

## ASYMPTOTIC BEHAVIOUR OF SOLUTIONS TO THE KORTEWEG–DEVRIES–BURGERS EQUATION

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**Abstract.** In this work we study the asymptotic behaviour of solutions to the Korteweg–deVries–Burgers equation in the case when the initial data has different asymptotic limits at  $\pm\infty$ . The method used is the one developed by Kawashima and Matsumura to discuss the asymptotic behaviour of travelling-wave solutions to Burgers equation.

**1. Introduction.** In this work we consider the Korteweg–deVries–Burgers equation

$$u_t + uu_x + u_{xxx} - \nu u_{xx} = 0, \quad (1.1)$$

where  $\nu$  is a positive constant and the initial data

$$u(x, 0) = u_0(x) \quad (1.2)$$

satisfies  $u_0(x) \rightarrow c_{\pm}$  as  $x \rightarrow \pm\infty$  with  $c_- > c_+$ . Under these conditions Bona and Schonbek ([1]) showed that (1.1) admits travelling-wave solutions  $\phi(x - ct)$  connecting  $c_-$  to  $c_+$ , which are monotone if  $\nu \gg 1$ . Moreover, in obtaining solutions to this problem one can look for solutions which are the sum of a travelling wave and a perturbation. It has been shown that this travelling wave is asymptotically stable for small perturbations (cf. [2]). In additional work in this direction, it was shown in [5] that if the perturbation lies in a suitable weighted class, then it decays. It is found that the  $L_2$ -norm of the perturbation decays at the rate of  $(1+t)^{-\frac{1}{4}}$  and the  $L_2$ -norm of the first order derivative decays at the rate of  $(1+t)^{-\frac{3}{4}}$  as  $t \rightarrow \infty$ . To obtain this rate of decay, one uses the properties of the underlying parabolic equation.

In this work we generalize the result obtained in [5]. Here we make use of the technique developed by Kawashima and Matsumura ([3]) and Matsumura and Nishihara ([4]) to obtain the decay rate of solutions to Burgers equation. The important idea used here is to make use of the decay properties of the underlying hyperbolic equation.

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The plan of the paper is as follows. In Section 2, we briefly discuss the notation used, and in Section 3 we obtain results on the asymptotic behaviour of the solution.

**2. Notation.** The notation used is mostly standard, but for the sake of completeness we present it here. We denote positive constants depending of the quantities  $a, b, c, \dots$  by  $C(a, b, c, \dots)$  or only by the letter  $C$  without confusion. Moreover, in the inequalities that follow, the constant  $C$  can change from one line to the next. For function spaces,  $L_2$  denotes the space of square integrable functions on the real line together with the norm

$$\|f\| = \left( \int_{-\infty}^{\infty} |f(x)|^2 dx \right)^{\frac{1}{2}}.$$

In addition,  $H^m$  denotes the  $m$ -th order Sobolev space of functions which together with their derivatives up to order  $m$  are square integrable and the space is equipped with the norm

$$\|f\|_m = \left( \sum_{j=0}^m \|\partial_x^j f\|^2 \right)^{\frac{1}{2}}.$$

For the weight function  $h(x)$ ,  $L_h^2$  denotes the space of measurable functions  $f$  satisfying  $\sqrt{h}f \in L_2$  together with the norm

$$\|f\|_h = \left( \int_{-\infty}^{\infty} h(x) |f(x)|^2 dx \right)^{\frac{1}{2}}.$$

In particular, when  $h(x) = \langle x \rangle^\alpha = (1+x^2)^{\alpha/2}$ , we write  $L_h^2 = L_\alpha^2$  and  $|\cdot|_h = |\cdot|_\alpha$  without confusion.

**3. Asymptotic behaviour of solutions.** We begin by considering the Korteweg–deVries–Burgers equation (henceforth referred to as the KdVB equation for short) together with initial data  $u_0(x)$  and assume that the initial data tends to constant states  $c_\pm$  as  $x \rightarrow \pm\infty$ . The only restriction imposed on these states at infinity is that  $c_- > c_+$ . There is a well-established global existence theory (cf. [2]) for the equation when additional smoothness conditions are imposed on the initial data. This result is stated here for completeness.

**Theorem 3.1** ([2]). *Suppose that  $u_0$  satisfies the following conditions:*

- i)  $u_0(x) \rightarrow c_\pm$  as  $x \rightarrow \pm\infty$
- ii)  $u_0' \in H^k$
- iii)  $(u_0 - c_+) \in L_2([0, \infty))$  and  $(u_0 - c_-) \in L_2((-\infty, 0])$ ,

*for some nonnegative integer  $k$ . Then there exists a unique solution  $u$  to (1.1) with data  $u_0$  such that  $u - \psi \in C(0, \infty, H^k)$ , where  $\psi$  is given by  $\psi = u_0 \star \rho$  where  $\star$  denotes convolution and  $\rho$  is a  $C^\infty$ -function with compact support.*

It was also shown in [2] that perturbations to the travelling wave which lie in a suitable weighted function class are stable. We state this result here:

**Theorem 3.2** ([1]). *Let  $\phi$  be a monotone decreasing, bore-like travelling-wave solution of the KdVB equation with a speed of propagation  $c > 0$ . Let  $X = L^2_{2\gamma} \cap H^k(\mathbb{R})$ , where  $k \geq 2$  and  $\gamma > \frac{1}{2}$ . Then there is an  $\epsilon > 0$  such that corresponding to any initial data  $u_0$  with  $\|u_0 - \phi\|_X \leq \epsilon$ , there is a real number  $x_0$  with  $|x_0| \leq \epsilon$  having the property*

$$\|u(\cdot, t) - \phi(\cdot - ct + x_0)\|_k \rightarrow 0$$

as  $t \rightarrow +\infty$ , where  $u$  is the solution to (1.1) with initial data  $u_0$ .

In addition, we consider solutions to the KdVB equation which are of the form

$$u(x, t) = \phi(\xi + x_0) + w_\xi(\xi, t), \quad \xi = x - ct,$$

where  $\phi$  is a monotone travelling-wave solution of the KdVB equation and  $w_\xi$  denotes the perturbation. Using the properties of the underlying parabolic equation, one can show that the perturbation to the travelling wave decays at a certain rate, provided that the initial data for the perturbation lies in a certain weighted function class. More precisely, since the travelling wave is a solution to the equation,  $w$  satisfies the equation

$$w_t - cw_\xi + \phi w_\xi + w_{\xi\xi\xi} - \nu w_{\xi\xi} + \frac{1}{2}w_\xi^2 = 0. \quad (3.1)$$

If the initial data for  $w$  satisfies  $(1 + x^2)^{1+\epsilon}w(x, 0) \in L_2$ , it was shown in [5] that the  $L_2$ -norm of the solution  $w$  and the  $L_2$ -norm of the first-order derivative decays at the rate  $(1 + t)^{-\frac{3}{4}}$ . This result is stated in the following theorem:

**Theorem 3.3** ([5]). *Let  $u$  be a solution to (1.1) with initial data  $u_0$  such that*

$$w(x, 0) = \int_{-\infty}^x [u_0(y) - \phi(y - ct + x_0)] dy \in L^2_\alpha \cap H^k$$

with  $\alpha = 2(1 + \epsilon)$  and for some  $k \geq 2$ . Then it follows that

$$\sup_{0 \leq t < \infty} t^{\frac{3}{2}} \int_{-\infty}^{\infty} (w_x^4 + w^4 + w^2 + w_x^2 + w_{xx}^2) dx < \infty.$$

In this work we generalize this rate of decay for the perturbation to the travelling wave. The main idea of the proof is to make use of the properties of the underlying hyperbolic equation. This technique was first developed by Kawashima and Matsumura ([3]—see also [4]), to obtain the decay for perturbations to travelling waves to the Burgers equation. We now state the main theorem of the paper.

**Theorem 3.4.** *Let  $w_\xi$  be a perturbation to the monotone travelling-wave solution of the KdVB equation which satisfies the zero mass condition. Then  $w$  satisfies (3.1) and the decay estimate:*

*For  $0 \leq k \leq \alpha$  we have*

$$\|\langle \xi - \xi_\star \rangle^{\frac{\alpha-k}{2}} w(t)\| \leq C(1+t)^{-\frac{(k-\epsilon)}{2}},$$

where  $\xi_\star$  is a constant to be suitably chosen. Here  $w(x, 0) \in L_\alpha^2$ , where  $\epsilon = 0$  if  $k$  is an integer and is any positive constant otherwise.

**Proof.** Note that here we consider a more restrictive class of perturbations than those considered in [5]. We begin by multiplying equation (3.1) by  $(1+t)^\gamma \langle \xi - \xi^\star \rangle^\beta w(\xi, t)$  and integrate the result over space with respect to the  $\xi$  variable. Here  $\langle \xi - \xi^\star \rangle$  is given by the expression  $\langle \xi - \xi^\star \rangle = \sqrt{1 + (\xi - \xi^\star)^2}$ , where  $\xi^\star$  is a fixed constant to be suitably chosen. On integrating by parts the term arising from  $(-c + \phi)w_\xi$  yields

$$(1+t)^\gamma \langle \xi - \xi^\star \rangle^{\beta-1} [-\phi' \langle \xi - \xi^\star \rangle - \beta(-c + \phi) \frac{\xi - \xi^\star}{\langle \xi - \xi^\star \rangle}] \frac{1}{2} w^2. \quad (3.2)$$

We now choose  $\xi^\star$  so that  $c = \phi(\xi^\star)$ . With this choice of  $\xi^\star$  the term  $A_\beta(\xi)$  in the square parentheses in (3.2) is bounded below by  $c_0\beta$ . In particular, note that this coefficient is always positive. Integrating all the remaining terms by parts, we obtain the inequality

$$\begin{aligned} & \left[ \frac{1}{2} (1+t)^\gamma \int_{-\infty}^{\infty} \langle \xi - \xi^\star \rangle^\beta w^2 d\xi \right]_t + \frac{c_0\beta}{4} (1+t)^\gamma \int_{-\infty}^{\infty} \langle \xi - \xi^\star \rangle^{\beta-1} w^2 d\xi \\ & + \frac{\nu}{2} (1+t)^\gamma \int_{-\infty}^{\infty} \langle \xi - \xi^\star \rangle^\beta w_\xi^2 d\xi \leq \frac{\gamma}{2} (1+t)^{\gamma-1} \int_{-\infty}^{\infty} \langle \xi - \xi^\star \rangle^\beta w^2 d\xi \\ & + C\beta (1+t)^\gamma \int_{-\infty}^{\infty} (\langle \xi - \xi^\star \rangle^{\beta-1} + \langle \xi - \xi^\star \rangle^{\beta-3}) w_\xi^2 d\xi. \end{aligned} \quad (3.3)$$

In obtaining this inequality, we bound the nonlinear term by the quantity

$$|w|_\infty (1+t)^\gamma \int_{-\infty}^{\infty} \langle \xi - \xi^\star \rangle^\beta w_\xi^2 d\xi,$$

which is small if the initial data is sufficiently small. Fix  $\alpha \geq 0$  and let  $\beta \leq \alpha$ ; then, the last term in (3.3) is estimated by dividing the integral over space into two sets, one the set  $B = \{\xi : |\xi - \xi^\star| \leq R\}$  and the other  $B^c$ , the complement of  $B$ . The radius  $R$  is chosen so that  $\frac{C\beta}{\langle \xi - \xi^\star \rangle} \leq \frac{\nu}{4}$  for all  $\xi$  in the set  $B^c$ . Then the term is bounded above by

$$\frac{\nu}{2} (1+t)^\gamma \int_{-\infty}^{\infty} \langle \xi - \xi^\star \rangle^\beta w_\xi^2 d\xi + C_R \beta (1+t)^\gamma \int_{-\infty}^{\infty} w_\xi^2 d\xi.$$

Here  $C_R$  is given by  $C_R = C(R^{\alpha-1} + R^{\alpha-3})$ . Hence, rearranging the constants we obtain for  $\beta \leq \alpha$

$$\begin{aligned} & \left[ \frac{1}{2}(1+t)^\gamma \int_{-\infty}^{\infty} \langle \xi - \xi^* \rangle^\beta w^2 d\xi \right]_t + \frac{C_0 \beta}{4} (1+t)^\gamma \int_{-\infty}^{\infty} \langle \xi - \xi^* \rangle^{\beta-1} w^2 d\xi \\ & + \nu (1+t)^\gamma \int_{-\infty}^{\infty} \langle \xi - \xi^* \rangle^\beta w_\xi^2 d\xi \\ & \leq \frac{\gamma}{2} (1+t)^{\gamma-1} \int_{-\infty}^{\infty} \langle \xi - \xi^* \rangle^\beta w^2 d\xi + C \beta (1+t)^\gamma \int_{-\infty}^{\infty} w_\xi^2 d\xi. \end{aligned}$$

On integration, this inequality yields

$$\begin{aligned} & (1+t)^\gamma |w(t)|_\beta^2 + \int_0^t [\beta(1+\tau)^\gamma |w(\tau)|_{\beta-1}^2 + (1+\tau)^\gamma |w_\xi(\tau)|_\beta^2] d\tau \\ & \leq C \left[ |w_0|_\beta^2 + \int_0^t [\gamma(1+\tau)^{\gamma-1} |w(\tau)|_\beta^2 + \beta(1+\tau)^\gamma \|w_\xi(\tau)\|^2] d\tau \right], \end{aligned} \quad (3.4)$$

which is the main inequality in this procedure. As in [3] and [4], by induction on  $\beta$  and  $\alpha$  we have, for  $k = 0, 1, 2, \dots, [\alpha]$ ,

$$\begin{aligned} & (1+t)^{k_0} |w(t)|_{\alpha-k_0}^2 \\ & + \int_0^t [(\alpha-k_0)(1+\tau)^{k_0} |w(\tau)|_{\alpha-k_0-1}^2 + (1+\tau)^{k_0} |w_\xi(\tau)|_{\alpha-k_0}^2] d\tau \leq C |w_0|_\alpha^2. \end{aligned} \quad (3.5)$$

This inequality proves Theorem 3.4 when  $k$  is an integer. We next consider the case when both  $\alpha$  and  $k$  are not integers. Then,

**Claim 1.** *We have*

$$(1+t)^\gamma \|w(t)\|^2 + \int_0^t (1+\tau)^\gamma \|w_\xi(\tau)\|^2 d\tau \leq C |w_0|_\alpha^2$$

for  $\gamma = \alpha - \epsilon$ , where  $\epsilon$  is an arbitrarily small positive constant.

**Claim 2.** *Let  $k \leq \alpha$ . Then*

$$(1+t)^\gamma |w(t)|_{\alpha-k}^2 + \int_0^t (1+\tau)^\gamma |w_\xi(\tau)|_{\alpha-k}^2 d\xi \leq C |w_0|_\alpha^2$$

for  $\gamma = k - \epsilon$  where  $\epsilon$  is an arbitrarily small constant.

Note that if  $\alpha = k$  then Claim 2 is the same as Claim 1. In particular, if  $k = 0$  we can take  $\epsilon = 0$  by virtue of (3.5).

**Proof of Claim 1.** In (3.4) we take  $\beta = 0$  and estimate the second term on the right-hand side. We have

$$\begin{aligned} & \gamma \int_0^t (1+\tau)^{\gamma-1} |w(\tau)|_0^2 d\tau \\ & = \gamma \int_0^t (1+\tau)^{\gamma-1} \int_{-\infty}^{\infty} \langle \xi - \xi^* \rangle^{(\alpha-[\alpha])\frac{1}{p} - (\alpha-[\alpha])\frac{1}{p}} (w^2)^{\frac{1}{p} + \frac{1}{p'}} d\xi d\tau. \end{aligned}$$

We do this in order to use (3.5) with  $k_0 = [\alpha]$ . Hence, we have

$$\begin{aligned}
& \gamma \int_0^t (1+\tau)^{\gamma-1} |w(\tau)|_0^2 d\tau \\
& \leq \gamma \int_0^t (1+\tau)^{\gamma-1} \left( \int_{-\infty}^{\infty} \langle \xi - \xi^* \rangle^{\alpha-[\alpha]} w^2 \right)^{\frac{1}{p}} \left( \int_{-\infty}^{\infty} \langle \xi - \xi^* \rangle^{-(\alpha-[\alpha])\frac{p'}{p}} w^2 d\xi \right)^{\frac{1}{p'}} d\tau \\
& = \gamma \int_0^t \left\{ (1+\tau)^{\gamma-[\alpha]-1} \left( (1+\tau)^{[\alpha]} \int_{-\infty}^{\infty} \langle \xi - \xi^* \rangle^{\alpha-[\alpha]} w^2 \right)^{\frac{1}{p}} \right. \\
& \quad \times \left. \left( (1+\tau)^{[\alpha]} \int_{-\infty}^{\infty} \langle \xi - \xi^* \rangle^{-(\alpha-[\alpha])\frac{p'}{p}} w^2 d\xi \right)^{\frac{1}{p'}} \right\} d\tau \\
& \leq |w_0|_{\alpha}^{\frac{2}{p}} \int_0^t (1+\tau)^{-[\alpha]+1-\gamma} \left( (1+\tau)^{[\alpha]} \int_{-\infty}^{\infty} \langle \xi - \xi^* \rangle^{-(\alpha-[\alpha])\frac{p'}{p}} w^2 d\xi \right)^{\frac{1}{p'}} d\tau \\
& \leq C |w_0|_{\alpha}^{\frac{2}{p}} \left( \int_0^t (1+\tau)^{-([\alpha]+1-\gamma)p} d\tau \right)^{\frac{1}{p}} \left( (1+\tau)^{[\alpha]} \int_{-\infty}^{\infty} \langle \xi - \xi^* \rangle^{-(\alpha-[\alpha])\frac{p'}{p}} w^2 d\xi \right)^{\frac{1}{p'}} d\tau.
\end{aligned}$$

Hence, in order to use (3.5) with  $k_0 = [\alpha]$ , we choose  $p$  to satisfy  $-(\alpha - [\alpha])\frac{p'}{p} = \alpha - k_0 - 1 = \alpha - [\alpha] - 1$ . This means that we choose  $p = \frac{1}{[\alpha]+1-\alpha}$ . The integral

$$\int_0^t (1+\tau)^{-([\alpha]+1-\gamma)p} d\tau$$

converges provided  $\gamma$  is less than  $\alpha$ ; that is,  $\gamma = \alpha - \epsilon$  for an arbitrarily small  $\epsilon > 0$ . This proves Claim 1.

**Proof of Claim 2.** Let  $k < \alpha$  and  $k_0 < k < k_0 + 1$ . We take  $\beta = \alpha - k$  in (3.4). If  $\gamma < \alpha$ , then the last term in (3.4) has already been estimated. Hence, we need to estimate the second term, in the same way as in Claim 1. In this case we can choose  $p = \frac{1}{1+k_0-k}$  and  $p' = \frac{1}{k-k_0}$ . This yields the result

$$\gamma \int_0^t (1+\tau)^{\gamma-1} |w(\tau)|_{\alpha-k}^2 d\tau \leq C |w_0|^2 \left( \int_0^t (1+\tau)^{-\frac{k_0+1-\gamma}{k_0+1-k}} d\tau \right)^{\frac{1}{p}},$$

where  $\gamma = k - \epsilon$ . This completes the proof of Claim 2 and hence completes the proof of the theorem.  $\square$

We now obtain the decay rate for the derivatives of  $w$ . To this end we prove the following theorem:

**Theorem 3.5.** *Let  $w$  satisfy the same conditions as in Theorem 3.4. Then the first-order derivative of  $w$  satisfies the following estimate:*

*For  $0 \leq k \leq \alpha$  we have*

$$\| \langle \xi - \xi^* \rangle^{\frac{\alpha+1-k}{2}} w_{\xi}(t) \|_{L_2} \leq C(1+t)^{-\frac{(k-\epsilon)}{2}}$$

if  $w_\xi(x, 0) \in L_{\alpha+1}^2$  and  $\epsilon = 0$  if  $k$  is an integer and is any small positive constant otherwise.

**Proof.** To obtain the decay rate for the first-order derivative of  $w$  we first differentiate the  $w$  equation with respect to  $\xi$ , then multiply the equation by  $\langle \xi_K \rangle^\beta w_\xi$ , where  $\langle \xi_K \rangle = \sqrt{K^2 + \xi^2}$ , with  $K$  a positive constant to be chosen appropriately. Integrating the result over space we obtain, after several integrations by parts, the inequality

$$\begin{aligned} & \left( \int_{-\infty}^{\infty} \frac{1}{2} \langle \xi_K \rangle^\beta w_\xi^2 d\xi \right)_t + \int_{-\infty}^{\infty} \left( \nu \langle \xi_K \rangle^\beta + \frac{3}{2} (\langle \xi_K \rangle^\beta)_\xi \right) w_{\xi\xi}^2 d\xi \\ &= - \int_{-\infty}^{\infty} \frac{1}{2} \left[ \langle \xi_K \rangle^\beta \phi'(\xi) + (c - \phi) (\langle \xi_K \rangle^\beta)_\xi - (\langle \xi_K \rangle^\beta)_{\xi\xi\xi} - \frac{1}{2} \nu (\langle \xi_K \rangle^\beta)_{\xi\xi} \right] w_\xi^2 d\xi \\ &+ \int_{-\infty}^{\infty} \langle \xi_K \rangle^\beta w_\xi (w_\xi^2)_\xi d\xi. \end{aligned}$$

Note that here

$$\begin{aligned} & \nu \langle \xi_K \rangle^\beta + \frac{3}{2} (\langle \xi_K \rangle^\beta)_\xi = \nu \langle \xi_K \rangle^\beta + \frac{3}{2} \beta \langle \xi_K \rangle^{\beta-1} \frac{\xi}{\langle \xi_K \rangle} \\ &= \langle \xi_K \rangle^{\beta-2} (\nu(K^2 + \xi^2) + \frac{3}{2} \beta \xi) = \langle \xi_K \rangle^{\beta-2} (\nu \xi^2 + \frac{3}{2} \beta \xi + \nu K^2) \geq C_0 \langle \xi_K \rangle^\beta, \end{aligned}$$

where  $C_0$  is a constant which depends only on  $\nu, K$  and  $\alpha$  where  $\beta \leq \alpha + 1$ . Since  $\phi'(\xi)$  tends to zero at an exponential order as  $\xi \rightarrow \pm\infty$ , we have

$$|\langle \xi_K \rangle^\beta \phi'(\xi)| = \langle \xi_K \rangle^\beta |\phi'(\xi)| \leq C \langle \xi_K \rangle^{\beta-1}$$

if  $\beta \leq \alpha + 1$ . Here  $C$  depends on  $\alpha$  and  $K$ . Finally,

$$|(c - \phi) (\langle \xi_K \rangle^\beta)_\xi - (\langle \xi_K \rangle^\beta)_{\xi\xi\xi} - \frac{1}{2} \nu (\langle \xi_K \rangle^\beta)_{\xi\xi}| \leq C \langle \xi_K \rangle^{\beta-1}.$$

Taking the nonlinear term into consideration, we have for  $\beta \leq \alpha + 1$  and  $K > \frac{3(\alpha+1)}{2\nu}$  the inequality

$$(|w_\xi|_\beta^2)_t + |w_{\xi\xi}(t)|_\beta^2 \leq C |w_\xi(t)|_{\beta-1}^2. \quad (3.6)$$

Multiply (3.6) by  $(1+t)^\delta$  and integrate the resulting equation with respect to time to get

$$\begin{aligned} & (1+t)^\delta |w_\xi(t)|_\beta^2 + \int_0^t (1+\tau)^\delta |w_{\xi\xi}(\tau)|_\beta^2 d\tau \\ & \leq C [ |w_{0\xi}|_\beta^2 + \int_0^t \delta (1+\tau)^{\delta-1} |w_\xi(\tau)|_\beta^2 + (1+\tau)^\delta |w_\xi(\tau)|_{\beta-1}^2 d\tau ]. \end{aligned} \quad (3.7)$$

We now use Claim 2 to obtain the desired estimate. First, let  $1 \leq \beta \leq \alpha$  and  $\beta - 1 = \alpha - k \geq 0$ . Then, using  $\delta = \gamma = k - \epsilon$  in (3.7), we have

$$(1+t)^\gamma |w_\xi(t)|_{\alpha+1-k}^2 + \int_0^t (1+\tau)^\gamma |w_{\xi\xi}(\tau)|_{\alpha+1-k}^2 d\tau \leq C(|w_0|_\alpha^2 + |w_{0\xi}|_\alpha^2), \quad (3.8)$$

which is the desired inequality for  $1 \leq k \leq \alpha$ . If  $w_{0\xi} \in L_{\alpha+1}^2$ , then (3.7) with  $\delta = 0$  and  $\beta = \alpha + 1$  shows that

$$\begin{aligned} |w_\xi(t)|_{\alpha+1}^2 + \int_0^t |w_{\xi\xi}(\tau)|_{\alpha+1}^2 d\tau &\leq C(|w_{0\xi}|_{\alpha+1}^2 + \int_0^t |w_\xi|_\alpha^2 d\tau) \\ &\leq C(|w_0|_\alpha^2 + |w_{0\xi}|_{\alpha+1}^2), \end{aligned} \quad (3.9)$$

which is the desired inequality for  $k = 0$ . For  $0 < k < 1$ , i.e., for  $\alpha < \beta < \alpha + 1$ ,  $\beta = \alpha + 1 - k$  and  $\gamma = k - \epsilon$  inequality (3.7) shows that

$$\begin{aligned} (1+\tau)^\gamma |w_\xi(t)|_{\alpha+1-k}^2 + \int_0^t (1+\tau)^\gamma |w_{\xi\xi}(\tau)|_{\alpha+1-k}^2 d\tau \\ \leq C(|w_{0\xi}|_{\alpha+1}^2 + \int_0^t \gamma(1+\tau)^{\gamma-1} |w_\xi(\tau)|_{\alpha+1-k}^2 d\tau + \int_0^t (1+\tau)^\gamma |w_\xi(\tau)|_{\alpha-k}^2 d\tau) \\ \leq C(|w_0|_\alpha^2 + |w_{0\xi}|_{\alpha+1}^2 + \int_0^t (1+\tau)^{\gamma-1} |w_\xi(\tau)|_{\alpha+1-k}^2 d\tau). \end{aligned}$$

By (3.4) with  $\gamma = 0$  and  $\beta = \alpha$  we have

$$\int_0^t |w_\xi(\tau)|_\alpha^2 d\tau \leq C|w_0|_\alpha. \quad (3.10)$$

Interpolating (3.9) and (3.10) we can estimate

$$\int_0^t (1+\tau)^{\gamma-1} |w_\xi(\tau)|_{\alpha+1-k}^2 d\tau$$

for  $0 < k < 1$  and  $\gamma < k$ . In fact,

$$\begin{aligned} \int_0^t (1+\tau)^{\gamma-1} |w_\xi|_\beta^2 d\tau &= \int_0^t (1+\tau)^{\gamma-1} \int_{-\infty}^\infty \langle \xi_K \rangle^\beta w_\xi(\tau)^2 d\xi d\tau \\ &\leq C \int_0^t \left\{ (1+\tau)^{\gamma-1} \left( \int_{-\infty}^\infty \langle \xi_K \rangle^\alpha w_\xi(\tau)^2 d\xi \right)^{\frac{1}{p}} \left( \int_{-\infty}^\infty \langle \xi_K \rangle^{\alpha+1} w_\xi^2 d\xi \right)^{\frac{1}{p'}} \right\} d\tau \\ &\leq C \left( \int_0^t (1+\tau)^{-(1-\gamma)p'} d\tau \right)^{1/p'} \left( \int_0^t |w_\xi(\tau)|_\alpha^2 d\tau \right)^{1/p}, \end{aligned}$$



where we choose  $p$  so that  $\beta = \alpha \frac{1}{p} + (\alpha + 1) \frac{1}{p'}$  with  $\frac{1}{p} + \frac{1}{p'} = 1$ . Hence  $\frac{1}{p'} = \beta - \alpha$  and  $\frac{1}{p} = 1 + \alpha - \beta = k$ .

Combining the estimates obtained above, we get

$$(1+t)^\gamma |w_\xi(t)|_{\alpha+1-k}^2 + \int_0^t (1+\tau)^\gamma |w_{\xi\xi}(\tau)|_{\alpha+1-k}^2 d\tau \leq C(|w_{0\xi}|_\alpha^2 + |w_{0\xi}|_{\alpha+1}^2),$$

where  $0 \leq k \leq \alpha$  if  $\gamma = k - \epsilon$  for any positive  $\epsilon$ . (In particular, we can take  $\epsilon = 0$  if  $k = 0$ .)

Thus, we have obtained the following results, including the case when  $\alpha$  and  $k$  are integers.

**Claim 2'.** Let  $0 \leq k \leq \alpha$ . Then

$$(1+t)^\gamma |w(t)|_{\alpha-k}^2 + \int_0^t (1+\tau)^\gamma |w_\xi(\tau)|_{\alpha-k}^2 d\tau \leq C|w_0|_\alpha^2$$

for  $\gamma = k - \epsilon$  where  $\epsilon = 0$  if  $k$  is an integer and is any positive constant otherwise.

**Claim 3.** Let  $0 \leq k \leq \alpha$  and  $\epsilon$  be as in Claim 2'. Then

$$(1+t)^\gamma |w_\xi(t)|_{\alpha+1-k}^2 + \int_0^t (1+\tau)^\gamma |w_{\xi\xi}(\tau)|_{\alpha+1-k}^2 d\tau \leq C(|w_0|_\alpha^2 + |w_0|_{\alpha+1}^2),$$

which completes the proof of the theorem.  $\square$

We can now prove an analogous result for all higher-order derivatives of  $w$ .

**Theorem 3.6.** *Let  $w$  satisfy the same conditions as in Theorem 3.4. Then the solution of (3.1) and its derivatives decay in time and satisfy the following estimate:*

*For each  $0 \leq k \leq \alpha$  the estimate*

$$\|\langle \xi - \xi^* \rangle^{\frac{\alpha+s-k}{2}} w_{(s)}(t)\|_{L_2} \leq C(1+t)^{-\frac{(k-\epsilon)}{2}},$$

where  $w_{(s)}$  denotes the derivative of order  $s$  of  $w$  with respect to  $\xi$ .

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