

ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF LUBRICATION
PROBLEM IN A THIN DOMAIN WITH A ROUGH BOUNDARY
AND TRESCA FLUID-SOLID INTERFACE LAW

BY

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Abstract. We study the asymptotic behavior of the solution of a Stokes flow in a thin domain, with a thickness of order ε , and a rough surface. The roughness is defined by a quasi-periodic function with period ε . We suppose that the flow is subject to a Tresca fluid-solid interface condition. We prove a new result on the lower-semicontinuity for the two-scale convergence, which allows us to obtain rigorously the limit problem and to establish the uniqueness of its solution.

1. Introduction. We consider in this paper a Stokes system with given exterior forces in a three-dimensional thin domain where the thickness is of order $\lambda\varepsilon$, for $\lambda > 0$ a fixed constant and ε a small parameter which will tend to zero. The boundary of this thin domain consists of three parts: the bottom, the lateral part, and the top surface. This research is motivated by a free boundary problem from lubrication theory where the domain of the flow is usually very thin and the roughness of the boundary strongly affects the flow. Moreover, in many problems of lubrication theory the widely assumed no-slip conditions when the fluid has the same velocity as the surrounding solid boundary seem not respected since the shear rate becomes too high [1], [2]; see also [3], [4], [5], [6], [7]. We consider then that on the bottom, the normal velocity is equal to zero but the tangential velocity is unknown and satisfies Tresca-type fluid-solid boundary conditions, with friction coefficient a given positive function. On the lateral and the top parts, we have adhering boundary conditions. The top surface is assumed to be rough with a roughness defined by a quasi-periodic function with period ε .

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The weak form of the problem is a variational inequality [8], see also [3], where the roughness of the top surface was neglected. We use the same scaling as in [9], where only the adhering boundary conditions are considered. See also [10], [11], [12] for related work.

The main difficulty here is to pass to the two-scale limit in the variational inequality, due to the term coming from the Tresca fluid-solid boundary conditions. This difficulty induces us to prove a needed result of lower-semicontinuity for the two-scale convergence, using some results on the subdifferential and regularization of convex functions.

This article is organized as follows. In Section 2 we give a precise strong and weak formulation of the considered problem, and recall from [3] the existence and uniqueness of solutions, which are the velocity and the pressure of the flow. In Section 3 we establish some estimates, independent of the parameter ε , for the velocity and the pressure. In Section 4 we prove some convergence results. In Section 5 we establish a first relation between the two-scale limit of the velocity and the pressure. In Section 6 we obtain the variational inequality between the two-scale limit of the velocity and the pressure, using the result of lower-semicontinuity for the two-scale convergence. In Section 7 we study the limit problem and obtain the uniqueness of its solution. The needed result of lower-semicontinuity for the two-scale convergence is proved in the last part of this work, Section 8.

2. Formulation of the problem. Let ω be an open set in \mathbb{R}^2 with Lipschitz boundary and $Y = [0, 1]^2$ the basic cell in \mathbb{R}^2 . We consider a smooth function $(z, y) \mapsto h(z, y)$, such that $h \in C^2(\omega \times \mathbb{R}^2)$,

$$\left. \begin{aligned} h \text{ and } \frac{\partial h}{\partial y_i} \text{ for } i = 1, 2 \text{ are periodic in } y, \\ \exists a, b > 0 \quad a \leq h(z, y) \leq b \quad \forall (z, y) \in \omega \times Y. \end{aligned} \right\} \quad (2.1)$$

We set

$$h_M = \max_{(z, y) \in \omega \times Y} h(z, y), \quad h_m = \min_{(z, y) \in \omega \times Y} h(z, y),$$

and we define the domains

$$\Omega_M = \omega \times]0, h_M[, \quad \Omega_m = \omega \times]0, h_m[.$$

Let ε be a small parameter, related to the roughness wavelength and also to the thickness of the gap between the surfaces $z_3 = 0$ and $z_3 = \lambda\varepsilon h(z, \frac{z}{\varepsilon})$, such that the domain occupied by the fluid is

$$\Omega^\varepsilon = \{(z, z_3) \in \mathbb{R}^3 : z \in \omega, \quad 0 < z_3 < \lambda\varepsilon h^\varepsilon(z)\},$$

where

$$h^\varepsilon(z) = h(z, \frac{z}{\varepsilon}), \quad z \in \omega, \quad (2.2)$$

and $\lambda > 0$ is a fixed constant.

Let Γ^ε be the boundary of Ω^ε . We have $\Gamma^\varepsilon = \bar{\omega} \cap \bar{\Gamma}_1^\varepsilon \cap \bar{\Gamma}_L^\varepsilon$, where Γ_L^ε is the lateral boundary, Γ_1^ε is the upper surface defined by $z_3 = \lambda\varepsilon h^\varepsilon(z)$, and ω is the bottom of the fluid domain defined by $z_3 = 0$.

We need in the following to transform the domain Ω^ε into

$$\Omega_\varepsilon = \{(x, x_3) \in \mathbb{R}^3 : x \in \omega, \quad 0 < x_3 < h^\varepsilon(x)\}$$

using the scaling

$$x = z \quad \text{and} \quad x_3 = \frac{z_3}{\lambda\varepsilon}, \tag{2.3}$$

and also to transform the domain Ω_ε into $\Omega = \omega \times]0, 1[$ using the scaling

$$t = x \quad \text{and} \quad t_3 = \frac{x_3}{h^\varepsilon}. \tag{2.4}$$

Let $s^0 = (s_1^0, s_2^0) \in \mathbb{R}^2$ be the velocity of the lower surface and g be a given function in $(H^{\frac{1}{2}}(\partial\Omega_m))^3$ which does not depend on ε , such that

$$g(x, h_m) = 0, \quad g_k(x, 0) = s_k^0 \quad \text{for } k = 1, 2 \text{ and } g_3 = 0 \text{ on } \partial\Omega_m, \tag{2.5}$$

$$\int_{\partial\Omega_m} g \cdot n d\sigma = 0. \tag{2.6}$$

We consider the Stokes system

$$\frac{\partial \sigma_{ij}^\varepsilon}{\partial z_j} + f_i^\varepsilon = 0 \quad \text{in } \Omega^\varepsilon, \tag{2.7}$$

where

$$\sigma_{ij}^\varepsilon = -p^\varepsilon \delta_{ij} + 2\mu d_{i,j}(u^\varepsilon) \quad \text{with} \quad d_{i,j}(u^\varepsilon) = \frac{1}{2} \left(\frac{\partial u_i^\varepsilon}{\partial z_j} + \frac{\partial u_j^\varepsilon}{\partial z_i} \right),$$

$$\text{div}(u^\varepsilon) = 0 \quad \text{in } \Omega^\varepsilon, \tag{2.8}$$

where $u^\varepsilon, p^\varepsilon, \mu$ are the velocity, the pressure and the viscosity of the fluid.

On the upper surface, which is assumed to be fixed, no slip condition is given, so that

$$u^\varepsilon = 0 \quad \text{on } \Gamma_1^\varepsilon. \tag{2.9}$$

The velocity is known and parallel to the ω -plane,

$$u^\varepsilon = g\left(z, \frac{z_3}{\lambda\varepsilon}\right) \quad \text{on } \Gamma_L^\varepsilon, \tag{2.10}$$

extended by 0 on $\Gamma_L^\varepsilon - \partial\omega \times]0, \lambda\varepsilon h_m[$.

There is a no flux condition across ω so that

$$u_3^\varepsilon = 0 \quad \text{on } \omega. \tag{2.11}$$

The tangential velocity on ω is *unknown* and satisfies the Tresca friction law with a friction coefficient k^ε [8],

$$\left. \begin{aligned} |\sigma_t^\varepsilon| < k^\varepsilon &\Rightarrow u_t^\varepsilon = s^0 \\ |\sigma_t^\varepsilon| = k^\varepsilon &\Rightarrow \exists r \geq 0 \text{ such that } u_t^\varepsilon = s^0 - r\sigma_t^\varepsilon \end{aligned} \right\} \text{ on } \omega, \tag{2.12}$$

where $|\cdot|$ denotes the \mathbb{R}^2 Euclidean norm. Here $n = (n_1, n_2, n_3)$ is the unit outward normal vector to Γ^ε , and

$$u_n^\varepsilon = u^\varepsilon \cdot n = u_i^\varepsilon n_i; \quad u_{t_i}^\varepsilon = u_i^\varepsilon - u_n^\varepsilon n_i, \tag{2.13}$$

$$\sigma_n^\varepsilon = (\sigma^\varepsilon \cdot n) \cdot n = \sigma_{ij}^\varepsilon n_i n_j; \quad \sigma_{t_i}^\varepsilon = \sigma_{ij}^\varepsilon n_j - \sigma_n^\varepsilon n_i \tag{2.14}$$

are, respectively, the normal, the tangential velocity on Γ^ε , the components of the normal and the tangential stress tensor on Γ^ε .

Due to (2.6) it is known [13] (lemma 2.2, p.24) that there exists a function G such that

$$G \in (H^1(\Omega_m))^3 : \quad \operatorname{div} G = 0 \quad \text{in } \Omega_m, \quad G = g \quad \text{on } \partial\Omega_m. \quad (2.15)$$

We still denote by G its extension by zero to Ω_ε . We set G^ε such that

$$G_i^\varepsilon(z, z_3) = G_i(z, \frac{z_3}{\lambda^\varepsilon}) \quad \text{for } i = 1, 2, \quad G_3^\varepsilon(z, z_3) = \lambda^\varepsilon G_3(z, \frac{z_3}{\lambda^\varepsilon}), \quad (2.16)$$

which belongs to $(H^1(\Omega^\varepsilon))^3$ and satisfies also $\operatorname{div} G^\varepsilon = 0$ in Ω^ε .

We set

$$K^\varepsilon = \left\{ v \in (H^1(\Omega^\varepsilon))^3 : \quad v = G^\varepsilon \text{ on } \Gamma_L^\varepsilon \cup \Gamma_1^\varepsilon, \quad v_3 = 0 \text{ on } \omega \right\},$$

$$K_{\operatorname{div}}^\varepsilon = \left\{ v \in K^\varepsilon : \quad \operatorname{div} v = 0 \quad \text{in } \Omega^\varepsilon \right\}.$$

For any three-dimensional vector $\varphi = (\varphi_1, \varphi_2, \varphi_3)^T$ we denote by $\underline{\varphi}$ the projected bi-dimensional vector $\underline{\varphi} = (\varphi_1, \varphi_2)^T$. A formal application of (2.7)–(2.12) leads to the following variational problem [8]:

PROBLEM 2.1. Find $u^\varepsilon \in K_{\operatorname{div}}^\varepsilon(\Omega^\varepsilon)$ and $p^\varepsilon \in L_0^2(\Omega^\varepsilon)$ such that

$$\begin{aligned} a(u^\varepsilon, \varphi^\varepsilon - u^\varepsilon) - (p^\varepsilon, \operatorname{div} \varphi^\varepsilon) + j(\varphi^\varepsilon) - j(u^\varepsilon) \\ \geq (f^\varepsilon, \varphi^\varepsilon - u^\varepsilon), \quad \forall \varphi^\varepsilon \in K^\varepsilon, \end{aligned} \quad (2.17)$$

where

$$a(u, \varphi) = 2\mu \sum_{i,j=1}^3 \int_{\Omega^\varepsilon} d_{ij}(u) d_{ij}(\varphi) \, dz dz_3,$$

$$j(\varphi) = \int_{\omega} k^\varepsilon |\underline{\varphi} - s^0| \, dz, \quad (f^\varepsilon, \varphi) = \sum_{i=1}^3 \int_{\Omega^\varepsilon} f_i^\varepsilon \varphi_i \, dz dz_3,$$

$$(p, \operatorname{div} \varphi) = \sum_{i=1}^3 \int_{\Omega^\varepsilon} p \frac{\partial \varphi_i}{\partial z_i} \, dz dz_3.$$

THEOREM 2.1. [3] Assume that f^ε in $(L^3(\Omega^\varepsilon))^3$ and the friction coefficient k^ε is a nonnegative function in $L^\infty(\omega)$. There exists a unique u^ε in $K_{\operatorname{div}}^\varepsilon$ and there exists a unique p^ε in $L_0^2(\Omega^\varepsilon)$, such that $(u^\varepsilon, p^\varepsilon)$ is a solution to Problem 2.1.

3. Estimates of the velocity and pressure. Observe first that as $u^\varepsilon(z, \lambda^\varepsilon h^\varepsilon(z)) = 0$, we can extend it by zero to the fixed domain Ω_M ($\bar{u} = u$ in Ω^ε and $\bar{u} = 0$ in $\Omega_M - \Omega^\varepsilon$), such that Korn's inequality

$$C_1 \|\nabla \bar{u}\|_{L^2(\Omega_M)} \leq a(\bar{u}, \bar{u}) \quad (C_1 \text{ independent of } \varepsilon)$$

holds. Moreover we can easily check that $C_1 = \frac{1}{2\mu}$. So we have also the same inequality for u^ε in Ω^ε .

Throughout this paper, for the sake of simplicity, the same notation will be used for a function defined on the three different domains Ω^ε , Ω_ε and Ω .

We assume the following dependence of data on ε :

$$\varepsilon^2 f^\varepsilon(z, z_3) = f(x, x_3), \quad \varepsilon k^\varepsilon(z) = k(x) \quad (3.1)$$

with f and k independent of ε , $f \in (L^2(\Omega_m))^3$ (extended by 0 on Ω^ε) and $k \in L^2(\omega)$ with $k > 0$ a.e. We set

$$\begin{aligned} \hat{K}^\varepsilon &= \{v \in (H^1(\Omega_\varepsilon))^3; v = (G_1, G_1, 0) \text{ on } \hat{\Gamma}_L^\varepsilon \cap \hat{\Gamma}_1^\varepsilon, \quad v_3 = 0 \text{ on } \omega\} \\ &\text{with } \hat{\Gamma}_L^\varepsilon = \{(x, x_3) : x \in \partial\omega, \quad 0 \leq x_3 \leq h^\varepsilon(x)\} \\ &\text{and } \hat{\Gamma}_1^\varepsilon = \{(x, h^\varepsilon(x)) : x \in \omega\}. \end{aligned}$$

Then passing to Ω_ε , and dividing by $\lambda\varepsilon$, (2.17) is equivalent to

$$\begin{aligned} \hat{a}(u^\varepsilon, \varphi^\varepsilon - u^\varepsilon) - \int_{\Omega_\varepsilon} p^\varepsilon \left(\frac{\partial \varphi_1^\varepsilon}{\partial x_1} + \frac{\partial \varphi_2^\varepsilon}{\partial x_2} + \frac{1}{\lambda\varepsilon} \frac{\partial \varphi_3^\varepsilon}{\partial x_3} \right) dx dx_3 \\ + \frac{1}{\lambda\varepsilon^2} \left(\hat{j}(\varphi^\varepsilon) - \hat{j}(u^\varepsilon) \right) \geq \frac{1}{\varepsilon^2} \int_{\Omega_\varepsilon} f \cdot (\varphi^\varepsilon - u^\varepsilon) dx dx_3 \quad \forall \varphi^\varepsilon \in \hat{K}^\varepsilon \end{aligned} \quad (3.2)$$

with the notation

$$\begin{aligned} \hat{a}(u^\varepsilon, \varphi^\varepsilon) &= \frac{\mu}{2} \sum_{i,j=1}^2 \int_{\Omega_\varepsilon} \left(\frac{\partial u_i^\varepsilon}{\partial x_j} + \frac{\partial u_j^\varepsilon}{\partial x_i} \right) \left(\frac{\partial \varphi_i^\varepsilon}{\partial x_j} + \frac{\partial \varphi_j^\varepsilon}{\partial x_i} \right) dx dx_3 \\ &+ \mu \sum_{i=1}^2 \int_{\Omega_\varepsilon} \left(\frac{1}{\lambda\varepsilon} \frac{\partial u_i^\varepsilon}{\partial x_3} + \frac{\partial u_3^\varepsilon}{\partial x_i} \right) \left(\frac{1}{\lambda\varepsilon} \frac{\partial \varphi_i^\varepsilon}{\partial x_3} + \frac{\partial \varphi_3^\varepsilon}{\partial x_i} \right) dx dx_3 \\ &+ \frac{2\mu}{\lambda^2 \varepsilon^2} \int_{\Omega_\varepsilon} \frac{\partial u_3^\varepsilon}{\partial x_3} \frac{\partial \varphi_3^\varepsilon}{\partial x_3} dx dx_3 \end{aligned} \quad (3.3)$$

and

$$\hat{j}(\varphi^\varepsilon) = \int_\omega k(x) |\varphi^\varepsilon(x, 0) - s^0| dx.$$

For any domain D in \mathbb{R}^3 we introduce the Banach space

$$V_{x_3}(D) = \{v \in L^2(D), \quad \frac{\partial v}{\partial x_3} \in L^2(D)\}$$

with the norm

$$\|v\|_{V_{x_3}(D)} = \|v\|_{L^2(D)} + \left\| \frac{\partial v}{\partial x_3} \right\|_{L^2(D)}.$$

First we give the estimates of the velocity in the rescaled domain Ω_ε ; then we give the estimates of the velocity and pressure in the domain Ω .

PROPOSITION 3.1. The following estimates, for the velocity, are satisfied in Ω_ε , with C denoting a constant which does not depend on ε :

$$\|u_i^\varepsilon\|_{V_{x_3}(\Omega_\varepsilon)} \leq C, \quad \left\| \frac{\partial u_i^\varepsilon}{\partial x_j} \right\|_{L^2(\Omega_\varepsilon)} \leq \frac{C}{\lambda\varepsilon}, \quad 1 \leq i \leq 3, \quad 1 \leq j \leq 2. \quad (3.4)$$

Proof. Taking $\varphi^\varepsilon = G^\varepsilon$ in (2.17) we obtain

$$a(u^\varepsilon, u^\varepsilon) + j(u^\varepsilon) \leq a(u^\varepsilon, G^\varepsilon) + j(G^\varepsilon) - (f^\varepsilon, G^\varepsilon) + (f^\varepsilon, u^\varepsilon)$$

but $j(G^\varepsilon) = j(s^0) = 0$. Then using also the Korn inequality we have

$$2\mu\|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 \leq \mu\|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)}\|\nabla G^\varepsilon\|_{L^2(\Omega^\varepsilon)} + \|u^\varepsilon\|_{L^2(\Omega^\varepsilon)}\|f^\varepsilon\|_{L^2(\Omega^\varepsilon)} + \|G^\varepsilon\|_{L^2(\Omega^\varepsilon)}\|f^\varepsilon\|_{L^2(\Omega^\varepsilon)}.$$

From

$$-u^\varepsilon(z, z_3) = \int_{z_3}^{\lambda\varepsilon h^\varepsilon(z)} \frac{\partial u^\varepsilon}{\partial z_3}(z, \xi) d\xi$$

we have

$$|u^\varepsilon(z, z_3)|^2 \leq \left(\int_0^{\lambda\varepsilon h^\varepsilon(z)} \left| \frac{\partial u^\varepsilon}{\partial z_3}(z, \xi) \right| dt \right)^2 \leq (\lambda\varepsilon h_M) \left(\int_0^{\lambda\varepsilon h^\varepsilon(z)} \left| \frac{\partial u^\varepsilon}{\partial z_3}(z, \xi) \right|^2 d\xi \right),$$

so

$$\begin{aligned} \int_{\Omega^\varepsilon} |u^\varepsilon(z, z_3)|^2 &\leq \lambda\varepsilon h_M \int_0^{\lambda\varepsilon h^\varepsilon(z)} \int_{\Omega^\varepsilon} \left| \frac{\partial u^\varepsilon}{\partial z_3}(z, \xi) \right|^2 dz dz_3 d\xi \\ &\leq \lambda^2 \varepsilon^2 h_M^2 \left\| \frac{\partial u^\varepsilon}{\partial z_3} \right\|_{L^2(\Omega^\varepsilon)}^2; \end{aligned}$$

thus

$$\|u^\varepsilon\|_{L^2(\Omega^\varepsilon)} \leq \lambda\varepsilon h_M \|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)}.$$

So

$$\begin{aligned} \|f^\varepsilon\|_{L^2(\Omega^\varepsilon)} \|u^\varepsilon\|_{L^2(\Omega^\varepsilon)} &\leq \left(\frac{\lambda\varepsilon h_M}{\sqrt{\mu}} \cdot \|f^\varepsilon\|_{L^2(\Omega^\varepsilon)} \right) (\sqrt{\mu} \|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)}) \\ &\leq \frac{h_M^2 \lambda^2 \varepsilon^2}{2\mu} \|f^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 + \frac{\mu}{2} \|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 \end{aligned}$$

and

$$\mu\|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)}\|\nabla G^\varepsilon\|_{L^2(\Omega^\varepsilon)} \leq \frac{\mu}{2}\|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 + \frac{\mu}{2}\|\nabla G^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2.$$

Then

$$\mu\|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 \leq \mu\|\nabla G^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 + \left(\frac{1}{2\mu} + \frac{\lambda^2 h_M^2 \varepsilon^2}{2\mu} \right) \|f^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2. \quad (3.5)$$

Using now the dependence of G^ε and f^ε on ε , and using the Poincaré inequality, we deduce (3.4). \square

PROPOSITION 3.2. The following estimates, for the velocity and the pressure, are satisfied in Ω , with C denoting a constant which does not depends on ε :

$$\left\| \frac{\partial u_i^\varepsilon}{\partial t_3} \right\|_{L^2(\Omega)} \leq C \quad \text{for } i = 1, 2, 3, \quad (3.6)$$

$$\varepsilon \left\| \frac{\partial u_i^\varepsilon}{\partial t_j} \right\|_{L^2(\Omega)} \leq C \quad \text{for } i = 1, 2, 3 \quad j = 1, 2, \quad (3.7)$$

$$\varepsilon^2 \left\| \frac{\partial p^\varepsilon}{\partial t_i} \right\|_{H^{-1}(\Omega)} \leq C, \quad i = 1, 2, \quad \varepsilon \left\| \frac{\partial p^\varepsilon}{\partial t_3} \right\|_{H^{-1}(\Omega)} \leq C. \quad (3.8)$$

Proof. Using the change of variables (2.3) and (2.4) we have

$$\frac{D(x_1, x_2, x_3)}{D(t_1, t_2, t_3)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_3 \frac{\partial h^\varepsilon}{\partial t_1} & t_3 \frac{\partial h^\varepsilon}{\partial t_2} & h^\varepsilon \end{pmatrix},$$

$$\frac{D(t_1, t_2, t_3)}{D(x_1, x_2, x_3)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{t_3}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial t_1} & -\frac{t_3}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial t_2} & \frac{1}{h^\varepsilon} \end{pmatrix}.$$

As

$$\left\| \frac{\partial h^\varepsilon}{\partial t_j} \right\|_{L^\infty(\omega)} \leq \frac{C}{\varepsilon} \quad \text{for } 1 \leq j \leq 2, \tag{3.9}$$

for $1 \leq i \leq 3$ and $1 \leq j \leq 2$, we deduce

$$\left\| \frac{\partial u_i^\varepsilon}{\partial t_3} \right\|_{L^2(\Omega)}^2 = \int_{\Omega_\varepsilon} \left(h^\varepsilon \frac{\partial u_i^\varepsilon}{\partial x_3} \right)^2 \frac{1}{h^\varepsilon} dx dx_3 \leq h_M \left\| \frac{\partial u_i^\varepsilon}{\partial x_3} \right\|_{L^2(\Omega_\varepsilon)}^2, \tag{3.10}$$

$$\begin{aligned} \left\| \frac{\partial u_i^\varepsilon}{\partial t_j} \right\|_{L^2(\Omega)}^2 &= \int_{\Omega_\varepsilon} \left(\frac{\partial u_i^\varepsilon}{\partial x_j} + t_3 \frac{\partial h^\varepsilon}{\partial x_j} \frac{\partial u_i^\varepsilon}{\partial x_3} \right)^2 \frac{1}{h^\varepsilon} dx dx_3 \\ &\leq C \left(\left\| \frac{\partial u_i^\varepsilon}{\partial x_j} \right\|_{L^2(\Omega_\varepsilon)}^2 + \frac{1}{\varepsilon^2} \left\| \frac{\partial u_i^\varepsilon}{\partial x_3} \right\|_{L^2(\Omega_\varepsilon)}^2 \right), \end{aligned} \tag{3.11}$$

so from (3.10) and (3.4) we deduce (3.6), and from (3.9), (3.11) and (3.4) we deduce (3.7).

To prove (3.8) we rewrite (2.17) with $\varphi^\varepsilon = u^\varepsilon \pm \psi$ for all $\psi \in (H_0^1(\Omega^\varepsilon))^3$; thus

$$\int_{\Omega^\varepsilon} p^\varepsilon \operatorname{div} \psi dz dz_3 = a(u^\varepsilon, \psi) - (f^\varepsilon, \psi) \quad \forall \psi \in (H_0^1(\Omega^\varepsilon))^3. \tag{3.12}$$

On the other hand, the change of variables (2.3) and (2.4) gives

$$\frac{D(z_1, z_2, z_3)}{D(t_1, t_2, t_3)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda \varepsilon t_3 \frac{\partial h^\varepsilon}{\partial t_1} & \lambda \varepsilon t_3 \frac{\partial h^\varepsilon}{\partial t_2} & \lambda \varepsilon h^\varepsilon \end{pmatrix}$$

and

$$\frac{D(t_1, t_2, t_3)}{D(z_1, z_2, z_3)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{t_3}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial t_1} & -\frac{t_3}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial t_2} & \frac{1}{\lambda \varepsilon h^\varepsilon} \end{pmatrix},$$

so

$$\int_{\Omega^\varepsilon} p^\varepsilon \operatorname{div} \psi = \int_{\Omega} p^\varepsilon \left[\sum_{i=1}^2 \left[\frac{\partial}{\partial t_i} (\lambda \varepsilon h^\varepsilon \psi_i) + \frac{\partial}{\partial t_3} (-\lambda \varepsilon t_3 \frac{\partial h^\varepsilon}{\partial t_i} \psi_i) \right] + \frac{\partial \psi_3}{\partial t_3} \right]. \tag{3.13}$$

Taking $\psi = (0, 0, \varphi)$ in (3.12) and (3.13) with $\varphi \in H_0^1(\Omega)$ we get

$$\begin{aligned} \int_{\Omega} p^\varepsilon \frac{\partial \varphi}{\partial t_3} dt dt_3 &= \frac{2\mu}{\lambda \varepsilon} \int_{\Omega} \frac{\partial u_3^\varepsilon}{\partial t_3} \frac{\partial \varphi}{\partial t_3} \frac{1}{h^\varepsilon} dt dt_3 - \frac{\lambda}{\varepsilon} \int_{\Omega} f_3 \varphi h^\varepsilon dt dt_3 \\ &+ \sum_{i=1}^2 \int_{\Omega} \mu \left(\frac{\partial u_i^\varepsilon}{\partial t_3} \frac{1}{\lambda \varepsilon h^\varepsilon} + \frac{\partial u_3^\varepsilon}{\partial t_i} - \frac{t_3}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial t_i} \frac{\partial u_3^\varepsilon}{\partial t_3} \right) \left(\frac{\partial \varphi}{\partial t_i} - \frac{t_3}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial t_i} \frac{\partial \varphi}{\partial t_3} \right) \lambda \varepsilon h^\varepsilon dt dt_3, \end{aligned}$$

so from (3.6)-(3.7) and (3.9) we deduce

$$| \int_{\Omega} p^\varepsilon \frac{\partial \varphi}{\partial t_3} | \leq \frac{C}{\varepsilon} \|\varphi\|_{H_0^1(\Omega)}. \tag{3.14}$$

Taking $\psi = (\frac{\varphi}{h^\varepsilon}, 0, 0)$ in (3.12) and (3.13) with $\varphi \in H_0^1(\Omega)$ we obtain

$$\begin{aligned} & \int_{\Omega} p^\varepsilon \frac{\partial(\lambda \varepsilon \varphi)}{\partial t_1} dt dt_3 - \int_{\Omega} p^\varepsilon \frac{\partial}{\partial t_3} \left(\lambda \varepsilon t_3 \frac{\partial h^\varepsilon}{\partial t_1} \frac{\varphi}{h^\varepsilon} \right) dt dt_3 = - \int_{\Omega} \frac{\lambda}{\varepsilon} f_1 \varphi h^\varepsilon dt dt_3 \\ & + 2\mu \int_{\Omega} \left[\frac{\partial u_1^\varepsilon}{\partial t_1} - \frac{t_3}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial t_1} \frac{\partial u_1^\varepsilon}{\partial t_3} \right] \left[\frac{\partial}{\partial t_1} \left(\frac{\varphi}{h^\varepsilon} \right) - \frac{t_3}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial t_1} \frac{1}{h^\varepsilon} \frac{\partial \varphi}{\partial t_3} \right] \lambda \varepsilon h^\varepsilon dt dt_3 \\ & + \mu \int_{\Omega} \left[\frac{\partial u_1^\varepsilon}{\partial t_2} - \frac{t_3}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial t_2} \frac{\partial u_1^\varepsilon}{\partial t_3} + \frac{\partial u_2^\varepsilon}{\partial t_1} - \frac{t_3}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial t_1} \frac{\partial u_2^\varepsilon}{\partial t_3} \right] \\ & \times \left[\frac{\partial}{\partial t_2} \left(\frac{\varphi}{h^\varepsilon} \right) - \frac{t_3}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial t_2} \frac{1}{h^\varepsilon} \frac{\partial \varphi}{\partial t_3} \right] \lambda \varepsilon h^\varepsilon dt dt_3; \end{aligned}$$

hence from (3.6), (3.7) and (3.9) we get

$$| \int_{\Omega} p^\varepsilon \frac{\partial(\varepsilon \varphi)}{\partial t_1} dt dt_3 - \int_{\Omega} p^\varepsilon \frac{\partial}{\partial t_3} \left(\varepsilon t_3 \frac{\partial h^\varepsilon}{\partial t_1} \frac{\varphi}{h^\varepsilon} \right) dt dt_3 | \leq \frac{C}{\varepsilon} \|\varphi\|_{H_0^1(\Omega)}. \tag{3.15}$$

Now taking $\psi = (0, 0, t_3 \frac{\partial h^\varepsilon}{\partial t_1} \frac{\varphi}{h^\varepsilon})$ in (3.12) and (3.13), with $\varphi \in H_0^1(\Omega)$, we get

$$\begin{aligned} & \int_{\Omega} p^\varepsilon \frac{\partial}{\partial t_3} \left(\frac{t_3}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial t_1} \varphi \right) dt dt_3 = 2\mu \int_{\Omega} \frac{\partial u_3^\varepsilon}{\partial t_3} \frac{1}{\lambda \varepsilon h^\varepsilon} \frac{\partial}{\partial t_3} \left(\frac{t_3}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial t_1} \varphi \right) dt dt_3 \\ & - \int_{\Omega} \frac{\lambda}{\varepsilon} f_3 t_3 \frac{\partial h^\varepsilon}{\partial t_1} \varphi dt dt_3 + \mu \sum_{i=1}^2 \int_{\Omega} \left[\frac{\partial u_i^\varepsilon}{\partial t_3} \frac{1}{\lambda \varepsilon h^\varepsilon} + \frac{\partial u_3^\varepsilon}{\partial t_i} - \frac{t_3}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial t_i} \frac{\partial u_3^\varepsilon}{\partial t_3} \right] \\ & \times \left[\frac{\partial}{\partial t_i} \left(\frac{t_3}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial t_1} \varphi \right) - \left(\frac{t_3}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial t_i} \right) \left(\frac{\partial \varphi}{\partial t_3} \frac{t_3}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial t_1} + \frac{1}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial t_1} \varphi \right) \right] \lambda \varepsilon dt dt_3. \end{aligned}$$

Then, from (3.6), (3.7) and (3.9), we easily obtain

$$| \int_{\Omega} p^\varepsilon \frac{\partial}{\partial t_3} \left(\frac{t_3}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial t_1} \varphi \right) dt dt_3 | \leq \frac{C}{\varepsilon^2} \|\varphi\|_{H_0^1(\Omega)}. \tag{3.16}$$

So, from (3.15) and (3.16) we obtain (for $i = 1$ but it is the same for $i = 2$)

$$| \int_{\Omega} p^\varepsilon \frac{\partial \varphi}{\partial t_i} dt dt_3 | \leq \frac{C}{\varepsilon^2} \|\varphi\|_{H_0^1(\Omega)} \quad \text{for } i = 1, 2. \tag{3.17}$$

So from (3.14), (3.17) we deduce (3.8). \square

For the next sections we need the following notation: for any function ψ , we denote its extension from Ω_ε to Ω_M by

$$\bar{\psi} = \begin{cases} \psi & \text{on } \Omega_\varepsilon \\ 0 & \text{on } \Omega_M - \Omega_\varepsilon. \end{cases}$$

We define also the sets

$$B_x = \{(y, x_3), \quad y \in Y, \quad 0 < x_3 < h(x, y)\}, \quad Q_M = \omega \times Y \times]0, h_M[,$$

$$Q = \bigcup_{x \in \omega} \{x\} \times B_x = \{(x, y, x_3), \quad x \in \omega, \quad y \in Y, \quad 0 < x_3 < h(x, y)\}$$

and the spaces

$$C_{\#}^{\infty}(B_x) = \{\varphi \in C^{\infty}(B_x), \quad \varphi \text{ is } y\text{-periodic, } \varphi(y, h(x, y)) = 0\},$$

$$H_{\#}^1(Y) = \{\varphi \in H^1(Y) : \quad \varphi \text{ is periodic}\},$$

$$H_{0\#}^1(B_x) = \{\varphi \in H^1(B_x) : \varphi \text{ is } y\text{-periodic and } \varphi(y, 0) = \varphi(y, h(x, y)) = 0\},$$

$$H_{1\#}^1(B_x) = \{\varphi \in H^1(B_x) : \varphi \text{ is } y\text{-periodic and } \varphi(y, h(x, y)) = 0\}.$$

REMARK 3.1. We choose $p^\varepsilon \in L^2(\Omega^\varepsilon)/\mathbb{R}$ such that

$$\int_{\Omega^\varepsilon} \frac{p^\varepsilon}{h^\varepsilon} dz dz_3 = 0, \quad \text{which implies } \int_{\Omega} p^\varepsilon dt dt_3 = 0.$$

We deduce from (3.8) and [14] that

$$\varepsilon^2 \|p^\varepsilon\|_{L^2(\Omega)} \leq C, \tag{3.18}$$

where C is a constant independent of ε .

4. Convergence results.

THEOREM 4.1. There exists $p \in L^2(\omega)$ with $\int_{\omega} p(x) dx = 0$ such that the two-scale convergence holds in Ω_M for a subsequence of ε which is still denoted by ε :

$$\varepsilon^2 \bar{p}^\varepsilon \rightharpoonup \xi(x, x_3, y) = \begin{cases} p & \text{if } x_3 < h(x, y) \\ 0 & \text{if } x_3 > h(x, y). \end{cases} \tag{4.1}$$

Proof. We have

$$\|p^\varepsilon\|_{L^2(\Omega_\varepsilon)} = \int_{\Omega} |p^\varepsilon|^2 h^\varepsilon dt dt_3 \leq h_M \|p^\varepsilon\|_{L^2(\Omega)}^2,$$

so

$$\varepsilon^2 \|\bar{p}^\varepsilon\|_{L^2(\Omega_M)} \leq C.$$

We can extract a subsequence such that $\varepsilon^2 \bar{p}^\varepsilon$ two-scale converges to some ξ in $L^2(\Omega_M \times [0, 1]^3)$. We choose an arbitrary φ in $\mathcal{D}(\Omega_M, C_{\#}^{\infty}(Y))$ with $\varphi = 0$ for $x_3 \leq h(x, y)$, and we get

$$\int_{\Omega_M} \varepsilon^2 \bar{p}^\varepsilon \varphi(x, \frac{x}{\varepsilon}, x_3) dx dx_3 = 0.$$

Passing now to the scale limit, we obtain

$$\int_{\Omega_M} \int_Y \xi \varphi(x, y, x_3) dy dx dx_3 = 0,$$

so $\xi = 0$ for $x_3 \geq h(x, y)$. We denote by p the restriction of ξ for $x_3 \leq h(x, y)$. Following [9], we prove that ξ depends only on x for $x_3 \leq h(x, y)$. For any ψ in $\mathcal{D}(\Omega_M; C_{\#}^{\infty}([0, 1]^3))$ with $\psi = 0$ for $x_3 \geq h(x, y)$, taking

$$\psi^\varepsilon(x, x_3) = \psi(x, x_3, \frac{x}{\varepsilon}, \frac{x_3}{\varepsilon}),$$

and $(u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon \pm \psi^\varepsilon)$ as a test function in (3.2), we have

$$\begin{aligned} \frac{1}{\lambda\varepsilon} \int_{\Omega_\varepsilon} p^\varepsilon \left(\frac{\partial\psi}{\partial x_3} + \frac{1}{\varepsilon} \frac{\partial\psi}{\partial y_3} \right) dx dx_3 &= \mu \sum_{i=1}^2 \int_{\Omega_\varepsilon} \left[\frac{1}{\lambda\varepsilon} \frac{\partial u_i^\varepsilon}{\partial x_3} + \frac{\partial u_3^\varepsilon}{\partial x_i} \right] \left[\frac{\partial\psi}{\partial x_i} + \frac{1}{\varepsilon} \frac{\partial\psi}{\partial y_i} \right] \\ &+ \frac{2\mu}{\lambda^2\varepsilon} \int_{\Omega_\varepsilon} \frac{\partial u_3^\varepsilon}{\partial x_3} \left(\frac{\partial\psi}{\partial x_3} + \frac{1}{\varepsilon} \frac{\partial\psi}{\partial y_3} \right) dx dx_3 - \int_{\Omega_\varepsilon} \frac{f_3}{\varepsilon^2} \psi dx dx_3. \end{aligned}$$

Then extending u^ε and p^ε by zero to Ω_M and passing to the two-scale limit after multiplication by ε^4 we deduce

$$\int_{[0,1]^3} \int_{\Omega_M} \xi \frac{\partial\psi}{\partial y_3} dx dx_3 dy dy_3 = 0, \quad (4.2)$$

so p does not depend on y_3 . To prove that ξ does not depend on x_3 , we take

$$\psi^\varepsilon(x, x_3) = \psi\left(x, x_3, \frac{x}{\varepsilon}\right),$$

with ψ in $\mathcal{D}(\Omega_M, C_{\#}^\infty(Y))$, $\psi = 0$ for $x_3 \geq h(x, y)$ and $(u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon \pm \psi^\varepsilon)$ as a test function in (3.2). Then

$$\begin{aligned} \frac{1}{\lambda\varepsilon} \int_{\Omega_\varepsilon} p^\varepsilon \frac{\partial\psi}{\partial x_3} dx dx_3 &= \mu \sum_{i=1}^2 \int_{\Omega_\varepsilon} \left[\frac{1}{\lambda\varepsilon} \frac{\partial u_i^\varepsilon}{\partial x_3} + \frac{\partial u_3^\varepsilon}{\partial x_i} \right] \left[\frac{\partial\psi}{\partial x_i} + \frac{1}{\varepsilon} \frac{\partial\psi}{\partial y_i} \right] dx dx_3 \\ &+ \frac{2\mu}{\lambda^2\varepsilon^2} \int_{\Omega_\varepsilon} \frac{\partial u_3^\varepsilon}{\partial x_3} \frac{\partial\psi}{\partial x_3} dz dz_3 - \int_{\Omega_\varepsilon} \frac{f_3}{\varepsilon^2} \psi dx dx_3. \end{aligned}$$

Passing to the two-scale limit after multiplication by ε^3 we deduce

$$\int_Y \int_{\Omega_\varepsilon} \xi \frac{\partial\psi}{\partial x_3} dx dx_3 dy = 0.$$

So p does not depend on x_3 . To prove that p does not depend on y_i for $i = 1, 2$ we take

$$\psi^\varepsilon(x, x_3) = \psi\left(x, x_3, \frac{x}{\varepsilon}\right)$$

with ψ in $\mathcal{D}(\Omega_M, C_{\#}^\infty(Y))$, where $\psi = 0$ for $x_3 \geq h(x, y)$ and $(u_1^\varepsilon \pm \psi^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)$ as a test function in (3.2), and we obtain

$$\begin{aligned} \int_{\Omega_\varepsilon} p^\varepsilon \left(\frac{\partial\psi}{\partial x_1} + \frac{1}{\varepsilon} \frac{\partial\psi}{\partial y_1} \right) dx dx_3 &= 2\mu \int_{\Omega_\varepsilon} \frac{\partial u_1^\varepsilon}{\partial x_1} \left(\frac{\partial\psi}{\partial x_1} + \frac{1}{\varepsilon} \frac{\partial\psi}{\partial y_1} \right) dx dx_3 \\ &+ \mu \int_{\Omega_\varepsilon} \left(\frac{\partial u_1^\varepsilon}{\partial x_2} + \frac{\partial u_2^\varepsilon}{\partial x_1} \right) \left(\frac{\partial\psi}{\partial x_2} + \frac{1}{\varepsilon} \frac{\partial\psi}{\partial y_2} \right) dx dx_3 \\ &+ \mu \int_{\Omega_\varepsilon} \left(\frac{1}{\lambda\varepsilon} \frac{\partial u_1^\varepsilon}{\partial x_3} + \frac{\partial u_3^\varepsilon}{\partial x_1} \right) \frac{1}{\lambda\varepsilon} \frac{\partial\psi}{\partial x_3} dx dx_3 - \frac{1}{\varepsilon^2} \int_{\Omega_\varepsilon} f_1 \psi dx dx_3. \end{aligned}$$

Passing to the two-scale limit after multiplication by ε^3 we deduce

$$\int_{\Omega_M} \int_Y \xi \frac{\partial\psi}{\partial y_1} dy dx dx_3 = 0.$$

So p does not depend on y_1 . In the same manner p does not depend on y_2 . It remains to prove that $\int_{\omega} p dx = 0$. Passing to the two-scale limit in the relation

$$\int_{\Omega_M} \frac{\varepsilon^2 p^\varepsilon}{h^\varepsilon} dx dx_3 = 0,$$

we find

$$\int_Q \frac{p}{h} dy dx dx_3 = 0,$$

which gives the expected result and finishes the proof. \square

THEOREM 4.2. There exists $u \in (L^2(\Omega_M; H^1_{\#}(Y)))^3$ with

$$\frac{\partial u}{\partial x_3} \in (L^2(\Omega_M \times Y))^3,$$

and there exists $v \in (L^2(\Omega_M; H^1_{\#}([0, 1]^3)))^3$ such that the two-scale convergences hold, up to a subsequence of ε denoted also by ε :

$$\bar{u}_i^\varepsilon \rightharpoonup u_i, \quad \varepsilon \frac{\partial \bar{u}_i^\varepsilon}{\partial x_j} \rightharpoonup \frac{\partial u_i}{\partial y_j} \quad \text{for } 1 \leq i \leq 3, \quad 1 \leq j \leq 2, \tag{4.3}$$

$$\frac{\partial \bar{u}_i^\varepsilon}{\partial x_3} \rightharpoonup \frac{\partial u_i}{\partial x_3} + \frac{\partial v_i}{\partial y_3} \quad \text{for } 1 \leq i \leq 3. \tag{4.4}$$

Moreover

$$\frac{\partial u_1}{\partial y_1} + \frac{\partial u_2}{\partial y_2} + \frac{1}{\lambda} \frac{\partial u_3}{\partial x_3} = 0 \quad \text{in } B_x \quad \text{a.e. } x \in \omega, \tag{4.5}$$

$$u_i = 0 \quad \text{for } x_3 \geq h(x, y) \quad \text{for } i = 1, 2, 3, \tag{4.6}$$

$$u_3 = 0 \quad \text{for } x_3 = 0, \tag{4.7}$$

$$\bar{u}_i^\varepsilon(x, 0) \rightharpoonup u_i(x, 0), \quad \text{for } \omega \quad \text{for } 1 \leq i \leq 2. \tag{4.8}$$

We have also the following relation satisfied by the y, x_3 -average of u :

$$\int_{\omega} \sum_{i=1}^2 \frac{\partial \theta}{\partial x_i} \left(\int_Y \int_0^{h_M} u_i dy dx_3 \right) dx = \int_{\partial \omega} \tilde{g}_i n_i \theta d\sigma \quad \text{for any } \theta \in H^1(\omega), \tag{4.9}$$

where

$$\tilde{g}_i(x) = \int_0^{h_m} g dx_3.$$

Proof. (4.3)-(4.4) follow from the estimate (3.4) using the two-scale convergence results [15], [16] as in [9]. For (4.5), note that u^ε is divergence free in Ω_ε , so

$$\frac{\partial \bar{u}_1^\varepsilon}{\partial x_1} + \frac{\partial \bar{u}_2^\varepsilon}{\partial x_2} + \frac{1}{\lambda \varepsilon} \frac{\partial \bar{u}_3^\varepsilon}{\partial x_3} = 0.$$

Multiplying by ε , passing to the two-scale limit in Ω_M and integrating on y_3 we easily obtain (4.5).

For (4.6), choosing φ in $C^\infty(\Omega_M, C^\infty_{\#}(Y))$ with $\varphi = 0$ for $x_3 \leq h(x, y)$, we get

$$\int_{\Omega_M} \bar{u}_i^\varepsilon \varphi(x, \frac{x}{\varepsilon}, x_3) dx dx_3 = 0.$$

Passing now to the two-scale limit for $i = 1, 2, 3$, we obtain

$$\int_{\Omega_M} \int_Y \bar{u}_i \varphi(x, y, x_3) dy dx dx_3 = 0.$$

So $\bar{u}_i = 0$ for $x_3 \geq h(x, y)$, for $i = 1, 2, 3$.

From (2.9), (2.11), and for all φ in $L^2(\omega; \mathcal{C}_\#(Y))$ we have

$$\int_{\Omega_M} \frac{\partial \bar{u}_3^\varepsilon}{\partial x_3} \varphi(x, \frac{x}{\varepsilon}) dx dx_3 = 0.$$

Passing to the two-scale limit we obtain

$$\int_{\Omega_M} \int_{[0,1]^3} \left(\frac{\partial u_3}{\partial x_3} + \frac{\partial v_3}{\partial y_3} \right) \varphi(x, y) dy dy_3 dx dx_3 = 0.$$

As

$$\int_{[0,1]^3} \frac{\partial v_3}{\partial y_3} \varphi(x, y) dy dy_3 = 0,$$

and using Green's formula we deduce

$$\int_{\omega} \int_Y u_3(x, y, 0) \varphi(x, y) dy dx = 0,$$

which proves (4.7).

In order to prove (4.8) let us remark from (3.4) that $u_i^\varepsilon(x, 0)$ is bounded in $L^2(\omega)$. Then there exists $\xi_i \in L^2(\omega \times Y)$ such that

$$u_i^\varepsilon(x, 0) \rightharpoonup \xi_i(x, 0) \quad \text{in } \omega.$$

Let us identify ξ . Passing to the two-scale limit in the obvious identity

$$\int_{\Omega_M} \frac{\partial u_i^\varepsilon}{\partial x_3} \varphi(x, \frac{x}{\varepsilon}) dx dx_3 = - \int_{\omega} u_i^\varepsilon(x, 0) \varphi(x, \frac{x}{\varepsilon}) dx \quad \text{for any } \varphi \in L^2(\omega; \mathcal{C}_\#(Y))$$

we obtain

$$\int_{\Omega_M} \int_Y \int_0^1 \left(\frac{\partial u_i}{\partial x_3} + \frac{\partial v_i}{\partial y_3} \right) \varphi(x, y) dy dy_3 dx dx_3 = - \int_{\omega} \int_Y \xi_i(x, y) \varphi(x, y) dy dx.$$

The term in v_i vanishes, so by Green's identity we find

$$\int_{\omega} \int_Y u_i(x, y, 0) \varphi(x, y) dy dx = \int_{\omega} \int_Y \xi_i(x, y) \varphi(x, y) dy dx$$

for an arbitrary φ , so $\xi_i(x, y) = u_i(x, y, 0)$ for $1 \leq i \leq 3$, which ends the proof of (4.8).

To prove (4.9) we write

$$\int_{\Omega_M} \left(\sum_{i=1}^2 \frac{\partial \bar{u}_i^\varepsilon}{\partial x_i} + \frac{1}{\lambda \varepsilon} \frac{\partial \bar{u}_3^\varepsilon}{\partial x_3} \right) \theta(x) dx dx_3 = 0 \quad \forall \theta \in H^1(\omega),$$

so by Green's formula we have

$$\int_{\Omega_M} \left(\sum_{i=1}^2 \bar{u}_i^\varepsilon \right) \frac{\partial \theta}{\partial x_i} dx dx_3 = \int_{\omega} \int_0^{h_M} \left(\sum_{i=1}^2 g_i n_i \right) \theta(x) dx dx_3.$$

Passing to the two-scale limit we obtain (4.9). □

5. A first relation between p and u . In the following we need to transform the domain B_x into $Y \times]0, 1[$ using the scaling

$$\xi = y, \quad \xi_3 = \frac{x_3}{h(x, \xi)}.$$

For any function ψ defined on B_x we define $\hat{\psi}$ defined on $Y \times]0, 1[$ by

$$\hat{\psi}(\xi, \xi_3) = \psi(y, x_3).$$

We have

$$\frac{D(y_1, y_2, x_3)}{D(\xi_1, \xi_2, \xi_3)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \xi_3 \frac{\partial h}{\partial \xi_1} & \xi_3 \frac{\partial h}{\partial \xi_2} & h \end{pmatrix} = A = A(x, \xi, \xi_3),$$

$$\frac{D(\xi_1, \xi_2, \xi_3)}{D(y_1, y_2, x_3)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{\xi_3}{h} \frac{\partial h}{\partial \xi_1} & -\frac{\xi_3}{h} \frac{\partial h}{\partial \xi_2} & \frac{1}{h} \end{pmatrix} = B = B((x, \xi, \xi_3)).$$

We will also use the following notation: $\mathcal{V} = \{v \in (C^1(\bar{Q}))^3 \text{ such that } \frac{\partial v_1}{\partial y_1} + \frac{\partial v_2}{\partial y_2} + \frac{1}{\lambda} \frac{\partial v_3}{\partial x_3} = 0, \text{ } v \text{ is } y\text{-periodic, and } v = 0 \text{ for } x_3 = 0 \text{ or } x_3 = h(x, y)\}$. $\mathcal{V}^0 = \{v \in \mathcal{V} : v = 0 \text{ for } x \in \partial\omega\}$.

For any $x \in \omega$ we set $\mathcal{V}_x = \{v \in (C^1(\bar{B}_x))^3 \text{ such that } \frac{\partial v_1}{\partial y_1} + \frac{\partial v_2}{\partial y_2} + \frac{1}{\lambda} \frac{\partial v_3}{\partial x_3} = 0, \text{ } v \text{ is } y\text{-periodic, } v = 0 \text{ for } x_3 = 0 \text{ or } x_3 = h(x, y)\}$. $V_{0,x} = \{v \in (H_{0\#}^1(B_x))^3 : \frac{\partial v_1}{\partial y_1} + \frac{\partial v_2}{\partial y_2} + \frac{1}{\lambda} \frac{\partial v_3}{\partial x_3} = 0\}$.

We also define the matrices

$$A_\lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda \xi_3 \frac{\partial h}{\partial \xi_1} & \lambda \xi_3 \frac{\partial h}{\partial \xi_2} & \lambda h \end{pmatrix}, \quad B_\lambda = A_\lambda^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{\xi_3}{h} \frac{\partial h}{\partial \xi_1} & -\frac{\xi_3}{h} \frac{\partial h}{\partial \xi_2} & \frac{1}{\lambda h} \end{pmatrix}.$$

We have the following technical lemmas.

LEMMA 5.1. For any $x \in \omega$ and any $\psi \in (L^2(B_x))^3$ the following two assertions are equivalent:

$$\frac{\partial \psi_1}{\partial y_1} + \frac{\partial \psi_2}{\partial y_2} + \frac{1}{\lambda} \frac{\partial \psi_3}{\partial x_3} = 0, \tag{5.1}$$

$$\operatorname{div}_{\xi, \xi_3}(h B_\lambda \hat{\psi}) = 0. \tag{5.2}$$

Proof. The relation (5.1) is equivalent to

$$\int_{B_x} \left(\sum_{k=1}^2 \psi_k \frac{\partial \varphi}{\partial y_k} + \frac{1}{\lambda} \psi_3 \frac{\partial \varphi}{\partial y_3} \right) dy_1 dy_2 dx_3 = 0 \quad \forall \psi \in \mathcal{D}(B_x).$$

Passing to $Y \times [0, 1]$ this gives

$$\int_{Y \times [0, 1]} \hat{\psi}^T \cdot B_\lambda^T \cdot \nabla_{\xi, \xi_3} \hat{\varphi} h d\xi d\xi_3 = 0,$$

which is equivalent to (5.2). □

LEMMA 5.2. For any $x_0 \in \omega$ and any $v \in \mathcal{V}_{x_0}$ there exists $\varphi \in \mathcal{V}$ such that $\varphi(x_0, y, x_3) = v(y, x_3)$.

Proof. We set $\hat{v}(\xi, \xi_3) = v(\xi, h(x_0, \xi)\xi_3)$ defined in $Y \times [0, 1]$. We choose $\hat{\varphi}(x, \xi, \xi_3)$ defined in $\omega \times Y \times [0, 1]$ such that

$$h(x, \xi)B_\lambda(x, \xi, \xi_3) \hat{\varphi} = h(x_0, \xi)B_\lambda(x_0, \xi, \xi_3) \hat{v}, \tag{5.3}$$

that is,

$$\hat{\varphi}(x, \xi, \xi_3) = \frac{h(x_0, \xi)}{h(x, \xi)} A_\lambda(x, \xi, \xi_3) B_\lambda(x_0, \xi, \xi_3) \hat{v}(\xi, \xi_3).$$

Finally we set

$$\varphi(x, y, x_3) = \hat{\varphi}(x, y, \frac{x_3}{h(x, y)}).$$

Since by hypothesis

$$\frac{\partial v_1}{\partial y_1} + \frac{\partial v_2}{\partial y_2} + \frac{1}{\lambda} \frac{\partial v_3}{\partial x_3} = 0,$$

this implies by Lemma 5.1 that

$$\operatorname{div}_{\xi, \xi_3} (h(x_0, \xi)B_\lambda(x_0, \xi, \xi_3) \hat{v}) = 0,$$

so by (5.3) and again Lemma 5.1,

$$\frac{\partial \varphi_1}{\partial y_1} + \frac{\partial \varphi_2}{\partial y_2} + \frac{1}{\lambda} \frac{\partial \varphi_3}{\partial x_3} = 0.$$

The remaining conditions on φ are obvious. □

We now have a first relation between p and u .

THEOREM 5.3. The limit pressure p belongs to $H^1(\omega)$ and the following relation yields a.e. x in ω ,

$$\begin{aligned} 2\mu \sum_{i,j=1}^3 \int_{B_x} D_{i,j}^B(u) D_{i,j}^B(\psi) dy dx_3 &= - \sum_{i=1}^2 \int_{B_x} \psi_i \frac{\partial p}{\partial x_i} dy dx_3 \\ &+ \sum_{i=1}^3 \int_{B_x} f_i \psi_i dy dx_3 \quad \forall \psi \in V_{0,x}, \end{aligned} \tag{5.4}$$

where for $1 \leq i, j \leq 2$,

$$D_{i,j}^B(u) = \frac{1}{2} \left(\frac{\partial u_i}{\partial y_j} + \frac{\partial u_j}{\partial y_i} \right), \quad D_{i,3}^B(u) = \frac{1}{2} \left(\frac{1}{\lambda} \frac{\partial u_i}{\partial x_3} + \frac{\partial u_3}{\partial y_i} \right), \quad D_{3,3}^B(u) = \frac{1}{\lambda} \frac{\partial u_3}{\partial x_3}.$$

Proof. We proceed as in [9]. From (3.12) passing to Ω_ε we obtain, dividing by $\lambda\varepsilon$,

$$\begin{aligned} \int_{\Omega_M} p^\varepsilon \left(\sum_{i=1}^2 \frac{\partial \bar{\psi}_i^\varepsilon}{\partial x_i} + \frac{1}{\lambda\varepsilon} \frac{\partial \bar{\psi}_3^\varepsilon}{\partial x_3} \right) &= \frac{\mu}{2} \sum_{i,j=1}^2 \int_{\Omega_M} \left(\frac{\partial \bar{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \bar{u}_j^\varepsilon}{\partial x_i} \right) \left(\frac{\partial \bar{\psi}_i^\varepsilon}{\partial x_j} + \frac{\partial \bar{\psi}_j^\varepsilon}{\partial x_i} \right) \\ &+ \sum_{i=1}^2 \mu \int_{\Omega_M} \left(\frac{1}{\lambda\varepsilon} \frac{\partial \bar{u}_i^\varepsilon}{\partial x_3} + \frac{\partial \bar{u}_3^\varepsilon}{\partial x_i} \right) \left(\frac{1}{\lambda\varepsilon} \frac{\partial \bar{\psi}_i^\varepsilon}{\partial x_3} + \frac{\partial \bar{\psi}_3^\varepsilon}{\partial x_i} \right) + \frac{2\mu}{\lambda^2\varepsilon^2} \int_{\Omega_M} \frac{\partial \bar{u}_3^\varepsilon}{\partial x_3} \frac{\partial \bar{\psi}_3^\varepsilon}{\partial x_3} \\ &- \frac{1}{\varepsilon^2} \int_{\Omega_M} f \cdot \bar{\psi}^\varepsilon \quad \forall \bar{\psi}^\varepsilon \in (H_0^1(\Omega_\varepsilon))^3. \end{aligned} \tag{5.5}$$

We now take in (5.5) $\psi^\varepsilon(x, x_3) = \psi(x, \frac{x}{\varepsilon}, x_3)$ with $\psi \in \mathcal{V}^0$ and obtain

$$\begin{aligned} \int_{\Omega_M} \bar{p}^\varepsilon \sum_{i=1}^2 \frac{\partial \psi_i}{\partial x_i} &= \frac{\mu}{2} \sum_{i,j=1}^2 \int_{\Omega_M} \left[\frac{\partial \bar{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \bar{u}_j^\varepsilon}{\partial x_i} \right] \left[\frac{\partial \psi_i}{\partial x_j} + \frac{1}{\varepsilon} \frac{\partial \psi_i}{\partial y_j} + \frac{\partial \psi_j}{\partial x_i} + \frac{1}{\varepsilon} \frac{\partial \psi_j}{\partial y_i} \right] \\ &+ \mu \sum_{i=1}^2 \int_{\Omega_M} \left[\frac{1}{\lambda \varepsilon} \frac{\partial \bar{u}_i^\varepsilon}{\partial x_3} + \frac{\partial \bar{u}_3^\varepsilon}{\partial x_i} \right] \left[\frac{1}{\lambda \varepsilon} \frac{\partial \psi_i}{\partial x_3} + \frac{\partial \psi_3}{\partial x_i} + \frac{1}{\varepsilon} \frac{\partial \psi_3}{\partial y_i} \right] + \frac{2\mu}{\lambda^2 \varepsilon^2} \int_{\Omega_M} \frac{\partial \bar{u}_3^\varepsilon}{\partial x_3} \frac{\partial \psi_3}{\partial x_3} \\ &- \frac{1}{\varepsilon^2} \int_{\Omega_M} f \cdot \bar{\psi}^\varepsilon. \end{aligned}$$

Passing to the two-scale limit in Ω_M , after multiplying by ε^2 , and using Theorems 4.1 and 4.2, we deduce

$$\begin{aligned} \int_Q p(x) \left(\sum_{i=1}^2 \frac{\partial \psi_i}{\partial x_i} \right) &= \frac{\mu}{2} \sum_{i,j=1}^2 \int_Q \left(\frac{\partial u_i}{\partial y_j} + \frac{\partial u_j}{\partial y_i} \right) \left(\frac{\partial \psi_i}{\partial y_j} + \frac{\partial \psi_j}{\partial y_i} \right) \\ &+ \mu \sum_{i=1}^2 \int_Q \left(\frac{1}{\lambda} \frac{\partial u_i}{\partial x_3} + \frac{1}{\lambda} \frac{\partial v_i}{\partial y_3} + \frac{\partial u_3}{\partial y_i} \right) \left(\frac{1}{\lambda} \frac{\partial \psi_i}{\partial x_3} + \frac{\partial \psi_3}{\partial y_i} \right) \\ &+ \frac{2\mu}{\lambda^2} \int_Q \left(\frac{\partial u_3}{\partial x_3} + \frac{\partial v_3}{\partial y_3} \right) \frac{\partial \psi_3}{\partial x_3} - \int_Q f \cdot \psi. \end{aligned}$$

As ψ, u do not depend on y_3 , and v_i is y_3 -periodic, we deduce

$$\begin{aligned} \int_Q p(x) \sum_{i=1}^2 \frac{\partial \psi_i}{\partial x_i} dy dx dx_3 &= 2\mu \sum_{i,j=1}^3 \int_Q D_{i,j}^B(u) D_{i,j}^B(\psi) dy dx dx_3 \\ &- \int_Q f \cdot \psi dy dx dx_3 \quad \forall \psi \in \mathcal{V}^0. \end{aligned} \tag{5.6}$$

Now we prove exactly as in [9], taking $\psi = (\theta(x)\bar{\xi}, 0, 0)$ as a test function in (5.6) with θ arbitrary in $\mathcal{D}(\omega)$ and $\bar{\xi}$ the extension on B_x of a function depending only on y_2 and x_3 defined on $[0, 1] \times [0, h_m]$ with nonzero average, that $\frac{\partial p}{\partial x_1} \in L^2(\omega)$ and the same holds for $\frac{\partial p}{\partial x_2} \in L^2(\omega)$. Then p is in $H^1(\omega)$.

Now taking in (5.6) $\psi(x, y, x_3) = \theta(x)\varphi(x, y, x_3)$ with θ in $\mathcal{D}(\omega)$ and φ in \mathcal{V} arbitrary, and using Green's formula in the left-hand side of (5.6), we obtain

$$\begin{aligned} - \int_{B_x} \sum_{i=1}^2 \frac{\partial p}{\partial x_i} \varphi_i dy dx x_3 &= -2\mu \sum_{i,j=1}^3 \int_{B_x} D_{i,j}^B(u) D_{i,j}^B(\varphi) dy dx x_3 \\ &- \sum_{i=1}^3 \int_{B_x} f_i \varphi_i dy dx x_3 \quad \text{a.e. } x \in \omega. \end{aligned} \tag{5.7}$$

(Remark that $D_{i,j}^B u(x, \cdot) \in L^2(B_x)$ a.e. $x \in \omega$ by Fubini's theorem.) Using Lemma 5.2 we deduce that (5.7) is true for any $\varphi \in \mathcal{V}_x$. Now using the density of \mathcal{V}_x in $V_{0,x}$ we have the expected result. \square

6. The variational inequality between p and u . In the following we propose to obtain another variational relation between p and u , more general than in Theorem 5.3. Now the test functions will not vanish on $x_3 = 0$ (except for the third component) and the equality becomes a variational inequality. We first need to introduce the following spaces: $\mathcal{C}_{0\#}^1(\overline{\omega \times Y}) = \{v \in \mathcal{C}^1(\overline{\omega \times Y}) \text{ with } v(x, \cdot) = 0 \text{ for } x \in \partial\omega \text{ and } v(x, \cdot) \text{ } y\text{-periodic on } Y\}$, which is a Banach space as a closed subspace of $\mathcal{C}^1(\overline{\omega \times Y})$.

We then introduce $\mathcal{U} = \{v \in (L^2(]0, h_M[; \mathcal{C}_{0\#}^1(\overline{\omega \times Y})))^3 \text{ with } v = 0 \text{ for } x_3 \geq h(x, y), v_3 = 0 \text{ for } x_3 = 0, \frac{\partial v}{\partial x_3} \in (L^2(]0, h_M[; \mathcal{C}_{0\#}^1(\overline{\omega \times Y})))^3, \text{ and } \frac{\partial v_1}{\partial y_1} + \frac{\partial v_2}{\partial y_2} + \frac{1}{\lambda} \frac{\partial v_3}{\partial x_3} = 0\}$.

We need also the set $U = \{v \in (L^2(Q_M))^3 : v = 0 \text{ for } x_3 \geq h(x, y); v_3 = 0 \text{ for } x_3 = 0, \frac{\partial v}{\partial y_i} \in (L^2(Q_M))^3, \quad i = 1, 2; \frac{\partial v}{\partial x_3} \in (L^2(Q_M))^3; v \text{ is } y\text{-periodic, and } \frac{\partial v_1}{\partial y_1} + \frac{\partial v_2}{\partial y_2} + \frac{1}{\lambda} \frac{\partial v_3}{\partial x_3} = 0\}$.

U is a Hilbert space endowed with the norm

$$\|v\|_U^2 = \left\| \frac{\partial v}{\partial x_3} \right\|_{(L^2(Q_M))^3}^2 + \sum_{i=1}^2 \left\| \frac{\partial v}{\partial y_i} \right\|_{(L^2(Q_M))^3}^2.$$

It is clear from Theorem 4.2 that $u \in U$. We have the following density result.

PROPOSITION 6.1. \mathcal{U} is dense in U for the norm of U .

Proof. We can suppose that 0 is in the interior of ω . We set for any $\sigma > 1$, σ near 1 ,

$$Q_{m\sigma} = \{(x, y, x_3) : \sigma x \in \omega; \quad \sigma x_3 \in]0, h_M[; \quad y \in Y\} \subset Q_M.$$

For any $v \in U$ we set $v^\sigma = (v_1^\sigma, v_2^\sigma, v_3^\sigma)$ such that, for $k = 1, 2$,

$$v_k^\sigma(x, y, x_3) = \begin{cases} v_k(\sigma x, y, \sigma x_3) & \text{for } (x, y, x_3) \in Q_{m\sigma}, \\ 0 & \text{for } (x, y, x_3) \in Q_M - Q_{m\sigma}, \end{cases}$$

$$v_3^\sigma(x, y, x_3) = \begin{cases} \frac{1}{\sigma} v_3(\sigma x, y, \sigma x_3) & \text{for } (x, y, x_3) \in Q_{m\sigma}, \\ 0 & \text{for } (x, y, x_3) \in Q_M - Q_{m\sigma}. \end{cases}$$

It is clear that $v^\sigma \in U$ and $v^\sigma \rightarrow v$ in U for $\sigma \rightarrow 1$. We extend v^σ by y -periodicity on $\omega \times]-1, 2[\times]0, h_M[$. Then we extend it by 0 on $\mathbb{R}^4 \times]0, h_M[$. We then introduce, for any $\delta > 0$ small enough, the function $v^{\sigma\delta} = v^\sigma \star_{x,y} \rho_\delta$ (defined by convolution in x, y only) where $\rho_\delta \in \mathcal{D}(\mathbb{R}^4)$ is a sequence of mollifier functions. We have

$$v^{\sigma\delta}(x, y, x_3) = \int_{\mathbb{R}^4} v^\sigma(x-t, y-z, x_3) \rho_\delta(t, z) dt dz$$

$$= \int_{\mathbb{R}^4} v^\sigma(t, z, x_3) \rho_\delta(x-t, y-z) dt dz.$$

For $\sigma > 1$ we have that $v^{\sigma\delta}$ is an element of \mathcal{U} for δ small enough. (Remark that

$$\frac{\partial v^{\sigma\delta}}{\partial y_k} = \frac{\partial v^\sigma}{\partial y_k} \star_{x,y} \rho_\delta \quad \text{for } k = 1, 2 \quad \text{and} \quad \frac{\partial v^{\sigma\delta}}{\partial x_3} = \frac{\partial v^\sigma}{\partial x_3} \star_{x,y} \rho_\delta.$$

We also prove classically that for fixed $\sigma > 1$ we have $v^{\sigma\delta} \rightarrow v^\sigma$ in U for $\delta \rightarrow 0$. This ends the proof. □

We can now obtain the following variational inequality.

THEOREM 6.1. The two-scale limits p and u satisfy the following relation:

$$\begin{aligned} & 2\mu \sum_{i,j=1}^3 \int_Q D_{i,j}^B(u) D_{i,j}^B(\varphi) dy dx dx_3 + j_0(u + \varphi) - j_0(u) \\ & + \sum_{i=1}^2 \int_Q \varphi_i \frac{\partial p}{\partial x_i} dy dx dx_3 \geq \int_Q f \cdot \varphi dy dx dx_3 \quad \forall \varphi \in \mathcal{U}, \end{aligned} \quad (6.1)$$

where

$$j_0(\psi) = \frac{1}{\lambda} \int_{\omega} \int_Y k(x) |\underline{\psi}(x, y, 0) - s^0| dy dx \quad \forall \psi \in \mathcal{U}.$$

Proof. Taking φ^ε as a test function in (3.2) such that

$$\varphi^\varepsilon(x, x_3) = (G_1, G_2, \lambda \varepsilon G_3)(x, x_3) + (\psi_1, \psi_2, \psi_3)\left(x, \frac{x}{\varepsilon}, x_3\right)$$

with $\psi \in \mathcal{U}$, and using the fact that $\operatorname{div}_{x,x_3} G = 0$,

$$\frac{\partial \psi_1}{\partial y_1} + \frac{\partial \psi_2}{\partial y_2} + \frac{1}{\lambda} \frac{\partial \psi_3}{\partial x_3} = 0, \quad \text{and} \quad G_i(x, 0) = s_i^0,$$

we obtain

$$\begin{aligned} & \frac{\mu}{2} \sum_{i,j=1}^2 \int_{\Omega_M} \left(\frac{\partial \bar{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \bar{u}_j^\varepsilon}{\partial x_i} \right) \left(\frac{\partial \psi_i}{\partial x_j} + \frac{1}{\varepsilon} \frac{\partial \psi_i}{\partial y_j} + \frac{\partial \psi_j}{\partial x_i} + \frac{1}{\varepsilon} \frac{\partial \psi_j}{\partial y_i} \right) \\ & + \frac{\mu}{2} \sum_{i=1}^2 \int_{\Omega_M} \left(\frac{\partial \bar{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \bar{u}_j^\varepsilon}{\partial x_i} \right) \left(\frac{\partial G_i}{\partial x_j} + \frac{\partial G_j}{\partial x_i} \right) \\ & + \mu \sum_{i=1}^2 \int_{\Omega_M} \left(\frac{1}{\lambda \varepsilon} \frac{\partial \bar{u}_i^\varepsilon}{\partial x_3} + \frac{\partial \bar{u}_3^\varepsilon}{\partial x_i} \right) \left(\frac{1}{\lambda \varepsilon} \frac{\partial \psi_i}{\partial x_3} + \frac{\partial \psi_3}{\partial x_i} + \frac{1}{\varepsilon} \frac{\partial \psi_3}{\partial y_i} \right) \\ & + \mu \sum_{i=1}^2 \int_{\Omega_M} \left(\frac{1}{\lambda \varepsilon} \frac{\partial \bar{u}_i^\varepsilon}{\partial x_3} + \frac{\partial \bar{u}_3^\varepsilon}{\partial x_i} \right) \left(\frac{1}{\lambda \varepsilon} \frac{\partial G_i}{\partial x_3} + \lambda \varepsilon \frac{\partial G_3}{\partial x_i} \right) \\ & + \frac{2\mu}{\lambda^2 \varepsilon^2} \int_{\Omega_M} \frac{\partial \bar{u}_3^\varepsilon}{\partial x_3} \frac{\partial \psi_3}{\partial x_3} + \frac{2\mu}{\lambda \varepsilon} \int_{\Omega_M} \frac{\partial \bar{u}_3^\varepsilon}{\partial x_3} \frac{\partial G_3}{\partial x_3} - \int_{\Omega_M} p^\varepsilon \sum_{i=1}^2 \frac{\partial \psi_i}{\partial x_i} \\ & + \frac{1}{\lambda \varepsilon^2} \int_{\omega} k(x) |\underline{\psi}(x, \frac{x}{\varepsilon}, 0)| \geq \frac{\mu}{2} \sum_{i=1}^2 \int_{\Omega_M} \left(\frac{\partial \bar{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \bar{u}_j^\varepsilon}{\partial x_i} \right)^2 + \frac{2\mu}{\lambda^2 \varepsilon^2} \int_{\Omega_M} \left(\frac{\partial \bar{u}_3^\varepsilon}{\partial x_3} \right)^2 \\ & + \mu \int_{\Omega_M} \left(\frac{1}{\lambda \varepsilon} \frac{\partial \bar{u}_i^\varepsilon}{\partial x_3} + \frac{\partial \bar{u}_i^\varepsilon}{\partial x_i} \right)^2 + \frac{1}{\lambda \varepsilon^2} \int_{\omega} k(x) |\underline{u}^\varepsilon(x, 0) - s^0| dx \\ & + \frac{1}{\varepsilon^2} \int_{\Omega_M} f \cdot (\psi - \bar{u}^\varepsilon) + \frac{1}{\varepsilon^2} \int_{\Omega_M} f \cdot (G_1, G_2, \lambda \varepsilon G_3)^T. \end{aligned} \quad (6.2)$$

We multiply by ε^2 and we pass to the limit in ε using Theorems 4.1, 4.2 and the fact that $p \in H^1(\omega)$. From the lower semicontinuity result, Theorem 8.5 and the fact that the application $(x, z) \in \omega \times \mathbb{R}^2 \mapsto k(x)|z - s^0| \in \mathbb{R}$ satisfies the hypotheses (H1) – (H5)

of Section 8, we obtain

$$\liminf_{\varepsilon \rightarrow 0} \int_{\omega} k(x)|\underline{u}^\varepsilon(x, 0) - s^0|dx \geq \int_{\omega} \int_Y k(x)|\underline{u}(x, y, 0) - s^0|dydx. \tag{6.3}$$

We then have

$$\begin{aligned} & 2\mu \sum_{i,j=1}^3 \int_Q D_{i,j}^B(u)D_{i,j}^B(\psi) + \frac{2\mu}{\lambda} \sum_{i=1}^2 \int_Q D_{i,3}^B(u) \frac{\partial G_i}{\partial x_3} + \sum_{i=1}^2 \int_Q \psi_i \frac{\partial p}{\partial x_i} \\ & + \frac{1}{\lambda} \int_{\omega} \int_Y k|\underline{\psi}(x, y, 0)| \geq 2\mu \sum_{i,j=1}^3 \int_Q (D_{i,j}^B(u))^2 + \frac{1}{\lambda} \int_{\omega} \int_Y k|\underline{u}(x, y, 0) - s^0| \tag{6.4} \\ & \quad + \int_Q f \cdot (\psi - u + (G_1, G_2, 0)^T) \quad \forall \psi \in \mathcal{U}. \end{aligned}$$

By the density of \mathcal{U} in U (Proposition 6.1) we deduce that the above variational inequality is satisfied for any ψ in U . Then we take in (6.4) $\psi = u + \varphi - (G_1, G_2, 0)^T$ with φ an arbitrary element of U and we obtain the desired result, provided that we prove

$$\int_Q \sum_{i=1}^2 u_i \frac{\partial p}{\partial x_i} = \int_Q \sum_{i=1}^2 G_i \frac{\partial p}{\partial x_i}. \tag{6.5}$$

From (4.9) we have

$$\int_Q \sum_{i=1}^2 u_i \frac{\partial p}{\partial x_i} = \int_{\partial\omega} p \sum_{i=1}^2 \tilde{g}_i n_i.$$

On the other hand, using Green’s formula we can write

$$\int_Q p \sum_{i=1}^2 \frac{\partial G_i}{\partial x_i} = - \int_Q \sum_{i=1}^2 G_i \frac{\partial p}{\partial x_i} + \int_{\partial\omega} \int_Y p \sum_{i=1}^2 g_i n_i.$$

Taking into account that $\operatorname{div}_{x,x_3} G = 0$ we deduce

$$\int_Q \sum_{i=1}^2 G_i \frac{\partial p}{\partial x_i} = \int_{\partial\omega} \int_Y p \sum_{i=1}^2 \tilde{g}_i n_i,$$

which gives (6.5). □

REMARK 6.2. A direct method to obtain the inequality (6.1) would be to take, as a test function, $\varphi^\varepsilon = \psi(x, \frac{x}{\varepsilon}, x_3) + u^\varepsilon$ in (3.2) with arbitrary ψ . But if we do that we obtain in the left side of (6.2),

$$\frac{1}{\lambda\varepsilon^2} \int_{\omega} k(x)|\underline{u}^\varepsilon(x, 0) + \underline{\psi}(x, \frac{x}{\varepsilon}, 0) - s^0|dx$$

instead of

$$\frac{1}{\lambda\varepsilon^2} \int_{\omega} k(x)|\underline{\psi}(x, \frac{x}{\varepsilon}, 0)|dx.$$

So, using the two-scale limit, we need to obtain

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{\omega} k(x)|\underline{u}^\varepsilon(x, 0) + \underline{\psi}(x, \frac{x}{\varepsilon}, 0) - s^0|dx \\ & = \int_{\omega} \int_Y k(x)|\underline{u}(x, y, 0) + \underline{\psi}(x, y, 0) - s^0|dx \end{aligned}$$

and a strong convergence result such as

$$\|\underline{u}^\varepsilon(x, 0) - \underline{u}(x, \frac{x}{\varepsilon}, 0)\|_{L^2(\omega \times Y)} \rightarrow 0$$

would be necessary. But such a result seems very difficult to prove since the function $y \mapsto u(x, y, 0)$ is not continuous. Then

$$(x, y) \mapsto k(x)|\underline{u}(x, y, 0) + \underline{\psi}(x, y, 0) - s^0|$$

is not a Carathéodory function.

7. The limit problem. We shall denote $s^*(x, y) = u(x, y, 0)$, which is one of the unknowns of our limit problem. For any $x \in \omega$ we recall the following space:

$$V_{0,x} = \{\varphi \in (H_{0\#}^1(B_x))^3 \quad \text{with} \quad \frac{\partial \varphi_1}{\partial y_1} + \frac{\partial \varphi_2}{\partial y_2} + \frac{1}{\lambda} \frac{\partial \varphi_3}{\partial x_3} = 0\}$$

and introduce

$$V_{1,x} = \{\varphi \in (H_{1\#}^1(B_x))^3 \quad \text{with} \quad \frac{\partial \varphi_1}{\partial y_1} + \frac{\partial \varphi_2}{\partial y_2} + \frac{1}{\lambda} \frac{\partial \varphi_3}{\partial x_3} = 0\}.$$

We define the following local problems:

(L_k) : Find $w^k \in V_{0,x}$ (for $k = 1, 2$) such that

$$2\mu \sum_{i,j=1}^3 \int_{B_x} D_{ij}^B(w^k) D_{ij}^B(\varphi) dy dx_3 = - \int_{B_x} \varphi_k dy dx_3 \quad \forall \varphi \in V_{0,x}. \tag{7.1}$$

(L_3) : Find $w^* \in V_{0,x}$ such that

$$2\mu \sum_{i,j=1}^3 \int_{B_x} D_{ij}^B(w^*) D_{ij}^B(\varphi) dy dx_3 = \int_{B_x} f \cdot \varphi dy dx_3 \quad \forall \varphi \in V_{0,x}. \tag{7.2}$$

On the other hand, we introduce the following space of traces on $x_3 = 0$: $S_x = \{\theta \in (L^2(Y))^2 : \exists \varphi \in V_{1,x}$ such that $\varphi_k(y, 0) = \theta_k(y), k = 1, 2, \varphi_3(y, 0) = 0\}$ and $S = \{\tilde{\theta} \in (L^2(\omega \times Y))^2 : \exists \tilde{\varphi} \in U$ such that $\tilde{\varphi}_k(x, y, 0) = \tilde{\theta}_k(x, y), k = 1, 2\}$.

It is clear that $s^*(x, \cdot) \in S_x$ a.e. $x \in \omega$ and $s^* \in S$. For any $s \in S$ we define the problem

(L) : Find $w \in V_{1,x}$ such that $w = (s(x, \cdot), 0)$ for $x_3 = 0$ and

$$2\mu \sum_{i,j=1}^3 \int_{B_x} D_{ij}^B(w) D_{ij}^B(\varphi) dy dx_3 = 0 \quad \forall \varphi \in V_{0,x}. \tag{7.3}$$

It is clear from the Korn inequality that $(L_k)_{k=1,2,3}$ and (L) are well-posed problems. We denote in the following by \mathcal{R} the linear operator from S to U such that $\mathcal{R}(s) = w$, where $w(x, \cdot)$ is the solution of the problem (L) a.e. x in ω . We introduce the following scalar product on S :

$$(s, \tau)_S = (\mathcal{R}(s), \mathcal{R}(\tau))_U. \tag{7.4}$$

We have

LEMMA 7.1. S endowed with the scalar product (7.4) is a Hilbert space.

Proof. Since \mathcal{R} is injective from S to U (since $\text{Ker}(\mathcal{R}) = \{0\}$) it suffices to prove that $\mathcal{R}(s)$ is closed in U . Let $w^k = \mathcal{R}(s^k)$ be a sequence in $\mathcal{R}(S)$ such that $w^k \rightarrow w$ in U . Then for any $\varphi \in \mathcal{V}$ we have (7.3) with w replaced by w^k . Multiplying by an arbitrary $\theta \in \mathcal{D}(\omega)$, integrating in ω and passing to the limit in k we obtain

$$\sum_{i,j=1}^3 \int_{\omega} \int_{B_x} D_{ij}^B(w) D_{ij}^B(\varphi) dy dx_3 \theta(x) dx = 0,$$

which implies, since θ is arbitrary, that

$$\sum_{i,j=1}^3 \int_{B_x} D_{ij}^B(w) D_{ij}^B(\varphi) dy dx_3 = 0, \quad \text{a.e. } x \in \omega. \tag{7.5}$$

Now set $s(x, y) = \underline{w}(x, y, 0)$, so $s \in S$. On the other hand, the equality (7.5) is true for any $\varphi \in \mathcal{V}$. Using Lemma 5.2 and the density of \mathcal{V}_x in $V_{0,x}$ we deduce that $w = \mathcal{R}(s)$, which ends the proof. \square

REMARK 7.2. It is clear from Lemma 7.1 that $\mathcal{R} \in \mathcal{L}(S, U)$.

We have the following result whose proof is elementary.

PROPOSITION 7.1. There exists a constant $C > 0$ such that

$$\|w^k\|_{(H^1(B_x))^3} \leq C \quad \text{for } k = 1, 2, 3 \quad \text{a.e. } x \in \omega,$$

where w^k is the solution of the problem (L_k) .

We now have

PROPOSITION 7.2. The limit u can be written in the following manner: a.e. $x \in \omega$ we have

$$u = \sum_{k=1}^2 w^k \frac{\partial p}{\partial x_k} + w^3 + w^* \quad \text{a.e. } (y, x_3) \in B_x \tag{7.6}$$

with w^k solutions of the problem $(L_k)_{k=1,2,3}$ and $w^* = \mathcal{R}(s^*)$.

Moreover p satisfies the following homogenized Reynolds equation:

$$\int_{\omega} A \nabla p \cdot \nabla v = \int_{\omega} \sum_{j=1}^2 \left[\int_{B_x} (w_j^* + w_j^3) dy dx_3 \right] \frac{\partial v}{\partial x_j} dx - \int_{\partial\omega} \tilde{g} \cdot \nu v \quad \forall v \in H^1(\omega), \tag{7.7}$$

where A is the matrix defined by

$$A_{j,k} = - \int_{B_x} w_j^k dy dx_3 \quad j = 1, 2, \quad k = 1, 2.$$

Proof. The relation (7.6) is easily obtained from (7.1)-(7.3), (5.4) by linearity and uniqueness and (7.7) is an immediate consequence of (7.6) and (4.9). \square

We shall need to consider the equality (7.7) with the supplementary condition $\int_{\omega} p dx = 0$ (which is proved in Theorem 4.1) as a well-posed problem in p . Remark also that by hypothesis (2.6) we have $\int_{\partial\omega} \tilde{g} \cdot \nu = 0$. This will be a classical consequence of the following result.

LEMMA 7.3. The matrix A belongs to $(L^\infty(\omega))^4$ and is symmetric and positive definite uniformly in x ; that is, there exists a constant $\alpha > 0$ such that

$$\xi^T A \xi \geq \alpha \|\xi\|^2 \quad \forall \xi \in \mathbb{R}^2. \quad (7.8)$$

Proof. We proceed as in [9]. The fact that A belongs to $(L^\infty(\omega))^4$ is an immediate consequence of Proposition 7.1. Taking w^l as a test function in (L_k) ($k = 1, 2$), we obtain

$$A_{k,l} = 2\mu \sum_{i,j=1}^3 \int_{B_x} D_{ij}^B(w^k) D_{ij}^B(w^l),$$

which gives the symmetry of A . We also deduce

$$\sum_{k,l=1}^2 A_{k,l} \xi_k \xi_l = 2\mu \sum_{i,j=1}^2 \int_{B_x} |D_{ij}^B(\tilde{w})|^2 \quad \text{with } \tilde{w} = \xi_1 w_1 + \xi_2 w_2. \quad (7.9)$$

On the other hand, multiplying (L_k) by ξ_k and adding in k we get

$$2\mu \sum_{i,j=1}^3 \int_{B_x} D_{ij}^B(\tilde{w}) D_{ij}^B(\varphi) = - \int_{B_x} (\xi_1 \varphi_1 + \xi_2 \varphi_2) \quad \forall \varphi \in V_{0,x}. \quad (7.10)$$

For any fixed ξ we choose φ in the following manner:

$$\varphi(y, x_3) = (-\xi_1 x_3 (h_m - x_3), -\xi_2 x_3 (h_m - x_3), 0)$$

for $x_3 \in [0, h_m]$ and extended by 0 on B_x . It is clear that $\varphi \in V_{0,x}$ for any $x \in \omega$; then it can be taken as a test function in (7.10). We obtain, using also the Cauchy-Schwarz inequality,

$$\frac{h_m^3}{6} \|\xi\|^2 \leq 2\mu \sqrt{\sum_{i,j=1}^3 \int_{B_x} |D_{ij}^B(\tilde{w})|^2} \sqrt{\sum_{i,j=1}^3 \int_{B_x} |D_{ij}^B(\varphi)|^2}. \quad (7.11)$$

With the help of (7.9) we easily obtain (7.8). \square

For any $s \in S$ let us define q as the unique solution of the problem

(P_0) : Find $q \in H^1(\omega)/\mathbb{R}$ such that

$$\int_{\omega} A \nabla q \nabla \varphi = \int_{\omega} \sum_{j=1}^2 \left(\int_{B_x} w_j dy dx_3 \right) \frac{\partial \varphi}{\partial x_j} \quad \forall \varphi \in H^1(\omega)$$

with $w = \mathcal{R}(s)$ (the solution of (L)).

We denote by \mathcal{Q} the linear operator from U to $H^1(\omega)$ such that $\mathcal{Q}(w) = q$ is a solution of (P_0) for any $w \in U$. It is obvious that \mathcal{Q} is also continuous, that is,

$$\mathcal{Q} \in \mathcal{L}(U, H^1(\omega)). \quad (7.12)$$

On the other hand, we define p_1 as the unique solution of the problem

(P_1) : Find $p_1 \in H^1(\omega)/\mathbb{R}$ such that

$$\int_{\omega} A \nabla p_1 \cdot \nabla \varphi = \int_{\omega} \sum_{j=1}^2 \left(\int_{B_x} w_j^3 dy dx_3 \right) \frac{\partial \varphi}{\partial x_j} - \int_{\partial \omega} \tilde{g} \cdot \nu \varphi \quad \forall \varphi \in H^1(\omega)/\mathbb{R}. \quad (7.13)$$

By linearity and uniqueness we easily see that p can be expressed as a function of s^* in the following manner:

$$p = p_0 + p_1 \quad (7.14)$$

with $p_0 = \mathcal{Q} \circ \mathcal{R}(s^*)$ and p_1 the solution of (P_1) .

It is clear from (7.6) that u can also be expressed as a function of s^* :

$$u = \sum_{k=1}^2 w^k \frac{\partial p_0}{\partial x_k} + w^* + \sum_{k=1}^2 w^k \frac{\partial p_1}{\partial x_k} + w^3. \quad (7.15)$$

Now substituting (7.14) and (7.15) into (6.1) we obtain that s^* satisfies the following variational inequality:

$$\begin{aligned} & 2\mu \int_Q \sum_{i,j=1}^3 \left[\sum_{k=1}^2 D_{ij}^B(w^k) \frac{\partial p_0}{\partial x_k} + D_{ij}^B(w^*) \right] \cdot D_{ij}^B(\varphi) + \int_Q \sum_{j=1}^2 \frac{\partial p_0}{\partial x_j} \varphi_j \\ & + 2\mu \int_Q \sum_{i,j=1}^3 \left[\sum_{k=1}^2 D_{ij}^B(w^k) \frac{\partial p_1}{\partial x_k} + D_{ij}^B(w^3) \right] \cdot D_{ij}^B(\varphi) + \int_Q \sum_{j=1}^2 \frac{\partial p_1}{\partial x_j} \varphi_j \\ & + \frac{1}{\lambda} \int_{\omega} \int_Y k(x) [|\varphi(x, y, 0) + s^*(x, y) - s^0| - |s^*(x, y) - s^0|] \\ & - \int_Q f \cdot \varphi \geq 0 \quad \forall \varphi \in U. \end{aligned} \quad (7.16)$$

The first two integrals in the above inequality will be seen as a bilinear form while the three subsequent terms as a linear form. It is then natural to introduce the following bilinear form b and linear form T respectively:

$$\begin{aligned} b(s, \theta) &= 2\mu \int_Q \sum_{i,j=1}^3 \left[\sum_{k=1}^2 D_{ij}^B(w^k) \frac{\partial q}{\partial x_k} + D_{ij}^B(w) \right] \cdot D_{ij}^B(\psi) \\ & + \int_Q \sum_{j=1}^2 \frac{\partial q}{\partial x_j} \psi_j, \quad \forall (s, \theta) \in S \times S, \end{aligned} \quad (7.17)$$

$$\begin{aligned} T(\theta) &= 2\mu \int_Q \sum_{i,j=1}^3 \left[\sum_{k=1}^2 D_{ij}^B(w^k) \frac{\partial p_1}{\partial x_k} + D_{ij}^B(w^3) \right] \cdot D_{ij}^B(\psi) \\ & + \int_Q \sum_{j=1}^2 \frac{\partial p_1}{\partial x_j} \psi_j - \int_Q f \cdot \psi \quad \forall \theta \in S \end{aligned} \quad (7.18)$$

with $w = \mathcal{R}(s)$, $q = \mathcal{Q}(\mathcal{R}(s))$, and ψ an arbitrary relevation of θ ; that is, $\psi \in U$ is such that $\psi|_{x_3=0} = \theta$.

Now we easily see that b and T are well-defined; that is, they are independent of the choice of the relevation ψ of θ . Indeed if ψ^1 and ψ^2 are two different relevements of θ , then $\psi^1 - \psi^2$ is an element of $V_{0,x}$. Moreover the expressions of the right side and of (7.16) and (7.17) vanish for $\psi \in V_{0,x}$ since w^k satisfies (L_k) for $k = 1, 2, 3$. This proves that b and T are well defined. With the above notation the variational inequality (7.16)

can be written in the following form:

$$\text{Find } s^* \in S : \quad b(s^*, \theta) + T(\theta) + \hat{j}(s^* + \theta) - \hat{j}(s^*) \geq 0 \quad \forall \theta \in S \quad (7.19)$$

where

$$\hat{j}(\theta) = \frac{1}{\lambda} \int_{\omega} \int_Y k(x) |\theta(x, y) - s^0| dy dx.$$

The expression of b and T can be simplified in the following manner.

PROPOSITION 7.3. We have

$$b(s, \theta) = 2\mu \int_Q \sum_{i,j=1}^3 D_{ij}^B(w) D_{ij}^B(\psi) + \int_{\omega} A \nabla q \cdot \nabla r,$$

$$T(\theta) = 2\mu \int_Q \sum_{i,j=1}^3 D_{ij}^B(w^3) D_{ij}^B(\psi) + \int_Q \sum_{j=1}^2 \frac{\partial p_1}{\partial x_j} \psi_j - \int_Q f \cdot \psi$$

with $w = \mathcal{R}(s)$, $\psi = \mathcal{R}(\theta)$, $q = \mathcal{Q}(\mathcal{R}(s))$ and $r = \mathcal{Q}(\mathcal{R}(\theta))$.

Proof. We take in (7.17) and (7.18) the relevent $\psi = \mathcal{R}(\theta)$. We then have

$$\sum_{i,j=1}^3 \int_Q D_{ij}^B(w^k) D_{ij}^B(\psi) = 0, \quad k = 1, 2$$

since $w^k \in V_{0,x}$ and ψ is the solution of (L) with s replaced by θ . On the other hand, considering $q = \mathcal{Q}(\mathcal{R}(s))$ as a test function in the problem (P_0) satisfied by $r = \mathcal{Q}(\mathcal{R}(\theta))$ we find

$$\int_Q \sum_{j=1}^2 \frac{\partial q}{\partial x_j} \psi_j = \int_{\omega} \sum_{j=1}^2 \left(\int_{B_x} \psi_j \right) \frac{\partial q}{\partial x_j} = \int_{\omega} A \nabla r \cdot \nabla q.$$

This ends the proof. \square

We can now state the main result of this paper.

THEOREM 7.4. The limit problem (7.19) satisfied by $s^* = u(x, y, 0)$ has a unique solution $s^* \in S$. Moreover for the entire sequence of ε we have in Ω_M ,

$$\varepsilon^2 \bar{p}^\varepsilon \rightarrow p_0 + p_1,$$

$$\bar{u}^\varepsilon \rightarrow \sum_{k=1}^2 w^k \frac{\partial p_0}{\partial x_k} + w^* + \sum_{k=1}^2 w^k \frac{\partial p_1}{\partial x_k} + w^3,$$

where w^k , for $k = 1, 2, 3$, are solutions of (L_k) , p_1 is a solution of (P_1) , $w^* = \mathcal{R}(s^*)$, $p_0 = \mathcal{Q}(\mathcal{R}(s^*))$. All the limit functions are defined on Q and extended by 0 to Q_M .

Proof. We use the form of b and T given in Proposition 7.3. The symmetry of b is obvious and the continuity of b and T are immediate from the fact that $A \in (L^\infty(\omega))^4$ (Proposition 7.1) and that \mathcal{R} and \mathcal{Q} are linear and continuous (Remark 7.2) and (7.12). Moreover using positive definiteness of A we get

$$b(s, s) \geq \int_Q \sum_{i,j}^3 |D_{ij}^B(w)|^2.$$

By Korn's inequality and since $\|w\|_U = \|s\|_S$ we deduce that there exists a constant $\alpha_1 > 0$ such that

$$b(s, s) \geq \alpha_1 \|s\|_S^2.$$

Since \hat{j} is convex and continuous in S (we have the continuous embedding of S in $L^2(\omega \times Y)$), and S is a Hilbert space we deduce classically the existence and uniqueness of a solution of the variational inequality (7.19) (Proposition 2.2 chap. II of [17]). We deduce from the uniqueness that the entire sequences (in ε) u^ε and $\varepsilon^2 p^\varepsilon$ converge to the indicated limits. \square

8. Lower-semicontinuity result for the two-scale convergence. The aim of this chapter is to prove (6.3), in a generalized case, in Theorem 8.5, which is still an open problem, using some results on the subdifferential and regularization of convex functions.

Let $\Omega \subset \mathbb{R}^m$ be an open bounded domain, $Y = [0, 1]^m$, and $\varphi : \Omega \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that the following hypotheses hold:

- (H1) $\varphi(x, \cdot, \cdot)$ is continuous a.e. $x \in \Omega$,
- (H2) $\varphi(\cdot, y, z)$ is measurable for all $(y, z) \in \mathbb{R}^m \times \mathbb{R}^n$,
- (H3) $\varphi(x, \cdot, z)$ is 1-periodic in y ,
- (H4) $\varphi(x, y, \cdot)$ is convex in z ,
- (H5) $\exists C_1 > 0$ and $\exists C_2 \in L^2(\Omega)$ such that

$$|\varphi(x, y, z)| \leq C_1 \|z\|^2 + C_2(x) \quad \text{a.e. } x \in \Omega \quad \forall (y, z) \in \mathbb{R}^m \times \mathbb{R}^n.$$

REMARK 8.1. A particular case of Theorem 8.5, where φ depends only on z , was proved in [18] and [19] (see also [20] for an analogous result for the stochastic two-scale convergence). All these results are valid only in the particular case where φ is an affine function. Another particular case, with $\varphi(z) = z^2$, can be found in [15]. In [16] the case $\varphi(z) = |z|^p$ for $p \in]1, +\infty[$ was also considered, which is a particular case of our only for $p \in]1, 2]$.

For $\delta > 0$ we define the partial Yosida-Moreau approximation of φ :

$$\varphi_\delta(x, y, z) = \inf_{s \in \mathbb{R}^n} \left\{ \varphi(x, y, s) + \frac{\|z - s\|^2}{2\delta} \right\}, \quad (8.1)$$

which is well defined (see for example [21], chapter 2, page 121, theorem 2.3) and for $(x, y) \in \Omega \times \mathbb{R}^m$ we consider the multivalued operator $A_{x,y}$ from \mathbb{R}^n to $2^{\mathbb{R}^n}$ such that $A_{x,y}(z) = \partial_z \varphi(x, y, z)$, where $\partial_z \varphi$ is the subdifferential, in z , of φ defined by

$$z \mapsto \partial_z \varphi(x, y, z) = \{w \in \mathbb{R}^n : \varphi(x, y, s) \geq \varphi(x, y, z) + w \cdot (s - z) \quad \forall s \in \mathbb{R}^n\}.$$

From (H1) and (H4) φ is continuous and convex in z , so we have (see for example [17], page 22, Proposition 5.2) that

$$\partial_z \varphi(x, y, z) \neq \emptyset; \quad \text{then} \quad \text{Dom}(A_{x,y}) = \mathbb{R}^n.$$

We have also (see for example [22], page 25, [21], theorem 2.1) that $A_{x,y}$ is a maximal monotone operator. Then (see for example [22], Proposition 2.2) we get

$$\mathfrak{S}(I + \delta A_{x,y}) = \mathbb{R}^n,$$

so for all $\delta > 0$, and for all $s \in \mathbb{R}^n$, there exists a unique $z \in \mathbb{R}^n$, such that

$$(I + \delta A_{x,y})(z) \ni s.$$

We denote $z = (I + \delta A_{x,y})^{-1}(s)$ and introduce the following application:

$$J_{x,y,\delta} : \mathbb{R}^n \rightarrow \mathbb{R}^n \quad \text{such that } J_{x,y,\delta}(s) = (I + \delta A_{x,y})^{-1}(s) \quad (8.2)$$

and the partial Yosida approximation of the operator $A_{x,y}$,

$$A_{x,y,\delta} : \mathbb{R}^n \rightarrow \mathbb{R}^n \quad \text{such that } A_{x,y,\delta}(s) = \frac{1}{\delta}(s - J_{x,y,\delta}(s)). \quad (8.3)$$

LEMMA 8.2. We have

$$\varphi_\delta(x, y, \cdot) \in \mathcal{C}^1(\mathbb{R}^n) \text{ and is convex in } z, \quad (8.4)$$

$$\lim_{\delta \rightarrow 0} \varphi_\delta(x, y, z) = \varphi(x, y, z), \quad (8.5)$$

$$\nabla_z \varphi_\delta = A_{x,y,\delta}, \quad (8.6)$$

$$\lim_{\delta \rightarrow 0} J_{x,y,\delta}(z) = z \quad \forall z \in \mathbb{R}^n, \quad (8.7)$$

$$\|J_{x,y,\delta}(z_1) - J_{x,y,\delta}(z_2)\| \leq \|z_1 - z_2\| \quad \forall (z_1, z_2) \in \mathbb{R}^n \times \mathbb{R}^n, \quad (8.8)$$

$$\|A_{x,y,\delta}(z_1) - A_{x,y,\delta}(z_2)\| \leq \frac{1}{\delta} \|z_1 - z_2\| \quad \forall (z_1, z_2) \in \mathbb{R}^n \times \mathbb{R}^n, \quad (8.9)$$

$$\varphi(x, y, J_{x,y,\delta}(z)) \leq \varphi_\delta(x, y, z) \leq \varphi(x, y, z) \quad \forall z \in \mathbb{R}^n. \quad (8.10)$$

Proof. The proof is similar to the well-known results in [21], page 121; see also [22]. \square
We have the following technical result.

PROPOSITION 8.1. For any ξ such that $\xi(x, y) \in \partial_z \varphi(x, y, 0)$ we have

$$\nabla_z \varphi_\delta(x, y, \delta \xi(x, y)) = \xi(x, y) \quad \text{a.e. } x \in \omega \quad \forall y \in Y, \quad (8.11)$$

$$|\nabla_z \varphi_\delta(x, y, \delta \xi(x, y))| \leq \sqrt{n}(C_1 + 2C_2(x)). \quad (8.12)$$

Proof. Since $\delta \xi \in \delta A_{x,y}(0) + 0$ we deduce $J_{x,y,\delta}(\delta \xi) = 0$, which implies

$$\nabla_z \varphi_\delta(x, y, \delta \xi(x, y)) = A_{x,y,\delta}(\delta \xi) = \xi(x, y),$$

which proves (8.11). On the other hand, we have

$$\varphi(x, y, z) - \varphi(x, y, 0) \geq \xi \cdot z \quad \forall z \in \mathbb{R}^n.$$

Taking successively $z = \pm e_k$ in the above inequality where e_k is the k -th element in the canonical basis in \mathbb{R}^n , we deduce using also (H5),

$$|\xi_k| \leq C_1 + 2C_2(x) \quad k = 1, \dots, n.$$

With the help of (8.11) we obtain (8.12). \square

PROPOSITION 8.2. We have

$$|\varphi_\delta(x, y, z)| \leq 2C_1 \|z\|^2 + C_2(x) \quad \text{a.e. } x \in \Omega, \quad \forall (y, z) \in \mathbb{R}^m \times \mathbb{R}^n, \quad (8.13)$$

for all $0 < \delta < (4C_1)^{-1}$, and also

$$|\nabla_z \varphi_\delta(x, y, z)| \leq 2\sqrt{n}(C_1 + 2C_2(x)) + \frac{\|z\|}{\delta}. \quad (8.14)$$

Proof. From (8.10), $\varphi_\delta(x, y, z) \leq \varphi(x, y, z)$; then with (H5),

$$\varphi_\delta(x, y, z) \leq 2C_1\|z\|^2 + C_2(x).$$

On the other hand, from the definition of φ_δ and (H5) we deduce

$$\varphi_\delta(x, y, z) \geq -C_2(x) + \inf_{s \in \mathbb{R}^n} \left\{ \frac{\|z - s\|^2}{2\delta} - C_1\|s\|^2 \right\}.$$

Denoting $B(s) = \frac{\|z - s\|^2}{2\delta} - C_1\|s\|^2$ we have

$$B(s) \geq B(s_{min}) \quad \forall s \in \mathbb{R}^n \quad \text{with} \quad s_{min} = \frac{z}{1 - 2\delta C_1}$$

and we also have

$$B(s_{min}) = -\frac{C_1}{1 - 2\delta C_1}\|z\|^2.$$

As $\delta < (4C_1)^{-1}$, then $\frac{C_1}{1 - 2\delta C_1} \leq 2C_1$, so

$$\varphi_\delta(x, y, z) \geq -2C_1\|z\|^2 - C_2(x).$$

Thus (8.13) follows. To prove (8.14), we write

$$\nabla_z \varphi_\delta(x, y, z) = \nabla_z \varphi_\delta(x, y, \delta \xi(x, y)) + \nabla_z \varphi_\delta(x, y, z) - \nabla_z \varphi_\delta(x, y, \delta \xi(x, y)),$$

so from (8.9), (8.11) and (8.12) we deduce (8.14). \square

We introduce now the following operators:

$$S_0 : L^2(\Omega \times Y) \rightarrow \mathbb{R} : \quad S_0(v) = \int_{\Omega} \int_Y \varphi(x, y, v(x, y)) dy dx,$$

$$S_0^\delta : L^2(\Omega \times Y) \rightarrow \mathbb{R} : \quad S_0^\delta(v) = \int_{\Omega} \int_Y \varphi_\delta(x, y, v(x, y)) dy dx,$$

which are well defined from (H5) and (8.13). Recall the definition of the epigraph of an operator S from X to \mathbb{R} , $\text{epi}(S) := \{(v, a) \in X \times \mathbb{R} : a \geq S(v)\}$.

PROPOSITION 8.3.

$$\text{epi}(S_0) = \bigcap_{\delta > 0} \text{epi}(S_0^\delta).$$

Proof. From Lemma 8.2 (8.10) we have $S_0^\delta(v) \leq S_0(v)$ for all $v \in (L^2(\Omega))^n$, so

$$\text{epi}(S_0) \subseteq \bigcap_{\delta > 0} \text{epi}(S_0^\delta).$$

Let $(v, a) \in \text{epi}(S_0^\delta)$ for all $\delta > 0$. Then $a \geq S_0^\delta(v)$. So taking $\delta \rightarrow 0$ using (8.5) and (8.13), we get the result by Lebesgue's dominated theorem. \square

We define now for all $v_0 \in (L^2(\Omega \times Y))^n$ and $\delta > 0$

$$D_{v_0, \delta} = \{(v, a) \in (L^2(\Omega \times Y))^n \times \mathbb{R} : \\ a \geq \int_{\Omega} \int_Y [\varphi_\delta(x, y, v_0(x, y)) + \nabla_z \varphi_\delta(x, y, v_0(x, y))\{v(x, y) - v_0(x, y)\}] dy dx\}.$$

The second integral is well defined from (8.13) and (8.14).

LEMMA 8.3. For any $\delta > 0$ we have

$$\text{epi}(S_0^\delta) = \bigcap_{v_0 \in (\mathcal{D}(\Omega \times Y))^n} D_{v_0, \delta},$$

where $(\mathcal{D}(\Omega \times Y))^n$ is the space of functions in $(C^\infty(\Omega \times Y))^n$ with compact supports in $\Omega \times Y$.

Proof. As φ_δ is a convex function in z we have

$$\begin{aligned} \varphi_\delta(x, y, v(x, y)) &\geq \varphi_\delta(x, y, v_0(x, y)) + \nabla_z \varphi_\delta(x, y, v_0(x, y)) \cdot [(v(x, y)) - (v_0(x, y))] \\ &\quad \forall v \in (L^2(\Omega \times Y))^n, \quad \forall v_0 \in (\mathcal{D}(\Omega \times Y))^n, \quad \forall \delta > 0. \end{aligned}$$

So by integration we easily obtain

$$\begin{aligned} a &\geq \int_\Omega \int_Y \varphi_\delta(x, y, v_0(x, y)) \\ &\quad + \int_\Omega \int_Y \nabla_z \varphi_\delta(x, y, v_0(x, y)) \cdot \{v(x, y) - v_0(x, y)\} dy dx; \end{aligned} \tag{8.15}$$

thus we deduce the first inclusion.

Let $(v, a) \in (L^2(\Omega \times Y))^n \times \mathbb{R}$ satisfying (8.15) for all $v_0 \in (\mathcal{D}(\Omega \times Y))^n$ and for a fixed $\delta > 0$. By density there exists a sequence $(v_k) \in (\mathcal{D}(\Omega \times Y))^n$ such that $v_k \rightarrow v \in (L^2(\Omega \times Y))^n$. Taking $v_0 = v_k$ in (8.15) we obtain

$$\begin{aligned} a &\geq \int_\Omega \int_Y \varphi_\delta(x, y, v_k(x, y)) dy dx \\ &\quad + \int_\Omega \int_Y \nabla_z \varphi_\delta(x, y, v_k(x, y)) \cdot \{v(x, y) - v_k(x, y)\} dy dx. \end{aligned} \tag{8.16}$$

We now pass to the limit $k \rightarrow +\infty$ in the above inequality. For the first integral we have

$$\begin{aligned} &\int_\Omega \int_Y |\varphi_\delta(x, y, v_k) - \varphi_\delta(x, y, v)| dy dx \\ &\leq \int_\Omega \int_Y |\nabla_z \varphi_\delta(x, y, v + \theta(v_k - v))| \cdot |v_k - v| dy dx, \end{aligned}$$

with $0 < \theta < 1$. Then using (8.14) we have

$$\int_\Omega \int_Y |\varphi_\delta(x, y, v_k) - \varphi_\delta(x, y, v)| dy dx \rightarrow 0 \quad \text{as } k \rightarrow +\infty.$$

Using also (8.14), $|\nabla_z \varphi_\delta(x, y, v_k(x, y))|$ is bounded in $L^2(\Omega \times Y)$, so the second integral of (8.16) tends to zero. Thus we get

$$a \geq \int_\Omega \int_Y \varphi_\delta(x, y, v(x, y)) dy dx,$$

so $(v, a) \in \text{epi}(S_0^\delta)$. □

LEMMA 8.4. For any δ such that $0 < \delta < \frac{1}{4C_1}$, we have

i): The functions $y \mapsto \varphi_\delta(x, y, z)$ and $y \mapsto \nabla_z \varphi_\delta(x, y, z)$ are 1-periodic a.e. $x \in \Omega$, for all $z \in \mathbb{R}^n$.

- ii): The functions $(y, z) \mapsto \varphi_\delta(x, y, z)$ and $(y, z) \mapsto \nabla_z \varphi_\delta(x, y, z)$ are continuous a.e. $x \in \Omega$.
- iii): The functions, from Ω to \mathbb{R} : $x \mapsto \varphi_\delta(x, y, z)$ and $x \mapsto \nabla_z \varphi_\delta(x, y, z)$ are measurable for all $(y, z) \in Y \times \mathbb{R}^n$.
- iv): For any compact $K \subset \mathbb{R}^n$ we have that

$$x \mapsto \sup_{(y,z) \in Y \times K} \varphi_\delta(x, y, z) \quad \text{and} \quad x \mapsto \sup_{(y,z) \in Y \times K} \nabla_z \varphi_\delta(x, y, z)$$

have a finite $L^2(\Omega)$ -norm.

Proof. We recall that (see for example [22], p.25)

$$\varphi_\delta(x, y, z) = \inf_{s \in \mathbb{R}^n} \left\{ \varphi(x, y, s) + \frac{\|z - s\|^2}{2\delta} \right\} = \varphi(x, y, \tilde{s}) + \frac{\|z - \tilde{s}\|^2}{2\delta}, \tag{8.17}$$

where $\tilde{s} = J_{x,y,\delta}(z)$. From the y -periodicity of φ we easily deduce the y -periodicity of φ_δ . Since from (8.6) and (8.3) we have

$$\nabla_z \varphi_\delta(x, y, z) = \frac{1}{\delta}(z - J_{x,y,\delta}(z)), \tag{8.18}$$

we deduce from (8.18) the y -periodicity of $\nabla_z \varphi_\delta$.

Let us fix $y^0 \in \mathbb{R}^m$, $z^0 \in \mathbb{R}^n$ and let $y_k \rightarrow y^0$ and $z_k \rightarrow z^0$. We set

$$s_k = J_{x,y_k,\delta}(z_k), \quad s^0 = J_{x,y^0,\delta}(z^0)$$

and we have

$$\begin{aligned} \varphi_\delta(x, y_k, z_k) &= \inf_{s \in \mathbb{R}^n} \left\{ \varphi(x, y_k, s) + \frac{\|z_k - s\|^2}{2\delta} \right\} = \varphi(x, y_k, s_k) + \frac{\|z_k - s_k\|^2}{2\delta}, \tag{8.19} \\ \varphi_\delta(x, y^0, z^0) &= \inf_{s \in \mathbb{R}^n} \left\{ \varphi(x, y^0, s) + \frac{\|z^0 - s\|^2}{2\delta} \right\} = \varphi(x, y^0, s^0) + \frac{\|z^0 - s^0\|^2}{2\delta}. \end{aligned}$$

We then have

$$\varphi(x, y_k, s_k) + \frac{\|z_k - s_k\|^2}{2\delta} \leq \varphi(x, y_k, s^0) + \frac{\|z_k - s^0\|^2}{2\delta}. \tag{8.20}$$

Using (H5) we deduce

$$-C_1 \|s_k\|^2 - C_2(x) + \frac{\|s_k\|^2}{2\delta} - \frac{\|z_k\|^2}{2\delta} \leq C_1 \|s^0\|^2 + C_2(x) + \frac{\|s^0\|^2}{2\delta} + \frac{\|z_k\|^2}{2\delta}.$$

Taking $\delta < \frac{1}{4C_1}$ we deduce that s_k is bounded since z_k is bounded. We can extract a subsequence denoted also s_k such that $s_k \rightarrow \xi$ with $\xi \in \mathbb{R}^n$.

Passing to the limit in (8.20) we deduce

$$\varphi(x, y^0, \xi) + \frac{\|z^0 - \xi\|^2}{2\delta} \leq \varphi(x, y^0, s^0) + \frac{\|z^0 - s^0\|^2}{2\delta},$$

so $\xi = s^0$ since s^0 is the minimum of

$$s \mapsto \varphi(x, y^0, s) + \frac{\|z^0 - s\|^2}{2\delta}.$$

From the uniqueness of the minimum s^0 , we deduce that all the subsequences of s_k converge to s^0 . We then have

$$J_{x,y_k,\delta}(z_k) \rightarrow J_{x,y^0,\delta}(z^0),$$

which gives from (8.18) the desired continuity for $\nabla_z \varphi_\delta$. Passing to the limit in (8.19) we deduce the continuity for φ_δ .

The measurability of the function $x \mapsto \varphi_\delta(x, y, z)$ is immediate from the measurability of the function $x \mapsto \varphi(x, y, z)$ and the definition of φ_δ . We now have for any fixed $y \in \mathbb{R}^m$, $z \in \mathbb{R}^n$,

$$\frac{\partial \varphi_\delta}{\partial z_j}(x, y, z) = \lim_{k \rightarrow \infty} k \left[\varphi_\delta(x, y, z + \frac{1}{k} e_k) - \varphi_\delta(x, y, z) \right].$$

Since the function $x \in \Omega \rightarrow k [\varphi_\delta(x, y, z + \frac{1}{k} e_k) - \varphi_\delta(x, y, z)]$ is measurable, then the function $x \rightarrow \frac{\partial \varphi_\delta}{\partial z_j}(x, y, z)$ is measurable. The last result (iv) is an immediate consequence of Proposition 8.2. \square

We now give the main result of this section:

THEOREM 8.5. Let u^ε in $(L^2(\Omega))^n$, which two-scale converges to some u in $(L^2(\Omega \times Y))^n$, and φ be such that (H1) – (H5) hold. Then we have

$$\liminf_{\varepsilon \rightarrow 0} \int_{\Omega} \varphi(x, \frac{x}{\varepsilon}, u^\varepsilon(x)) dx \geq \int_{\Omega} \int_Y \varphi((x, y, u(x, y))) dy dx.$$

Proof. Let

$$b = \liminf_{\varepsilon \rightarrow 0} \int_{\Omega} \varphi(x, \frac{x}{\varepsilon}, u^\varepsilon(x)) dx,$$

so there exists a subsequence of ε , still denoted by ε , such that

$$\int_{\Omega} \varphi(x, \frac{x}{\varepsilon}, u^\varepsilon(x)) dx \rightarrow b \quad \text{when } \varepsilon \rightarrow 0.$$

We also have

$$\begin{aligned} \int_{\Omega} \varphi(x, \frac{x}{\varepsilon}, u^\varepsilon(x)) dx &\geq \int_{\Omega} \varphi_\delta(x, \frac{x}{\varepsilon}, u^\varepsilon(x)) dx \\ &\geq \int_{\Omega} \varphi_\delta(x, \frac{x}{\varepsilon}, v_0(x, \frac{x}{\varepsilon})) dx \\ &+ \int_{\Omega} \nabla_z \varphi_\delta(x, \frac{x}{\varepsilon}, v_0(x, \frac{x}{\varepsilon})) \cdot [u^\varepsilon(x) - v_0(x, \frac{x}{\varepsilon})] dx \end{aligned} \quad (8.21)$$

for any $v_0 \in (\mathcal{D}(\Omega \times Y))^n$.

From Lemma 8.4 and using also Lemma 5.3 in [15], we deduce that

- (1) $(x, y) \mapsto \varphi_\delta(x, y, v_0(x, y))$ is in $L^1(\Omega, \mathcal{C}_\#(Y))$,
- (2) $(x, y) \mapsto \nabla_z \varphi_\delta(x, y, v_0(x, y))$ is in $(L^2(\Omega, \mathcal{C}_\#(Y)))^n$.

We can now pass to the limit, $\varepsilon \rightarrow 0$, in (8.21), using Theorem 2, p.40 of [16] and also Theorem 7, p.48 of [16] and that u^ε two-scale converges to u , to get

$$\begin{aligned} b &\geq \int_{\Omega} \int_Y \varphi_\delta(x, y, v_0(x, y)) dy dx \\ &+ \int_{\Omega} \int_Y \nabla_z \varphi_\delta(x, y, v_0(x, y)) \cdot [u(x, y) - v_0(x, y)] dy dx. \end{aligned} \quad (8.22)$$

Thus

$$(u, b) \in \mathcal{D}_{v_0, \delta} \quad \forall \delta > 0 \quad \text{and} \quad \forall v_0 \in (\mathcal{D}(\Omega \times Y))^n.$$

From Lemma 8.3 we deduce that

$$(u, b) \in \text{epi}(S_0^\delta) \quad \forall \delta > 0,$$

and from Proposition 8.3 we deduce that $(u, b) \in \text{epi}(S_0)$. Thus

$$b = \liminf_{\varepsilon \rightarrow 0} \int_{\Omega} \varphi(x, \frac{x}{\varepsilon} u^\varepsilon(x)) dx \geq S_0(u) = \int_{\Omega} \int_Y \varphi(x, y, u(x, y)) dy dx.$$

□

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