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Asymptotic behaviour of solutions of the Tjon–Wu equation

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Abstract. The paper is devoted to the problem of the asymptotical stability of the Tjon–Wu version of the Boltzmann equation. It is proved that for every initial density with the finite second moment the corresponding unique solution converges strongly to the appropriate stationary solution. The result is based on the expansion of the solutions in the form of Wild's sums.

1. Introduction. The Boltzmann energy equation in the Tjon–Wu form may be written as follows (see [8]):

$$\frac{\partial U(t, x)}{\partial t} + U(t, x) = \int_x^\infty \frac{1}{y} \int_0^y U(t, y-z)U(t, z)\,dz\,dy. \tag{1.1}$$

Here, $U(t, x)$ is the unknown function, t the time and x the energy. Thus (1.1) is always considered for $t \geqslant 0$ and $x \geqslant 0$. This model has recently been studied from the mathematical and physical point of view (see [1]–[5], [7]). In particular, T. Dłotko and A. Lasota have proved the existence of the unique solution of (1.1) with the initial condition

$$U(0, x) = u_0(x) \quad \text{for } x \geqslant 0, \tag{1.2}$$

in the space $L^1(0, \infty)$ (with the weight function $1+x$). They also proved the weak asymptotical stability in the subspace of $L^1[0, \infty)$ consisting of the densities with finite moment of all orders. Recently, new existence theorems for (1.1), (1.2) were proved by Barnsley *et al.* [1] (in the space L^2 with the weight e^x) and Herod [5] (in the space $BC[0, \infty)$ of bounded continuous functions). Equation (1.1) may also be treated in the weak form as the equation for Borel measures on $[0, \infty)$. In this case, the existence and stability theorems were proved by Ferland and Giroux [4].

From physical point of view, it is natural to consider (1.1) as an evolution equation in the space $L^1(0, \infty)$. Thus, setting

$$P(u, v)(x) = \int_x^\infty \frac{1}{y} \int_0^y u(y-z)v(z)\,dz\,dy \tag{1.3}$$

for $u, v \in L^1(0, \infty)$, we may write (1.1) in the form

$$\frac{dU}{dt}+U = P(U, U), \qquad U(0) = u_0. \tag{1.4}$$

It is easy to verify that (1.3) defines a bounded bilinear operator acting on L^1.

We shall assume that the initial function $u_0 \in L^1$ satisfies the conditions:

(i) $u_0(x) \geqslant 0$ a.e.,

(ii) $\int_0^\infty u_0(x)dx = 1$,

(iii) $\int_0^\infty xu_0(x)dx = 1, \quad \int_0^\infty x^2 u_0(x)dx < \infty$.

Now, we may formulate our main result:

THEOREM 1.1. *For every $u_0 \in L^1$ satisfying* (i), (ii), *there exists a unique strong solution $U(t)$ of* (1.4) *defined for all $t \geqslant 0$. This solution fulfils conditions analogous to* (i) *and* (ii). *If, in addition, u_0 satisfies* (iii), *then $U(t)$ converges strongly in L^1 (as $t \to \infty$) to the stationary solution w given by $w(x) = e^{-x}$.*

2. Preliminaries. Let

$$M^n(v) = \int_0^\infty x^n v(x)dx$$

for $v \in L^1$ and for $n \geqslant 0$. If $M^n(v)$ and $M^n(w)$ are finite we have

$$M^n(P(v, w)) = \frac{1}{n+1}\sum_{k=0}^{n}\binom{n}{k}M^k(v)M^{n-k}(w). \tag{2.1}$$

We omit the simple proof of this equality (see [3]). Let D be the set of all probability densities, i.e., $D = \{v \in L^1\colon\ M^0(v) = 1,\ v \geqslant 0\}$. According to positivity of P ($v, w \geqslant 0$ implies $P(v,w) \geqslant 0$) and to (2.1), we have $P(D, D) \subset D$ and consequently

$$\|P(v, w)\| \leqslant \|v\| \times \|w\| \qquad \text{for } v, w \in L^1. \tag{2.2}$$

Let $\{u_n\}_{n=0}^{\infty}$ be the sequence defined by the recurrent equalities

$$u_n = \frac{1}{n}\sum_{k=0}^{n-1} P(u_k, u_{n-1-k}). \tag{2.3}$$

The sequence (2.3) will be called the *sequence of the convolution iterates of P*. Following [6], the solution of problem (1.4) exists and has the expansion

$$U(t) = e^{-t}\sum_{n=0}^{\infty}(1-e^{-t})^n u_n. \tag{2.4}$$

If $u_0 \in D$ then $u_n \in D$ for every n and the series (2.4) are convergent to an element of D for $t \geqslant 0$. Moreover, the convergence of the sequence $\{u_n\}_{n=0}^{\infty}$ implies the convergence of the $U(t)$ (for t tending to infinity) to the same limit. Thus it only remains to prove the convergence of the convolution iterates under additional assumptions (iii) posed on the function u_0.

To end this section, we shall make two remarks.

Remark 2.1. If $M^0(u_0) = M^1(u_0) = 1$, then $M^1(u_n) = 1$ for every n.

This immediately follows from (2.1) and (2.3).

Remark 2.2. If $M^0(u_0) = M^1(u_0) = 1$, $M^2(u_0) < \infty$, then

$$\lim_{n\to\infty} M^2(u_n) = 2. \tag{2.5}$$

Indeed, according to (2.1) and (2.3) we have

$$\begin{aligned}
M^2(u_n) &= \frac{1}{n}\sum_{k=0}^{n-1} M^2(P(u_k, u_{n-1-k}))\\
&= \frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{3}\{M^2(u_k)+2+M^2(u_{n-1-k})\} = \frac{2}{3}+\frac{2}{3n}\sum_{k=0}^{n-1} M^2(u_k)\\
&= \frac{n-1}{n}\left\{\frac{2}{3}+\frac{2}{3(n-1)}\sum_{k=0}^{n-2} M^2(u_k)\right\}+\frac{2}{3n}+\frac{2}{3n}M^2(u_{n-1})\\
&= \frac{n-1}{n}M^2(u_{n-1})+\frac{2}{3n}+\frac{2}{3n}M^2(u_{n-1}) = \frac{2}{3n}+\frac{3n-1}{3n}M^2(u_{n-1}).
\end{aligned}$$

Hence

$$M^2(u_n)-2 = \frac{3n-1}{3n}(M^2(u_{n-1})-2) = \left(\prod_{k=1}^{n}\frac{3k-1}{3k}\right)(M^2(u_0)-2).$$

Since

$$\lim_{n\to\infty}\prod_{k=1}^{n}\frac{3k-1}{3k} = 0,$$

we get the desired result.

3. The power iterates. Let E be a real Banach space, Q a bilinear operator from $E\times E$ into E and $\{u_n\}_{n=0}^{\infty}$ the sequence of the convolution iterates of Q. If Z is a subset of space E, then the *convex hull of* Z, denoted by co(Z), is the smallest convex set that contains Z. We have the following

Lemma 3.1. *Let $Z_0 \subset W \subset E$, where W is a convex set such that $Q(W, W) \subset W$ and Z_0 is a subset of W such that $u_n \in Z_0$ for every n. Let us define a sequence of subsets of W as*

$$Z_{m+1} = \mathrm{co}(Q(Z_m, Z_m)) \quad \text{for } m \geqslant 0. \tag{3.1}$$

Then for every $m = 0, 1, \ldots$ there are sequences $\{v_{m,n}\}_{n=0}^{\infty}$, $\{w_{m,n}\}_{n=0}^{\infty}$, $\{a_{m,n}\}_{n=0}^{\infty}$ satisfying for every n the relations

$$v_{m,n} \in Z_m, \quad w_{m,n} \in W, \quad a_{m,n} \in (0,\, 1) \tag{3.2}$$

and

$$u_n = a_{m,n} v_{m,n} + (1 - a_{m,n}) w_{m,n}. \tag{3.3}$$

Moreover, the sequence $\{a_{m,n}\}_{n=0}^{\infty}$ *for every* m *satisfies the condition*

$$\lim_{n \to \infty} a_{m,n} = 1. \tag{3.4}$$

Proof. Setting $v_{0,n} = w_{0,n} = u_n$ and $a_{0,n} = (n+1)/(n+2)$, we may define these sequences for $m \geqslant 1$ by the following recurrent equalities:

$$a_{m+1,n} = \frac{1}{n} \sum_{k=0}^{n-1} a_{m,k} a_{m,n-1-k},$$

$$v_{m+1,n} = (n a_{m+1,n})^{-1} \sum_{k=0}^{n-1} a_{m,k} a_{m,n-1-k} Q(v_{m,k}, v_{m,n-1-k}),$$

$$v_{m+1,n} = (n - n a_{m+1,n})^{-1} \{2 \sum_{k=0}^{n-1} a_{m,k}(1 - a_{m,n-1-k}) Q(v_{m,k},\, w_{m,n-1-k})$$
$$+ \sum_{k=0}^{n-1} (1 - a_{m,k})(1 - a_{m,n-1-k}) Q(w_{m,k},\, w_{m,n-1-k})\}.$$

These equalities are assumed to hold for $n \geqslant 1$. To complete the definition of these sequences, we set $v_{m+1,0} = v_{m+1,1}$, $w_{m+1,0} = w_{m+1,1}$ and $a_{m+1,0} = a_{m+1,1}$. It is easy to check by induction that (3.2) and (3.3) hold. We shall derive only (3.4). We have

$$|1 - a_{m+1,n}| \leqslant \left|1 - \frac{1}{n} \sum_{k=0}^{n-1} a_{m,k}\right| + \left|\frac{1}{n} \sum_{k=0}^{n-1} a_{m,k} - \frac{1}{n} \sum_{k=0}^{n-1} a_{m,k} a_{m,n-1-k}\right|$$
$$\leqslant \frac{1}{n} \sum_{k=0}^{n-1} |1 - a_{m,k}| + \frac{1}{n} \sum_{k=0}^{n-1} |a_{m,k}|\, |1 - a_{m,n-1-k}| \leqslant \frac{2}{n} \sum_{k=0}^{n-1} |1 - a_{m,k}|.$$

Since the last inequality holds for $n \geqslant 1$ and $m \geqslant 0$, by an induction argument it is easy to show (3.4). This completes the proof. ∎

For notational convenience, let GC_{-1} be the set of all real-valued measurable nonnegative functions defined on the interval $\boldsymbol{R}_+ = [0, \infty)$ (we do not exclude the possibility that the value of a function is equal to infinity for some $x \in \boldsymbol{R}_+$), let GC_0 be the set of all nonincreasing elements of GC_{-1}, and let GC_m for $m \geqslant 1$ denote the set of all $v \in GC_0$ such that the inequality

$$a v^{1/m}(x_1) + (1-a) v^{1/m}(x_2) \geqslant v^{1/m}(a x_1 + (1-a) x_2)$$

holds for all $x_1,\, x_2 \in \boldsymbol{R}_+$ and $a \in [0,\, 1]$. The following lemmas describe some properties of the sets GC_m.

LEMMA 3.2. *Let $\mathfrak{f}$ be a measurable real-valued function defined on* $[0, 1] \times \boldsymbol{R}_+$ *such that* $\mathfrak{f}(t, \cdot) \in GC_m$ *for all* $t \in [0, 1]$ *and some* $m \geqslant -1$. *Then*

$$\mathfrak{F}(x) = \int_0^1 \mathfrak{f}(t, x)\,dt$$

belongs to GC_m.

Proof. For $m \geqslant -1$, this is a part of the Fubini Theorem. In the case of $m = 0$, this lemma is due to the monotonicity of the integral. Let $m \geqslant 1$ and $\alpha \in [0, 1]$. Then for $x_1, x_2 \in \boldsymbol{R}_+$ we have

$$\begin{aligned}\mathfrak{F}^{1/m}(\alpha x_1 + (1-\alpha)x_2) &= \Big\{\int_0^1 \mathfrak{f}(t, \alpha x_1 + (1-\alpha)x_2)\,dt\Big\}^{1/m} \\ &\leqslant \Big\{\int_0^1 (\alpha \mathfrak{f}^{1/m}(t, x_1) + (1-\alpha)\mathfrak{f}^{1/m}(t, x_2))^m\,dt\Big\}^{1/m} \\ &\leqslant \Big\{\int_0^1 (\alpha \mathfrak{f}^{1/m}(t, x_1))^m\,dt\Big\}^{1/m} + \Big\{\int_0^1 ((1-\alpha)\mathfrak{f}^{1/m}(t, x_2))^m\,dt\Big\}^{1/m} \\ &= \alpha \mathfrak{F}^{1/m}(x_1) + (1-\alpha)\mathfrak{F}^{1/m}(x_2).\end{aligned}$$

The first inequality follows from the definition of the set GC_m and the second is the Minkowski inequality. This completes the proof. ∎

Note that the function $\mathfrak{f}$ is nonnegative, so that $\mathfrak{F}(x)$ always exists, even though it might be infinite.

LEMMA 3.3. *Let functions v and w belong to GC_m for some $m \geqslant -1$. Then the function u given by $u(x) = v(x)w(x)$ also belongs to GC_m.*

Proof. For $m = -1$ and $m = 0$, this lemma is obvious. Notice that for $m \geqslant 1$ the function $v \in GC_m$ if and only if $v^{1/m} \in GC_1$. Then it is sufficient to prove the lemma in the case $m = 1$. Let $v, w \in GC_m$. Thus they are nonincreasing functions and for $x_1, x_2 \in \boldsymbol{R}_+$ we have

$$0 \leqslant [v(x_1) - v(x_2)]\,[w(x_1) - w(x_2)].$$

This is equivalent to the inequality

$$\begin{aligned}\alpha u(x_1) + (1-\alpha)u(x_2) \geqslant\ & \alpha^2 v(x_1)w(x_1) + \alpha(1-\alpha)v(x_1)w(x_2) \\ & + \alpha(1-\alpha)v(x_2)w(x_1) + (1-\alpha)^2 v(x_1)w(x_2),\end{aligned}$$

where $\alpha \in [0, 1]$. Hence we have

$$\begin{aligned}\alpha u(x_1) + (1-\alpha)u(x_2) &\geqslant [\alpha v(x_1) + (1-\alpha)v(x_2)]\,[\alpha w(x_1) + (1-\alpha)w(x_2)] \\ &\geqslant v(\alpha x_1 + (1-\alpha)x_2)\,w(\alpha x_1 + (1-\alpha)x_2) \\ &= u(\alpha x_1 + (1-\alpha)x_2)\end{aligned}$$

which completes the proof. ∎

Let T be a linear transformation defined as follows:

$$Tv(x) = \int_x^\infty v(x)dx. \tag{3.5}$$

Note that for $v \in GC_{-1}$ the integral (3.5) is well defined even though it might be infinite.

LEMMA 3.4. *Let $v \in GC_m$ for some $m \geqslant -1$. Then $Tv \in GC_{m+1}$.*

Proof. For $m = -1$, the lemma is obvious. The proof for $m \geqslant 0$ will be based on the closedness of the class GC_m under taking supremum. Namely, we shall construct for $v \in GC_{m+1}$ and every $x \in \boldsymbol{R}_+$ the function $f_x \in GC_{m+1}$ such that $Tv(x) = f_x(x)$ and $Tv \geqslant f_x$ on $\boldsymbol{R}_+$. Hence $Tv(y) = \sup\{f_x(y)\colon x \in \boldsymbol{R}_+\}$ for every $y \in \boldsymbol{R}_+$ and consequently $Tv \in GC_{m+1}$. We shall construct the function f_x in a different way for three kinds of points. If the point x is such that $Tv(y) = \infty$ for $y < x$, then let $f_x(y) = Tv(y)$ for $y \leqslant x$ and $f_x(y) = 0$ for $y > x$. If the point x is such that $Tv(x) = 0$, then let $f_x \equiv 0$ on $\boldsymbol{R}_+$. It may easily be verified that the function f_x constructed in both of these cases has the desired properties. Now we assume that the inequalities $0 < Tv(x) < \infty$ and $0 < v(x) < \infty$ hold. Under these assumptions there exists a unique function w such that

$$w(y) = \begin{cases} a(b-y)^m & \text{for } y \leqslant b, \\ 0 & \text{for } y > b, \end{cases}$$

where nonnegative parameters a and b are so chosen such that the equalities $v(x) = w(x)$ and $Tv(x) = Tw(x)$ hold. In accordance with the above equalities and the inclusion $v \in GC_m$, there exists a finite real number $x_1 > x$ such that $v \leqslant w$ on $[x, x_1]$ and $v \geqslant w$ on $\boldsymbol{R}_+ \setminus [x, x_1]$. We claim that $Tw \leqslant Tv$ on $\boldsymbol{R}_+$. Indeed, this is obvious for $y > x_1$. For $y \in [x, x_1]$ we have

$$Tv(y) = Tv(x) - \int_x^y v(s)ds \geqslant Tw(x) - \int_x^y w(s)ds = Tw(y)$$

and for $y < x$

$$Tv(y) = Tv(x) + \int_y^x v(s)ds \geqslant Tw(x) + \int_y^x w(s)ds = Tw(y).$$

Thus the claim is proved. Let $f_x(y) = Tw(y)$. Note that $Tw \in GC_{m+1}$. Thus we have constructed the desired function f_x for every $x \in \boldsymbol{R}_+$ and the proof is complete. ■

Let $D_0 = \{v \in D\colon M^1(v) = 1,\ M^2(v) < \infty\}$ and $D_m = \mathrm{co}(P(D_{m-1}, D_{m-1}))$ for $m \geqslant 1$. We shall assume that the initial point of (1.4) belongs to D_0, i.e., $u_0 \in D_0$. In accordance with Remarks 2.1 and 2.2 we have $P(D_0, D_0) \subset D_0$. This inclusion and the convexity of the set D_0 imply that $D_1 \subset D_0$, and consequently, $D_{m+1} \subset D_m$ for $m \geqslant 0$. From (2.1) it follows that

$u_n \in D_0$ for every n. The following proposition states a very important property of the sets D_m.

PROPOSITION 3.1. *For all* $m \geqslant 0$,

$$D_m \subset GC_{m-1} \cap L^1. \tag{3.6}$$

Proof. For $m = 0$ and $m = 1$ the proposition is obvious. Let $m > 1$. According to the convexity of the sets GC_m we prove the inclusion (3.6) by an induction argument if we show that for $v, w \in GC_{m-2} \cap L^1$ the function $P(v, w)$ belongs to $GC_{m-1} \cap L^1$. We have

$$P(v, w)(x) = \int_x^\infty \frac{1}{y} \int_0^y v(y-z)w(z)dzdy = \int_x^\infty \int_0^1 v((1-t)y)w(ty)dtdy$$

and the desired result follows immediately from Lemmas 3.3, 3.2 and 3.4. Then, since the inclusion $P(v, w) \in L^1$ follows from (2.1), the proof is complete. ∎

4. Convergence of the convolution iterates. Using Proposition 3.1 and Lemma 3.4, we immediately have for $m \geqslant 0$

$$TD_m \subset GC_m \tag{4.1}$$

and, for the same reason,

$$T^2 D_m \subset GC_{m+1}. \tag{4.2}$$

Moreover, for $v \in D_0$ we have

$$T^2 v(0) = M^1(v) = 1, \tag{4.3}$$

$$(T^2 v)'(0) = -M^0(v) = -1. \tag{4.4}$$

Let for $m \geqslant 1$

$$w_m(x) = \begin{cases} m^{-m}(m-x)^m & \text{for } x \leqslant m, \\ 0 & \text{for } x > m. \end{cases} \tag{4.5}$$

The inclusion (4.2) and equalities (4.3), (4.4) imply that

$$T^2 v \geqslant w_{m+1} \quad \text{for } v \in D_m. \tag{4.6}$$

We claim that for every $m \geqslant 0$ the function w_m is a *lower function* (see [7]) for the sequence $\{T^2 u_n\}_{n=0}^{\infty}$, i.e.,

$$\lim_{n\to\infty} \|(w_m - T^2 u_n)^+\| = 0. \tag{4.7}$$

We use the form of u_m given by Lemma 3.1, where $W = Z_0 = D_0$ and consequently $Z_m = D_m$. Let us notice that in accordance with (3.2) we have for every m and n that u_n, $v_{m,n}$, $w_{m,n}$ are elements of GC_{-1} so the transformation T is well defined on these sequences. Thus for $m \geqslant 0$

$$\|(w_{m+1}-T^2 u_n)^+\| = \|(w_{m+1}, a_{m,n} T^2 v_{m,n} - (1-a_{m,n}) T^2 w_{m,n})^+\|$$
$$= a_{m,n}\|(w_{m+1}-T^2 v_{m,n})^+\| + (1-a_{m,n})\|(w_{m+1}-T^2 w_{m,n})^+\|.$$

Hence, since $\|(w_{m+1}-T^2 v_{m,n})^+\| = 0$ (see (4.6)) and

$$\|(w_{m+1}-T^2 v_{m,n})^+\| \leqslant \|w_{m+1}\| = \frac{m+1}{m+2} < 1$$

(this follows from positivity of $w_{m,n}$) we have

$$\|(w_{m+1}-T^2 u_n)^+\| \leqslant 1-a_{m,n}.$$

According to (3.4) we obtain (4.7) for every m.

COROLLARY 4.1. *Let $w(x) = e^{-x}$. Then w is the lower function for the sequence $\{T^2 u_n\}_{n=0}^{\infty}$.*

It immediately follows from the corollary that the function w is a limit of the increasing sequence $\{w_m\}_{m=1}^{\infty}$. Since $\|T^2 u_n\| = \frac{1}{2}M^2(u_n)$ and $\|w\| = 1$, Remark 2.2 and the previous corollary imply the following:

COROLLARY 4.2.

$$s\text{-}\lim_{n\to\infty} T^2 u_n = w. \tag{4.8}$$

5. Strong convergence. Having weak convergence of the sequence of the convolution iterates, we will prove *via* the Fréchet–Kolmogorov theorem that this convergence is exactly strong.

FRÉCHET–KOLMOGOROV THEOREM. *Let S be the real line, $\mathscr{B}$ the σ-ring of Baire subsets B of S and $m(B) = \int_B dx$ the ordinary Lebesgue measure of B. Then a subset K of $L^p(S, \mathscr{B}, m)$, $1 \leqslant p < \infty$, is strongly pre-compact iff it satisfies the conditions:*

$$\sup_{x\in K}\|x\| = \sup_{x\in K}\left(\int_S |x(s)|^p ds\right)^{1/p} < \infty, \tag{1}$$

$$\lim_{t\to 0}\int_S |x(t+s)-x(s)|^p ds = 0 \quad \textit{uniformly in } x\in K, \tag{2}$$

$$\lim_{\alpha\uparrow\infty}\int_{|s|>\alpha} |x(s)|^p ds = 0 \quad \textit{uniformly in } x\in K. \tag{3}$$

Condition (1) is obvious, because $\|u_n\| = 1$ for all $n\in N$. In order to derive (2), we first notice that the functions u_n are nonincreasing for $n \geqslant 1$. Hence

$$\int_{R_+} |u_n(t+s)-u_n(s)|ds = \int_{R_+} [u_n(s)-u_n(s+t)]ds = \int_0^t u_n(s)ds.$$

If (2) did not hold then there would exist $\varepsilon > 0$, $\delta > 0$ and an infinite set N_0 contained in N such that

$$\int_0^{\varepsilon} u_n(s)ds \geqslant \varepsilon \quad \text{for all } n \in N_0. \tag{5.1}$$

But this implies that

$$\int_c^{t} u_n(s)ds \geqslant t \quad \text{for } t \leqslant \varepsilon,\ n \in N_0$$

because the functions u_n are nonincreasing. Then we have for $t < \varepsilon$, $n \in N_0$

$$(T^2 u_n)(t) = -Tu_n(t) \geqslant -\left(1 - \int_c^{t} u_n(s)ds\right) \geqslant t - 1.$$

This differential inequality and the relation $T^2 u_n(0) = 1$ imply that

$$T^2 u_n(t) \geqslant 1 - t + t^2 > w(t)$$

which contradicts the weak convergence of the sequence $\{u_n\}_{n=0}^{\infty}$.

Now we complete the proof if we check that property (3) holds. For all $n \geqslant 1$, we have

$$\int_{R_+} u_n(x)dx \leqslant 2/x^2;$$

then

$$\int_a^{\infty} |u_n(s)|ds \leqslant \int_a^{\infty} \frac{2}{s^2} ds = \frac{2}{a}$$

and (3) holds. This completes the proof.

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