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# Asymptotic behaviour of solutions to Laplace's tidal equations at low frequencies

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**SUMMARY**

The asymptotic behaviour of solutions to Laplace's tidal equations at low frequencies is considered. The method used is based on perturbation in small parameters, these being the ratios of tidal frequency and the coefficient of bottom friction to the angular frequency of the Earth's rotation. It is shown that the resulting solutions are unstable in that the functions involved in the zero-order approximation are not uniquely determined by the zero-order equations, but depend on first-order terms as well. Because of this instability, direct methods of numerical integration are inefficient. We propose a different procedure, replacing the original set of equations in partial derivatives by ordinary differential equations that have a stable solution. The equations are examined qualitatively. It is shown, in particular, that for the case of an ocean of uniform depth over the whole Earth, they coincide with the well-known Lamb's equations. The asymptotic behaviour of the solutions is examined as modified by basin shape, bottom topography and bottom friction.

**Key words:** ocean model, tides

## INTRODUCTION

In recent years the asymptotic behaviour of solutions to Laplace's tidal equations at low frequencies has attracted increased attention in connection with the problem of determining the rheological properties of the Earth's mantle at low frequencies (O'Connor & Starr 1983; Carton 1983; Dickman 1985; Molodensky 1985; O'Connor 1986; Carton & Wahr 1986). As shown by Anderson & Minster (1979), Zharkov & Molodensky (1979), Smith & Dahlen (1981), Molodensky & Zharkov (1982), mantle anelasticity has the effect of significantly increasing the Chandler wobble period well beyond the range of observational uncertainty. Comparison of the observed period with its theoretical value for an ideally elastic Earth model can isolate mantle anelasticity effects, thus permitting determination of parameters of the mechanical quality factor $Q_\mu$ for oscillations having the Chandler wobble period $T_1 = 1.2$ yr.

It is mentioned in Merriam (1985) that the modern accuracy attainable in the measurement of satellite coordinates and orbital parameters by the LAGEOS program makes it possible to determine tidal variations in the velocity of the Earth's rotation and the Love number $k$ for oscillations of a still longer period, $T_2 = 18.6$ yr. Comparison of observed and theoretical values of Love numbers can yield the $Q_\mu$ for oscillations of period $T_2$ as well.

Reliable estimation of anelasticity effects for the mantle requires accurate calculations of the oceanic tide whose influence, both on the Chandler wobble period and on the amplitude of tidal variations in the velocity of rotation, is very great.

Carton (1983), O'Connor & Starr (1983), Dickman (1985), O'Connor (1986) and Carton & Wahr (1986) made both analytical evaluations of the pole tide for the simplest models (an ocean of uniform depth over an absolutely rigid Earth) and direct numerical calculations for more complex ocean models.

It should be noted that direct numerical calculations of the long-period tides are much more difficult than those for the diurnal and semidiurnal period range. The reasons for this are as follows:

(1) Because the long-period tidal amplitudes are very small, it is practically impossible to correct calculation results by comparison with observations.

(2) As will be shown below, the solutions of Laplace's tidal equations at low frequency significantly depend on terms of the order of the ratio of the frequency $\sigma$ to the angular velocity of the Earth's rotation $\omega$. The solutions become unstable in the limit $\sigma/\omega \to 0$; hence, even very small errors of numerical integration significantly affect the final results.

It has been shown (Molodensky 1985) that the pole tide can be described by a finite set of ordinary differential equations whose solution significantly depends on depth distribution and the coastline shape. Thus, with no bottom friction and an axially symmetric distribution of depth, the deviation of the ocean surface from the equipotential surface $\xi$ approaches zero; for realistic, axially asymmetric ocean models we generally have $\xi \neq 0$ even in the limiting case $\sigma/\omega \to 0$.

This paper presents a rigorous derivation of the ordinary differential equations that determine the tides of both the first and the second class (according to Laplace's classification) at low frequencies. This problem is solved by a perturbation method in small parameters, these being the ratios of tidal frequency and the coefficient of bottom friction to the angular frequency of the Earth's rotation. The resulting equations are examined qualitatively and quantitatively. It is shown, in particular, that these equations are the well-known Lamb's equations for the case of an ocean of uniform depth over the whole of the Earth (Lamb 1932, section 218). We consider the effects that basin shape, bottom topography and bottom friction have on the asymptotic behaviour of the solutions. The results should be taken into account in constructing rheological Earth models at low frequency on the basis of astrometric and satellite data.

## 2. BASIC EQUATIONS

Our treatment will be based on Laplace's tidal equations (Lamb 1932, section 214) written in the form

$$\dot{v}_\theta - 2\omega v_\varphi \cos\theta - F_\theta = -\frac{g}{a}\frac{\partial(\zeta - \bar{\zeta})}{\partial\theta} \tag{1a}$$

$$\dot{v}^0_\varphi + 2\omega v_\theta \cos\theta - F_\varphi = -\frac{g}{a\sin\theta}\frac{\partial(\zeta - \bar{\zeta})}{\partial\varphi} \tag{1b}$$

$$\dot{\zeta} = -\mathrm{div}_2(\mathbf{v}h), \tag{1c}$$

where $\theta$ and $\varphi$ are colatitude and longitude, respectively; $v_\theta$, $v_\varphi$, $F_\theta$, and $F_\varphi$ are the components of tidal flow velocity and force of bottom friction, the dot above a symbol denotes the time derivative, $a$ is the Earth's radius, $g$ the acceleration due to gravity at the Earth's surface, $\omega$ the angular velocity of the Earth's diurnal rotation, $h = h(\theta, \varphi)$ ocean depth and $\zeta$ tidal height.

$$\bar{\zeta} = \Phi/g \tag{2}$$

is the static tidal height, $\Phi$ the tide-generating potential equal to

$$\Phi = \Phi_0(\tfrac{3}{2}\cos^2\theta - \tfrac{1}{2}) \tag{3a}$$

in the case of long-period lunisolar tides (tides of the first species according to Laplace's classification) and

$$\Phi = \Phi_0 \sin\theta\cos\theta\cos(\sigma t - \varphi) \tag{3b}$$

in the case of the pole tide (or tide of the second species according to the same classification).

In (3a) and (3b), $\Phi_0$ is a constant that does not depend on $\theta$, $\varphi$, $t$; $t$ is the time; $\sigma$ the angular tidal frequency. The divergence of the two-dimensional vector $\mathbf{v}h(v_\theta h, v_\varphi h)$ entering (1c) is defined in spherical coordinates as

$$\mathrm{div}_2(\mathbf{v}h) = \frac{1}{a\sin\theta}\left\{\frac{\partial(hv_\theta\sin\theta)}{\partial\theta} + \frac{\partial(hv_\varphi)}{\partial\varphi}\right\}. \tag{4}$$

Equations (1a) and (1b) are the $\theta$ and $\varphi$ components in the equations of motion and (1c) is a continuity condition (constancy of the volume of an element of fluid under tidal flow).

As will be shown in section 6, the pole and long-period tide can be considered to be laminar. In such a case, the force of bottom friction is proportional to the velocity:

$$F_\theta = -\kappa v_\theta; \quad F_\varphi = -\kappa v_\varphi, \tag{5a}$$

where $\kappa$ is the coefficient of bottom friction. Its numerical value is, as will be shown in section 7, of the order of

$$\kappa \sim 10^{-8}\,\mathrm{s}^{-1}. \tag{5b}$$

Equations (1) must satisfy boundary conditions; namely, the velocity component $v_n$ that is normal to $\Gamma_0$ is zero on the boundary $\Gamma_0$ between land and ocean:

$$v_n|_{\Gamma_0} = 0. \tag{6}$$

System (1a–c) can easily be reduced to a single scalar equation in tidal height, $\tilde{\zeta}$. To do this, we express $v_\theta$ and $v_\varphi$ in terms of $\tilde{\zeta} = \zeta - \bar{\zeta}$ using (1a) and (1b). Writing $v_\theta$, $v_\varphi$ in complex-valued form $v_\theta \sim \exp(i\sigma t)$, $v_\varphi \sim \exp(i\sigma t)$ and replacing $\dot{v}_\theta$, $\dot{v}_\varphi$ by $i\sigma v_\theta$ and $i\sigma v_\varphi$, respectively, we obtain

$$\begin{aligned} v_\varphi &= \frac{g}{a}\,\frac{2\omega\cos\theta\,\partial\tilde{\zeta}/\partial\theta - \dfrac{i\sigma+\kappa}{\sin\theta}\,\partial\tilde{\zeta}/\partial\varphi}{4\omega^2\cos^2\theta + (i\sigma+\kappa)^2} \\ v_\theta &= -\frac{g}{a}\,\frac{\dfrac{2\omega\cos\theta}{\sin\theta}\dfrac{\partial\tilde{\zeta}}{\partial\varphi} + (i\sigma+\kappa)\dfrac{\partial\tilde{\zeta}}{\partial\theta}}{4\omega^2\cos^2\theta + (i\sigma+\kappa)^2}. \end{aligned} \tag{7}$$

Substitution of these expressions in (1c) yields an equation in second-order partial derivatives in $\tilde{\zeta}(\theta, \varphi)$ which is equivalent to system (1).

The value of $\kappa$ determined through (5b) is four orders smaller than the angular velocity of the Earth's rotation $\omega = 0.7 \times 10^{-4}\,\mathrm{s}^{-1}$, while the ratio $\sigma/\omega$ for the pole and 19-yr tide is equal to $2 \times 10^{-3}$ and $10^{-4}$, respectively. For this reason, solutions to (1) under boundary conditions (6) can be sought as expansions in powers of the small parameters $\sigma/\omega$ and $\kappa/\omega$. It would seem that, the parameters being so small, the terms containing them may be discarded. However, that is not really the case because, when $\sigma = \kappa = 0$, the order of equations (1) is reduced from two to one; that is, system (1) is equivalent to a single second-order scalar equation with small coefficients in front of the higher derivatives. It is known from the general theory of differential equations that solutions of such systems may significantly depend on the small parameters, i.e. may be unstable. Below it will be shown that when $\sigma = \kappa = 0$, solutions to (1) are not determined uniquely. If the solutions are represented as expansions in powers of the small parameters $\sigma/\omega$, $\kappa/\omega$ and afterwards only linear terms are retained, then equations (1) in partial derivatives can be reduced to ordinary equations that have a unique solution and are not unstable. Because the solutions of equations (1) are unstable, a direct numerical integration is hardly possible; whereas integration of sets of ordinary equations does not pose any serious computational difficulties.

The principal result of the present paper, equation (34), was also derived by Molodensky (1985) but by a less rigorous approach. The author does not wish to pursue here the numerical calculation of the effect of the pole tide on the Chandler period which was considered in Molodensky (1985). Such calculations are very sensitive to small changes in depth (see discussion following equation (50) of the present paper). They may be re-considered in a later paper.

## 3. REPRESENTATION OF SOLUTIONS AS EXPANSIONS IN $\sigma/\omega$, $\kappa/\omega$

Expanding (7) in powers of $\sigma/\omega$ and $\kappa/\omega$ and retaining linear terms alone, we get

$$\mathbf{v} = \mathbf{v}^{(0)} + \mathbf{v}^{(1)}, \tag{8}$$

where

$$v_\varphi^{(0)} = \frac{g}{2a\omega \cos\theta} \frac{\partial \bar{\xi}}{\partial \theta} \tag{9a}$$

$$v_\theta^{(0)} = -\frac{g}{2a\omega \cos\theta \sin\theta} \frac{\partial \bar{\xi}}{\partial \varphi} \tag{9b}$$

are zero-order terms, while

$$\mathbf{v}^{(1)} = -\frac{g}{4\omega^2 \cos^2\theta} (\kappa + i\sigma) \nabla \bar{\xi} \tag{10}$$

are first-order terms.

Substitution of these expressions into (1c) gives

$$L_0(\bar{\xi}) + L_1(\bar{\xi}) = i\sigma\bar{\xi}, \tag{11}$$

where

$$L_0(\bar{\xi}) = -\mathrm{div}_2\,(h\mathbf{v}^{(0)}(\bar{\xi})) = \frac{g}{2\omega a^2 \sin\theta}\left(\frac{\partial(h/\cos\theta)}{\partial\theta}\frac{\partial\bar{\xi}}{\partial\varphi} - \frac{\partial(h/\cos\theta)}{\partial\varphi}\frac{\partial\bar{\xi}}{\partial\theta}\right); \tag{12}$$

$$L_1(\bar{\xi}) = -i\sigma\bar{\xi} - \mathrm{div}_2\,[h\mathbf{v}^{(1)}(\bar{\xi})] = -i\sigma\bar{\xi} + \frac{g}{4\omega^2}\left\{\frac{h(\kappa+i\sigma)}{\cos^2\theta}\Delta\bar{\xi} + \left[\nabla\bar{\xi}, \nabla\left(\frac{h(\kappa+i\sigma)}{\cos^2\theta}\right)\right]\right\}. \tag{13}$$

The boundary value problem (11) and (6) is an inhomogeneous equation in second partial derivatives with homogeneous boundary conditions. From (13) one can see that $L_1(\bar{\xi}) = 0$ when $\sigma = 0$ and $\kappa = 0$, so that the order of equation (11) is lowered from two to one. In addition, the right-hand side of (11) involves a small factor, $\sigma$. If this is set equal to zero, the result is a homogeneous equation with homogeneous boundary conditions whose solution is not determined uniquely. Since we do not know anything about the asymptotic behaviour of the ratio $\bar{\xi}/\bar{\xi}$ at low frequency beforehand, the right-hand side of (11) cannot, in the general case, be discarded, even when $\sigma \to 0$. For this reason the natural procedure would be to begin by examining equation (11) in the approximation $L_1(\bar{\xi}) = 0$; the factor $\sigma$ on the right-hand side of (11) is to be considered a small parameter which does not, however, vanish.

## 4. APPROXIMATION $L_1(\bar{\xi}) = 0$

In this approximation, eqn (11) becomes an inhomogeneous first-order equation

$$L_0(\bar{\xi}) = i\sigma\bar{\xi}. \tag{14}$$

The homogeneous equation corresponding to (14) reduces to the condition

$$L_0(\bar{\xi}) = 0 \tag{15a}$$

or

$$[\nabla(h/\cos\theta), \nabla\bar{\xi}] = 0. \tag{15b}$$

Relation (15b) shows that when there are no external tide-generating forces and $\sigma = \kappa = 0$, the vectors $\nabla\bar{\xi}$ and $\nabla(h/\cos\theta)$ are collinear, i.e. the isolines of $\bar{\xi} = \text{constant}$ coincide with those of $h/\cos\theta = \text{constant}$. When $\sigma = \kappa = 0$, the relations connecting $v_\varphi$, $v_\theta$ and $\bar{\xi}$ reduce to (9), from which it is seen that the vector $\mathbf{v} = (v_\theta, v_\varphi)$ is normal to $\nabla\bar{\xi}$; hence, isolines of $\bar{\xi} = \text{constant}$ coincide with lines of flow. It follows, therefore, from (15b) that

$$[\mathbf{v}, \nabla(h/\cos\theta)] = 0, \tag{16}$$

i.e. when $\sigma = \kappa = 0$, lines of flow coincide with isolines of $h/\cos\theta$, i.e. the 'geostrophic contours' of Greenspan (1969). Equation (16), of course, expresses the well-known law of conservation of potential vorticity.

Note that (16) can also be obtained as follows. Write the operator curl in spherical coordinates and calculate the radial divergence component of the left- and right-hand side of the equations of motion (1a) and (1b); when $v_\theta = v_\varphi = F_\theta = F_\varphi = 0$, we shall have

$$\frac{1}{a\sin\theta}\left(\frac{\partial}{\partial\varphi}(v_\varphi\cos\theta) + \frac{\partial}{\partial\theta}(v_\theta\sin\theta\cos\theta)\right) = \mathrm{div}_2\,(\mathbf{v}\cos\theta) = \cos\theta\,\mathrm{div}_2\,\mathbf{v} + (\mathbf{v}, \nabla\cos\theta) = 0. \tag{17a}$$

On the other hand, when $\sigma = 0$ and $\bar{\xi} = 0$, it follows from the incompressibility condition (1c) that

$$\mathrm{div}_2(\mathbf{v}h) = h\,\mathrm{div}_2\,\mathbf{v} + (\mathbf{v}, \nabla h) = 0. \tag{17b}$$

Substituting out $\mathrm{div}_2\,\mathbf{v}$ from these two relations, we get (16) once more.

The boundary condition (6) can easily be expressed in terms of $\bar{\xi}$. Since $\mathbf{v}$ and $\nabla\bar{\xi}$ are perpendicular in the zero approximation, it follows from (6) that

$$\bar{\xi}|_{\Gamma_0} = \text{constant}. \tag{18}$$

Obviously, the general solution to (15b) is

$$\bar{\xi} = f(h/\cos\theta), \tag{19}$$

where $f$ is an arbitrary function. Since the depth $h = 0$ at the coastline $\Gamma_0$, (18) is a straightforward consequence of (19).

To sum up, the homogeneous equation (15) has an infinity of solutions of the form (19), each automatically satisfying the boundary conditions.

Solutions to the inhomogeneous equation (14) can be found by the general method in use for integrating linear and quasilinear first-order equations (method of Cauchy characteristics (Kamke 1966)). Using the explicit form (12) of the operator $L_0(\bar{\xi})$, equation (14) can be represented in the form

$$\frac{\partial\alpha}{\partial\theta}\frac{\partial\bar{\xi}}{\partial\varphi} - \frac{\partial\alpha}{\partial\varphi}\frac{\partial\bar{\xi}}{\partial\theta} - i\sigma\sin\theta\bar{\xi} = 0 \tag{20}$$

where,

$$\alpha = \frac{g}{2\omega a^2}\frac{h}{\cos\theta}. \tag{21}$$

Relation (20) may be regarded as an orthogonality condition for vectors with the Cartesian coordinates

$$\mathbf{e}_1 = \left(\frac{\partial \bar{\xi}}{\partial \varphi}, \frac{\partial \bar{\xi}}{\partial \theta}, -1\right)$$

and

$$\mathbf{e}_2 = \left(\frac{\partial \alpha}{\partial \theta}, -\frac{\partial \alpha}{\partial \varphi}, i\sigma \sin\theta \bar{\xi}\right).$$

Since

$$d\bar{\xi} = \frac{\partial \bar{\xi}}{\partial \varphi} d\varphi + \frac{\partial \bar{\xi}}{\partial \theta} d\theta,$$

the vector $\mathbf{e}_1$ is also orthogonal to the vector

$$\mathbf{e}_3 = (d\varphi, d\theta, d\bar{\xi})$$

which is tangent to the desired surface $\bar{\xi}(\theta, \varphi)$. The vector $\mathbf{e}_2$ is thus perpendicular to the normal to the surface $\bar{\xi}(\theta, \varphi)$, hence itself lies in a plane that is tangent to that surface. Consequently, the curves defined by the ordinary differential equations

$$\frac{d\varphi}{\partial \alpha/\partial \theta} = \frac{d\theta}{-\partial \alpha/\partial \varphi} = \frac{d\bar{\xi}}{i\sigma \sin\theta \bar{\xi}} \tag{22}$$

belong to the desired surface $\bar{\xi}(\theta, \varphi)$.

Equations (22) impose two restrictions on the three differentials $d\varphi, d\theta, d\bar{\xi}$, hence describe a one-parameter family of curves called characteristics of the original equation (20). If each characteristic has a single point of intersection with the contour $\Gamma_0$, the conditions for $\bar{\xi}$ at $\Gamma_0$ can be regarded as initial conditions for equations (22). In such a case, the family of characteristics (22) fills the entire surface $\zeta(\theta, \varphi)$ and the solution of the original equation in first-order partial derivatives can be considered as known.

It is easy to show, however, that the case under consideration is degenerate, no characteristic $\Gamma$ having a common point with $\Gamma_0$. To see this, note that the first part of equation (22) yields

$$\frac{\partial \alpha}{\partial \theta} d\theta + \frac{\partial \alpha}{\partial \varphi} d\varphi = 0, \tag{23}$$

i.e. the characteristics $\Gamma$ coincide with the isolines of $\alpha = \text{constant}$ or $h/\cos\theta = \text{constant}$ defined above. Since $\alpha = 0$ at the coastline $\Gamma_0$, the characteristics $\Gamma$ corresponding to $\alpha \neq 0$ do not intersect $\Gamma_0$.

The second part of (22) defines the increment $d\bar{\xi}$ along $\Gamma$. Introducing an element of length $dl$ of the contour $\Gamma$ and remembering that (23) holds on that contour, we get

$$\begin{aligned} dl &= a(d\theta^2 + \sin^2\theta\, d\varphi^2)^{1/2} \\ &= a\left(\frac{(\partial \alpha/\partial \varphi)^2}{(\partial \alpha/\partial \theta)^2} d\varphi^2 + \sin^2\theta\, d\varphi^2\right)^{1/2} \\ &= a^2 \left| \frac{\sin\theta}{\partial \alpha/\partial \theta} \frac{\partial \alpha}{\partial n} d\varphi \right|, \end{aligned}$$

where

$$\begin{aligned} \partial \alpha/\partial n &= \pm|\nabla \alpha| \\ &= \pm\frac{1}{a}\left[\left(\frac{\partial \alpha}{\partial \theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial \alpha}{\partial \varphi}\right)^2\right]^{1/2}, \end{aligned}$$

$\partial/\partial n$ being the derivative along the normal to $\Gamma$.

If $\theta$ is measured from the north pole, $\varphi$ is western longitude, as we move clockwise along $\Gamma$, and $\partial/\partial n$ is the derivative along the outer normal, then the expression enclosed within the modulus lines is always positive. Hence

$$\frac{d\varphi}{\partial \alpha/\partial \theta} = \frac{dl}{a^2 \sin\theta\, \partial \alpha/\partial h}. \tag{24}$$

Consequently, the second part of equations (22) can also be represented in the form

$$d\bar{\xi} = \frac{i\sigma \bar{\xi}}{a^2\, \partial \alpha/\partial n} dl. \tag{25}$$

Equation (25) is fully equivalent to the original equation (20), in the sense that any integral curve of (25) belongs to the surface $\bar{\xi}(\theta, \varphi)$ defined by (20) and, conversely, any solution of (20) can be represented as a family of integral curves of (25). For this reason, the existence condition for solutions of the partial equation is equivalent to that for solutions of the ordinary equation (25). It is easy to see that this latter reduces to the single requirement

$$\oint_\Gamma d\bar{\xi} = \frac{i\sigma}{a^2} \oint_\Gamma \frac{\bar{\xi}\, dl}{\partial \alpha/\partial n} = 0. \tag{26}$$

When (26) does not hold, the increment of $\bar{\xi}$ along a closed contour $h/\cos\theta = \text{constant}$ does not vanish, which is incompatible with the assumption of $\bar{\xi}$ being a single-valued function of the coordinates.

For the general case of an arbitrary depth distribution $h = h(\theta, \varphi)$, condition (26) is not true, hence there will be no unique solution to the inhomogeneous equation (20).

An exception to this rule is the case of the pole tide in an ocean with axially symmetric distributions of land, sea surface and depth. When $h = h(\theta, \varphi)$ and $\bar{\xi}$ has values given by (2) and (3b), the contours $\Gamma$ are identical with the parallels $\theta = \text{constant}$, and the left-hand side of (26) reduces to the vanishing integrals

$$\int_0^{2\pi} \bar{\xi}|_{\theta=\text{constant}}\, d\varphi. \tag{27}$$

In such a case, equation (20) has an elementary solution

$$\bar{\xi} = -\frac{\bar{\zeta}\sigma \sin\theta}{\partial \alpha/\partial \theta} \tag{28}$$

which falls off as $\sigma \to 0$.

To sum up, we have shown that the inhomogeneous equation (11) in the general case has no solution in the approximation $L_1(\bar{\xi}) = 0$. This means that the term $L_1(\bar{\xi})$ cannot be discarded even in the limiting case $\sigma \to 0$, $\kappa \to 0$ in which $L_1(\bar{\xi})/L_0(\bar{\xi}) \to 0$.

## 5. SOLUTION WITH $L_1(\bar{\xi})$ TAKEN INTO ACCOUNT

When the term $L_1(\bar{\xi})$ has been inserted into equation (20), this takes the form

$$\frac{\partial \alpha}{\partial \theta}\frac{\partial \bar{\xi}}{\partial \varphi} - \frac{\partial \alpha}{\partial \varphi}\frac{\partial \bar{\xi}}{\partial \theta} + \sin\theta[L_1(\bar{\xi}) - i\sigma \bar{\xi}] = 0. \tag{29}$$

An exact imitation of what we did in Section 3 gives an equation that is similar to (25), except that $i\sigma\bar{\xi}$ has been replaced by $i\sigma\bar{\zeta} - L_1(\zeta)$:

$$\left.\frac{d\bar{\zeta}}{dl}\right|_\Gamma = \frac{1}{a^2\,\partial\alpha/\partial n}[i\sigma\bar{\zeta} - L_1(\bar{\zeta})]. \tag{30}$$

As far as form is concerned, (30) is still a second-order partial-derivative equation but, considering that the coefficients in front of the higher derivatives are small, it can be reduced to a set of ordinary differential equations. To see this, we make use of a perturbation method in the small parameters $\sigma/\omega$, $\kappa/\omega$. Since the right-hand side of (30) involves the functions $\bar{\zeta}$, $\zeta$ and derivatives of $\bar{\zeta}$ with respect to $\theta$, $\varphi$ with small factors in front of them, their values may be taken from the zero-order approximation (19). Setting

$$\bar{\zeta} = \bar{\zeta}_0 + \bar{\zeta}_1$$

where $\bar{\zeta}_0$, $\bar{\zeta}_1$ are the zero- and first-order approximation, respectively, and putting $\bar{\zeta}_0 = \bar{\zeta}_0\ (h/\cos\theta)$ in accordance with (19) or, which amounts to the same thing,

$$\bar{\zeta}_0 = \bar{\zeta}_0(\alpha), \tag{31}$$

we obtain

$$\nabla\bar{\zeta} \approx \nabla\bar{\zeta}_0 = \bar{\zeta}_0'\,\nabla\alpha, \quad \Delta\bar{\zeta} \approx \Delta\bar{\zeta}_0 = \bar{\zeta}_0''(\nabla\alpha)^2 + \bar{\zeta}_0'\,\Delta\alpha,$$

where the prime denotes differentiation with respect to $\alpha$.

Substitution of these in (13) yields

$$\begin{aligned} L_1(\bar{\zeta}) \approx L_1[\bar{\zeta}_0(\alpha)] = -i\sigma\bar{\zeta}_0 + \frac{\alpha a^2}{2\omega\cos\theta} \\ \times [(\kappa + i\sigma) \times (\bar{\zeta}_0''(\nabla\alpha)^2 + \bar{\zeta}_0'\,\Delta\alpha)] \\ + \frac{a^2}{2\omega}\bar{\zeta}_0'\frac{\partial\alpha}{\partial n}\frac{\partial}{\partial n}\left(\frac{(\kappa + i\sigma)\alpha}{\cos\theta}\right). \end{aligned} \tag{32}$$

The right-hand side of (32) involves an unknown function, $\bar{\zeta}_0(\alpha)$. As was mentioned in Section 3, that function cannot be found from the zero-order approximation, and its determination requires first-order terms.

To begin with, we consider the case in which the coastline does not intersect the equator, and all contours $\Gamma$ of the form $h/\cos\theta = \text{constant}$ are closed ones. To find $\bar{\zeta}_0(\alpha)$, substitute (31) into (30) and integrate $d\bar{\zeta}/dl$ along $\Gamma$. Since $\bar{\zeta}$ is a single-valued function of the coordinates, its increment along a closed contour $\Gamma$ is equal to zero. Therefore, from (30) it follows that

$$\begin{aligned} \oint_\Gamma d\bar{\zeta} = \oint_\Gamma d\bar{\zeta}_1 = \oint_\Gamma \frac{d\bar{\zeta}}{dl}\,dl \\ = \frac{1}{a^2}\oint_\Gamma \frac{dl}{\partial\alpha/\partial n}\{i\sigma\bar{\zeta} - L_1[\bar{\zeta}_0(\alpha)]\} = 0. \end{aligned} \tag{33}$$

Since we have $\alpha = \text{constant}$ on $\Gamma$, $\bar{\zeta}_0(\alpha)$ and its derivatives with respect to $\alpha$ are also constant, so they may be put before the integration sign. The result is an ordinary differential equation in $\bar{\zeta}_0(\alpha)$:

$$c_1(\alpha)\bar{\zeta}_0''(\alpha) + c_2(\alpha)\bar{\zeta}_0'(\alpha) + c_3(\alpha)\bar{\zeta}_0(\alpha) = b(\alpha), \tag{34}$$

where

$$c_1(\alpha) = \alpha\oint_\Gamma \frac{(\kappa + i\sigma)\,\partial\alpha/\partial n}{\cos\theta}\,dl,$$

$$\begin{aligned} c_2(\alpha) = \oint\left[\alpha\frac{(\kappa + i\sigma)\,\Delta\alpha}{\cos\theta\,\partial\alpha/\partial n} + \frac{\partial}{\partial n}\left(\frac{(\kappa + i\sigma)\alpha}{\cos\theta}\right)\right]dl \\ = \oint_\Gamma \frac{\operatorname{div}_2\left(\dfrac{\alpha(\kappa + i\sigma)\,\nabla\alpha}{\cos\theta}\right)}{\partial\alpha/\partial n}\,dl; \end{aligned}$$

$$c_3(\alpha) = -\frac{2i\sigma\omega}{a^2}\oint_\Gamma \frac{dl}{\partial\alpha/\partial n};$$

$$b(\alpha) = \frac{2i\sigma\omega}{a^2}\oint_\Gamma \frac{\zeta\,dl}{\partial\alpha/\partial n}. \tag{35}$$

It should be noted that $c_1(\alpha)$ and $c_2(\alpha)$ are related as follows:

$$c_2(\alpha) = \frac{dc_1(\alpha)}{d\alpha}. \tag{36}$$

To see this, multiply $c_2(\alpha)$ by the differential $d\alpha$ and recall that

$$\frac{d\alpha\,dl}{\partial\alpha/\partial n} = \delta s,$$

where $\delta s$ is an element of area between two infinitesimally close isolines of $\alpha = \alpha_0$ and $\alpha = \alpha_0 + d\alpha$. Transforming the integral over a surface into one along a line by Gauss's formula, we find

$$\begin{aligned} d\alpha\,c_2(\alpha) = \iint_{\delta s}\operatorname{div}_2\left(\frac{\alpha(\kappa + i\sigma)\,\nabla\alpha}{\cos\theta}\right)ds \\ = d\left(\alpha\oint_\Gamma \frac{(\kappa + i\sigma)\,\partial\alpha/\partial n}{\cos\theta}\,dl\right) = dc_1(\alpha). \end{aligned}$$

Dividing both sides by $d\alpha$ yields (36).

Combining (36) and (34), one can rewrite the latter in the form

$$[c_1(\alpha)\bar{\zeta}_0'(\alpha)]' + c_3(\alpha)\bar{\zeta}_0(\alpha) = b(\alpha). \tag{34a}$$

Integrating (34a) within the limits $\alpha_1$ and $\alpha_2$ gives

$$\begin{aligned} c_1(\alpha)\bar{\zeta}_0'(\alpha)\big|_{\alpha_1}^{\alpha_2} = \int_{\alpha_1}^{\alpha_2}[c(\alpha) - c_3(\alpha)\bar{\zeta}_0(\alpha)]\,d\alpha \\ = \frac{2i\sigma\omega}{a^2}\int_{\alpha_1}^{\alpha_2}\oint_\Gamma \frac{\zeta + \bar{\zeta}_0}{\partial\alpha/\partial n}\,dl\,d\alpha \\ = \frac{2i\sigma\omega}{a^2}\iint_{s_{12}}\zeta\,ds, \end{aligned} \tag{34b}$$

where $s_{12}$ is the area enclosed between the isolines of $\alpha = \alpha_1$ and $\alpha = \alpha_2$.

The integral $i\sigma\iint_{s_{12}}\zeta\,ds$ determines the rate of change of the volume of sea water within the region between the isolines of $\alpha = \alpha_1$ and $\alpha = \alpha_2$, while

$$\frac{a^2}{2\omega}c_1\bar{\zeta}_0'(\alpha)\big|_{\alpha_1}^{\alpha_2} = \oint_{\Gamma_2} hv_n^{(1)}\,dl - \oint_{\Gamma_1} hv_n^{(1)}\,dl$$

is the inflow through these boundaries. Therefore, (34b) is the condition of mass conservation. Thus, the condition for conservation of total mass is a straightforward consequence of (34).

After separating the real and imaginary parts in (34a), we get a set of ordinary fourth-order differential equations

$$\begin{aligned} (a_1\bar{\zeta}_0^{c\prime} + a_2\bar{\zeta}^{s\prime})' + a_3\bar{\zeta}_0^s = b_1 \\ (a_1\bar{\zeta}_0^{s\prime} - a_2\bar{\zeta}_0^{c\prime})' - a_3\bar{\zeta}_0^c = b_2, \end{aligned} \tag{37}$$

where

$a_1 = \mathrm{Re}\, c_1;\quad a_2 = \mathrm{Im}\, c_1;\quad a_3 = \mathrm{Im}\, c_3;$

$b_1 = \mathrm{Re}\, b;\quad b_2 = -\mathrm{Im}\, b;$

$\bar{\zeta}^c = \mathrm{Re}\, \bar{\zeta}_0;\quad \bar{\zeta}^s = -\mathrm{Im}\, \bar{\zeta}_0;$

$$\bar{\zeta} = \mathrm{Re}\,(\bar{\zeta}_0 e^{i\sigma t}) = \mathrm{Re}\, \bar{\zeta}_0 \cos \sigma t - \mathrm{Im}\, \bar{\zeta}_0 \sin \sigma t = \bar{\zeta}^c \cos \sigma t + \bar{\zeta}^s \sin \sigma t;$$

$$b_2 = \frac{2\sigma\omega}{a^2} \bar{\zeta}_0(\tfrac{3}{2} \cos^2 \theta - \tfrac{1}{2});$$

$$b_1 = 0$$

for the case of long-period lunisolar tides of the first species and

$$b_1 = \frac{2\sigma\omega}{a^2} \bar{\zeta}_0 \oint_\Gamma \frac{\sin \theta \cos \theta \sin \varphi}{\partial \alpha / \partial n}\, dl;$$

$$b_2 = -\frac{2\sigma\omega}{a^2} \bar{\zeta}_0 \oint_\Gamma \frac{\sin \theta \cos \theta \cos \varphi}{\partial \alpha / \partial n}\, dl$$

for the case of the pole tide (tide of the second species according to Laplace's classification); $\bar{\zeta}_0 = \Phi_0/g$; $\Phi_0$ is the amplitude of the tide-generating potential given by (3a) and (3b).

When $\kappa = 0$, we have $a_1 = 0$, and the set of equations (37) separates into two independent sets of second-order equations.

We discuss the conditions under which these equations can be used.

The operator $L_1(\bar{\zeta})$ can obviously be represented in the form (32) only in those cases in which $|\bar{\zeta}_1| \ll |\bar{\zeta}_0|$. To see when this is true, note that $\partial \bar{\zeta}_0 / \partial l|_\Gamma = 0$ because of (31); hence, the left-hand side of (30) equals $\partial \bar{\zeta}_1 / \partial l|_\Gamma$. Replacing $\partial \bar{\zeta} / \partial l$ by $\partial \bar{\zeta}_1 / \partial l$ in the left-hand side of (30) and integrating along the contour $\Gamma$ from $l_1$ to $l_2$, we get

$$\bar{\zeta}_1(l_2) - \bar{\zeta}_1(l_1) \sim \frac{l |i\sigma\bar{\zeta} - L_1(\bar{\zeta})|}{a^2\, \partial\alpha/\partial n},$$

where $l$ is the contour length. According to (13) we have

$$L_1(\bar{\zeta}) \sim \sigma |\bar{\zeta}_0| + \frac{gh(\sigma + \kappa)}{\omega^2 l_0^2} |\bar{\zeta}|,$$

where $l_0$ is the scale of distance over which the functions $\bar{\zeta}$, $h$, $\kappa$ experience significant variation. The condition $|\bar{\zeta}_1| \ll |\bar{\zeta}_0|$ can also be represented in the form

$$\begin{cases} l\sigma\bar{\zeta} \ll \dfrac{gh\bar{\zeta}}{\omega l_0} \\ \dfrac{gh(\sigma+\kappa)}{\omega^2 l_0^2}\, l\, \dfrac{\omega a^2 l_0}{a^2 gh} \ll 1 \end{cases} \tag{38a}$$

or

$$\sigma \ll \frac{gh}{\omega l l_0} \frac{\bar{\zeta}}{\bar{\zeta}}; \qquad \sigma + \kappa \ll \frac{\omega l_0}{l}. \tag{38b}$$

When $l_0 \sim l \sim a \sim 6 \times 10^3$ km, $h \sim 4$ km, and $\bar{\zeta} \sim \bar{\zeta}$, these conditions reduce to the requirements $\sigma \ll \omega$, $\kappa \ll \omega$ which, as we have seen, are true within a wide margin of safety. The condition (38b) can break down either when $|\bar{\zeta}| \gg \bar{\zeta}_0$ or when $h \ll 4$ km. We are not interested in the former of these cases, as the relevant asymptotic behaviour of $\bar{\zeta}$ is known. In the case $|\bar{\zeta}| \sim |\bar{\zeta}_0|$, condition (38b) breaks down at the Chandler wobble frequency when $h \lesssim 50$ m and at the frequency of the 19-yr tide when $h \lesssim 2$ m. Since $h$ is of the order of a few kilometres over most of the worlds ocean, one can assume (38) to be true and equations (34) to be applicable.

The boundary conditions for (34) require regularity of solution (finite velocity of tide flow in the vicinity of the coastline, the equator, and extrema of $\alpha(\theta, \varphi)$) and will be considered below.

## 6. EXAMPLES

### 6.1 The case $h = h(\theta)$, $\kappa = 0$

As a first example of the use of equations (34), we show that a particular case of these are the well-known Lamb's equations (Lamb 1932, section 218) for the long-period tide in an ocean of uniform depth over the whole Earth. Putting $\kappa = 0$, $h = $ constant in (35) and denoting $k = g/2\omega a^2$ for the sake of brevity, we get

$$\Delta\alpha = \frac{1}{a^2 \sin\theta} \frac{\partial}{\partial\theta} \left\{ \sin\theta \left[ \frac{\partial}{\partial\theta} \left( \frac{kh}{\cos\theta} \right) \right] \right\} = \frac{2kh}{a^2 \cos^3 \theta};$$

$$\frac{\partial\alpha}{\partial n} = \frac{1}{a} \frac{\partial}{\partial\theta} \left( \frac{kh}{\cos\theta} \right) = \frac{kh \sin\theta}{a \cos^2\theta} \frac{\partial}{\partial n} \left( \frac{\alpha}{\cos\theta} \right) = \frac{1}{a} \frac{\partial}{\partial\theta} \left( \frac{kh}{\cos^2\theta} \right) = \frac{2kh \sin\theta}{a \cos^3\theta}. \tag{39}$$

When $h = $ constant, contours $\Gamma$ of the form $\alpha = $ constant are identical with parallels of $\theta = $ constant. Substituting (39) into (35) and integrating over $l$ ($dl = a \sin\theta\, d\varphi$), we get

$$c_1 = 2\pi i\sigma k^2 h^2 \frac{\sin^2\theta}{\cos^4\theta} = 2\pi i\sigma \left( \frac{\alpha^4}{k^2 h^2} - \alpha^2 \right);$$

$$c_2 = \frac{dc_1}{d\alpha} = 2\pi i\sigma \left( \frac{4\alpha^3}{k^2 h^2} - 2\alpha \right); \tag{40}$$

$$c_3 = -\frac{4\pi i\sigma\omega}{\alpha} \cos\theta;$$

$$b = 0$$

for the pole tide and

$$b = \frac{4\pi i\sigma\omega}{\alpha} \bar{\zeta}_0 \left( \tfrac{3}{2} \cos^3\theta - \frac{\cos\theta}{2} \right)$$

for the long-period lunisolar tide of the first species.

Expressing the derivatives of $\bar{\zeta}_0$ with respect to $\alpha$ in terms of derivatives with respect to $\theta$, we get

$$\frac{d\bar{\zeta}_0}{d\alpha} = \frac{\cos^2\theta}{kh \sin\theta} \frac{d\bar{\zeta}_0}{d\theta};$$

$$\frac{d^2\bar{\zeta}_0}{d\alpha^2} = \left( \frac{\cos^2\theta}{kh \sin\theta} \right)^2 \frac{d\bar{\zeta}_0}{d\theta^2} - \left( \frac{2\cos\theta}{kh} + \frac{\cos^3\theta}{kh \sin^2\theta} \right) \times \frac{\cos^2\theta}{kh \sin\theta} \frac{d\bar{\zeta}_0}{d\theta}. \tag{41}$$

Substitution of (40) and (41) in (34) gives

$$\frac{d^2\bar{\xi}_0}{d\theta^2}+\frac{1+\sin^2\theta}{\sin\theta\cos\theta}\frac{d\bar{\xi}_0}{d\theta}-\frac{4\omega^2a^2\cos^2\theta}{gh}(\bar{\xi}_0+\delta\bar{\xi}_0)=0, \tag{42}$$

where

$$\delta=\begin{cases}1 & \text{for the long-period lunisolar tide (of the first species)}\\ 0 & \text{for the pole tide (of second species)}\end{cases}$$

For the case of an arbitrary axially symmetric depth distribution $h=h(\theta)$ and $\kappa=0$, equation (33) becomes

$$-\frac{\partial}{\partial\cos\theta}\left(\frac{h\sin^2\theta}{\cos^2\theta}\frac{d\bar{\xi}_0(\theta)}{d\cos\theta}\right)+\frac{4\omega^2a^2}{g}\bar{\xi}_0(\theta)=-\frac{2\omega^2a^2}{\pi g}\int_0^{2\pi}\bar{\xi}\,d\varphi. \tag{43}$$

When $\delta=1$, equations (42) and (43) are identical with Lamb's equation (Lamb 1932, section 218, formula (5)), if we put $f=\sigma/2\omega=0$ and $s=0$ in it; for $\bar{\xi}$ as given by (2), and (3a), the right-hand side of (43) and $\delta$ are zero, and equations (42) and (43) have only the trivial solution $\bar{\xi}=0$. That conclusion also follows from (28) when $\sigma$ is made to approach zero in it:

$$\lim_{\sigma\to 0}\bar{\xi}(\theta,\sigma)=0. \tag{44}$$

It is easy to see that for the general case of an arbitrary depth distribution the contours $h/\cos\theta=$ constant do not coincide with parallels of $\theta=$ constant, the integrals $\oint_\Gamma\bar{\xi}\,dl$ are generally speaking not equal to zero and (44) is not true.

To sum up, in the case $\kappa=0$ the height of the pole tide asymptotically approaches the equipotential surface only when the distributions of land, sea and depth are axially symmetric.

### 6.2 Small depth approximation

Equation (34) has a fairly simple solution for the case of sufficiently small depths. We have $\alpha\to 0$ when $h\to 0$, and the coefficients $c_1\to 0$ and $c_2\to 0$ in accordance with (35). Consequently, the differential equation (34) reduces to the non-differential condition

$$\bar{\xi}_0(\alpha)=-\frac{\oint_\Gamma\frac{\bar{\xi}\,dl}{\partial\alpha/\partial n}}{\oint_\Gamma\frac{dl}{\partial\alpha/\partial n}}. \tag{45}$$

From this formula one can see that the value of $\bar{\xi}_0$ on $\Gamma$ is minus the value of $\bar{\xi}$ on the same contour calculated with the weighting function $(\partial\alpha/\partial n)^{-1}$. The relation has a simple physical sense, viz. when $h\to 0$, the flow through $\Gamma$, $\int h(\mathbf{v},\mathbf{n})\,dl$, tends to zero, hence

$$\iint_s\zeta\,ds=\iint_s(\bar{\xi}_0+\bar{\xi})\,ds=0.$$

Taking into account that $ds=dl\,d\alpha/(\partial\alpha/\partial n)$ and $\bar{\xi}_0|_\Gamma=$ constant, one can easily see that this condition is equivalent to (45).

From (45) we have $\bar{\xi}_0=-\bar{\xi}$ near maxima and minima of $h/\cos\theta$, where the contour $\Gamma$ degenerates to a point, i.e. the height of the oceanic tide $\zeta=\bar{\xi}+\bar{\xi}=0$.

For the small depth approximation to be valid it is necessary that

$$|\mathrm{div}_2\,h\mathbf{v}^{(1)}(\bar{\xi})|\ll\sigma\,|\bar{\xi}|$$

or

$$\frac{gh}{\omega^2l^2}\ll 1,$$

where $l$ is the typical scale of distance over which the function $h/\cos\theta$ experiences significant variation. An example that illustrates the dependence of $\bar{\xi}$ on $l$ and $h$ will be considered in Section 6.5.

### 6.3 Model of a small basin that does not intersect the equator

Consider a circular basin that does not intersect the equator, has a centrally symmetric distribution of depths, and whose size is much less than the Earth's radius. Denote the coordinates of the centre of the basin by $(\theta_0,\varphi_0)$, the polar coordinates of a point relative to the centre by $r$, $\psi$, and assume $\theta\approx$ constant within the region occupied by the basin. The contours $\Gamma$ are then identical with circles of $r=$ constant, and $\bar{\xi}$ is a function of $r$ only. Putting $\bar{\xi}=\bar{\xi}(r)$, $\theta=$ constant, and $\kappa=0$ in (13), we get

$$L_1(\bar{\xi})=-i\sigma\bar{\xi}_0+\frac{i\sigma g}{4\omega^2\cos^2\theta}\left\{\frac{h}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\bar{\xi}_0}{\partial r}\right)+\frac{\partial\bar{\xi}_0}{\partial r}\frac{dh}{dr}\right\}. \tag{46}$$

Substitution of this in (33) yields

$$\bar{\xi}_0-\frac{g}{4\omega^2a^2\cos^2\theta_0}\left[h\frac{d^2\bar{\xi}_0}{dr^2}+\left(\frac{h}{r}+\frac{dh}{dr}\right)\frac{d\bar{\xi}_0}{dr}\right]=-\frac{1}{2\pi}\int_0^{2\pi}\bar{\xi}(r,\psi)\,d\psi. \tag{47}$$

As (47) is a second-order equation, it requires two boundary conditions. These are as follows.

(1) As we have $h/r\to\infty$ in (47) when $h\neq 0$ and $r\to 0$, it follows from the finiteness of the solutions that

$$d\bar{\xi}_0/dr|_{r=0}=0. \tag{48}$$

(2) The other boundary condition follows from the fact that $\bar{\xi}_0$ is finite at the basin boundary (for $r=r_0$). The depth $h\to 0$ when $r\to r_0$; but as $h'<0$, the coefficient in front of $\bar{\xi}_0'$ is positive, while that in front of $\bar{\xi}_0''$ is negative and tends to zero. It is easy to show that the solution will then increase without bound in the general case. Imposing the requirement that $\bar{\xi}$ be finite when $r=r_0$, together with (48), makes the solution unique.

### 6.4 Model of a basin intersecting the equator; singular points

Equations (34) are based on the assumption that the isolines $\Gamma$ of the form $h/\cos\theta=$ constant are closed. It is easy to see, however, that this is not always the case. If the coastline intersects the equator, then the ratio $h/\cos\theta$ is an indefinite expression of the form 0/0 in an infinitesimal vicinity of an intersection point, and can take any value between $-\infty$ and $+\infty$. Therefore, the intersections of $\Gamma_0$ and the equator are singular points. Contours $\Gamma$ may begin and end at singular points. To see this, consider the simplest model—an ocean of uniform depth with steep walls within a coastline of arbitrary configuration. In such a case, isolines of $h/\cos\theta = \text{constant}$ are identical with parallels far from the coast; $h$ falls off rapidly near the land boundary, displacing isolines towards the equator where $1/|\cos\theta|$ is greater. If the coastline $\Gamma_0$ intersects the equator, then the condition $h/\cos\theta = \text{constant}$ gives $h \to 0$ when $\theta \to \pi/2$, i.e. all isolines of $h/\cos\theta = \text{constant}$ begin and end at the points of intersection between the coastline $h = 0$ and the equator $\theta = \pi/2$.

Although the contour $\Gamma$ in that case is not closed, (34) is still true. To see this, note that, when $\theta \to \pi/2$, formula (9b) and the condition for $v$ to be finite give

$$\left.\frac{\partial\bar{\xi}}{\partial\varphi}\right|_{\theta=\pi/2} = 0,$$

i.e. $\bar{\xi}$ is constant at the equator. For this reason integrals $\int_\Gamma d\bar{\xi}$ are again zero for those $\Gamma$ which begin and end at the equator, and integrals along the closed contours involved in (35) should be replaced by integrals along unclosed contours that begin and end at singular points. Then it is easy to see that the integrals defining $c_1(\alpha)$ and $c_2(\alpha)$ diverge, i.e. the coefficients of $\bar{\xi}_0''(\alpha)$ and $\bar{\xi}_0'(\alpha)$ in (34) increase without bound, and the solution of (34) degenerates into $\bar{\xi}_0(\alpha) = \text{constant}$. The value of this constant is determined by the condition for conservation of mass

$$\iint_s (\bar{\xi} + \bar{\xi})\, ds = 0,$$

whence

$$\text{constant} = -\frac{1}{s}\iint \bar{\xi}\, ds,$$

where $s$ is basin area.

One can see from this expression that the deviation of the dynamic tide from the static, in regions where the characteristics of $\Gamma$ begin and end at singular points, vanishes in the limit $\sigma \to 0$.

### 6.5 Realistic ocean models

Below, we consider several ocean models with realistic land and sea distribution and various distributions of depth.

In the simplest case—an ocean of uniform depth—all characteristics begin and end at the points of intersection between coastline and the equator; hence, as has been shown in Section 6.4, $\bar{\xi} = 0$.

To evaluate the effects of bottom topography, take a depth distribution of the form

$$h = h_0(1 + \varepsilon \sin n\theta \sin n\varphi),$$

where $h_0 = \text{constant}$, and $\varepsilon$ is the amplitude of depth fluctuations. When $n \gg 1$, this distribution corresponds to isolines of $\alpha = \text{constant}$, which have the shape of cells with radius $r_0 \sim \pi a/2n$. In the equatorial zone (when $|\cos\theta| \lesssim 1/2n\varepsilon$), the isolines of $\alpha = \text{constant}$ cease to be closed, they begin and end at singular points. Treating all closed cells as circles in a first approximation, one can use equation (47), Section 6.3, to describe the tide in a cell.

When $n \gg 1$, the functions $\bar{\xi}(\theta, \varphi)$ as given by (3) can be regarded as constant in a cell, i.e. one can put in (47)

$$\frac{1}{2\pi}\int_0^{2\pi} \bar{\xi}(r, \psi)\, d\psi = \bar{\xi}(\theta_0, \varphi_0).$$

Condition (48a) for the centre of a cell obviously remains true. In contrast to the case considered in Section 6.2, we have $h \neq 0$ at cell boundaries, so that the finiteness condition for $\bar{\xi}_0$ is true under any initial conditions for $\bar{\xi}_0|_{r=0}$. Since characteristics lying at cell boundaries are identical with parallels of $\theta = \text{constant}$, $\bar{\xi}_0$ is zero at them. Consequently, another boundary condition should be added to (48), viz.

$$\bar{\xi}_0(r_0) = 0.$$

When $\varepsilon \ll 1$, we have $dh/dr \ll h/r$ in equation (47), and this can be written in dimensionless form

$$H\bar{\xi}_0'' + \frac{H}{x}\bar{\xi}_0' - \zeta = 0, \tag{49}$$

where $\zeta = \bar{\xi} + \bar{\xi}_0$ and

$$H = \frac{g}{4\omega^2\cos^3\theta_0}\frac{h}{r_0^2} = \frac{73}{\cos^2\theta_0} = \frac{ha}{r_0^2}$$

is dimensionless depth, $r_0$ is cell radius, the prime denotes a derivative with respect to the dimensionless radius $x = r/r_0$ $(0 \leqslant x \leqslant 1)$.

The boundary conditions in dimensionless form are

$$\zeta|_{x=1} = \bar{\xi}, \tag{50a}$$

$$\zeta'|_{x=0} = 0. \tag{50b}$$

Expanding $\zeta$ in powers of $x$, one can easily show that the solution of (49) with the boundary conditions (50) has the form

$$\zeta = \bar{\xi}(1 + c_1 + c_2 + \cdots)^{-1}(1 + c_1x^2 + c_2x^4 + \cdots),$$

where

$$c_k = \frac{1}{2^2 \times 4^2 \times \cdots (2k)^2}\frac{1}{H^k}.$$

The ratio of the mean height of the dynamic tide to that of the static one is

$$\gamma = \frac{2}{\bar{\xi}}\int_0^1 x\zeta(x)\, dx = \frac{1 + c_1/2 + c_2/3 + \cdots}{1 + c_1 + c_2 + \cdots}.$$

The numerical values of this ratio for a set of values of $H$ are as follows:

| | | | | | |
|---|---|---|---|---|---|
| $H = 0.1$ | 0.2 | 0.5 | 1.0 | 2.0 | 5.0 |
| $\gamma = 0.520$ | 0.656 | 0.812 | 0.893 | 0.944 | 0.976 |

From the table one can see that when depth increases (or, which amounts to the same thing, when the horizontal dimensions of cells decrease), the dynamic tide approaches the static one. When $\cos^2\theta_0 = 0.5$ and $h = 4$ km, $H = 0.1$ corresponds to $r_0 = 6 \times 10^3$ km, and $H = 5$ to $r_0 = 8 \times 10^2$ km. The typical horizontal topographic scale is $r_0 \sim (1\text{–}5) \times 10^3$ km for the real ocean, so the dynamic tide deviates considerably from the static.

It should be noted that small changes in depth significantly affect the asymptotic behaviour of the solutions: when $\varepsilon$ is small, but not zero, the above solution does not depend on $\varepsilon$ and is not zero; while $\bar{\zeta}=0$ when $\varepsilon=0$, as shown in Section 6.4.

## 7 EFFECT OF BOTTOM FRICTION

When $\sigma\to 0$ and $x\neq 0$, the $c_3(\alpha)$ and $b(\alpha)$ given by (35) tend to zero, while $c_1(\alpha)$ and $c_2(\alpha)$ are different from zero, so that equation (34) takes the form

$$c_1(\alpha)\bar{\zeta}_0''(\alpha)+c_2(\alpha)\bar{\zeta}'(\alpha)=0. \tag{51}$$

One can easily see that this equation has a single finite solution

$$\bar{\zeta}_0(\alpha)=\text{constant}, \tag{52}$$

because we have $\Delta\alpha<0$ in (35) near maxima of $\alpha(\theta,\varphi)$, so that

$$\frac{c_1(\alpha)}{c_2(\alpha)}\to\beta(\alpha-\alpha_{\max}), \tag{53}$$

where $\beta$ is a positive constant. Substitution of (53) in (52) determines another solution of (51)

$$\bar{\zeta}_0'=(\alpha_{\max}-\alpha)^{-1/\beta}$$

which increases without bound when $\alpha\to\alpha_{\max}$.

Further, it follows from the condition of conservation of total mass that the constant in (52) is zero. Thus, when the coefficient of bottom friction is not zero, the deviation of the ocean surface from equilibrium vanishes asymptotically.

To see for what values of $\sigma$ and $\kappa$ approximation (51) is valid, one should obviously compare the terms entering (51) with $c_3(\alpha)\bar{\zeta}_0(\alpha)$. Assuming, for order of magnitude estimates,

$$\Delta\alpha\sim\alpha/l^2,\quad \partial\alpha/\partial n\sim\alpha/l,\quad \bar{\zeta}_0'\sim\bar{\zeta}_0/\alpha,$$
$$\bar{\zeta}_0''\sim\bar{\zeta}_0/\alpha^2,\quad \alpha\sim gh/\omega a^2,$$

where $l$ is the typical scale of distance over which depth changes significantly, we get

$$\frac{c_1\bar{\zeta}_0''+c_2\bar{\zeta}_0'}{c_3\bar{\zeta}_0}\sim\frac{\kappa gh}{l^2\sigma\omega^2}.$$

The effect of bottom friction can be disregarded only when this ratio is much less than one, i.e.

$$\frac{\kappa}{\sigma}\ll\frac{l^2\omega^2}{gh}. \tag{54}$$

To obtain an order of magnitude estimate for $\kappa$, we first evaluate the thickness of the bottom boundary layer, $D$, in which viscous friction is significant. When $\sigma\ll\omega$, motion within the layer is determined by the balance of the Coriolis force and viscous friction. The force due to the latter is of the order of $\nu v\delta S/D$ for a laminar flow (here $\nu$ is viscosity, $\delta S$ is an element of area), so

$$|2\rho[\boldsymbol{\omega}\mathbf{v}]D\,\delta S|\sim|\nu v\delta S/D|$$

whence

$$D\sim\left(\frac{\nu}{\omega\rho}\right)^{1/2}\sim 10\,\text{cm}$$

for $\nu\sim 2\times 10^{-2}$ poise.

For the pole and long-period lunisolar tides

$$v\sim(10^{-4}\text{–}10^{-5})\,\text{cm s}^{-1},$$

so that the Reynolds number

$$R=\rho Dv/\nu\sim 10^{-1}\text{–}10^{-2}$$

is far below the critical value; hence, the assumption of laminar flow within the bottom layer is valid.

We thus see that the force of friction acting on an element of area is of the order of $\nu v\delta S/D\sim v\delta S'\times(\nu\omega\rho)^{1/2}$, and

$$\kappa\sim\frac{(\nu\omega\rho)^{1/2}}{h}\sim 10^{-8}\,\text{s}^{-1}$$

for $h\sim 1$ km. Substitution of this into (54) yields

$$\sigma\gg\frac{(\nu\rho)^{1/2}g}{\omega^{3/2}l^2}\sim\frac{3\times 10^8\,\text{cm}^2\,\text{s}^{-1}}{l^2}. \tag{55}$$

When $l\sim 10^3$ km $=10^8$ cm, this gives

$$\sigma\gg 3\times 10^{-8}\,\text{s}^{-1}\quad\text{or}\quad T=\frac{2\pi}{\sigma}\ll 8\,\text{yr}.$$

For the opposite limiting case, $T\gg 8$ yr, the deviation of the dynamic tide from the static can be ignored.

From this estimate one can see that the effect of laminar bottom friction is not so great that tides of 14-month or even 19-yr period can be treated as static. Yet one cannot also wholly ignore bottom friction and assume $\kappa=0$. As was to be expected, bottom friction is significant in shelves and shallow-water areas where the parameter $l$ may be well below $10^3$ km and the right-hand side of (55) increases like $1/l^2$.

We conclude by noting that the term $k_h\,\Delta\mathbf{v}$ is sometimes added (Kagan & Monin 1978; Marchuk & Kagan 1983) to Laplace's equations (1); here $k_h$ is the so-called 'coefficient of turbulent horizontal friction' having the dimension of kinematic viscosity $\nu/\rho$. Schwiderski (1980) finds the numerical values of $k_h$ by trial and error in such a way as to obtain the best fit between theoretical and observed tides. With $h$ varying between 10 and 7000 m he assumes $10^7\,\text{cm}^2\,\text{s}^{-1}<k_h<10^{10}\,\text{cm}^2\,\text{s}^{-1}$ which is 9–12 orders greater than the kinematic viscosity $\nu/\rho$. When $l\sim 10^8$ cm,

$$|k_h\,\Delta\mathbf{v}|\sim\kappa'\,|\mathbf{v}|,$$

where the value $\kappa'\sim k_h/l^2\sim(10^{-6}\text{–}10^{-9})\,\text{s}^{-1}$ is comparable to the value of $\kappa$ assumed in the above discussion.

Flow velocity and the Reynolds number for the long-period tide are about four orders smaller than those for the short-period components; it is not ruled out, however, that horizontal turbulent friction may, in some cases, exist here too.

Equations (37) can obviously be extended to the case $k_h\neq 0$, provided $\kappa$ is understood as the operator

$$\kappa=\kappa_0-k_h\,\Delta.$$

Then

$$L_1(\bar{\zeta}) = -i\sigma\bar{\zeta} - \frac{g}{4\omega^2}\mathrm{div}_2 \times \left(\frac{h}{\cos^2\theta}[(\kappa_0 + i\sigma)\nabla\bar{\zeta} - k_h\nabla(\Delta\bar{\zeta})]\right)$$

and the order of (37) becomes six instead of four.

## 8 CONCLUSIONS

Our main conclusions are as follows:

I. The asymptotic behaviour of solutions to Laplace's tidal equations at low frequencies is described by a single scalar equation in second-order partial derivatives (11) with small coefficients of the higher (second-order) derivatives. Solutions of that equation significantly depend on the small coefficients and are therefore unstable. Because of the instability, direct methods of numerical integration do not work.

II. Solution of the problem by the method of perturbations in small parameters equal to the ratios of tidal frequency $\sigma$ and the coefficient of bottom friction $\kappa$ to the Earth's rotation rate $\omega$ shows the following:

(1) It follows from the zero approximation $\sigma = \kappa = 0$ that lines of flow, lines of equal $\bar{\zeta} = \text{constant}$, and isolines of $h/\cos\theta = \text{constant}$ coincide, and $\bar{\zeta}$ is a function of a single variable, $\alpha$.

(2) The dependence $\bar{\zeta}(\alpha)$ is not determined by the zero approximation, and can be obtained when first-order terms are retained. The first-order theory of perturbations defines the function $\bar{\zeta}(\alpha)$ through a set of ordinary differential equations of the fourth order, (37); its numerical integration for realistic ocean models does not pose serious computational difficulties.

III. An analysis of equations (37) permits one to draw the following conclusions:

(1) When the case of a basin that does not intersect the equator is considered, system (37) has a single regular solution.

(2) The points of intersection between the equator and the coastline are singularities. Characteristics of (14) of the form $\alpha = \text{constant}$ may begin and end at singularities. The deviation of ocean surface from the equipotential surface $\bar{\zeta}$ vanishes asymptotically within regions belonging to these characteristics. In regions belonging to closed characteristics, the asymptotic behaviour of $\bar{\zeta}$ depends on the ratio of two small parameters, $\sigma/\kappa$, on typical horizontal dimensions of the region and on the mean depth: in particular, $\bar{\zeta}$ tends asymptotically to zero when $\kappa \gg \sigma$; when $\kappa \sim \sigma$ and $\kappa \ll \sigma$, the asymptotic behaviour of $\bar{\zeta}(\alpha)$ can be evaluated using the relations obtained in Section 6.5. When the simplest case of an ocean of uniform depth over the whole Earth is considered and $\kappa \ll \sigma \ll \omega$, equation (37) is equivalent to Lamb's equation (Lamb 1932, section 218).

(3) Numerical estimates for the simplest models show the significance of dynamic effects in the theory of long-period tides, and these have to be taken into account in the interpretation of recent data on the rotation of the Earth.

It seems that the physical interpretation of our results is as follows:

1. The influence of tide-generating and Coriolis forces on the real ocean leads to the generation of large-scale vortical currents. The origin of these currents is fully similar to the origin of cyclones and anticyclones in the atmosphere. If the level of the ocean in the centre of the vortex is lower than the equipotential surface ($\bar{\zeta} < 0$), then the circulation is clockwise in the southern hemisphere and counter-clockwise in the northern hemisphere; in the case $\bar{\zeta} > 0$ the directions of circulation are opposite. In contrast to the case of cyclones in the atmosphere, the configuration of currents in the ocean are determined not by the configuration of external forces, but by the distribution of depths: in a first approximation, the lines of flow coincide with the isolines $h/\cos\theta = \text{constant}$.

In the case of periodical variations of external forces in time, the directions of circulation in each vortex and sign of $\bar{\zeta}$ oscillate with a period which is equal to the period $T$ of oscillation of external forces. Obviously, if $T = 2\pi/\sigma \to \infty$ and $\kappa = 0$, then the vortical currents with fixed velocity $\mathbf{v}$ can be generated by infinitesimal external forces. Nevertheless, these currents render sufficient influence on the values $\bar{\zeta}$ (by analogy with the influence of cyclones on the atmospheric pressure).

From the mathematical point of view, the finite variation of $\bar{\zeta}$ under the influence of infinitesimal external forces is evidence of instability of solutions of corresponding differential equations.

If $\kappa$ is small enough, then the time of dissipative attenuation of vortical currents may be comparable to, or more than, the period of oscillation. In this case, the external fluid has no possibility of penetrating into the central part of the vortex, and amplitudes of oscillations of level in this region are smaller, to some extent, than their values in the static approximation. In the opposite case when $T \to \infty$, and $\kappa \neq 0$, the time of attenuation of the vortex is small in comparison with $T$, and $\lim_{T\to\infty} \bar{\zeta} = 0$.

In the vicinities of singular points the distances between different lines of flow tend to zero and the fluid, which participates in different vortical motions, is mixed. As a result, $\bar{\zeta}$ is constant in the regions where lines of flow begin and end at singular points.

In any region which is bounded by closed lines of flow, the variation of the full volume of water $\iint_s \zeta\, ds = 0$. If $\iint_s \bar{\zeta}\, ds \neq 0$, then the static solution $\zeta = \bar{\zeta}$ is in contradiction with this condition. In the simplest case $h = h(\theta)$, the region $s$ is bounded by circles $\theta = \text{constant}$, and condition $\iint_s \bar{\zeta}\, ds = 0$ is valid only for the tides of second class (2) and (3b). This is why the case of the tides of second class in the ocean with axially symmetrical distributions of lands and depths is a special one, for which $\lim_{\sigma\to 0} \bar{\zeta} = 0$.

In conclusion we would like to point out that Carton & Wahr (1986) have considered the resonant excitation of barotropic Rossby waves as a possible cause of deflection of a dynamic pole tide from a static one. The dispersion relation for these waves is

$$\sigma = -\frac{2\omega}{a}\frac{k_\varphi \sin\theta}{k^2}$$

(Carton & Wahr 1986; Kagan & Monin 1978), where $k_\varphi$, $k$ are the $\varphi$-component of wave vector $k$ and its modulus. In the case $k_\varphi \sim k$, $\sin\theta \sim 1$ and $\sigma/\omega \sim 2 \times 10^{-3}$, one can see from this formula that scale of length

$$l_0 \sim |k^{-1}| \sim \frac{a\sigma}{\omega} \sim 10\,\text{km},$$

i.e. the Rossby waves with Chandler period have very small wavelength. In addition, Carton & Wahr (1986) have proved that, for this period, Rossby waves are very susceptible to the effects of friction. For a reasonable value of the diffusive friction parameter, the Rossby wave has an $e$-folding distance which is smaller than the undamped wavelength. As a result, adding this wave has very little effect on the solution except at points very close to the western wall.

In our approximation $L_1 \ll L_0$ the solutions of equations (1), which describe the resonant Rossby waves excitation, are excluded automatically. Indeed, in the case $l_0 \sim a\sigma/\omega$ and smooth distribution of depths, we have from (12) and (13):

$$\frac{|L_1|}{|L_0|} \sim \frac{a\,|\kappa + i\sigma|}{\omega} \frac{|\Delta \xi|}{|\nabla \xi|} \sim \frac{|\kappa + i\sigma|}{\sigma}$$

and the condition $L_1 \ll L_0$ is not fulfilled. It is easy to see that, in the case $\kappa = 0$, solutions of equations (34) are independent of $\sigma$ and, consequently, the resonant excitation of Rossby waves is not described by these equations.

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## NOTE ADDED IN PROOF

After this paper was accepted, an anonymous referee pointed out that for the case $k = 0$ another form of our equation (34a) was obtained earlier by J. Proudman (Proudman, J., 1913, *Proc. Lond. math. Soc.*, **13**, 273–306). Indeed, using in equation (22) of Proudman the coordinates $\xi = h \sec \theta$ and $\eta$ which is equal to the length along the contour $\xi =$ constant, we shall have in Proudman's notations:

$$A = \left[\frac{\partial (h \sec \theta)}{\partial s}\right]^{-1}, \quad B = 1.$$

After substitution of these expressions into Proudman's equation (22), we obtain our equation (34a) in the case $k = 0$.

Proudman's approach is based on the use of the continuity condition in integral form, which is equivalent to our equation (34b). Because (34b) is equivalent to (34a), it is clear that equation (34a) may be considered as a direct consequence of the mass conservation condition in the region which is bounded by the geostrophical contour $h \sec \theta =$ constant. From the formal point of view, such an approach is correct, but it does not give the answers to the questions: (i) why continuity conditions in the regions, bounded by geostrophycal contours, have a special role? and (ii) is the process of direct numerical integration of equations (1) reduced to the integration of (34a) and, if so, then in what way? Probably, our approach gives the answers to these questions and has the following advantages.

(i) It is shown that equation (34a) is the result of the solution of equations (1) by the method of perturbations. The geostrophic contours are the characteristics of unperturbed equations, and (34a) must be considered as the condition of existence of solution in the first-order approximation. As the solutions involved in the zero-order approximation depend on first-order terms, it is clear that solutions are unstable and the direct methods of their numerical calculation do not work [the process of numerical integration of equations (1) is not reduced to the integration of (34a)].

(ii) The method described here provides the possibility to obtain not only solutions in the zero-order approximation, but in the first-order too [i.e. the substitution of equation (32) in the right-hand part of equation (30) give values of $\partial \xi_1/\partial l|_\Gamma$ and, after integration along $\Gamma$, the values $\xi_1$].

(iii) Our approach is applicable not only to the case when $\mathbf{v}$ can be expressed in terms of $\xi$ (when $L_1$ is a simple algebraic function of $\mathbf{v}$), but in the general case, when $L_1$ is the arbitrary differential operator with small coefficients (for example, in the form given in the end of Section 7).