

For $s = 1, 2, \ldots$ and $\alpha \in I_s$ we define a set of multi-indices $I_s(\alpha)$ with integer coordinates and a set of points $\{x_{\alpha\beta}^{(s)}: \beta \in I_s(\alpha)\}$ such that $x_{\alpha\beta}^{(s)} = x_0^{(s)} + 2\varrho_s\beta$ and

$$K_s(\alpha)\setminus \overset{\circ}{K_s'}(\alpha) = \bigcup_{\beta\in I_s(\alpha)} K_s(\alpha,\beta), \qquad K_s(\alpha,\beta) = K(x_{\alpha\beta}^{(s)}, \varrho_s). \tag{4.19}$$

We define also the functions

$$\begin{cases} w_{\alpha\beta}^{(s)}(x, q) = v(x, F, q) & \text{for } F = K_s(\alpha,\beta)\setminus\Omega_s, \\ v_{\alpha\beta}^{(s)}(x, q) = v(x, F, q) & \text{for } F = K_s'(\alpha,\beta)\setminus\Omega_s, \end{cases} \tag{4.20}$$

where $v(x, F, q)$ is the function defined in (1.15) and

$$K_s'(\alpha,\beta) = K(x_{\alpha\beta}^{(s)}, 2\varrho_s). \tag{4.21}$$

Let $|I_s|$ and $|I_s(\alpha)|$ be the numbers of multi-indices of the sets I_s and $I_s(\alpha)$ respectively. It is easy to see that

$$|I_s| \leq (2\lambda_s\varrho_s)^{-n}\,\text{meas}(\Omega), \qquad |I_s(\alpha)| \leq 2n\lambda_s^{n-1}.$$

Let us define the cut-off functions $\varphi_{\alpha\beta}^{(s)}(x)$ by

$$\varphi_{\alpha\beta}^{(s)}(x) = \frac{2}{\mu_s}\min\left\{\left(w_{\alpha\beta}^{(s)}(x, 1) - \frac{\mu_s}{2}\right)_+, \frac{\mu_s}{2}\right\} \tag{4.22}$$

for $s = 1, 2, \ldots$, $\alpha \in I_s$, $\beta \in I_s(\alpha)$, and let $G_{\alpha\beta}^{(s)}$ be the set where $\varphi_{\alpha\beta}^{(s)}(x)$ is different from zero. By (4.12) we have $\varphi_{\alpha\beta}^{(s)}(x) = 1$ for $x \in K_s(\alpha,\beta)\setminus\Omega_s$.

For future use we state the following estimates for the functions $\varphi_\alpha^{(s)}(x)$ and $\varphi_{\alpha\beta}^{(s)}(x)$.

LEMMA 4.4. – *Assume that conditions* A_1, A_2, A_3, B_1, B_2 *are satisfied. Then for every* $\alpha \in I_s$ *and* $\beta \in I_s(\alpha)$ *we have*

$$\int_{G_\alpha^{(s)}} \left|\frac{\partial\varphi_\alpha^{(s)}(x)}{\partial x}\right|^m dx \leq k_{10}\mu_s^{1-m}\nu(K_s''(\alpha)) \leq k_{10}\mu_s^{1-m}\omega(\lambda_s\varrho_s), \tag{4.23}$$

$$\int_{G_\alpha^{(s)}} \left| \frac{\partial \varphi_\alpha^{(s)}(x)}{\partial x} \right|^2 dx \leq k_{11} \mu_s^{-1} \nu(K_s''(\alpha)) , \tag{4.24}$$

$$\int_{G_{\alpha\beta}^{(s)}} \left| \frac{\partial \varphi_{\alpha\beta}^{(s)}(x)}{\partial x} \right|^m dx \leq k_{12} \mu_s^{1-m} \nu(K_s'(\alpha, \beta)) \leq k_{12} \mu_s^{1-m} \omega(2\varrho_s), \tag{4.25}$$

with constants k_{10}, k_{11}, k_{12} depending only on n, m, α_1, α_2, H, and M.

PROOF. – Let $E_\alpha^{(s)}(\mu) = \{x \in \Omega_0 : |w_\alpha^{(s)}(x)| \leq \mu\}$. Using Lemma 2.1 we have

$$\int_{G_\alpha^{(s)}} \left| \frac{\partial \varphi_\alpha^{(s)}(x)}{\partial x} \right|^m dx \leq \left(\frac{2}{\mu_s} \right)^m \int_{E_\alpha^{(s)}(\mu_s)} \left| \frac{\partial w_\alpha^{(s)}(x)}{\partial x} \right|^m dx \leq c_{15} \mu_s^{1-m} C_m(K_s'(\alpha) \setminus \Omega_s) ,$$

and we obtain (4.23) from conditions B_1 and B_2. The other inequalities are proved in a similar way. ■

LEMMA 4.5. – *Assume that conditions* A_1, A_2, A_3, B_1, B_2 *are satisfied. Then*

$$\lim_{s\to\infty} \sum_{\alpha \in I_s} \operatorname{meas}(G_\alpha^{(s)}) = 0 . \tag{4.26}$$

PROOF. – We introduce the auxiliary functions

$$\overline{\varphi}_\alpha^{(s)}(x) = \frac{4}{\mu_s} \min\left\{ \left(|w_\alpha^{(s)}(x)| - \frac{\mu_s}{4} \right)_+ , \frac{\mu_s}{4} \right\} .$$

As in (4.17) we can prove that $\overline{\varphi}_\alpha^{(s)}(x) = 0$ for $x \notin K_s(\alpha)$ and s large enough. Since (4.23) holds also for $\overline{\varphi}_\alpha^{(s)}(x)$, from Poincaré inequality we obtain

$$\int_{K_s(\alpha)} |\overline{\varphi}_\alpha^{(s)}(x)|^m dx \leq (2\lambda_s \varrho_s)^m \int_{K_s(\alpha)} \left| \frac{\partial \overline{\varphi}_\alpha^{(s)}(x)}{\partial x} \right|^m dx \leq \tag{4.27}$$

$$\leq c_{16} \mu_s^{1-m} (\lambda_s \varrho_s)^m \nu(K_s''(\alpha)) .$$

Observing that $\overline{\varphi}_\alpha^{(s)}(x) = 1$ for $x \in G_\alpha^{(s)}$, from the last inequality and from (4.5) we obtain

$$\sum_{\alpha \in I_s} \operatorname{meas}(G_\alpha^{(s)}) \leq c_{16} \mu_s^{1-m} (\lambda_s \varrho_s)^m \sum_{\alpha \in I_s} \nu(K_s''(\alpha)) \leq c_{17} \mu_s (\lambda_s \varrho_s)^{(1-\eta)m} \nu(\Omega) , \tag{4.28}$$

and the right hand side of (4.28) tends to zero as $s \to \infty$ by Lemma 4.1. ■

LEMMA 4.6. – *Assume that conditions* A_1, A_2, A_3, B_1, B_2 *are satisfied. Then there exists an integer s_2, that we may assume larger than the constant s_1 in Lemma 4.8,* *such that*

$$G^{(s)}_{\alpha\beta} \subset K\left(x^{(s)}_{\alpha\beta}, \frac{4}{3}\varrho_s\right) \tag{4.29}$$

for every $s \geq s_2$, *for every* $\alpha \in I_s$, *and for every* $\beta \in I_s(\alpha)$.

PROOF. – As in the proof of (4.18) it is possible to obtain the estimate

$$w^{(s)}_{\alpha\beta}(x, 1) \leq c_{18}\left(\frac{\omega(2\varrho_s)}{\varrho_s^{n-m}}\right)^{1/(m-1)} \qquad \text{for } x \notin K\left(x^{(s)}_{\alpha\beta}, \frac{4}{3}\varrho_s\right),$$

and by (4.5) and (4.12) we have

$$c_{18}\left(\frac{\omega(2\varrho_s)}{\varrho_s^{n-m}}\right)^{1/(m-1)} \leq 2^{(n-m)/(m-1)} c_{18}\mu_s^{m/(m-1)} < \frac{\mu_s}{2}$$

for sufficiently large s. This implies that $w^{(s)}_{\alpha\beta}(x, 1) \leq \mu_s/2$ for $x \notin K(x^{(s)}_{\alpha\beta}, (4/3)\varrho_s)$, and the conclusion follows from (4.22). ■

REMARK 4.7. – From the inclusions (4.17) and (4.29) it follows that

$$\varphi^{(s)}_{\alpha\beta}(x)\,\varphi^{(s)}_{\gamma}(x) = 0 \tag{4.30}$$

for $s \geq s_2$, $\alpha, \gamma \in I_s$, $\beta \in I_s(\alpha)$, $\alpha \neq \gamma$.

REMARK 4.8. – Denote by $\chi(G^{(s)}_{\alpha\beta}, x)$ the characteristic function of the set $G^{(s)}_{\alpha\beta}$. Then from (4.29) we obtain

$$\sum_{\alpha \in I_s} \sum_{\beta \in I_s(\alpha)} \chi(G^{(s)}_{\alpha\beta}, x) \leq 2^n$$

for every $x \in \Omega$ and for every $s \geq s_2$.

REMARK 4.9. – For $\alpha \in I_s$ and $\beta \in I_s(\alpha)$ let $I_s(\alpha, \beta)$ be the set of all pairs (γ, δ) of multi-indices such that $\gamma \in I_s$, $\delta \in I_s(\gamma)$, and $G^{(s)}_{\gamma\delta} \cap G^{(s)}_{\alpha\beta} \neq \emptyset$, and let $|I_s(\alpha, \beta)|$ be the number of elements of the set $I_s(\alpha, \beta)$. The from (4.29) we have $|I_s(\alpha, \beta)| \leq 3^n$ for $s \geq s_2$.

LEMMA 4.10. – *Assume that conditions* A_1, A_2, A_3, B_1, B_2 *are satisfied. Then there exists a positive constant* k_{13}, *depending only on* n, m, α_1, α_2, *such that, if* s *is sufficiently large,*

$$\operatorname{meas}(G^{(s)}_{\alpha\beta}) \leq k_{13}\varrho_s^m \mu_s^{1-m} \nu(K'_s(\alpha, \beta)) \tag{4.31}$$

for every $\alpha \in I_s$ *and for every* $\beta \in I_s(\alpha)$. *Moreover*

$$\lim_{s\to\infty} \sum_{\alpha \in I_s} \sum_{\beta \in I_s(\alpha)} \operatorname{meas}(G^{(s)}_{\alpha\beta}) = 0\,. \tag{4.32}$$

PROOF. – We introduce the auxiliary functions

$$\overline{\varphi}^{(s)}_{\alpha\beta}(x) = \frac{4}{\mu_s} \min\left\{\left(w^{(s)}_{\alpha\beta}(x, 1) - \frac{\mu_s}{4}\right)_+, \frac{\mu_s}{4}\right\}.$$

As in the proof of (4.27) we ovtain the inequality

$$\int_{K_s'(\alpha,\beta)} |\overline{\varphi}^{(s)}_{\alpha\beta}(x)|^m \, dx \leq (4\varrho_s)^m \int_{K_s'(\alpha,\beta)} \left|\frac{\partial \overline{\varphi}^{(s)}_{\alpha\beta}(x)}{\partial x}\right|^m dx \leq c_{19}\varrho_s^m \mu_s^{1-m} \nu(K_s'(\alpha,\beta)).$$

As $\overline{\varphi}^{(s)}_{\alpha\beta}(x) = 1$ for every $x \in G^{(s)}_{\alpha\beta}$, from the last inequality we obtain (4.31). Using Lemma 4.2 and the choice of μ_s we get

$$\sum_{\alpha\in I_s} \sum_{\beta \in I_s(\alpha)} \operatorname{meas}(G^{(s)}_{\alpha\beta}) \leq c_{20}\varrho_s^m \mu_s^{1-m} \frac{1}{\lambda_s} \nu(\Omega) \leq c_{20}\mu_s \varrho_s^{m(1-\eta)} \lambda_s^{-1-m\eta} \nu(\Omega),$$

and the right hand side of this inequality tends to zero as $s \to \infty$ by (4.2) and (4.6). ■

LEMMA 4.11. – *Assume that conditions* A_1, A_2, A_3, B_1, B_2 *are satisfied. Then there exist three positive constants* k_{14}, k_{15}, *and* k_{16}, *depending only on* n, m, α_1, α_2, H, *and* M, *such that the inequalities*

$$\int_{G^{(s)}_{\gamma\delta}} \left|\frac{\partial \varphi^{(s)}_{\alpha}(x)}{\partial x}\right|^m (\varphi^{(s)}_{\gamma\delta}(x))^m \, dx \leq k_{14}\mu_s^{1-m}\nu(K_s'(\gamma,\delta)), \tag{4.33}$$

$$\int_{G^{(s)}_{\gamma\delta}} \left|\frac{\partial v^{(s)}_{\alpha}(x,q)}{\partial x}\right|^m (\varphi^{(s)}_{\gamma\delta}(x))^m \, dx \leq k_{15}\mu_s^{1-m}\nu(K_s'(\gamma,\delta)), \tag{4.34}$$

$$\int_{G^{(s)}_{\gamma\delta}} \left|\frac{\partial v^{(s)}_{\alpha\beta}(x,q)}{\partial x}\right|^m (\varphi^{(s)}_{\gamma\delta}(x))^m \, dx \leq k_{16}\mu_s^{1-m}\nu(K_s'(\gamma,\delta)), \tag{4.35}$$

hold for s large enough $\alpha, \beta \in I_s$, $\beta \in I_s(\alpha)$, $\delta \in I_s(\gamma)$, $|q| \leq H + M$.

PROOF. – Let $w^{(s)}_{\alpha}(x)$ be the function defined in (4.14) and let $E^{(s)}_{\alpha\beta\gamma} = \{x \in G^{(s)}_{\gamma\delta} : |w^{(s)}_{\alpha}(x)| \leq \mu_s\}$. By the definition of I_s' and I_s'' given in (4.13) we have $\mu_s \leq |w^{(s)}_{\alpha}(x)|$ in $K_s'(\alpha)\backslash\Omega_s$. This implies that the function $\psi^{(s)}_{\alpha}(x) = \mu_s - \min\{|w^{(s)}_{\alpha}(x)|, \mu_s\}$ vanishes in $K_s'(\alpha)\backslash\Omega_s$. Consequently we can use the test function $\varphi^{(s)}_{\alpha\gamma\delta}(x) = \psi^{(s)}_{\alpha}(x)(\varphi^{(s)}_{\gamma\delta}(x))^m$ in the integral identity (1.15) for the function $w^{(s)}_{\alpha}(x)$. Since

$0 \leqslant \psi_\alpha^{(s)}(x) \leqslant \mu_s$, using (1.4) and (1.8) we obtain

$$\int_{E_{\alpha\gamma\delta}^{(s)}} \left(\left| \frac{\partial w_\alpha^{(s)}(x)}{\partial x} \right|^2 + \left| \frac{\partial w_\alpha^{(s)}(x)}{\partial x} \right|^m \right) (\varphi_{\gamma\delta}^{(s)}(x))^m \, dx \leqslant$$

$$\leqslant c_{21} \mu_s \int_{E_{\alpha\gamma\delta}^{(s)}} \left(\left| \frac{\partial w_\alpha^{(s)}(x)}{\partial x} \right| + \left| \frac{\partial w_\alpha^{(s)}(x)}{\partial x} \right|^{m-1} \right) \left| \frac{\partial \varphi_{\gamma\delta}^{(s)}(x)}{\partial x} \right| (\varphi_{\gamma\delta}^{(s)}(x))^{m-1} \, dx \,,$$

which, by Young's inequality, implies

$$\int_{E_{\alpha\gamma\delta}^{(s)}} \left(\left| \frac{\partial w_\alpha^{(s)}(x)}{\partial x} \right|^2 + \left| \frac{\partial w_\alpha^{(s)}(x)}{\partial x} \right|^m \right) (\varphi_{\gamma\delta}^{(s)}(x))^m \, dx \leqslant$$

$$\leqslant c_{22} \mu_s^2 \int_{G_{\gamma\delta}^{(s)}} \left| \frac{\partial \varphi_{\gamma\delta}^{(s)}(x)}{\partial x} \right|^2 dx + c_{22} \mu_s^m \int_{G_{\gamma\delta}^{(s)}} \left| \frac{\partial \varphi_{\gamma\delta}^{(s)}(x)}{\partial x} \right|^m dx \,.$$

Using (4.25), (4.31), and Hölder's inequality we obtain

$$\int_{E_{\alpha\gamma\delta}^{(s)}} \left| \frac{\partial w_\alpha^{(s)}(x)}{\partial x} \right|^m (\varphi_{\gamma\delta}^{(s)}(x))^m \, dx \leqslant c_{23} \mu_s \nu(K_s'(\gamma, \delta)) + c_{23} \mu_s^{3-m} \varrho_s^{m-2} \nu(K_s'(\gamma, \delta)) \,.$$

By (4.5) and (4.12) we have $\varrho_s \leqslant \lambda_s \varrho_s \leqslant (\lambda_s \varrho_s)^\eta \leqslant \mu_s$, so that

$$\int_{E_{\alpha\gamma\delta}^{(s)}} \left| \frac{\partial w_\alpha^{(s)}(x)}{\partial x} \right|^m (\varphi_{\gamma\delta}^{(s)}(x))^m \, dx \leqslant 2 c_{23} \mu_s \nu(K_s'(\gamma, \delta)) \,.$$

Since $\partial\varphi_\alpha^{(s)} / \partial x = 0$ a.e. in $G_{\gamma\delta}^{(s)} \setminus E_{\alpha\gamma\delta}^{(s)}$ and $|\partial \varphi_\alpha^{(s)} / \partial x| = (2/\mu_s) |\partial w_\alpha^{(s)} / \partial x|$ a.e. in $E_{\alpha\gamma\delta}^{(s)}$, inequality (4.33) follows easily from the previous estimate.

Let us prove (4.34). In the integral identity (1.15) for the function $v_\alpha^{(s)}(x, q)$ we use the test function $\varphi(x) = (v_\alpha^{(s)}(x, q) - q)(\varphi_{\gamma\delta}^{(s)}(x))^m$. Using (1.4), (1.8), and Young's inequality we obtain the estimate

$$\int_{G_{\gamma\delta}^{(s)}} \left(\left| \frac{\partial v_\alpha^{(s)}(x, q)}{\partial x} \right|^2 + \left| \frac{\partial v_\alpha^{(s)}(x, q)}{\partial x} \right|^m \right) (\varphi_{\gamma\delta}^{(s)}(x))^m \, dx \leqslant$$

$$\leqslant c_{24} \int_{G_{\gamma\delta}^{(s)}} \left(\left| \frac{\partial \varphi_{\gamma\delta}^{(s)}(x)}{\partial x} \right|^2 + \left| \frac{\partial \varphi_{\gamma\delta}^{(s)}(x)}{\partial x} \right|^m \right) dx \,.$$

Inequality (4.34) follows now from this estimate, by using (4.25), (4.31), and Hölder's inequality. The proof of (4.35) is analogous. ∎

Let us construct now a sequence of functions $\chi^{(s)}_{\alpha\beta}(x)$ such that

$$0 \leq \chi^{(s)}_{\alpha\beta}(x) \leq 1 \quad \text{for } x \in \boldsymbol{R}^n , \tag{4.36}$$

$$\chi^{(s)}_{\alpha\beta}(x) = 0 \quad \text{for } x \notin G^{(s)}_{\alpha\beta} , \tag{4.37}$$

$$\sum_{\alpha \in I_s} \sum_{\beta \in I_s(\alpha)} \chi^{(s)}_{\alpha\beta}(x) = 1 \quad \text{for } x \in \bigcup_{\alpha \in I_s} \bigcup_{\beta \in I_s(\alpha)} (K_s(\alpha, \beta) \backslash \Omega_s) . \tag{4.38}$$

To this aim we order the set J of all pairs (α, β) of multi-indices with $\alpha \in I_s$, $\beta \in I_s(\alpha)$ in a lexicographic way. We write $(\alpha, \beta) < (\gamma, \delta)$ if, in the sequence of numbers

$$\gamma_1 - \alpha_1, \ldots, \gamma_n - \alpha_n , \qquad \delta_1 - \beta_1, \ldots, \delta_n - \beta_n ,$$

the first non-zero difference is positive.

If $(\widehat{\alpha}, \widehat{\beta})$ is the minimum element of J, then we put $\chi^{(s)}_{\widehat{\alpha}\widehat{\beta}}(x) = \varphi^{(s)}_{\widehat{\alpha}\widehat{\beta}}(x)$. Assume, by induction, that for every $(\alpha, \beta) \in J$ with $(\alpha, \beta) < (\gamma, \delta)$ we can define a function $\chi^{(s)}_{\alpha\beta}(x)$ such that (4.36) and (4.37) hold, and

$$\sum_{(\alpha, \beta) < (\gamma, \delta)} \chi^{(s)}_{\alpha\beta}(x) = 1 \quad \text{for } x \in \bigcup_{(\alpha, \beta) < (\gamma, \delta)} (K_s(\alpha, \beta) \backslash \Omega_s) \tag{4.39}$$

$$0 \leq \sum_{(\alpha, \beta) < (\gamma, \delta)} \chi^{(s)}_{\alpha\beta}(x) \leq 1 \quad \text{for } x \in \boldsymbol{R}^n . \tag{4.40}$$

We define now the function $\chi^{(s)}_{\gamma\delta}(x)$ by the equality

$$\chi^{(s)}_{\gamma\delta}(x) = \varphi^{(s)}_{\gamma\delta}(x) \Big(1 - \sum_{(\alpha, \beta) < (\gamma, \delta)} \chi^{(s)}_{\alpha\beta}(x) \Big) . \tag{4.41}$$

Then (4.36) and (4.37) hold for $\chi^{(s)}_{\gamma\delta}(x)$. Let us prove that

$$\sum_{(\alpha, \beta) \leq (\gamma, \delta)} \chi^{(s)}_{\alpha\beta}(x) = 1 \quad \text{for } x \in \bigcup_{(\alpha, \beta) \leq (\gamma, \delta)} (K_s(\alpha, \beta) \backslash \Omega_s) , \tag{4.42}$$

$$0 \leq \sum_{(\alpha, \beta) \leq (\gamma, \delta)} \chi^{(s)}_{\alpha\beta}(x) \leq 1 \quad \text{for } x \in \boldsymbol{R}^n . \tag{4.43}$$

If $x \in \bigcup_{(\alpha, \beta) < (\gamma, \delta)} (K_s(\alpha, \beta) \backslash \Omega_s)$, then from (4.39) and (4.41) we obtain $\chi^{(s)}_{\gamma\delta}(x) = 0$ and consequently equality (4.42) follows from (4.39). If $x \in K_s(\gamma, \delta) \backslash \Omega_s$, then

$$\sum_{(\alpha, \beta) \leq (\gamma, \delta)} \chi^{(s)}_{\alpha\beta}(x) = \sum_{(\alpha, \beta) < (\gamma, \delta)} \chi^{(s)}_{\alpha\beta}(x) + \varphi^{(s)}_{\gamma\delta}(x) \Big(1 - \sum_{(\alpha, \beta) < (\gamma, \delta)} \chi^{(s)}_{\alpha\beta}(x) \Big) = 1 , \tag{4.44}$$

since $\varphi^{(s)}_{\gamma\delta}(x) = 1$ in $K_s(\gamma, \delta) \backslash \Omega_s$. This proves (4.42). Inequalities (4.43) follows from (4.40) and (4.44).

Proceeding by induction we construct a sequence $\chi^{(s)}_{\alpha\beta}(x)$ such that (4.36), (4.37), and (4.38) hold for every $(\alpha, \beta) \in J$.

REMARK 4.12. – Since $\chi_{\alpha\beta}^{(s)}(x) = 0$ for $x \notin G_{\alpha\beta}^{(s)}$ by (4.37), from the inclusions (4.17) and (4.29) it follows that

$$\chi_{\alpha\beta}^{(s)}(x)\, \varphi_{\gamma}^{(s)}(x) = 0 \tag{4.45}$$

for $s \geq s_2$, $\alpha, \gamma \in I_s$, $\beta \in I_s(\alpha)$, $\alpha \neq \gamma$.

LEMMA 4.13. – *Assume that conditions* A_1, A_2, A_3, B_1, B_2 *are satisfied. Then there exist two positive constants* k_{17} *and* k_{18}, *depending only on* n, *such that*

$$\begin{cases} \chi_{\alpha\beta}^{(s)}(x) \leq k_{17} \sum\limits_{(\gamma,\,\delta) \in I_s(\alpha,\,\beta)} \varphi_{\gamma\delta}^{(s)}(x), \\ \left| \dfrac{\partial \chi_{\alpha\beta}^{(s)}(x)}{\partial x} \right| \leq k_{18} \sum\limits_{(\gamma,\,\delta) \in I_s(\alpha,\,\beta)} \left| \dfrac{\partial \varphi_{\gamma\delta}^{(s)}(x)}{\partial x} \right| \end{cases} \tag{4.46}$$

for every $s \geq s_2$, *for every* $\alpha \in I_s$, *and for every* $\beta \in I_s(\alpha)$.

PROOF. – From the construction of the function $\chi_{\alpha\beta}^{(s)}(x)$ it follows that $\chi_{\alpha\beta}^{(s)}(x)$ is equal to a sum of terms of the form

$$\pm \varphi_{\alpha^{(1)}\beta^{(1)}}^{(s)}(x) \ldots \varphi_{\alpha^{(N)}\beta^{(N)}}^{(s)}(x), \tag{4.47}$$

where $(\alpha^{(i)}, \beta^{(i)}) \neq (\alpha^{(j)}, \beta^{()})$ if $i \neq j$. From Remark 4.8 it follows that the function in (4.47) is identically zero if $N > 2^n$. Therefore we have the estimate

$$\left| \frac{\partial \chi_{\alpha\beta}^{(s)}(x)}{\partial x} \right| \leq k_{18} \sum_{\gamma \in I_s} \sum_{\delta \in I_s(\gamma)} \left| \frac{\partial \varphi_{\gamma\delta}^{(s)}(x)}{\partial x} \right|, \tag{4.48}$$

with a constant k_{18} depending only on n. By (4.37) the left hand side of (4.48) is equal to zero outside $G_{\alpha\beta}^{(s)}$ and consequently, by Remark 4.9, in the right hand side of (4.48) we can omit the terms with $(\gamma, \delta) \notin I_s(\alpha, \beta)$. The proof of the first inequality in (4.46) is analogous. ∎

Let us consider now the functions $\psi_{\alpha\beta}^{(s)}(x)$ defined by the equality

$$\psi_{\alpha\beta}^{(s)}(x) = \chi_{\alpha\beta}^{(s)}(x)(1 - \varphi_{\alpha}^{(s)}(x)). \tag{4.49}$$

Then $0 \leq \psi_{\alpha\beta}^{(s)}(x) \leq 1$ in $\boldsymbol{R}^n$ and $\psi_{\alpha\beta}^{(s)}(x) = 0$ for $x \notin G_{\alpha\beta}^{(s)}$. Let us verify that

$$\sum_{\alpha \in I_s} \varphi_{\alpha}^{(s)}(x) + \sum_{\alpha \in I_s} \sum_{\beta \in I_s(\alpha)} \psi_{\alpha\beta}^{(s)}(x) = 1 \qquad \text{for } x \in \bigcup_{\alpha \in I_s} (K_s(\alpha) \backslash \Omega_s). \tag{4.50}$$

First of all we note that by (4.30) and (4.41) we have

$$\psi_{\alpha\beta}^{(s)}(x) = \chi_{\alpha\beta}^{(s)}(x)\left(1 - \sum_{\gamma \in I_s} \varphi_{\gamma}^{(s)}(x)\right). \tag{4.51}$$

If $x \in \bigcup_{\alpha \in I_s} ((K_s(\alpha) \backslash K_s'(\alpha)) \backslash \Omega_s)$, then from (4.38) and (4.51) we obtain

$$\sum_{\alpha \in I_s} \varphi_\alpha^{(s)}(x) + \sum_{\alpha \in I_s} \sum_{\beta \in I_s(\alpha)} \psi_{\alpha\beta}^{(x)}(x) = \sum_{\alpha \in I_s} \varphi_\alpha^{(s)}(x) + \sum_{\alpha \in I_s} \sum_{\beta \in I_s(\alpha)} \chi_{\alpha\beta}^{(s)}(x) \left(1 - \sum_{\gamma \in I_s} \varphi_\gamma^{(s)}(x)\right) = 1 .$$

For $x \in K_s(\alpha) \backslash \Omega_s$ we have

$$\varphi_\alpha^{(s)}(x) = 1 , \qquad \varphi_\gamma^{(s)}(x) = 0 \qquad \text{for } \gamma \neq \alpha \tag{4.52}$$

in virtue of (4.15) and (4.17). So we obtain from (4.51)

$$\psi_{\alpha\beta}^{(s)}(x) = \chi_{\alpha\beta}^{(s)}(x) \left(1 - \sum_{\gamma \in I_s} \varphi_\gamma^{(s)}(x)\right) = 0 \qquad \text{for } x \in \bigcup_{\alpha \in I_s} (K_s'(\alpha) \backslash \Omega_s) , \tag{4.53}$$

and hence identity (4.50) is proved.

LEMMA 4.14. – *Assume that conditions* A_1, A_2, A_3, B_1, B_2 *are satisfied. Then there exists a positive constant* k_{19}, *depending only on* n, m, α_1, α_2, H, *and* M, *such that*

$$\int_{G_{\alpha\beta}^{(s)}} \left| \frac{\partial \psi_{\alpha\beta}^{(s)}(x)}{\partial x} \right|^m dx \leq k_{19} \mu_s^{1-m} \nu(K(x_{\alpha\beta}^{(s)}, 4\varrho_s)) \leq k_{19} \mu_s^{1-m} \omega(4\varrho_s) \tag{4.54}$$

for every $s \geq s_2$, *for every* $\alpha \in I_s$, *and for every* $\beta \in I_s(\alpha)$.

PROOF. – From the definition of $\psi_{\alpha\beta}^{(s)}(x)$ given in (4.49) and from Lemma 4.13 we obtain

$$\int_{G_{\alpha\beta}^{(s)}} \left| \frac{\partial \psi_{\alpha\beta}^{(s)}(x)}{\partial x} \right|^m dx \leq c_{25} \sum_{(\gamma, \delta) \in I_s(\alpha, \beta)} \int_{G_{\alpha\beta}^{(s)}} \left(\left| \frac{\partial \varphi_{\gamma\delta}^{(s)}(x)}{\partial x} \right|^m + |\varphi_{\gamma\delta}^{(s)}(x)|^m \left| \frac{\partial \varphi_\alpha^{(s)}(x)}{\partial x} \right|^m \right) dx .$$

The first inequality in (4.54) is now a consequence of (4.25), (4.33), and of the inclusion

$$\bigcup_{(\gamma, \delta) \in I_s(\alpha, \beta)} K_s'(\gamma, \delta) \subset K(x_{\alpha\beta}^{(s)}, 4\varrho_s) , \tag{4.55}$$

which follows from (4.29). The second inequality in (4.54) is a consequence of condition B_2. ■

5. – Asymptotic expansion of the sequence of solutions.

In this section we investigate an asymptotic expansion of the solutions $u_s(x)$ that will be fundamental in our study:

$$u_s(x) = u_0^{(s)}(x) + \sum_{i=1}^{5} r_s^{(i)}(x) + w_s(x), \tag{5.1}$$

where

$$r_s^{(1)}(x) = \sum_{\alpha \in I_s} ((f(x) - f_\alpha^{(s)}) - (u_0^{(s)}(x) - u_\alpha^{(s)}))\, \varphi_\alpha^{(s)}(x),$$

$$r_s^{(2)}(x) = \sum_{\alpha \in I_s} \sum_{\beta \in I_s(\alpha)} ((f(x) - f_{\alpha\beta}^{(s)}) - (u_0^{(s)}(x) - u_{\alpha\beta}^{(s)}))\, \psi_{\alpha\beta}^{(s)}(x),$$

$$r_s^{(3)}(x) = \sum_{\alpha \in I_s} v_\alpha^{(s)}(x, f_\alpha^{(s)} - u_\alpha^{(s)})\, \varphi_\alpha^{(s)}(x),$$

$$r_s^{(4)}(x) = \sum_{\alpha \in I_s} \sum_{\beta \in I_s(\alpha)} v_{\alpha\beta}^{(s)}(x, f_{\alpha\beta}^{(s)} - u_{\alpha\beta}^{(s)})\, \psi_{\alpha\beta}^{(s)}(x),$$

$$r_s^{(5)}(x) = \sum_{\alpha \in I_s} \sum_{\beta \in I_s(\alpha)} ((f_\alpha^{(s)} - u_\alpha^{(s)}) - v_\alpha^{(s)}(x, f_\alpha^{(s)} - u_\alpha^{(s)}))\, \varphi_\alpha^{(s)}(x)\, \chi_{\alpha\beta}^{(s)}(x),$$

and $w_s(x)$ is the remainder. Here $u_0^{(s)}(x)$ is the averaging of the function $u_0(x)$ defined in (1.16), $f_\alpha^{(s)}$ and $u_\alpha^{(s)}$ are the mean values of the functions $f(x)$ and $u_0^{(s)}(x)$ in the cube $K_s(\alpha)$ defined in (4.10), $f_{\alpha\beta}^{(s)}$ and $u_{\alpha\beta}^{(s)}$ are the mean values of the same functions in the cube $K_s'(\alpha, \beta)$ defined in (4.21), $v_\alpha^{(s)}(x, q)$ and $v_{\alpha\beta}^{(s)}(x, q)$ are the functions introduced in (4.11) and (4.20), $\varphi_\alpha^{(s)}(x)$ are the functions introduced in (4.15), $\chi_{\alpha\beta}^{(s)}(x)$ are the functions introduced in (4.41), and $\psi_{\alpha\beta}^{(s)}(x)$ are the functions introduced in (4.49).

The study of the behaviour of the terms of the asymptotic expansion (5.1) is the main purpose of this section.

LEMMA 5.1. – *Assume that conditions* A_1, A_2, A_3, B_1, B_2 *are satisfied. Let* $g(x)$ *be a function in* $C_0^\infty(\Omega)$ *and let* $w_s(x)$ *be the remainder of the asymptotic expansion* (5.1). *Then there exists a number* s_3, *depending on* $g(x)$, *such that* $g(x)\, w_s(x)$ *belongs to* $\overset{\circ}{W}{}_m^1(\Omega_s)$ *for* $s \geq s_3$.

PROOF. – See [59], Lemma 4.6. ■

LEMMA 5.2. – *Assume that conditions* A_1, A_2, A_3, B_1, B_2 *are satisfied. Then the sequences* $r_s^{(1)}(x)$, $r_s^{(2)}(x)$, $r_s^{(4)}(x)$, *and* $r_s^{(5)}(x)$ *converge to zero strongly in* $W_m^1(\Omega)$ *as* $s \to \infty$.

PROOF. – Since the functions $f(x)$ and $u_0(x)$ are bounded, from Lemmas 4.5 and 4.10 we obtain immediately that the sequence $r_s^{(i)}(x)$, $i = 1, 2, 4, 5$ converge to zero strongly in $L_m(\Omega)$ as $s \to \infty$.

Let us estimate the norm of the gradient of $r_s^{(i)}(x)$ in $L_m(\Omega)$. For $i=1$ we have

$$\int_\Omega \left| \frac{\partial r_s^{(1)}(x)}{\partial x} \right|^m dx \leqslant c_{26} \sum_{\alpha \in I_s} \int_{G_\alpha^{(s)}} \left(\left| \frac{\partial f(x)}{\partial x} \right|^m + \left| \frac{\partial u_0(x)}{\partial x} \right|^m \right) dx +$$

$$+ c_{26} \sum_{\alpha \in I_s} \int_{G_\alpha^{(s)}} \left| \frac{\partial u_0^{(s)}(x)}{\partial x} - \frac{\partial u_0(x)}{\partial x} \right|^m dx +$$

$$+ c_{26} \sum_{\alpha \in I_s} \int_{G_\alpha^{(s)}} (|f(x) - f_\alpha^{(s)}|^m + |u_0^{(s)}(x) - u_\alpha^{(s)}|^m) \left| \frac{\partial \varphi_\alpha^{(s)}(x)}{\partial x} \right|^m dx .$$

The first term in the right hand side of the previous inequality tends to zero as $s \to \infty$ by Lemma 4.5 and by the absolute continuity of the integral. The second term tends to zero by the properties of the averaging functions.

Since the function $f(x)$ belongs to the space $C^{0,\eta}(\overline{\Omega})$ with $\eta > 0$, Lemma 2.7, together with (4.17) and (4.23), yields

$$(5.2) \quad \sum_{\alpha \in I_s} \int_{G_\alpha^{(s)}} (|f(x) - f_\alpha^{(s)}|^m + |u_0^{(s)}(x) - u_\alpha^{(s)}|^m) \left| \frac{\partial \varphi_\alpha^{(s)}(x)}{\partial x} \right|^m dx \leqslant$$

$$\leqslant c_{27} \mu_s^{1-m} (\lambda_s \varrho_s)^{m\eta} \, \nu(\Omega) + c_{27} \mu_s^{1-m} \frac{\omega(\lambda_s \varrho_s)}{(\lambda_s \varrho_s)^{n-m}} \int_\Omega \left| \frac{\partial u_0(x)}{\partial x} \right|^m dx \leqslant$$

$$\leqslant c_{27} \mu_s \nu(\Omega) + c_{27} \mu_s \int_\Omega \left| \frac{\partial u_0(x)}{\partial x} \right|^m dx ,$$

where in the last inequality we use the definition of μ_s given in (4.5). Both terms in the last line of (5.2) tend to zero as $s \to \infty$ by Lemma 4.1. This completes the proof of the strong convergence to zero of the gradient of $r_s^{(1)}(x)$.

Let us estimate now the norm of the gradient of $r_s^{(2)}(x)$ in $L_m(\Omega)$. Recalling that the function $\psi_{\alpha\beta}^{(s)}(x)$ is zero outside $G_{\alpha\beta}^{(s)}$, we have

$$\int_\Omega \left| \frac{\partial r_s^{(2)}(x)}{\partial x} \right|^m dx \leqslant$$

$$\leqslant c_{28} \sum_{\alpha \in I_s} \sum_{\beta \in I_s(\alpha)} \int_{G_{\alpha\beta}^{(s)}} \left(\left| \frac{\partial f(x)}{\partial x} \right|^m + \left| \frac{\partial u_0(x)}{\partial x} \right|^m + \left| \frac{\partial u_0^{(s)}(x)}{\partial x} - \frac{\partial u_0(x)}{\partial x} \right|^m \right) dx +$$

$$+ c_{28} \sum_{\alpha \in I_s} \sum_{\beta \in I_s(\alpha)} \int_{G_{\alpha\beta}^{(s)}} (|f(x) - f_{\alpha\beta}^{(s)}|^m + |u_0^{(s)}(x) - u_{\alpha\beta}^{(s)}|^m) \left| \frac{\partial \psi_{\alpha\beta}^{(s)}(x)}{\partial x} \right|^m dx .$$

The first sum in the right hand side of the previous inequality tends to zero by (4.32), by the absolute continuity of the integral, and by the properties of the averaging functions.

By using (4.54), Lemma 2.7, and the Hölder continuity of the function $f(x)$, we obtain

$$\sum_{\alpha\in I_s}\sum_{\beta\in I_s(\alpha)}\int_{G^{(s)}_{\alpha\beta}}(|f(x)-f^{(s)}_{\alpha\beta}|^m+|u_0^{(s)}(x)-u^{(s)}_{\alpha\beta}|^m)\left|\frac{\partial\psi^{(s)}_{\alpha\beta}(x)}{\partial x}\right|^m dx\leqslant$$

$$\leqslant c_{29}\mu_s^{1-m}\varrho_s^{m\eta}\sum_{\alpha\in I_s}\sum_{\beta\in I_s(\alpha)}\nu(K(x^{(s)}_{\alpha\beta},4\varrho_s))+c_{29}\mu_s^{1-m}\frac{\omega(4\varrho_s)}{\varrho_s^{n-m}}\int_\Omega\left|\frac{\partial u_0(x)}{\partial x}\right|^m dx\leqslant$$

$$\leqslant c_{30}\mu_s\lambda_s^{-m\eta}\nu(\Omega)+c_{30}\mu_s\int_\Omega\left|\frac{\partial u_0(x)}{\partial x}\right|^m dx\,,$$

where in the last inequality we use (4.12) and the definition of μ_s given in (4.5). Both terms in the last line tend to zero as $s\to\infty$ by Lemma 4.1.

Let us estimate now the norm of the gradient of $r_s^{(4)}(x)$ in $L_m(\Omega)$. Using (4.12), (4.33), (4.35), (4.46), (4.49), (4.54), and (4.55) we obtain

$$\text{(5.3)}\qquad \int_\Omega\left|\frac{\partial r_s^{(4)}(x)}{\partial x}\right|^m dx\leqslant$$

$$\leqslant c_{31}\sum_{\alpha\in I_s}\sum_{\beta\in I_s(\alpha)}\sum_{(\gamma,\delta)\in I_s(\alpha,\beta)}\int_{G^{(s)}_{\gamma\delta}}\left|\frac{\partial v^{(s)}_{\alpha\beta}(x,f^{(s)}_{\alpha\beta}-u^{(s)}_{\alpha\beta})}{\partial x}\right|^m(\varphi^{(s)}_{\gamma\delta}(x))^m dx+$$

$$+c_{31}\sum_{\alpha\in I_s}\sum_{\beta\in I_s(\alpha)}\int_{G^{(s)}_{\alpha\beta}}\left|\frac{\partial\psi^{(s)}_{\alpha\beta}(x)}{\partial x}\right|^m dx\leqslant c_{32}\mu_s^{1-m}\sum_{\alpha\in I_s}\sum_{\beta\in I_s(\alpha)}\nu(K(x^{(s)}_{\alpha\beta},4\varrho_s))\,.$$

It is easy to verify that the following inclusion holds

$$\text{(5.4)}\qquad \bigcup_{\alpha\in I_s}\bigcup_{\beta\in I_s(\alpha)}K(x^{(s)}_{\alpha\beta},4\varrho_s)\subset F_s\cap\Omega\,,$$

where F_s is the set defined in (4.7). Consequently, from (4.5), (4.8), (5.3), and (5.4) we obtain

$$\text{(5.5)}\qquad \int_\Omega\left|\frac{\partial r_s^{(4)}(x)}{\partial x}\right|^m dx\leqslant c_{33}\mu_s^{1-m}\lambda_s^{-1}\nu(\Omega)\leqslant c_{33}\mu_s^{m+1}\nu(\Omega)\,,$$

and the right hand side of this inequality tends to zero by Lemma 4.1.

For the gradient of $r_s^{(5)}(x)$ we have the estimate

$$\int_\Omega \left| \frac{\partial r_s^{(5)}(x)}{\partial x} \right|^m dx \leqslant c_{34} \sum_{\alpha \in I_s} \sum_{\beta \in I_s(\alpha)} \left(\int_{G_{\alpha\beta}^{(s)}} \left| \frac{\partial \chi_{\alpha\beta}^{(s)}(x)}{\partial x} \right|^m dx + \right.$$

$$\left. + \int_{G_{\alpha\beta}^{(s)}} \left(\left| \frac{\partial v_\alpha^{(s)}(x, f_\alpha^{(s)} - u_\alpha^{(s)})}{\partial x} \right|^m + \left| \frac{\partial \varphi_\alpha^{(s)}(x)}{\partial x} \right|^m \right) (\chi_{\alpha\beta}^{(s)}(x))^m \, dx \right).$$

Using (4.46), (4.25), (4.33), (4.34), (4.55), (5.4), (4.8), and (4.5), we obtain

$$(5.6) \qquad \int_\Omega \left| \frac{\partial r_s^{(5)}(x)}{\partial x} \right|^m dx \leqslant c_{35} \mu_s^{1-m} \sum_{\alpha \in I_s} \sum_{\beta \in I_s(\alpha)} \sum_{(\gamma, \delta) \in I_s(\alpha, \beta)} \nu(K_s'(\gamma, \delta)) \leqslant$$

$$\leqslant c_{35} \mu_s^{1-m} \sum_{\alpha \in I_s} \sum_{\beta \in I_s(\alpha)} \nu(K(x_{\alpha\beta}^{(s)}, 4\varrho_s)) \leqslant 7nc_{35} \mu_s^{1-m} \lambda_s^{-1} \nu(\Omega) \leqslant 7nc_{35} \mu_s^{m+1} \nu(\Omega),$$

and the right hand side of this inequality tends to zero as $s \to \infty$ by Lemma 4.1. ■

LEMMA 5.3. – *Assume that conditions* A_1, A_2, A_3, B_1, B_2 *are satisfied. Then the sequence* $r_s^{(3)}(x)$ *is bounded in* $W_m^1(\Omega)$ *and converges to zero strongly in* $W_p^1(\Omega)$ *for* $p < m$.

PROOF. – The strong convergence to zero of $r_s^{(3)}(x)$ in $L_m(\Omega)$ follows from Lemma 4.5 and from the estimate

$$|v_\alpha^{(s)}(x, f_\alpha^{(s)} - u_\alpha^{(s)})| \leqslant H + M \qquad \text{for } x \in \Omega_0,$$

which is a consequence of (1.12) and (1.13) and Remark 2.2. We estimate the derivative of $r_s^{(3)}(x)$ in $L_m(\Omega)$ by means of Lemmas 2.1 and 4.4. Taking B_1 into account we obtain

$$(5.7) \qquad \int_\Omega \left| \frac{\partial r_s^{(3)}(x)}{\partial x} \right|^m dx \leqslant c_{36} \sum_{\alpha \in I_s} \int_\Omega \left| \frac{\partial v_\alpha^{(s)}(x, f_\alpha^{(s)} - u_\alpha^{(s)})}{\partial x} \right|^m dx +$$

$$+ c_{36} \sum_{\alpha \in I_s} \int_{E_\alpha^{(s)}(\mu_s)} |v_\alpha^{(s)}(x, f_\alpha^{(s)} - u_\alpha^{(s)})|^m \left| \frac{\partial \varphi_\alpha^{(s)}(x)}{\partial x} \right|^m dx \leqslant c_{37} \sum_{\alpha \in I_s} \nu(K_s''(\alpha)) \leqslant c_{37} \nu(\Omega),$$

where $E_\alpha^{(s)}(\mu) = \{x \in \Omega_0 : |w_\alpha^{(s)}(x)| \leqslant \mu\}$. Consequently in the third integral in (5.7) we have majorized $|v_\alpha^{(s)}(x, f_\alpha^{(s)} - u_\alpha^{(s)})|$ by $2\mu_s$ in both cases $\alpha \in I_s'$ and $\alpha \in I_s''$ (see (4.13) and (4.14)).

Observing that the function $r_s^{(3)}(x)$ vanishes outside $\bigcup_{\alpha \in I_s} G_\alpha^{(s)}$ and applying Hölder's inequality, we obtain for $1 < p < m$

$$\int_\Omega \left| \frac{\partial r_s^{(3)}(x)}{\partial x} \right|^p dx \leq \left(\int_\Omega \left| \frac{\partial r_s^{(3)}(x)}{\partial x} \right|^m dx \right)^{p/m} \left(\sum_{a \in I_s} \operatorname{meas}(G_a^{(s)}) \right)^{1-p/m} .$$

The right-hand side of this inequality tends to zero thanks to (4.26) and (5.7). This concludes the proof of the lemma. ■

LEMMA 5.4. – *Assume that conditions* A_1, A_2, A_3, B_1, B_2 *are satisfied. Let* $w_s(x)$ *be the remainder of the asymptotic expansion* (5.1) *and let* $g(x)$ *be a function in* $C_0^\infty(\Omega)$. *Then the sequence* $g(x)\, w_s(x)$ *converges strongly to zero in* $W_m^1(\Omega)$.

PROOF. – We may assume that $|g(x)| \leq 1$ for every $x \in \Omega$. By Lemma 5.1 the function $g(x)\, w_s(x)$ belongs to $\overset{\circ}{W}{}_m^1(\Omega_s)$ for s large enough. Moreover, we shall assume that $s \geq s_2$ and $s \geq s_3$, where s_2 and s_3 are the constants in Lemmas 4.6 and 5.1. So we can take $|g(x)|^m w_s(x)$ as test function in the integral identity (1.10) corresponding to the boundary value problem (0.1), (0.2), obtaining

$$(5.8) \qquad \sum_{j=1}^n \int_\Omega a_j\left(x, u_s(x), \frac{\partial u_s(x)}{\partial x}\right) \frac{\partial}{\partial x_j} \left(|g(x)|^m w_s(x) \right) dx =$$

$$= - \int_\Omega a_0\left(x, u_s(x), \frac{\partial u_s(x)}{\partial x}\right) |g(x)|^m w_s(x)\, dx .$$

Let us investigate the behaviour of the integrals on the left-hand side of (5.8) as $s \to \infty$. From Lemmas 5.2 and 5.3 and from (5.1) it follows that $w_s(x)$ converges to zero strongly in $L_m(\Omega)$. This convergence is also in $L_r(\Omega)$ for every $r > 1$, since the sequence $w_s(x)$ is uniformly bounded (Theorem 1.3). By (1.8) and (1.11) this implies that

$$\lim_{s \to \infty} \int_\Omega \left(\sum_{j=1}^n a_j\left(x, u_s(x), \frac{\partial u_s(x)}{\partial x}\right) \frac{\partial}{\partial x_j} |g(x)|^m + a_0\left(x, u_s(x), \frac{\partial u_s(x)}{\partial x}\right) |g(x)|^m \right) \cdot$$

$$\cdot\, w_s(x)\, dx = 0$$

and consequently

$$(5.9) \qquad \lim_{s \to \infty} \int_\Omega |g(x)|^m \sum_{j=1}^n a_j\left(x, u_s(x), \frac{\partial u_s(x)}{\partial x}\right) \frac{\partial w_s(x)}{\partial x_j}\, dx = 0 .$$

We rewrite the integral in (5.9) in the form

$$(5.10) \qquad \int_\Omega |g(x)|^m \sum_{j=1}^n a_j\left(x, u_s(x), \frac{\partial u_s(x)}{\partial x}\right) \frac{\partial w_s(x)}{\partial x_j}\, dx = I_1^{(s)} + I_2^{(s)} + I_3^{(s)} + I_4^{(s)} + I_5^{(s)} ,$$

where

$$I_1^{(s)} = \int_\Omega |g(x)|^m \cdot$$

$$\cdot \sum_{j=1}^n \left(a_j\left(x, u_s(x), \frac{\partial u_s(x)}{\partial x}\right) - a_j\left(x, u_s(x), \frac{\partial u_s(x)}{\partial x} - \frac{\partial w_s(x)}{\partial x}\right)\right) \frac{\partial w_s(x)}{\partial x_j}\, dx\,,$$

$$I_2^{(s)} = \int_\Omega |g(x)|^m \cdot$$

$$\cdot \sum_{j=1}^n \left(a_j\left(x, u_s(x), \frac{\partial u_s(x)}{\partial x} - \frac{\partial w_s(x)}{\partial x}\right) - a_j\left(x, u_s(x), \frac{\partial u_0(x)}{\partial x} + \frac{\partial r_s^{(3)}(x)}{\partial x}\right)\right) \frac{\partial w_s(x)}{\partial x_j}\, dx\,,$$

$$I_3^{(s)} = \int_\Omega |g(x)|^m \cdot$$

$$\cdot \sum_{j=1}^n \left(a_j\left(x, u_s(x), \frac{\partial u_s(x)}{\partial x} + \frac{\partial r_s^{(3)}(x)}{\partial x}\right) - a_j\left(x, u_s(x), \frac{\partial r_s^{(3)}(x)}{\partial x}\right)\right) \frac{\partial w_s(x)}{\partial x_j}\, dx\,,$$

$$I_4^{(s)} = \int_\Omega |g(x)|^m \sum_{j=1}^n \left(a_j\left(x, u_s(x), \frac{\partial r_s^{(3)}(x)}{\partial x}\right) - a_j\left(x, 0, \frac{\partial r_s^{(3)}(x)}{\partial x}\right)\right) \frac{\partial w_s(x)}{\partial x_j}\, dx\,,$$

$$I_5^{(s)} = \int_\Omega |g(x)|^m \sum_{j=1}^n a_j\left(x, 0, \frac{\partial r_s^{(3)}(x)}{\partial x}\right) \frac{\partial w_s(x)}{\partial x_j}\, dx\,.$$

By (1.5) we have the following estimate for $I_1^{(s)}$:

$$I_1^{(s)} \geq \alpha_1 \int_\Omega |g(x)|^m \left| \frac{\partial w_s(x)}{\partial x} \right|^m dx\,. \tag{5.11}$$

The convergence to zero of $I_2^{(s)}$, $I_3^{(s)}$, and $I_4^{(s)}$ is proved as in Theorem 4.9 of [59]. Therefore (5.9), (5.10), and (5.11) imply that

$$\limsup_{s\to\infty} \int_\Omega |g(x)|^m \left| \frac{\partial w_s(x)}{\partial x} \right|^m dx \leq \frac{1}{\alpha_1} \limsup_{s\to\infty} |I_5^{(s)}|\,. \tag{5.12}$$

To study the behaviour of $I_5^{(s)}$ we introduce the function $\zeta_\alpha^{(s)}(x)$ defined by the equality

$$\zeta_\alpha^{(s)}(x) = \frac{1}{\mu_s} \min\{(|w_\alpha^{(s)}(x)| - \mu_s)_+, \mu_s\}\,. \tag{5.13}$$

As for the function $\varphi_\alpha^{(s)}(x)$ we can prove (Lemma 4.4) the estimate

$$\int_{H_\alpha^{(s)}} \left| \frac{\partial \zeta_\alpha^{(s)}(x)}{\partial x} \right|^m dx \leqslant k_{20} \mu_s^{1-m} \nu(K_s''(\alpha)) , \tag{5.14}$$

where $H_\alpha^{(s)}$ is the set of points x such that $\zeta_\alpha^{(s)}(x) \neq 0$. Moreover we have $H_\alpha^{(s)} \subset G_\alpha^{(s)}$ and

$$\varphi_\alpha^{(s)}(x) = 1 \quad \text{in } H_\alpha^{(s)} \tag{5.15}$$

for every $\alpha \in I_s$. Using this property and Lemma 4.3 we obtain

$$I_5^{(s)} = I_6^{(s)} + I_7^{(s)} + I_8^{(s)} , \tag{5.16}$$

where

$$I_6^{(s)} = \sum_{\alpha \in I_s} \int_{H_\alpha^{(s)}} \sum_{j=1}^n a_j\left(x, 0, \frac{\partial v_\alpha^{(s)}(x)}{\partial x}\right) \frac{\partial}{\partial x_j} \left(|g(x)|^m \zeta_\alpha^{(s)}(x)\, w_s(x)\right) dx ,$$

$$I_7^{(s)} = - \sum_{\alpha \in I_s} \int_{H_\alpha^{(s)}} \sum_{j=1}^n a_j\left(x, 0, \frac{\partial r_s^{(3)}(x)}{\partial x}\right) \zeta_\alpha^{(s)}(x)\, w_s(x) \, \frac{\partial |g(x)|^m}{\partial x_j} \, dx ,$$

$$I_8^{(s)} = \sum_{\alpha \in I_s} \int_{G_\alpha^{(s)}} |g(x)|^m \sum_{j=1}^n a_j\left(x, 0, \frac{\partial}{\partial x} (\varphi_\alpha^{(s)} v_\alpha^{(s)})(x)\right) \frac{\partial}{\partial x_j} \left(w_s(x)(1 - \zeta_\alpha^{(s)}(x))\right) dx .$$

Here $v_\alpha^{(s)}(x) = v_\alpha^{(s)}(x, f_\alpha^{(s)} - u_\alpha^{(s)})$. By the definition of the function $v_\alpha^{(s)}(x, f_\alpha^{(s)} - u_\alpha^{(s)})$ we obtain that $I_6^{(s)} = 0$. Since $w_s(x)$ converges to zero strongly in $L_r(\Omega)$ for every $r > 1$, Lemma 5.3 implies that $I_7^{(s)}$ tends to zero as $s \to \infty$. So we have

$$I_6^{(s)} = 0 , \qquad \lim_{s \to \infty} I_7^{(s)} = 0 . \tag{5.17}$$

In order to estimate $I_8^{(s)}$ we introduce the sets

$$E_\alpha^{(s)} = \{x \in \Omega \colon |w_\alpha^{(s)}(x)| \leqslant 2\mu_s\} \cap G_\alpha^{(s)} .$$

Since $1 - \zeta_\alpha^{(s)}(x) = 0$ in $G_\alpha^{(s)} \backslash E_\alpha^{(s)}$, from (1.8) we obtain the inequality

$$|I_8^{(s)}| \leqslant c_{38} \left(\sum_{\alpha \in I_s} \int_{E_\alpha^{(s)}} \left(1 + \left| \frac{\partial}{\partial x} (v_\alpha^{(s)} \varphi_\alpha^{(s)})(x) \right| \right)^m dx \right)^{(m-2)/m} \cdot \tag{5.18}$$

$$\cdot \left(\sum_{\alpha \in I_s} \int_{E_\alpha^{(s)}} \left| \frac{\partial}{\partial x} (v_\alpha^{(s)} \varphi_\alpha^{(s)})(x) \right|^m dx \right)^{1/m} \left(\sum_{\alpha \in I_s} \int_{E_\alpha^{(s)}} \left| \frac{\partial}{\partial x} (w_s (1 - \zeta_\alpha^{(s)}))(x) \right|^m dx \right)^{1/m} .$$

If $\alpha \in I_s'$, then $|v_\alpha^{(s)}(x)| = |w_\alpha^{(s)}(x)| \leqslant 2\mu_s$ in $E_\alpha^{(s)}$. If $\alpha \in I_s''$, then $|v_\alpha^{(s)}(x)| \leqslant |f_\alpha^{(s)} - u_\alpha^{(s)}| \leqslant 2\mu_s$ in Ω. Therefore $|v_\alpha^{(s)}(x)| \leqslant 2\mu_s$ in $E_\alpha^{(s)}$ for every $\alpha \in I_s$. Consequently Lemma 2.1, condition B_1, and inequality (4.23) yield

$$\int_{E_\alpha^{(s)}} \left| \frac{\partial}{\partial x} (v_\alpha^{(s)} \varphi_\alpha^{(s)})(x) \right|^m dx \leqslant \tag{5.19}$$

$$\leqslant c_{39} \int_{E_\alpha^{(s)}} \left(\left| \frac{\partial v_\alpha^{(s)}(x)}{\partial x} \right|^m + \mu_s^m \left| \frac{\partial \varphi_\alpha^{(s)}(x)}{\partial x} \right|^m \right) dx \leqslant c_{40} \mu_s \nu(K_s''(\alpha)),$$

which, together with (4.12) and (4.28), gives

$$\sum_{\alpha \in I_s} \int_{E_\alpha^{(s)}} \left(1 + \left| \frac{\partial}{\partial x} (v_\alpha^{(s)} \varphi_\alpha^{(s)})(x) \right| \right)^m dx \leqslant c_{41} \mu_s \nu(\Omega). \tag{5.20}$$

For the last integral of the right-hand side of (5.18) we have the estimate

$$\sum_{\alpha \in I_s} \int_{E_\alpha^{(s)}} \left| \frac{\partial}{\partial x} (w_s(1 - \zeta_\alpha^{(s)}))(x) \right|^m dx \leqslant \tag{5.21}$$

$$\leqslant c_{42} \left(\int_\Omega \left| \frac{\partial w_s(x)}{\partial x} \right|^m dx + \sum_{\alpha \in I_s} \int_{H_\alpha^{(s)}} |w_s(x)|^m \left| \frac{\partial \zeta_\alpha^{(s)}(x)}{\partial x} \right|^m dx \right).$$

By (5.1) and by Lemmas 5.2 and 5.3 there exists a constant c_{34} such that

$$\int_\Omega \left| \frac{\partial w_s(x)}{\partial x} \right|^m dx \leqslant c_{43}. \tag{5.22}$$

We shall now evaluate the last integral in (5.21). By (5.15) for $x \in H_\alpha^{(s)}$ we have $\varphi_\alpha^{(s)}(x) = 1$ and, consequently, from (4.49) we get $\psi_\alpha^{(s)}(x) = 0$. Therefore from the asymptotic expansion (5.1) we obtain

$$w_s(x) = u_s(x) - u_0^{(s)}(x) - r_s^{(1)}(x) - r_s^{(3)}(x) - r_s^{(5)}(x) \qquad \text{for } x \in H_\alpha^{(s)},$$

and

$$\sum_{\alpha \in I_s} \int_{H_\alpha^{(s)}} |w_s(x)|^m \left| \frac{\partial \zeta_\alpha^{(s)}(x)}{\partial x} \right|^m dx \leqslant c_{44}(I_9^{(s)} + I_{10}^{(s)} + I_{11}^{(s)} + I_{12}^{(s)}), \tag{5.23}$$

where

$$I_9^{(s)} = \sum_{\alpha \in I_s} \int_{H_\alpha^{(s)}} |u_s(x) - u_0^{(s)}(x)|^m \left| \frac{\partial \zeta_\alpha^{(s)}(x)}{\partial x} \right|^m dx ,$$

$$I_{10}^{(s)} = \sum_{\alpha \in I_s} \int_{H_\alpha^{(s)}} (|f(x) - f_\alpha^{(s)}|^m + |u_0^{(s)}(x) - u_\alpha^{(s)}|^m) \left| \frac{\partial \zeta_\alpha^{(s)}(x)}{\partial x} \right|^m dx ,$$

$$I_{11}^{(s)} = \sum_{\alpha \in I_s} \int_{H_\alpha^{(s)}} |v_\alpha^{(s)}(x, f_\alpha^{(s)} - u_\alpha^{(s)})|^m \left| \frac{\partial \zeta_\alpha^{(s)}(x)}{\partial x} \right|^m dx ,$$

$$I_{12}^{(s)} = \sum_{\alpha \in I_s} \sum_{\beta \in I_s(\alpha)} \int_{H_\alpha^{(s)}} |\chi_{\alpha\beta}^{(s)}(x)|^m \left| \frac{\partial \zeta_\alpha^{(s)}(x)}{\partial x} \right|^m dx .$$

In order to estimate $I_9^{(s)}$ we fix a function $\sigma_\alpha^{(s)}(x)$ of class $C_0^\infty(\Omega)$, equal to one on $K_s(\alpha)$ and to zero outside $K(x_\alpha^{(s)}, 2\lambda_s \varrho_s)$, and satisfying $|\partial \sigma_\alpha^{(s)}/\partial x| \leqslant 2/(\lambda_s \varrho_s)$. Then we take the test function

$$|u_s(x) - u_0^{(s)}(x)|^m \min\{|w_\alpha^{(s)}(x)| - 2\mu_s, 0\}(\sigma_\alpha^{(s)}(x))^m$$

in the integral identity corresponding to the boundary value problem for the function $w_\alpha^{(s)}(x)$ defined by (4.14), and we obtain

$$\text{(5.24)} \qquad \int_{E_\alpha^{(s)}} |u_s(x) - u_0^{(s)}(x)|^m \left| \frac{\partial w_\alpha^{(s)}(x)}{\partial x} \right|^m dx \leqslant$$

$$\leqslant c_{45} \mu_s^2 \int_{K(x_\alpha^{(s)}, 2\lambda_s \varrho_s)} \left| \frac{\partial u_s(x)}{\partial x} - \frac{\partial u_0^{(s)}(x)}{\partial x} \right|^2 dx +$$

$$+ c_{45} \mu_s^m \int_{K(x_\alpha^{(s)}, 2\lambda_s \varrho_s)} \left| \frac{\partial u_s(x)}{\partial x} - \frac{\partial u_0^{(s)}(x)}{\partial x} \right|^m dx +$$

$$+ c_{45} \left(\frac{\mu_s}{\lambda_s \varrho_s} \right)^2 \int_{K(x_\alpha^{(s)}, 2\lambda_s \varrho_s)} |u_s(x) - u_0^{(s)}(x)|^2 dx +$$

$$+ c_{45} \left(\frac{\mu_s}{\lambda_s \varrho_s} \right)^m \int_{K(x_\alpha^{(s)}, 2\lambda_s \varrho_s)} |u_s(x) - u_0^{(s)}(x)|^m dx .$$

Using Lemma 2.6, (4.12), (5.24), and the choice of ϱ_s we have

$$(5.25)\qquad |I_9^{(s)}| \le c_{46}\mu_s^{-m}\left(\mu_s^2 R^{2/m} + \mu_s^m R + \left(\frac{\mu_s}{\lambda_s\varrho_s}\right)^2 \int_\Omega |u_s(x)-u_0(x)|^2\,dx + \right.$$

$$\left. + \left(\frac{\mu_s}{\lambda_s\varrho_s}\right)^m \int_\Omega |u_s(x)-u_0(x)|^m\,dx\right) \le c_{47}\mu_s^{2-m},$$

where R is the constant in inequality (1.11).

The estimate for $I_{10}^{(s)}$ is similar to (5.2) and can be obtained by the same arguments, using (5.14) instead of (4.23):

$$(5.26)\qquad |I_{10}^{(s)}| \le c_{48}\mu_s^{1-m}(\lambda_s\varrho_s)^{m\eta}\,\nu(\Omega) + c_{48}\mu_s^{1-m}\frac{\omega(\lambda_s\varrho_s)}{(\lambda_s\varrho_s)^{n-m}}R \le c_{49}\mu_s .$$

As $\partial\zeta_\alpha^{(s)}/\partial x = 0$ in $H_\alpha^{(s)}\setminus E_\alpha^{(s)}$ and $|v_\alpha^{(s)}(x, f_\alpha^{(s)} - u_\alpha^{(s)})| \le 2\mu_s$ in $E_\alpha^{(s)}$, from (5.14) we obtain

$$(5.27)\qquad |I_{11}^{(s)}| \le c_{50}\mu_s\nu(\Omega).$$

The estimate for $I_{12}^{(s)}$ is similar to (5.6) and can be obtained by the same arguments:

$$(5.28)\qquad |I_{12}^{(s)}| \le c_{51}\mu_s^{1-m}\lambda_s^{-1}\nu(\Omega) \le c_{52}\mu_s .$$

Using (5.25)-(5.28) we obtain

$$|I_9^{(s)} + I_{10}^{(s)} + I_{11}^{(s)} + I_{12}^{(s)}| \le c_{47}\mu_s^{2-m} + c_{53}\mu_s .$$

Therefore (4.12), (5.18), and (5.20)-(5.23) imply that

$$|I_8^{(s)}| \le c_{54}\mu_s^{(m-1)/m}(c_{43} + c_{47}\mu_s^{2-m} + c_{53}\mu_s)^{1/m} \le c_{55}\mu_s^{1/m}$$

and by virtue of Lemma 4.1 we have

$$(5.29)\qquad \lim_{s\to\infty} I_8^{(s)} = 0 .$$

From (5.12), (5.26), (5.17), and (5.29) it follows that the sequence $g(x)\,w_s(x)$ converges to zero strongly in $W_m^1(\Omega)$. ■

PROOF OF THEOREM 1.4. – If we compare the asymptotic expansions (1.17) and (5.1) we obtain

$$(5.30)\qquad R_s(x) = r_s^{(1)}(x) + r_s^{(2)}(x) + r_s^{(4)}(x) + r_s^{(5)}(x) + w_s(x).$$

Therefore Theorem 1.4 follows Lemmas 5.2 and 5.4. ■

6. – Choice of the decomposition.

So far ϱ_s is an arbitrary sequence which converges to zero and satisfies (4.1). In order to conclude the proof of Theorem 1.5 we need a very precise choice of ϱ_s. We begin with some lemmas about subadditive functions.

LEMMA 6.1. – *Let $\beta(B)$ be a non-negative increasing subadditive function defined for every Borel set $B \subset \Omega$. Assume that there exists a bounded Borel measure $\mu(B)$ such that $\beta(B) \leqslant \mu(B)$ for every Borel set $B \subset \Omega$. Then*

$$\beta(B) = \sup\{\beta(E)\colon E \text{ compact},\ E \subset B\} \tag{6.1}$$

for every Borel set $B \subset \Omega$.

PROOF. – Let us fix a Borel set $B \subset \Omega$ and let S be the right hand side of (6.1). By monotonicity it is enough to prove that $\beta(B) \leqslant S$. Since $\mu(B)$ is a bounded Borel measure, for every $\varepsilon > 0$ there exists a compact set $E \subset B$ such that $\mu(B \backslash E) < \varepsilon$. As $\beta(B\backslash E) \leqslant \mu/B\backslash E)$, by subadditivity we have

$$\beta(B) \leqslant \beta(E) + \beta(B\backslash E) \leqslant S + \mu(B\backslash E) < S + \varepsilon\,,$$

and letting ε tend to zero we obtain $\beta(B) \leqslant S$. ∎

LEMMA 6.2. – *Let $\beta(B)$ be a non-negative increasing function defined for every Borel set $B \subset \Omega$, and let $\lambda(B)$ be the function defined by*

$$\lambda(B) = \sup \sum_{i \in I} \beta(B_i)\,, \tag{6.2}$$

where the supremum is over all finite families $\{B_i\}_{i \in I}$ of disjoint Borel sets contained in B. Then $\lambda(B)$ is the smallest superadditive function such that $\lambda(B) \geqslant \beta(B)$ for every Borel set $B \subset \Omega$. If, in addition, $\beta(B)$ is countably subadditive, then $\lambda(B)$ is a Borel measure.

PROOF. – It is clear from (6.2) that $\lambda(B)$ is superadditive and that $\lambda(B) \geqslant \beta(B)$ for every Borel set $B \subset \Omega$. Let $\eta(B)$ be another superadditive function such that $\eta(B) \geqslant \beta(B)$ for every Borel set $B \subset \Omega$. Then $\eta(B)$ is non-negative, increasing, and

$$\eta(B) \geqslant \sum_{i \in I} \eta(B_i) \geqslant \sum_{i \in I} \beta(B_i)$$

for every finite family $\{B_i\}_{i \in I}$ of disjoint Borel sets contained in B. By (6.2) this implies that $\eta(B) \geqslant \lambda(B)$ for every Borel set $B \subset \Omega$.

If $\beta(B)$ is countably subadditive, it is easy to see that $\lambda(B)$ is countably subadditive too. Since $\lambda(B)$ is non-negative, increasing, and superadditive, we conclude that it is countably additive. Therefore $\lambda(B)$ is a Borel measure. ∎

LEMMA 6.3. – *Let $\beta(B)$ be a non-negative increasing subadditive function defined for every Borel set $B \subset \Omega$, and let $\lambda(B)$ be the function defined by* (6.2). *Assume that* (6.1) *holds for every Borel set $B \subset \Omega$. Then for every Borel set $B \subset \Omega$ and for every*

$t < \lambda(B)$ there exists $\delta > 0$ such that

$$t < \sum_{i \in I} \beta(B_i) \leq \lambda(B) \tag{6.3}$$

for every finite Borel partition $\{B_i\}_{i \in I}$ of B such that $\operatorname{diam}(B_i) < \delta$ *for every $i \in I$.*

PROOF. – Let us fix a Borel set $B \subset \Omega$ and a real number $t < \lambda(B)$. By (6.2) there exists a finite family $\{A_j\}_{j \in J}$ of disjoint Borel sets contained in B such that

$$t < \sum_{j \in J} \beta(A_j).$$

By (6.1) there exists a finite family $\{E_j\}_{j \in J}$ of compact sets such that $E_j \subset A_j$ for every $j \in J$ and

$$t < \sum_{j \in J} \beta(E_j).$$

As the compact sets E_j are pairwise disjoint, there exists $\delta > 0$ such that $\operatorname{dist}(E_{j_1}, E_{j_2}) > 2\delta$ for $j_1 \neq j_2$. Let $\{B_i\}_{i \in I}$ be a finite Borel partition of B with $\operatorname{diam}(B_i) < \delta$ for every $i \in I$. By subadditivity for every $j \in J$ we have

$$\beta(E_j) \leq \sum_{i \in I_j} \beta(B_i),$$

where $I_j = \{i \in I \colon B_i \cap E_j \neq \emptyset\}$. Since $\operatorname{dist}(E_{j_1}, E_{j_2}) > 2\delta$ for $j_1 \neq j_2$, the sets I_j are pairwise disjoint, hence

$$t < \sum_{j \in J} \sum_{i \in I_j} \beta(B_i) \leq \sum_{i \in I} \beta(B_i).$$

The second inequality in (6.3) follows from (6.2). ■

Condition B_1 is expressed in terms of cubes. The following lemma shows that it implies an inequality for every compact set.

LEMMA 6.4. – *Assume that condition* B_1 *is satisfied. Then*

$$\limsup_{s \to \infty} C_m(E \backslash \Omega_s) \leq \nu(E) \tag{6.4}$$

for every compact set $E \subset \Omega$.

PROOF. – Let us fix a compact set $E \subset \Omega$. For every $\varepsilon > 0$ there exists a finite family of closed cubes $K(x_i, \varrho_i)$, $1 \leq i \leq k$, such that

$$E \subset \bigcup_{i=1}^{k} K(x_i, \varrho_i) \quad \text{and} \quad \sum_{i=1}^{k} \nu(K(x_i, \varrho_i)) < \nu(E) + \varepsilon .$$

We may assume that $K(x_i, 2\varrho_i) \subset \Omega$. By the subadditivity of the capacity C_m and by condition B_1 we have

$$C_m(E\backslash\Omega_s) \leqslant \sum_{i=1}^{k} C_m(K(x_i, \varrho_i)\backslash\Omega_s) \leqslant \sum_{i=1}^{k} \nu(K(x_i, \varrho_i + r_s))$$

for every s such that $r_s \leqslant \min_{1\leqslant i\leqslant k} \varrho_i$. Since r_s tends to zero as $s \to \infty$ we obtain

$$\limsup_{s\to\infty} C_m(E\backslash\Omega_s) \leqslant \sum_{i=1}^{k} \nu(K(x_i, \varrho_i)) < \nu(E) + \varepsilon .$$

As ε tends to zero we obtain (6.4). ■

For every compact set $E \subset \Omega$ and for every real number q we define

$$\beta'(E, q) = \liminf_{s\to\infty} C_A(E\backslash\Omega_s, q), \qquad \beta''(E, q) = \limsup_{s\to\infty} C_A(E\backslash\Omega_s, q). \tag{6.5}$$

By Theorem 4.3 of [23] the functions $C_A(E\backslash\Omega_s, q)$ are increasing with respect to E. Therefore the functions $\beta'(E, q)$ and $\beta''(E, q)$ are increasing with respect to E. By Lemma 2.4 there exists a constant k_4 such that

$$\begin{cases} \left| \dfrac{1}{q'} C_A(E\backslash\Omega_s, q') - \dfrac{1}{q''} C_A(E\backslash\Omega_s, q'') \right| \leqslant k_4 |q' - q''|^{1/(m-1)} C_m(E\backslash\Omega_s), \\ \left| \dfrac{1}{q'} C_A(E\backslash\Omega_s, q') \right| \leqslant k_4 |q'|^{1/(m-1)} C_m(E\backslash\Omega_s), \end{cases} \tag{6.6}$$

for every compact set $E \subset \Omega$ and for every pair of real numbers q', q'', with $0 < |q'|$, $|q''| \leqslant H + M$. By (6.4) this implies that for every compact set $E \subset \Omega$ the functions $C_A(E\backslash\Omega_s, q)$ are equi-continuous with respect to q in $[-H-M, H+M]$. Therefore, from Theorem 8.15 of [24] we deduce that there exist a subsequence, still denoted by Ω_s, and a function $\beta(U, q)$ such that

$$\sup_{E\subset U} \beta'(E, q) = \sup_{E\subset U} \beta''(E, q) = \beta(U, q) \tag{6.7}$$

for every real number q and for every open set $U \subset \Omega$. The same result can also be obtained by applying Proposition 5.9 and Theorem 16.9 of [16], with $X = \boldsymbol{R}^1$. Let us extend $\beta(U, q)$ to every Borel set $B \subset \Omega$ by

$$\beta(B, q) = \inf\{\beta(U, q): U \text{ open}, U \supset B\}. \tag{6.8}$$

Note that

$$\beta'(E, q) \leqslant \beta''(E, q) \leqslant \beta(E, q) \tag{6.9}$$

for every compact set $E \subset \Omega$. By Theorem 5.7 of [23] the functions $C_A(E\backslash\Omega_s, q)$ are subadditive with respect to E, hence $\beta''(E, q)$ is subadditive with respect to E. This implies that $\beta(B, q)$ is countably subadditive with respect to B (see, e.g., [16], Propositions 14.19 and 14.22). By Proposition 6.6 of [23] for every compact set $E \subset \Omega$ we have

$$C_A(E\backslash\Omega_s, q) \leqslant k_{21} |q| (1 + |q|^{m-1}) C_m(E\backslash\Omega_s), \tag{6.10}$$

where k_{21} is a constant depending only on α_1, α_2, m, n, and $\operatorname{diam}(\Omega)$. By Lemma 6.4 this implies

$$\beta'(E, q) \leqslant \beta''(E, q) \leqslant k_{21} |q| (1 + |q|^{m-1}) \nu(E) \tag{6.11}$$

for every compact set $E \subset \Omega$, hence

$$\beta(B, q) \leqslant k_{21} |q| (1 + |q|^{m-1}) \nu(B) \tag{6.12}$$

for every Borel set $B \subset \Omega$ and for every real number q. Moreover, (6.6) implies that

$$\begin{cases} \left| \dfrac{1}{q'} \beta(B, q') - \dfrac{1}{q''} \beta(B, q'') \right| \leqslant k_4 |q' - q''|^{1/(m-1)} \nu(B), \\ \left| \dfrac{1}{q'} \beta(B, q') \right| \leqslant k_4 |q'|^{1/(m-1)} \nu(B), \end{cases} \tag{6.13}$$

for every Borel set $B \subset \Omega$ and for every pair of real numbers q', q'' such that $0 < |q'|$, $|q''| \leqslant H + M$.

For every real number q and for every Borel set $B \subset \Omega$ we define

$$\lambda(B, q) = \sup \sum_{i \in I} \beta(B_i, q), \tag{6.14}$$

where the supremum is over all finite families $\{B_i\}_{i \in I}$ of disjoint Borel sets contained in B. Since $\beta(B, q)$ is countably subadditive with respect to B, for every q the set function $B \mapsto \lambda(B, q)$ is the smallest Borel measure on Ω such that $\lambda(B, q) \geqslant \beta(B, q)$ for every Borel set $B \subset \Omega$ (Lemma 6.2).

By (6.3), (6.12), and (6.13) we have

$$\lambda(B, q) \leqslant k_{21} |q| (1 + |q|^{m-1}) \nu(B), \tag{6.15}$$

$$\left| \frac{1}{q'} \lambda(B, q') - \frac{1}{q''} \lambda(B, q'') \right| \leqslant k_4 |q' - q''|^{1/(m-1)} \nu(B), \tag{6.16}$$

$$\left| \frac{1}{q'} \lambda(B, q') \right| \leqslant k_4 |q'|^{1/(m-1)} \nu(B), \tag{6.17}$$

for every Borel set $B \subset \Omega$, for every real number q, and for every pair of real numbers q', q'' such that $0 < |q'|$, $|q''| \leqslant H + M$. By the Radon-Nikodym Theorem for every rational number $q \neq 0$ there exists a Borel function $g(x, q)$, defined for $x \in \Omega$, such that

$$\frac{1}{q} \lambda(B, q) = \int_B g(x, q) \, d\nu(x), \tag{6.18}$$

for every Borel set $B \subset \Omega$. By (6.15), (6.16), and (6.17) we have

$$|g(x, q)| \leq k_{21}(1 + |q|^{m-1}), \tag{6.19}$$

$$|g(x, q') - g(x, q'')| \leq k_4 |q' - q''|^{1/(m-1)}, \tag{6.20}$$

$$|g(x, q')| \leq k_4 |q'|^{1/(m-1)}, \tag{6.21}$$

for ν-almost every $x \in \Omega$, for every rational number $q \neq 0$, and for every pair of rational numbers q', q'' such that $0 < |q'|$, $|q''| \leq H + M$. This allows us to extend $g(x, q)$ to a Borel function defined on $\Omega \times \boldsymbol{R}^1$ such that (6.18), (6.19), and (6.20) hold also for real numbers q, q', and q''.

By Theorem 3.1 of [4] we have

$$\lim_{r \to 0} \frac{\beta(K(x, r), q)}{q\nu(K(x, r))} = g(x, q) \tag{6.22}$$

for ν-almost every $x \in \Omega$ and for every $q \neq 0$. Let us fix $q \in \boldsymbol{R}^1$. By (6.7) for every $x \in \Omega$ there exists a countable set $N(x) \subset \boldsymbol{R}^1$ such that

$$\beta'(K(x, r), q) = \beta''(K(x, r), q) = \beta(K(x, r), q) \tag{6.23}$$

for every $r \notin N(x)$ (see Proposition 4.8 of [24] or Proposition 14.15 of [16]). Since the function $r \mapsto \beta(K(x, r), q)/\nu(K(x, r))$ is right continuous, hypothesis (1.22), together with (6.22) and (6.23), implies that $g(x, q) = C(x, q)$ for ν-almost every $x \in \Omega$. Therefore (6.18) gives

$$\frac{1}{q} \lambda(B, q) = \int_B C(x, q)\, d\nu(x) \tag{6.24}$$

for every Borel set $B \subset \Omega$ and for every $q \neq 0$, while (6.19), (6.20), and (6.21) imply that

$$C(x, 0) = 0, \tag{6.25}$$

$$|C(x, q)| \leq k_{21}(1 + |q|^{m-1}), \tag{6.26}$$

$$|C(x, q') - C(x, q'')| \leq k_4 |q' - q''|^{1/(m-1)}, \tag{6.27}$$

for ν-almosty every $x \in \Omega$, for every real number q, and for every pair of real numbers q', q'' such that $0 < |q'|$, $|q''| \leq H + M$.

Let us define the sequence $\hat{\varrho}_i$ by

$$\hat{\varrho}_i = \sup_{s \geq i} \left(2r_s + \left(\int_\Omega |u_s(x) - u_0(x)|^m \, dx \right)^{1/m} \right), \tag{6.28}$$

and let $\hat{\lambda}_i$ be the corresponding sequence constructed using (4.3) and (4.4). By Lemma

4.1 we have

$$\lim_{i\to\infty} \hat{\lambda}_i = +\infty \quad \text{and} \quad \lim_{i\to\infty} \hat{\lambda}_i \hat{\varrho}_i = 0 . \tag{6.29}$$

For every i we fix a point $\hat{x}_0^{(i)}$ which satisfies Lemma 4.2 with λ_s and ϱ_s replaced by $\hat{\lambda}_i$ and $\hat{\varrho}_i$. For every multi-index α with integer coordinates we define $\hat{x}_\alpha^{(i)} = \hat{x}_0^{(i)} + 2\hat{\lambda}_i \hat{\varrho}_i \alpha$ we consider the sets

$$\begin{cases} \hat{K}_i(\alpha) = K(\hat{x}_\alpha^{(i)}, \hat{\lambda}_i \hat{\varrho}_i), \\ \hat{K}_i'(\alpha) = K(\hat{x}_\alpha^{(i)}, (\hat{\lambda}_i - 2)\hat{\varrho}_i), \\ \hat{K}_i'''(\alpha) = K(\hat{x}_\alpha^{(i)}, (\hat{\lambda}_i - 3)\hat{\varrho}_i), \end{cases} \tag{6.30}$$

The set (resp. the number) of all multi-indices α such that $K(\hat{x}_\alpha^{(i)}, 3\hat{\lambda}_i \hat{\varrho}_i) \subset \Omega$ is denoted by $\hat{I}_i$ (resp. by $|\hat{I}_i|$). For every $j \geq i$ and for every $\gamma \in \hat{I}_i$ we define

$$\hat{I}_j^{(i)}(\gamma) = \{\alpha \in \hat{I}_j : \hat{K}_j(\alpha) \subset \hat{K}_i(\gamma)\} . \tag{6.31}$$

It is clear that

$$\hat{\lambda}_j \hat{\varrho}_j \leq \hat{\varrho}_i \Rightarrow \hat{K}_i'(\gamma) \subset \bigcup_{\alpha \in \hat{I}_j^{(i)}(\gamma)} \hat{K}_j(\alpha) . \tag{6.32}$$

By (6.7) for every $j \geq i$ and for every $\gamma \in \hat{I}_i$ we have

$$\liminf_{s\to\infty} \sum_{\alpha \in \hat{I}_j^{(i)}(\gamma)} C_A(\hat{K}_j'(\alpha) \backslash \Omega_s, q) \geq \sum_{\alpha \in \hat{I}_j^{(i)}(\gamma)} \beta(\hat{K}_j'''(\alpha), q) . \tag{6.33}$$

As $\beta(B, q)$ is subadditive with respect to B, by (6.12) we have

$$\sum_{\alpha \in \hat{I}_j^{(i)}(\gamma)} \beta(\hat{K}_j(\alpha), q) \leq \sum_{\alpha \in \hat{I}_j^{(i)}(\gamma)} \beta(\hat{K}_j'''(\alpha), q) + c_{56}|q| \sum_{\alpha \in \hat{I}_j^{(i)}(\gamma)} \nu(\hat{K}_j(\alpha) \backslash \hat{K}_j'''(\alpha)) \tag{6.34}$$

for every $j \geq i$, for every $\gamma \in \hat{I}_i$, and for every q with $|q| \leq H + M$. From (6.33) and (6.34) we obtain

$$\liminf_{s\to\infty} \sum_{\alpha \in \hat{I}_j^{(i)}(\gamma)} C_A(\hat{K}_j'(\alpha) \backslash \Omega_s, q) \geq \sum_{\alpha \in \hat{I}_j^{(i)}(\gamma)} \beta(\hat{K}_j(\alpha), q) - c_{56}|q| \sum_{\alpha \in \hat{I}_j^{(i)}(\gamma)} \nu(\hat{K}_j(\alpha) \backslash \hat{K}_j'''(\alpha)) \tag{6.35}$$

for $|q| \leq H + M$. Similarly form (6.9) and (6.14) we obtain

$$\limsup_{s\to\infty} \sum_{\alpha \in \hat{I}_j^{(i)}(\gamma)} C_A(\hat{K}_j'(\alpha) \backslash \Omega_s, q) \leq \sum_{\alpha \in \hat{I}_j^{(i)}(\gamma)} \beta''(\hat{K}_j'(\alpha), q) \leq \sum_{\alpha \in \hat{I}_j^{(i)}(\gamma)} \beta(\hat{K}_j'(\alpha), q) \leq \lambda(\hat{K}_i(\gamma), q) . \tag{6.36}$$

Given a positive integer i and a real number q, with $|q| \leq H + M$, by Lemma 6.3 and by (6.32) for every $\varepsilon > 0$ there exists $\delta(\varepsilon, i, q) > 0$ such that

$$\lambda(\widehat{K}_i'(\gamma), q) - \varepsilon \frac{|q|}{|\hat{I}_i|} < \sum_{\alpha \in \hat{I}_j^{(i)}(\gamma)} \beta(\widehat{K}_j(\alpha), q) \tag{6.37}$$

for every $\gamma \in \hat{I}_i$ and for every j such that

$$2\sqrt{n}\,\hat{\lambda}_j \hat{\varrho}_j < \delta(\varepsilon, i, q) \quad \text{and} \quad \hat{\lambda}_j \hat{\varrho}_j < \hat{\varrho}_i . \tag{6.38}$$

By (6.35) and (6.37) we have

$$\liminf_{s \to \infty} \sum_{\alpha \in I_j^{(i)}(\gamma)} C_A(\widehat{K}_j'(\alpha) \backslash \Omega_s, q) > \tag{6.39}$$

$$> \lambda(\widehat{K}_i'(\gamma), q) - \varepsilon \frac{|q|}{|\hat{I}_i|} - c_{56}\,|q| \sum_{\alpha \in \hat{I}_j^{(i)}(\gamma)} \nu(\widehat{K}_j(\alpha) \backslash \widehat{K}_j'''(\alpha))$$

for every $\gamma \in \hat{I}_i$ and for every j satisfying (6.38). Let us fix $\varepsilon > 0$ and j satisfying (6.38). By (6.36) and (6.39) there exists $s(\varepsilon, i, q, j)$ such that

$$\lambda(\widehat{K}_i'(\gamma), q) - \varepsilon \frac{|q|}{|\hat{I}_i|} - c_{56}\,|q| \sum_{\alpha \in \hat{I}_j^{(i)}(\gamma)} \nu(\widehat{K}_j(\alpha) \backslash \widehat{K}_j'''(\alpha)) \leqslant \tag{6.40}$$

$$\leqslant \sum_{\alpha \in \hat{I}_j^{(i)}(\gamma)} C_A(\widehat{K}_j'(\alpha) \backslash \Omega_s, q) \leqslant \lambda(\widehat{K}_i(\gamma), q) + \varepsilon \frac{|q|}{|\hat{I}_i|}$$

for every $\gamma \in \hat{I}_i$ and for every $s \geqslant s(\varepsilon, i, q, j)$. We may assume that $s(\varepsilon, i, q, j+1) > s(\varepsilon, i, q, j)$.

We want to prove that condition (6.40) is uniform with respect to q, for $|q| \leqslant H + M$. By (0.3), (6.6), and (6.28) we have

$$\left| \frac{1}{q'} C_A(\widehat{K}_j'(\alpha) \backslash \Omega_s, q') - \frac{1}{q''} C_A(\widehat{K}_j'(\alpha) \backslash \Omega_s, q'') \right| \leqslant \tag{6.41}$$

$$\leqslant k_4\,|q' - q''|^{1/(m-1)}\, \nu(\widehat{K}(\hat{x}_\alpha^{(j)}, (\hat{\lambda}_j - 1)\,\hat{\varrho}_j))$$

for every $\alpha \in \hat{I}_j$, for every $s \geqslant j$, and for every q', q'' with $0 < |q'|, |q''| \leqslant H + M$. This implies

$$\sum_{\alpha \in \hat{I}_j^{(i)}(\gamma)} \left| \frac{1}{q'} C_A(\widehat{K}_j'(\alpha) \backslash \Omega_s, q') - \frac{1}{q''} C_A(\widehat{K}_j'(\alpha) \backslash \Omega_s, q'') \right| \leqslant \tag{6.42}$$

$$\leqslant k_4\,|q' - q''|^{1/(m-1)}\, \nu(\widehat{K}_i(\gamma))$$

$\gamma \in \hat{I}_i$, for every for every $s \geqslant j \geqslant i$, and for every q', q'' with $0 < |q'|, |q''| \leqslant H + M$. Given an integer i and $\varepsilon > 0$, we fix a finite sequence $q_0, q_1, \ldots, q_k$ of non-zero real numbers such that

$$(6.43)\qquad \begin{cases} -H-M=q_0<q_1<\ldots<q_k=H+M\,, \\ k_4\,|q_r-q_{r-1}|^{1/(m-1)}\,\nu(\widehat{K}_i(\gamma))<\dfrac{\varepsilon}{|\widehat{I}_i|} \qquad \text{for } r=1,\ldots,k\,. \end{cases}$$

Let $\delta(\varepsilon, i)=1/\sqrt{n}\min_{0\le r\le k}\delta(\varepsilon, i, q_r)$, where $\delta(\varepsilon, i, q)$ is defined before formula (6.37). By (6.29) there exists $j(\varepsilon, i)\geq i$ such that for every $j\geq j(\varepsilon, i)$

$$(6.44)\qquad 2\sqrt{n}\,\widehat{\lambda}_j\widehat{\varrho}_j<\delta(\varepsilon, i) \quad\text{ and }\quad \widehat{\lambda}_j\widehat{\varrho}_j<\widehat{\varrho}_i\,.$$

For every $j\geq j(\varepsilon, i)$ we set $s(\varepsilon, i, j)=\max\{j, \max_{0\le r\le k}s(\varepsilon, i, q_r, j)\}$. If $|q|\le H+M$, by (6.43) there exists q_r such that

$$k_4\,|q-q_r|^{1/(m-1)}\,\nu(\widehat{K}_i(\gamma))<\frac{\varepsilon}{|\widehat{I}_i|}\,.$$

From (6.16), (6.40), and (6.42) we obtain

$$(6.45)\qquad \lambda(\widehat{K}_i'(\gamma), q)-3\varepsilon\frac{|q|}{|\widehat{I}_i|}-c_{56}\,|q|\sum_{\alpha\in\widehat{I}_j^{(i)}(\gamma)}\nu(\widehat{K}_j(\alpha)\backslash\widehat{K}_j'''(\alpha))\leq$$

$$\leq\sum_{\alpha\in\widehat{I}_j^{(i)}(\gamma)}C_A(\widehat{K}_j'(\alpha)\backslash\Omega_s, q)\leq\lambda(\widehat{K}_i(\gamma), q)+3\varepsilon\frac{|q|}{|\widehat{I}_i|}$$

for every $\gamma\in\widehat{I}_i$, $j\geq j(\varepsilon, i)$, $s\geq s(\varepsilon, i, j)$, and q with $|q|\le H+M$.

Choice of ϱ_s. Let us fix ε with $0<\varepsilon<1$. By (6.29) there exists an integer i such that

$$(6.46)\qquad \frac{1}{\widehat{\lambda}_i}<\varepsilon \quad\text{ and }\quad \widehat{\lambda}_i\varrho_i<\varepsilon\,.$$

If i is large enough we have

$$\nu\Big(\Omega\backslash\bigcup_{\gamma\in I_i}\widehat{K}_i(\gamma)\Big)<\frac{\varepsilon}{2}\,.$$

By Lemma 4.2 we have also

$$\nu\Big(\bigcup_{\gamma\in\widehat{I}_i}\widehat{K}_i(\gamma)\backslash\bigcup_{\gamma\in\widehat{I}_i}\widehat{K}_i'(\gamma)\Big)<\frac{\varepsilon}{2}$$

for i large enough. Therefore we may assume that

$$(6.47)\qquad \nu\Big(\Omega\backslash\bigcup_{\gamma\in I_i}\widehat{K}_i'(\gamma)\Big)<\varepsilon\,.$$

Let $s(\varepsilon, i) = s(\varepsilon, i, j(\varepsilon, i))$. Since $s(\varepsilon, i, j+1) > s(\varepsilon, i, j)$ for every $j \geq j(\varepsilon, i)$, we define

$$\varrho_s = \widehat{\varrho}_j, \qquad \lambda_s = \widehat{\lambda}_j, \qquad x_\alpha^{(s)} = \widehat{x}_\alpha^{(j)} \tag{6.48}$$

for $s(\varepsilon, i, j) \leq s < s(\varepsilon, i, j+1)$ and $j \geq s(\varepsilon, i)$. Moreover we set $\varrho_s = \widehat{\varrho}_1$, $\lambda_s = \widehat{\lambda}_1$, and $x_\alpha^{(s)} = \widehat{x}_\alpha^{(1)}$ for $s < s(\varepsilon, i)$. Then ϱ_s is non-increasing and tends to zero as $s \to \infty$. Moreover, condition (4.1) follows from (6.28). It is easy to see that λ_s and ϱ_s satisfy (4.3) and (4.4). For every $\gamma \in \widehat{I}_i$ we define

$$I_s^{(i)}(\gamma) = \{\alpha \in I_s \colon K_s(\alpha) \subset \widehat{K}_i(\gamma)\}. \tag{6.49}$$

Then $I_s^{(i)}(\gamma) = \widehat{I}_j^{(i)}(\gamma)$ for $s(\varepsilon, i, j) \leq s < s(\varepsilon, i, j+1)$. By (6.44) this implies

$$2\sqrt{n}\,\lambda_s \varrho_s < \delta(\varepsilon, i) \quad \text{and} \quad \lambda_s \varrho_s < \widehat{\varrho}_i, \tag{6.50}$$

for every $s \geq s(\varepsilon, i)$. For every $\alpha \in I_s$ we define

$$K_s'''(\alpha) = K(x_\alpha^{(s)}, (\lambda_s - 3)\,\varrho_s).$$

Then, by (6.45), we have

$$\lambda(\widehat{K}_i'(\gamma), q) - 3\varepsilon \frac{|q|}{|\widehat{I}_i|} - c_{56}\,|q| \sum_{\alpha \in I_s^{(i)}(\gamma)} \nu(K_s(\alpha) \backslash K_s'''(\alpha)) \leq \tag{6.51}$$

$$\leq \sum_{\alpha \in I_s^{(i)}(\gamma)} C_A(K_s'(\alpha) \backslash \Omega_s, q) \leq \lambda(\widehat{K}_i(\gamma), q) + 3\varepsilon \frac{|q|}{|\widehat{I}_i|}$$

for every $\gamma \in \widehat{I}_i$, $s \geq s(\varepsilon, i)$, and q with $|q| \leq H + M$.

7. - The limit boundary value problem.

In this section we shall prove Theorem 1.5 about the boundary value problem satisfied by the limit function $u_0(x)$. Let us fix $0 < \varepsilon < 1$ and an integer i satisfying (6.46). We shall use the sequences ϱ_s and λ_s defined by (6.48) and the sequence μ_s defined by (4.5).

PROOF OF THEOREM 1.5. - The strong convergence of $u_s(x)$ to $u_0(x)$ in $W_p^1(\Omega')$ for $p < m$ and for subdomains Ω' such that $\overline{\Omega'} \subset \Omega$ is a consequence of the asymptotic expansion (5.1) proved in Section 5 together with Lemmas 5.2, 5.3, and 5.4. Since the sequence $u_s(x)$ is bounded in $W_m^1(\Omega)$, we immediately obtain the strong convergence of $u_s(x)$ in $W_p^1(\Omega)$ for every $p < m$.

Let $g(x)$ be an arbitrary function of class $C_0^1(\Omega)$ such that

$$\max_{x \in \Omega} |g(x)| + \max_{x \in \Omega} \left| \frac{\partial g(x)}{\partial x} \right| \leq 1. \tag{7.1}$$

Let us introduce the sequence

$$g_s(x) = g(x) + \sum_{j=1}^{5} \varrho_s^{(j)}(x), \tag{7.2}$$

where

$$\varrho_s^{(1)}(x) = \sum_{\alpha \in I_s} (g_\alpha^{(s)} - g(x))\, \varphi_\alpha^{(s)}(x),$$

$$\varrho_s^{(2)}(x) = \sum_{\alpha \in I_s} \sum_{\beta \in I_s(\alpha)} (g_{\alpha\beta}^{(s)} - g(x))\, \psi_{\alpha\beta}^{(s)}(x),$$

$$\varrho_s^{(3)}(x) = -\sum_{\alpha \in I_s'} \frac{g_\alpha^{(s)}}{f_\alpha^{(s)} - u_\alpha^{(s)}} w_\alpha^{(s)}(x)\, \varphi_\alpha^{(s)}(x) - \sum_{\alpha \in I_s''} \frac{g_\alpha^{(s)}}{2\mu_s} w_\alpha^{(s)}(x)\, \varphi_\alpha^{(s)}(x),$$

$$\varrho_s^{(4)}(x) = -\sum_{\alpha \in I_s} \sum_{\beta \in I_s(\alpha)} g_{\alpha\beta}^{(s)} v_{\alpha\beta}^{(s)}(x, 1)\, \psi_{\alpha\beta}^{(s)}(x),$$

$$\varrho_s^{(5)}(x) = -\sum_{\alpha \in I_s'} g_\alpha^{(s)} \left(1 - \frac{w_\alpha^{(s)}(x)}{f_\alpha^{(s)} - u_\alpha^{(s)}} \right) \varphi_\alpha^{(s)}(x) \sum_{\beta \in I_s(\alpha)} \chi_{\alpha\beta}^{(s)}(x) -$$

$$- \sum_{\alpha \in I_s''} g_\alpha^{(s)} \left(1 - \frac{w_\alpha^{(s)}(x)}{2\mu_s} \right) \varphi_\alpha^{(s)}(x) \sum_{\beta \in I_s(\alpha)} \chi_{\alpha\beta}^{(s)}(x).$$

Here $g_\alpha^{(s)}$ and $g_{\alpha\beta}^{(s)}$ are the mean values of the function $g(x)$ in the cubes $K_s(\alpha)$ and $K_s(\alpha, \beta)$, while $w_\alpha^{(s)}(x)$, $\varphi_\alpha^{(s)}(x)$, $\psi_{\alpha\beta}^{(s)}(x)$ and $\chi_{\alpha\beta}^{(s)}(x)$ are the functions defined by (4.14), (4.15), (4.49), and (4.41).

LEMMA 7.1. – *Assume that conditions* A_1, A_2, A_3, B_1, B_2 *are satisfied. Then there exists a number* s_4, *depending on* $g(x)$, *such that* $g(x)\, g_s(x)$ *belongs to* $\overset{\circ}{W}{}_m^1(\Omega)$ *for every* $s \geqslant s_4$.

PROOF. – See Lemma 5.2 in [59].

LEMMA 7.2. – *Assume that conditions* A_1, A_2, A_3, B_1, B_2 *are satisfied. Then the sequences* $\varrho_s^{(1)}(x)$, $\varrho_s^{(2)}(x)$, $\varrho_s^{(4)}(x)$, $\varrho_s^{(5)}(x)$ *converge to zero strongly in* $W_m^1(\Omega)$ *as* $s \to \infty$.

PROOF. – The strong convergence of $\varrho_s^{(j)}(x)$, $j = 1, 2, 4, 5$, in $W_m^1(\Omega)$ can be obtained as in the proof of the convergence of $r_s^{(j)}$ in Lemma 5.2. For the estimate of the derivatives of $\varrho_s^{(5)}(x)$ we use the inequality $|f_\alpha^{(s)} - u_\alpha^{(s)}| > 2\mu_s$ for $\alpha \in I_s'$, which follows from (4.13). Using also the arguments which lead to (5.6) we obtain

$$\int_\Omega \left| \frac{\partial \varrho_s^{(5)}(x)}{\partial x} \right|^m dx \leqslant C_{41} \mu_s^{1-2m} \lambda_s^{-1} \nu(\Omega) \leqslant \mu_s \nu(\Omega),$$

where the last inequality follows from (4.5). ∎

LEMMA 7.3. – *Assume that conditions* A_1, A_2, A_3, B_1, B_2 *are satisfied. Then the sequence* $\varrho_s^{(3)}(x)$ *is bounded in* $W_m^1(\Omega)$ *and converges strongly to zero in* $W_p^1(\Omega)$ *for* $p < m$.

PROOF. – The strong convergence of $\varrho_s^{(3)}(x)$ in $L_m(x)$ follows from (4.26) and from the inequality (Remark 2.2)

$$|v_\alpha^{(s)}(x, q)| \leq |q| \quad \text{for } x \in \Omega . \tag{7.3}$$

We estimate the derivative of $\varrho_s^{(3)}(x)$ in $L_m(x)$ by using (4.15) and we obtain

$$\int_\Omega \left| \frac{\partial \varrho_s^{(3)}(x)}{\partial x} \right|^m dx \leq \tag{7.4}$$

$$\leq c_{57} \sum_{\alpha \in I_s'} \left(\int_\Omega \frac{1}{|f_\alpha^{(s)} - u_\alpha^{(s)}|^m} \left| \frac{\partial w_\alpha^{(s)}(x)}{\partial x} \right|^m dx + \int_{E_\alpha^{(s)}(\mu_s)} \frac{|w_\alpha^{(s)}(x)|^m}{|f_\alpha^{(s)} - u_\alpha^{(s)}|^m} \left| \frac{\partial \varphi_\alpha^{(s)}(x)}{\partial x} \right|^m dx \right) +$$

$$+ c_{57} \sum_{\alpha \in I_s''} \left(\int_\Omega \frac{1}{\mu_s^m} \left| \frac{\partial w_\alpha^{(s)}(x)}{\partial x} \right|^m dx + \int_{E_\alpha^{(s)}(\mu_s)} \frac{|w_\alpha^{(s)}(x)|^m}{\mu_s^m} \left| \frac{\partial \varphi_\alpha^{(s)}(x)}{\partial x} \right|^m dx \right),$$

where $E_\alpha^{(s)}(\mu) = \{x \in \Omega_0 : |w_\alpha^{(s)}(x)| \leq \mu\}$. Thus in the first integral over $E_\alpha^{(s)}(\mu_s)$ we can majorize $|w_\alpha^{(s)}(x)|$ by μ_s.

Since $\lambda_s \varrho_s \leq (\lambda_s \varrho_s)^\eta \leq \mu_s$ by (4.5) and (4.12), from Lemma 2.1 we obtain

$$\int_{E_\alpha^{(s)}(\mu)} \left(1 + \left| \frac{\partial w_\alpha^{(s)}(x)}{\partial x} \right| \right)^{m-2} \left| \frac{\partial w_\alpha^{(s)}(x)}{\partial x} \right|^2 dx \leq \tag{7.5}$$

$$\leq c_{58} \mu (\max\{ |f_\alpha^{(s)} - u_\alpha^{(s)}|, 2\mu_s\})^{m-1} C_m(K_s'(\alpha) \setminus \Omega_s)$$

for every μ with $0 < \mu \leq \max\{|f_\alpha^{(s)} - u_\alpha^{(s)}|, 2\mu_s\}$. From condition B_1 and from (7.4) and (7.5) we obtain

$$\int_\Omega \left| \frac{\partial \varrho_s^{(3)}(x)}{\partial x} \right|^m dx \leq c_{59} \sum_{\alpha \in I_s} \nu(K_s''(\alpha)) \leq c_{59} \nu(\Omega) .$$

The proof of the strong convergence of $\varrho_s^{(3)}(x)$ in $W_p^1(\Omega)$ is totally analogous with the proof of the same property for $r_s^{(3)}(x)$ in Lemma 5.3. ■

PROOF OF THEOREM 1.5 (Continuation). – According to Lemma 7.1 we can take the test function $\varphi(x) = g(x) g_s(x)$ in the integral identity (1.10) corresponding to the boundary value problem (0.1), (0.2). We obtain that

$$J_1^{(s)} + J_2^{(s)} + J_3^{(s)} = 0 , \tag{7.6}$$

where

$$
(7.7)\quad \begin{cases}
J_1^{(s)} = \sum_{j=1}^{n} \int_{\Omega} a_j\left(x, u_s(x), \frac{\partial u_s(x)}{\partial x}\right) \frac{\partial}{\partial x_j} (g(x))^2\, dx + \\
\qquad\qquad + \int_{\Omega} a_0\left(x, u_s(x), \frac{\partial u_s(x)}{\partial x}\right) (g(x))^2\, dx\,, \\
J_2^{(s)} = \sum_{j=1}^{n} \int_{\Omega} a_j\left(x, u_s(x), \frac{\partial u_s(x)}{\partial x}\right) \sum_{k\neq 3} \frac{\partial}{\partial x_j} (g(x)\, \varrho_s^{(k)}(x))\, dx + \\
\qquad\qquad + \int_{\Omega} a_0\left(x, u_s(x), \frac{\partial u_s(x)}{\partial x}\right) \sum_{k=1}^{5} (g(x)\, \varrho_s^{(k)}(x))\, dx\,, \\
J_3^{(s)} = \sum_{j=1}^{n} \int_{\Omega} a_j\left(x, u_s(x), \frac{\partial u_s(x)}{\partial x}\right) \frac{\partial}{\partial x_j} (g(x)\, \varrho_s^{(3)}(x))\, dx\,.
\end{cases}
$$

The above mentioned strong convergence of $u_s(x)$ to $u_0(x)$ in $W_p^1(\Omega)$ for $p < m$ allows us to pass to the limit in $J_s^{(1)}$ and to obtain

$$
(7.8)\quad J_1^{(s)} = \sum_{j=1}^{n} \int_{\Omega} a_j\left(x, u_0(x), \frac{\partial u_0(x)}{\partial x}\right) \frac{\partial}{\partial x_j} (g(x))^2\, dx + \int_{\Omega} a_0\left(x, u_0(x), \frac{\partial u_0(x)}{\partial x}\right) (g(x))^2\, dx + \gamma_s^{(1)}\,,
$$

where $\gamma_s^{(1)}$ tends to zero as $s \to \infty$. Taking into account Lewmmas 7.2 and 7.3, the boundedness of $u_s(x)$ in $W_m^1(\Omega)$, and the boundedness in $L_\infty(\Omega)$ of the sequences $\varrho_s^{(k)}(x)$, $k = 1, \ldots, 5$, we infer that

$$
(7.9)\quad \lim_{s\to\infty} J_2^{(s)} = 0\,.
$$

It remains to study the behaviour of $J_3^{(s)}$. Using the asymptotic expansion (5.1) for $u_s(x)$ we obtain

$$
(7.10)\quad J_3^{(s)} = J_4^{(s)} + J_5^{(s)} + J_6^{(s)} + J_7^{(s)}\,,
$$

where

$$
J_4^{(s)} = \int_{\Omega} \sum_{j=1}^{n} \left(a_j\left(x, u_s(x), \frac{\partial u_s(x)}{\partial x}\right) - a_j\left(x, u_s(x), \frac{\partial u_0(x)}{\partial x} + \frac{\partial r_s^{(3)}(x)}{\partial x}\right)\right) \cdot \frac{\partial}{\partial x_j} (g(x)\, \varrho_s^{(3)}(x))\, dx\,,
$$

$$J_5^{(s)} = \int_\Omega \sum_{j=1}^n \left(a_j\left(x, u_s(x), \frac{\partial u_0(x)}{\partial x} + \frac{\partial r_s^{(3)}(x)}{\partial x}\right) - a_j\left(x, u_s(x), \frac{\partial r_s^{(3)}(x)}{\partial x}\right)\right) \cdot \frac{\partial}{\partial x_j} (g(x)\, \varrho_s^{(3)}(x))\, dx ,$$

$$J_6^{(s)} = \int_\Omega \sum_{j=1}^n \left(a_j\left(x, u_s(x), \frac{\partial r_s^{(3)}(x)}{\partial x}\right) - a_j\left(x, 0, \frac{\partial r_s^{(3)}(x)}{\partial x}\right)\right) \frac{\partial}{\partial x_j} (g(x)\, \varrho_s^{(3)}(x))\, dx ,$$

$$J_7^{(s)} = \int_\Omega \sum_{j=1}^n a_j\left(x, 0, \frac{\partial r_s^{(3)}(x)}{\partial x}\right) \frac{\partial}{\partial x_j} (g(x)\, \varrho_s^{(3)}(x))\, dx .$$

It is easy to prove that

$$\lim_{s\to\infty} (|J_4^{(s)}| + |J_5^{(s)}| + |J_6^{(s)}|) = 0 . \tag{7.11}$$

Since $\varrho_s^{(3)}(x)$ is bounded in $W_m^1(\Omega)$ (Lemma 7.3) and $w_s(x)$ converges to zero strongly in $W_m^1(\Omega')$ for every open set Ω' such that $\mathrm{supp}(g) \subset \Omega' \subset \overline{\Omega'} \subset \Omega$ (Lemma 5.4), the estimate for $J_4^{(s)}$ is analogous with the estimate for $I_2^{(s)}$ in the proof of Lemma 5.4 (see [59], Theorem 4.9). The estimate of $J_5^{(s)}$ is analogous with the estimate of $I_3^{(s)}$ in (5.10), while the estimate for $J_6^{(s)}$ is analogous with the estimate for $I_4^{(s)}$.

We deal now with $J_7^{(s)}$, writing this term in the form

$$J_7^{(s)} = - \sum_{\alpha \in I_s} \sum_{j=1}^n \frac{(g_\alpha^{(s)})^2}{f_\alpha^{(s)} - u_\alpha^{(s)}} \int_{\Omega_0} a_j\left(x, 0, \frac{\partial v_\alpha^{(s)}(x)}{\partial x}\right) \frac{\partial v_\alpha^{(s)}(x)}{\partial x_j}\, dx + \sum_{j=1}^6 R_j^{(s)} , \tag{7.12}$$

where

$$R_1^{(s)} = \sum_{\alpha \in I_s''} \sum_{j=1}^n \frac{(g_\alpha^{(s)})^2}{f_\alpha^{(s)} - u_\alpha^{(s)}} \int_{\Omega_0} a_j\left(x, 0, \frac{\partial v_\alpha^{(s)}(x)}{\partial x}\right) \frac{\partial v_\alpha^{(s)}(x)}{\partial x_j}\, dx ,$$

$$R_2^{(s)} = - \sum_{\alpha \in I_s'} \sum_{j=1}^n \frac{g_\alpha^{(s)}}{f_\alpha^{(s)} - u_\alpha^{(s)}} \int_{G_\alpha^{(s)}} g(x) \cdot \left(a_j\left(x, 0, \frac{\partial}{\partial x} (v_\alpha^{(s)} \varphi_\alpha^{(s)})(x)\right) \frac{\partial}{\partial x_j} (v_\alpha^{(s)} \varphi_\alpha^{(s)})(x) - a_j\left(x, 0, \frac{\partial v_\alpha^{(s)}(x)}{\partial x}\right) \frac{\partial v_\alpha^{(s)}(x)}{\partial x_j}\right) dx ,$$

$$R_3^{(s)} = \sum_{\alpha \in I_s'} \sum_{j=1}^n \frac{g_\alpha^{(s)}}{f_\alpha^{(s)} - u_\alpha^{(s)}} \int_{G_\alpha^{(s)}} (g_\alpha^{(s)} - g(x))\, a_j\left(x, 0, \frac{\partial v_\alpha^{(s)}(x)}{\partial x}\right) \frac{\partial v_\alpha^{(s)}(x)}{\partial x_j}\, dx ,$$

$$R_4^{(s)} = \sum_{\alpha \in I_s'} \sum_{j=1}^{n} \frac{(g_\alpha^{(s)})^2}{f_\alpha^{(s)} - u_\alpha^{(s)}} \int_{\Omega_0 \setminus G_\alpha^{(s)}} a_j\left(x, 0, \frac{\partial v_\alpha^{(s)}(x)}{\partial x}\right) \frac{\partial v_\alpha^{(s)}(x)}{\partial x_j}\, dx\,,$$

$$R_5^{(s)} = -\sum_{\alpha \in I_s''} \sum_{j=1}^{n} \frac{g_\alpha^{(s)}}{2\mu_s} \int_{G_\alpha^{(s)}} g(x)\, a_j\left(x, 0, \frac{\partial}{\partial x}(v_\alpha^{(s)} \varphi_\alpha^{(s)})(x)\right) \frac{\partial}{\partial x_j}(w_\alpha^{(s)} \varphi_\alpha^{(s)})(x)\, dx\,,$$

$$R_6^{(s)} = \sum_{j=1}^{n} \int_{\Omega} a_j\left(x, 0, \frac{\partial r_s^{(3)}(x)}{\partial x}\right) \varrho_s^{(3)}(x) \frac{\partial g(x)}{\partial x_j}\, dx\,.$$

In (7.12) and in the definition of $R_j^{(s)}$ the function $v_\alpha^{(s)}(x, f_\alpha^{(s)} - u_\alpha^{(s)})$ is denoted by $v_\alpha^{(s)}(x)$, while $w_\alpha^{(s)}(x)$ is the function defined by (4.14). As in Section 4, $G_\alpha^{(s)}$ is the set where $\varphi_\alpha^{(s)}(x)$ is different from zero.

LEMMA 7.4. – *Assume that conditions* A_1, A_2, A_3, B_1, B_2 *are satisfied. Then*

$$\lim_{s \to \infty} R_j^{(s)} = 0 \tag{7.13}$$

for $k = 1, \ldots, 6$.

PROOF. – We first prove (7.13) for $k = 2, 3, 4$. Since $\varphi_\alpha^{(s)}(x) = 1$ if $\alpha \in I_s'$ and $|v_\alpha^{(s)}(x, f_\alpha^{(s)} - u_\alpha^{(s)})| \geq \mu_s$, the integral in the definition of $R_2^{(s)}$ can be replaced by an integral on the set $E_\alpha^{(s)}(\mu_s) \cap G_\alpha^{(s)}$, where $E_\alpha^{(s)}(\mu_s)$ is the set defined after (7.4). Since $|v_\alpha^{(s)}(x)| \leq \mu_s$ in $E_\alpha^{(s)}(\mu_s)$, from (1.8), (7.5), and B_1 we obtain

$$|R_2^{(s)}| \leq c_{60} \sum_{\alpha \in I_s'} \frac{1}{|f_\alpha^{(s)} - u_\alpha^{(s)}|} \int_{E_\alpha^{(s)}(\mu_s)} \left(1 + \left|\frac{\partial v_\alpha^{(s)}(x)}{\partial x}\right|\right)^{m-2} \left|\frac{\partial v_\alpha^{(s)}(x)}{\partial x}\right|^2 dx \leq$$

$$\leq c_{61} \mu_s \sum_{\alpha \in I_s'} C_m(K_s'(\alpha) \setminus \Omega_s) \leq c_{61} \mu_s \nu(\Omega)\,.$$

By Lemma 4.1 this implies (7.13) for $k = 2$. The proof for $k = 4$ is similar. For $k = 3$ the result follows from (7.1) and from the estimate obtained in (7.5) with $\mu = |f_\alpha^{(s)} - u_\alpha^{(s)}|$.

Since $\lambda_s \varrho_s \leq (\lambda_s \varrho_s)^\eta \leq \mu_s$ by (4.5) and (4.12), from Lemma 2.1 we obtain

$$\int_{\Omega_0} \left(1 + \left|\frac{\partial v_\alpha^{(s)}(x)}{\partial x}\right|\right)^{m-2} \left|\frac{\partial v_\alpha^{(s)}(x)}{\partial x}\right|^2 dx \leq \tag{7.14}$$

$$\leq c_{62} |f_\alpha^{(s)} - u_\alpha^{(s)}| \mu_s^{m-1} C_m(K_s'(\alpha) \setminus \Omega_s)$$

for every $\alpha \in I_s''$. Using (1.8), (7.14), condition B_1, and Lemma 4.1 we obtain (7.13) for $k = 1$.

Since $|w_\alpha^{(s)}(x)| \leq 2\mu_s$ for $\alpha \in I_s''$, by (4.15) we have

$$\left| \frac{\partial}{\partial x} (w_\alpha^{(s)} \varphi_\alpha^{(s)})(x) \right| \leq c_{63} \left| \frac{\partial w_\alpha^{(s)}(x)}{\partial x} \right| \tag{7.15}$$

for every $\alpha \in I_s''$. From (1.8) and (7.15) we obtain that $|R_5^{(s)}|$ is less than or equal to

$$\frac{c_{64}}{\mu_s} \sum_{\alpha \in I_s''} \left(\int_{G_\alpha^{(s)}} \left(\left| \frac{\partial v_\alpha^{(s)}(x)}{\partial x} \right| + |v_\alpha^{(s)}(x)| \left| \frac{\partial \varphi_\alpha^{(s)}(x)}{\partial x} \right| \right)^m dx \right)^{(m-1)/m} \cdot$$

$$\cdot \left(\int_\Omega \left| \frac{\partial w_\alpha^{(s)}(x)}{\partial x} \right|^m dx \right)^{1/m} +$$

$$+ \frac{c_{64}}{\mu_s} \sum_{\alpha \in I_s''} \left(\int_{G_\alpha^{(s)}} \left(\left| \frac{\partial v_\alpha^{(s)}(x)}{\partial x} \right| + |v_\alpha^{(s)}(x)| \left| \frac{\partial \varphi_\alpha^{(s)}(x)}{\partial x} \right| \right)^2 dx \right)^{1/2} \left(\int_\Omega \left| \frac{\partial w_\alpha^{(s)}(x)}{\partial x} \right|^2 dx \right)^{1/2}.$$

Using (7.5) with $\mu = 2\mu_s$, together with the estimates (4.23), (4.24), (7.14), and recalling that $|v_\alpha^{(s)}(x, f_\alpha^{(s)} - u_\alpha^{(s)})| \leq 2\mu_s$ for $\alpha \in I_s''$, we obtain

$$|R_5^{(s)}| \leq c_{65} (\mu_s^{(m-1)/m} + \mu_s^{(m-1)/2}) \sum_{\alpha \in I_s''} \nu(K_s''(\alpha)) \leq c_{66} \mu_s^{(m-1)/m} \nu(\Omega),$$

and the right hand side of the last inequality tends to zero as $s \to \infty$ by Lemma 4.1. Therefore (7.13) holds for $k = 5$.

Finally the convergence of $R_6^{(s)}$ to zero follows from (1.8), from Lemmas 5.3 and 7.3, and from the fact that the sequence $\varrho_s^{(3)}(x)$ is bounded in $L_\infty(\Omega)$. ■

Now we return to (7.12) and we study the behaviour of the first term of the right hand side as $s \to \infty$. Let E_s be the sequence of real numbers defined by the equality

$$\sum_{\alpha \in I_s} \sum_{j=1}^{n} \frac{(g_\alpha^{(s)})^2}{f_\alpha^{(s)} - u_\alpha^{(s)}} \int_{\Omega_0} a_j\left(x, 0, \frac{\partial v_\alpha^{(s)}(x)}{\partial x} \right) \frac{\partial v_\alpha^{(s)}(x)}{\partial x_j} dx = \tag{7.16}$$

$$= \int_\Omega C(x, f(x) - u_0(x)) (g(x))^2 \, d\nu(x) + E_s .$$

LEMMA 7.5. - *Assume that conditions* A_1, A_2, A_3, B_1, B_2, C *are satisfied. Then there exists a constant* k_{22}, *independent of the constants* ε *and* i *used in* (6.48) *in the definition of the sequences* ϱ_s *and* λ_s, *such that for every* $s \geq s(\varepsilon, i)$ *we have*

$$|E_s| \leq k_{22} \varepsilon^{\eta/(m-1)} + k_{22} (\tau(\varepsilon))^{1/m} + \gamma_s^{(2)}, \tag{7.17}$$

where $\tau(r)$ *is the function defined in* (3.5) *and* $\gamma_s^{(2)}$ *tends to zero as* $s \to \infty$.

PROOF. – By the definition of capacity given in (1.21) we have

$$\sum_{\alpha\in I_s}\sum_{j=1}^{n}\frac{(g_\alpha^{(s)})^2}{f_\alpha^{(s)}-u_\alpha^{(s)}}\int_{\Omega_0} a_j\left(x,0,\frac{\partial v_\alpha^{(s)}(x)}{\partial x}\right)\frac{\partial v_\alpha^{(s)}(x)}{\partial x_j}\,dx = \tag{7.18}$$

$$=\sum_{\gamma\in\hat I_i}\sum_{\alpha\in I_s^{(i)}(\gamma)}\frac{(g_\alpha^{(s)})^2}{f_\alpha^{(s)}-u_\alpha^{(s)}}C_A(K_s'(\alpha)\backslash\Omega_s, f_\alpha^{(s)}-u_\alpha^{(s)})+E_s^{(1)},$$

where

$$E_s^{(1)}=\sum_{\alpha\in J_s^{(i)}}\frac{(g_\alpha^{(s)})^2}{f_\alpha^{(s)}-u_\alpha^{(s)}}C_A(K_s'(\alpha)\backslash\Omega_s, f_\alpha^{(s)}-u_\alpha^{(s)})$$

and

$$J_s^{(i)}=I_s\backslash\bigcup_{\gamma\in\hat I_i}I_s^{(i)}(\gamma).$$

By (0.3) and (6.10) for every $s\geqslant s(\varepsilon, i)$ we have

$$E_s^{(1)}\leqslant c_{67}\sum_{\alpha\in J_s^{(i)}}C_m(K_s'(\alpha)\backslash\Omega_s)\leqslant c_{67}\sum_{\alpha\in J_s^{(i)}}\nu(K_s''(\alpha))\leqslant\frac{7nc_{67}}{\hat\lambda_i}\nu(\Omega), \tag{7.19}$$

where, in the last inequality we use Lemma 4.2 and the inclusion

$$\bigcup_{\alpha\in J_s^{(i)}}K_s''(\alpha)\subset\bigcup_{\gamma}(K(\hat x_\gamma^{(i)},\hat\lambda_i\hat\varrho_i)\backslash K(\hat x_\gamma^{(i)},(\hat\lambda_i-6)\hat\varrho_i)), \tag{7.20}$$

which follows from (6.50). From (6.46) and (7.19) we obtain

$$E_s^{(1)}<7nc_{67}\,\varepsilon\nu(\Omega) \tag{7.21}$$

for every $s\geqslant s(\varepsilon, i)$.

Let $\hat g_\gamma^{(i)}, \hat f_\gamma^{(i)}, \hat u_\gamma^{(s,i)}$ be the mean values of the functions $g(x), f(x), u_0^{(s)}(x)$ in the cube $\hat K_i(\gamma)$. Then we have

$$\sum_{\gamma\in\hat I_i}\sum_{\alpha\in I_s^{(i)}(\gamma)}\frac{(g_\alpha^{(s)})^2}{f_\alpha^{(s)}-u_\alpha^{(s)}}C_A(K_s'(\alpha)\backslash\Omega_s, f_\alpha^{(s)}-u_\alpha^{(s)})= \tag{7.22}$$

$$=\sum_{\gamma\in\hat I_i}\sum_{\alpha\in I_s^{(i)}(\gamma)}\frac{(g_\gamma^{(i)})^2}{\hat f_\gamma^{(i)}-\hat u_\gamma^{(s,i)}}C_A(K_s'(\alpha)\backslash\Omega_s, \hat f_\gamma^{(i)}-\hat u_\gamma^{(s,i)})+E_s^{(2)},$$

where, by Lemma 2.4 and condition B_1,

$$|E_s^{(2)}|\leqslant k_4(H+M)^{1/(m-1)}\sum_{\gamma\in\hat I_i}\sum_{\alpha\in I_s^{(i)}(\gamma)}|(g_\alpha^{(s)})^2-(\hat g_\gamma^{(i)})^2|\,\nu(K_s''(\alpha))+$$

$$+k_4\sum_{\gamma\in\hat I_i}\sum_{\alpha\in I_s^{(i)}(\gamma)}(|f_\alpha^{(s)}-\hat f_\gamma^{(i)}|^{1/(m-1)}+|u_\alpha^{(s)}-\hat u_\gamma^{(s,i)}|^{1/(m-1)})\,\nu(K_s''(\alpha)).$$

As $g(x)$ is Lipschitz continuous, from (0.3), (1.12), and (6.46) we get

$$|E_s^{(2)}| \leq c_{68}(\widehat{\lambda}_i \widehat{\varrho}_i)^{\eta/(m-1)}\, \nu(\Omega) +$$

$$+ c_{68}(\nu(\Omega))^{(m^2-m-1)/(m(m-1))}\left(\sum_{\alpha \in I_s} \int_{K_s(\alpha)} |u_\alpha^{(s)} - u_0^{(s)}(x)|^m \, d\nu(x)\right)^{1/(m(m-1))} +$$

$$+ c_{68}(\nu(\Omega))^{(m^2-m-1)/(m(m-1))}\left(\sum_{\alpha \in \widehat{I}_s} \int_{\widehat{K}_s(\alpha)} |u_\alpha^{(s)} - \widehat{u}_\gamma^{(s,i)}(x)|^m \, d\nu(x)\right)^{1/(m(m-1))}.$$

From the Poincaré-Wirtinger inequality proved in (3.13) and from (6.46) we obtain

$$|E_s^{(2)}| \leq c_{69}\, \varepsilon^{\eta/(m-1)} + c_{69}(\tau(\varepsilon))^{1/m}\left(\int_\Omega \left|\frac{\partial u_0(x)}{\partial x}\right|^m dx\right)^{1/(m(m-1))} \tag{7.23}$$

for every $s \geq s(\varepsilon, i)$. Let $E_s^{(3)}$ be defined by

$$\sum_{\gamma \in \widehat{I}_i} \frac{(\widehat{g}_\gamma^{(i)})^2}{\widehat{f}_\gamma^{(i)} - \widehat{u}_\gamma^{(s,i)}} \sum_{\alpha \in I_s^{(i)}(\gamma)} C_A(K_s'(\alpha)\setminus\Omega_s, \widehat{f}_\gamma^{(i)} - \widehat{u}_\gamma^{(s,i)}) = \tag{7.24}$$

$$= \sum_{\gamma \in \widehat{I}_i} (\widehat{g}_\gamma^{(i)})^2 \int_{\widehat{K}_i'(\gamma)} C(x, \widehat{f}_\gamma^{(i)} - \widehat{u}_\gamma^{(s,i)})\, d\nu(x) + E_s^{(3)}.$$

By (6.15), (6.24), and (6.51) for every $s \geq s(\varepsilon, i)$ we have

$$|E_s^{(3)}| \leq 3\varepsilon + c_{70}\, \nu(F_s \cap \Omega) + c_{70}\, \nu(\widehat{F}_i \cap \Omega),$$

where F_s is the set defined in (4.7) and $\widehat{F}_i$ is the set which appears in the right hand side of (7.20). By Lemma 4.2 and by (6.46) we have

$$|E_s^{(3)}| \leq c_{71}\,\varepsilon + \frac{c_{71}}{\lambda_s} \tag{7.25}$$

for every $s \geq s(\varepsilon, i)$. Since the function $g(x)$ is Lipschitz continuous by (7.1), we have

$$\sum_{\gamma \in \widehat{I}_i} (\widehat{g}_\gamma^{(i)})^2 \int_{\widehat{K}_i'(\gamma)} C(x, \widehat{f}_\gamma^{(i)} - \widehat{u}_\gamma^{(s,i)})\, d\nu(x) = \tag{7.26}$$

$$= \sum_{\gamma \in \widehat{I}_i} \int_{\widehat{K}_i'(\gamma)} C(x, \widehat{f}_\gamma^{(i)} - \widehat{u}_\gamma^{(s,i)})(g(x))^2\, d\nu(x) + E_s^{(4)},$$

where, by (6.26) and (6.46),

$$|E_s^{(4)}| \leq c_{72}\widehat{\lambda}_i \widehat{\varrho}_i\, \nu(\Omega) \leq c_{73}\,\varepsilon \tag{7.27}$$

for every $s \geq s(\varepsilon, i)$. We now write

$$(7.28) \quad \sum_{\gamma \in \hat{I}_i} \int_{\widehat{K}_i'(\gamma)} C(x, \hat{f}_\gamma^{(i)} - \widehat{u}_\gamma^{(s,i)})(g(x))^2 \, d\nu(x) =$$

$$= \int_\Omega C(x, f(x) - u_0^{(s)}(x))(g(x))^2 \, d\nu(x) + E_s^{(5)} .$$

Since $C(x, f(x) - u_0^{(s)}(x))$ is bounded uniformly with respect to s, by (1.12), (3.13), (6.27), (6.46), (6.47) we have

$$(7.29) \quad |E_s^{(5)}| \leq c_{74} \sum_{\gamma \in \hat{I}_i} \int_{\widehat{K}_i'(\gamma)} (|\hat{f}_\gamma^{(i)} - f(x)|^{1/(m-1)} + |\widehat{u}_\gamma^{(s,i)} - u_0^{(s)}(x)|^{1/(m-1)}) d\nu(x) + c_{74}\varepsilon \leq$$

$$\leq c_{75}(\widehat{\lambda}_i \widehat{\varrho}_i)^{\eta/(m-1)} + c_{75}\left(\sum_{\gamma \in \hat{I}_i} \int_{\widehat{K}_i'(\gamma)} |\widehat{u}_\gamma^{(s,i)} - u_0^{(s)}(x)|^m \, d\nu(x)\right)^{1/(m(m-1))} + c_{74}\varepsilon \leq$$

$$\leq c_{76}\varepsilon^{\eta/(m-1)} + c_{76}(\tau(\varepsilon))^{1/m}\left(\int_\Omega \left|\frac{\partial u_0(x)}{\partial x}\right|^m dx\right)^{1/(m(m-1))}$$

for every $s \geq s(\varepsilon, i)$. Finally we write

$$(7.30) \quad \int_\Omega C(x, f(x) - u_0^{(s)}(x))(g(x))^2 \, d\nu(x) = \int_\Omega C(x, f(x) - u_0(x))(g(x))^2 \, d\nu(x) + E_s^{(6)} .$$

As $u_0(x)$ is C_m-quasi continuous, $u_0^{(s)}(x)$ converges to $u_0(x)$ for all $x \in \Omega$ except for a set of m-capacity zero (see [61], Theorem 3.3.3). By condition B_2 the measure ν belongs to the dual of the Sobolev space $W_m^1(\Omega)$ (see [62], Theorem 4.7.5), thus it vanishes on all sets of m-capacity zero. Therefore $u_0^{(s)}(x)$ converges to $u_0(x)$ almost everywhere with respect to the measure ν and, consequently,

$$(7.31) \quad \lim_{s \to \infty} E_s^{(6)} = 0$$

by (6.26), (6.27) and by the dominated convergence theorem. Inequality (7.17) follows now from (4.6), (7.18), and (7.21)-(7.31). ∎

PROOF OF THEOREM 1.5 (Conclusion). – Let us define E by the equality

$$(7.32) \quad \sum_{j=1}^n \int_\Omega a_j\left(x, u_0(x), \frac{\partial u_0(x)}{\partial x}\right) \frac{\partial (g(x))^2}{\partial x_j} \, dx + \int_\Omega a_0\left(x, u_0(x), \frac{\partial u_0(x)}{\partial x}\right)(g(x))^2 \, dx =$$

$$= \int_\Omega C(x, f(x) - u_0(x))(g(x))^2 \, d\nu(x) + E .$$

Using (7.6)-(7.13), (7.16), and (7.17) for every $s \geqslant s(\varepsilon, i)$ we obtain

$$|E| \leqslant k_{22}\varepsilon^{\eta/(m-1)} + k_{22}(\tau(\varepsilon))^{1/m} + \gamma_s^{(3)}, \tag{7.33}$$

where $\gamma_s^{(3)}$ tends to zero as $s \to \infty$. In this inequality the left hand side is independent of ε and s, while the right hand side can be made arbitrarily small for sufficiently large s and sufficiently small ε. This shows that $E = 0$ and that identity (1.23) is satisfied if $\varphi(x) = (g(x))^2$, with $g(x)$ in $C_0^\infty(\Omega)$. By a standard approximation argument we can establish (1.23) for every $\varphi(x)$ in $\overset{\circ}{W}{}^1_m(\Omega) \cap L_\infty(\Omega)$.

Finally, $u_0(x)$ belongs to the set $f(x) + \overset{\circ}{W}{}^1_m(\Omega)$, since this is true for $u_s(x)$ for every s. This shows that $u_0(x)$ is a solution of the boundary value problem (1.23) and concludes the proof of Theorem 1.5. ■

REFERENCES

[1] H. ATTOUCH, *Variational Convergence for Functions and Operators*, Pitman, London (1984).

[2] H. ATTOUCH - C. PICARD, *Variational inequalities with varying obstacles: the general form of the limit problem*, J. Funct. Anal., **50** (1983), pp. 329-386.

[3] M. BALZANO, *Random relaxed Dirichlet problems*, Ann. Mat. Pura Appl. (4), **153** (1988), pp. 133-174.

[4] M. BALZANO, *A derivation theorem for countably subadditive set functions*, Boll. Un. Mat. Ital. (7), **2-A** (1988), pp. 241-249.

[5] J. R. BAXTER - N. C. JAIN, *Asymptotic capacities for finely divided bodies and stopped diffusions*, Illinois J. Math., **31** (1987), pp. 469-495.

[6] A. BRAIDES - L. NOTARANTONIO, *Fractal relaxed Dirichlet problems*, Manuscripta Math., **81** (1993), pp. 41-56.

[7] J. CASADO DIAZ, *Homogenisation of Dirichlet problems for monotone operators in varying domains*, Proc. Roy. Soc. Edinburgh Sect. A, **127** (1997), pp. 457-478.

[8] J. CASADO DIAZ - A. GARRONI, *Asymptotic behaviour of nonlinear elliptic systems on varying domains*, Preprint SISSA, Trieste (1996).

[9] I. CHAVEL - E. A. FELDMAN, *The Lenz shift and Wiener sausage in Riemannian manifolds*, Compositio Math., **60** (1986), pp. 65-84.

[10] I. CHAVEL - E. A. FELDMAN, *Spectra of manifolds less a small hole*, Duke Math. J., **56** (1986), pp. 339-414.

[11] I. CHAVEL - E. A. FELDMAN, *The Wiener sausage, and a theorem of Spitzer, in Riemannian Manifolds*, in *Probability Theory and Harmonic Analysis (Cleveland, Ohio, 1983)*, Monographs Textbooks Pure Appl. Math., **98**, Dekker, New York (1986), pp. 45-60.

[12] D. CIORANESCU, *Calcul des variations sur des sous-espaces variables. Applications*, C. R. Acad. Sci. Paris Sér. A, **291** (1980), pp. 87-90.

[13] D. CIORANESCU - F. MURAT, *Un terme étrange venu d'ailleurs, I and II*, in *Nonlinear Partial Differential Equations and their Applications, College di France Seminar*, Vol. II, pp. 98-138, and Vol. III, pp. 154-178, Res. Notes in Math., **60** and **70**, Pitman, London (1982 and 1983), translated in *A strange term coming from nowhere*, Topics in the Mathematical Modelling of Composite Materials, Birkhäuser, Boston (1994).

[14] D. CIORANESCU - J. SAINT JEAN PAULIN, *Homogenization in open sets with holes*, J. Math. Anal. Appl., **71** (1979), pp. 590-607.

[15] G. DAL MASO, *Γ-convergence and μ-capacities*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), **14** (1987), pp. 423-464.

[16] G. DAL MASO, *An Introduction to Γ-Convergence*, Birkhäuser, Boston (1993).

[17] G. DAL MASO - A. DEFRANCESCHI, *Limits of nonlinear Dirichlet problems in varying domains*, Manuscripta Math., **61** (1988), pp. 251-278.

[18] G. DAL MASO - A. GARRONI, *New results on the asymptotic behaviour of Dirichlet problems in perforated domains*, Math. Models Methods Appl. Sci., 4 (1994), pp. 373-407.

[19] G. DAL MASO - A. GARRONI, *The capacity method for asymptotic Dirichlet problems*, Asymptotic Anal., to appear.

[20] G. DAL MASO - U. MOSCO, *Wiener's criterion and Γ-convergence*, Appl. Math. Optim., **15** (1987), pp. 15-63.

[21] G. DAL MASO - F. MURAT, *Dirichlet problems in perforated domains for homogeneous monotone operators on* H_0^1, in *Calculus of Variations, Homogenization and Continuum Mechanics (CIRM-Luminy, Marseille, 1993)*, World Scientific, Singapore (1994), pp. 177-202.

[22] G. DAL MASO - F. MURAT, *Asymptotic behaviour and correctors for Dirichlet problems in perforated domains with homogeneous monotone operators*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), to appear.

[23] G. DAL MASO - I. V. SYRYPNIK, *Capacity theory for monotone operators*, Potential Anal., to appear.

[24] E. DE GIORGI - G. LETTA, *Une notion générale de convergence faible pour des fonctions croissantes d'ensemble*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 4 (1977), pp. 61-99.

[25] L. C. EVANS - R. F. GARIEPY, *Measure Theory and Fine Properties of Functions*, CRC Press, Boca Raton (1992).

[26] R. FIGARI - E. ORLANDI - S. TETA, *The Laplacian in regions with many small obstacles: fluctuation around the limit operator*, J. Statist. Phys., **41** (1985), pp. 465-487.

[27] R. FIGARI - G. PAPANICOLAOU - J. RUBINSTEIN, *The point interaction approximation for diffusion in regions with many small holes*, in *Stochastic Methods in Biology*, Lecture Notes in Biomath., **70**, Springer-Verlag, Berlin (1987), pp. 202-220.

[28] D. GILBARG - N. S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin (1977).

[29] J. HEINONEN - T. KILPELÄINEN - O. MARTIO, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Clarendon Press, Oxford (1993).

[30] W. JÄGER - A. MIKELIĆ, *Homogenization of the Laplace equation in a partially perforated domain*, preprint Univ. Lyon-St. Etienne (1994).

[31] W. JÄGER - A. MIKELIĆ, *On the boundary conditions at the contact interface between a porous medium and a free fluid*, preprint Univ. Heidelberg (1994).

[32] M. KAC, *Probabilistic method in some problems of scattering theory*, Rocky Mountain J. Math., 4 (1974), pp. 511-538.

[33] E. YA. KHRUSLOV, *The method of orthogonal projections and the Dirichlet problems in domains with a fine-grained boundary*, Math. USSR-Sb., **17** (1972), pp. 37-59.

[34] E. YA. KHRUSLOV, *The first boundary value problem in domains with a complicated boundary for higher order equations*, Math. USSR-Sb., **32** (1977), pp. 535-549.

[35] E. YA. KHRUSLOV, *Homogenized models of composite media*, in *Composite Media and Homogenization Theory (Trieste, 1990)*, Birkhäuser, Boston (1991), pp. 159-182.

[36] A. A. KOVALEVSKY, *On G-convergence of operators of Dirichlet problem with varying domain*, Dokl. Akad. Nauk Ukraine Ser. A, **5** (1993), pp. 13-17.

[37] N. LABANI - C. PICARD, *Homogenization of nonlinear Dirichlet problem in a periodically perforated domain*, in *Recent Advances in Nonlinear Elliptic and Parabolic Problems (Nancy, 1988)*, Res. Notes in Math., **208**, Longman, Harlow (1989), pp. 294-305.

[38] V. A. MARCHENKO - E. YA. KHRUSLOV, *Boundary Value Problems in Domains with Finely Granulated Boundaries* (in Russian), Naukova Dumka, Kiev (1974).

[39] V. A. MARCHENKO - E. YA. KHRUSLOV, *New results in the theory of boundary value problems for regions with closed-grained boundaries*, Uspekhi Mat. Nauk, **33** (1978), pp. 127.
[40] V. G. MAZ'YA, *Sobolev Spaces*, Springer-Verlag, Berlin (1985).
[41] V. G. MAZ'YA - V. P. KHAVIN, *Nonlinear potential theory*, Russian Math. Surverys, **27** (1972), pp. 71-148.
[42] T. MEKKAOUI - C. PICARD, *Error estimates for the homogenization of a quasilinear Dirichlet problem in a periodically perforated domain*, in *Progress in Partial Differential Equations: The Metz Survey 2*, Wiley, New York (1993), pp. 185-193.
[43] L. NOTARANTONIO, *Asymptotic behaviour of Dirichlet problems on a Riemannian manifold*, Ricerche Mat., **41** (1992), pp. 327-367.
[44] S. OZAWA, *Surgery of domain and the Green's function of the Laplacian*, Proc. Japan Acad. Ser. A Math. Sci., **56** (1980), pp. 459-461.
[45] S. OZAWA, *Singular variation of domains and eigenvalues of the Laplacian*, Duke Math. J., **48** (1981), pp. 767-778.
[46] S. OZAWA, *Eigenvalues of the Laplacian on wildly perturbed domains*, Proc. Japan Acad. Ser. A Math. Sci., **58** (1982), pp. 419-421.
[47] S. OZAWA, *On an elaboration of M. Kac's theorem concerning eigenvalues of the Laplacian in a region with randomly distributed small obstacles*, Comm. Math. Phys., **91** (1983), pp. 473-487.
[48] S. OZAWA, *Point interaction poential approximation for* $(-\Delta + U)^{-1}$ *and eigenvalue of the Laplacian on wildly perturbed domain*, Osaka J. Math., **20** (1983), pp. 923-937.
[49] S. OZAWA, *Random media and the eigenvalues of the Laplacian*, Comm. Math. Phys., **94** (1984), pp. 421-437.
[50] L. S. PANKRATOV, *On convergence of solutions of variational problems in weakly connected domains*, preprint PTILT, Kharkov (1988).
[51] G. C. PAPANICOLAOU - S. R. S. VARADHAN, *Diffusion in regions with many small holes*, in *Stochastic Differential Systems, Filtering and Control. Proc. of the IFIP-WG 7/1 Wworking Conference (Vilnius, Lithuania, 1978)*, Lecture Notes in Control and Information Sci., **25**, Springer-Verlag, Berlin (1980), pp. 190-206.
[52] J. RAUCH - M. TAYLOR, *Potential and scattering theory on wildly perturbed domains*, J. Funct. Anal., **18** (1975), pp. 27-59.
[53] J. RAUCH - M. TAYLOR, *Electrostatic screening*, J. Math. Phys., **16** (1975), pp. 284-288.
[54] I. V. SKRYPNIK, *A quasilinear Dirichlet problem for a domain with fine-grained boundary*, Dokl. Akad. Nauk Ukrain. SSR Ser. A, **2** (1982), pp. 21-25.
[55] I. V. SKRYPNIK, *Nonlinear Elliptic Boundary Value Problems*, Teubner-Verlag, Leipzig (1986).
[56] I. V. SKRYPNIK, *Method for Analysis of Nonlinear Elliptic Boundary Value Problems*, Nauka, Moscow (1990). Translated in: Translation of Mathematical Monographs, **139**, American Mathematical Society, Providence (1994).
[57] I. V. SKRYPNIK, *Averaging nonlinear Dirichlet problems in domains with channels*, Soviet Math. Dokl., **42** (1991), pp. 853-857.
[58] I. V. SKRYPNIK, *Asymptotic behaviour of solutions of nonlinear elliptic problems in perforated domains*, Mat. Sb. (N.S.), **184** (1993), pp. 67-90.
[59] I. V. SKRYPNIK, *Homogenization of nonlinear Dirichlet problems in perforated domains of general structure*, Mat. Sb. (N.S.), **187** (1996), pp. 125-157.
[60] I. V. SKRYPNIK, *New conditions for the homogenization of nonlinear Dirichlet problems in perforated domains*, Ukrain. Mat. Zh., **48** (1996), pp. 675-694.
[61] W. P. ZIEMER, *Weakly Differentiable Functions*, Springer-Verlag, Berlin (1989).