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ASYMPTOTIC BEHAVIOUR OF THE SOLUTIONS OF SOME DISCRETE VOLTERRA EQUATIONS

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Abstract. Asymptotic properties and asymptotic equivalence of some Volterra difference equations are investigated.

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1. Introduction

In recent years, considerable attention has been paid to the development of a qualitative theory for difference equations. Recent contributions were made by many authors including [1-11].

In particular, paper [1] surveys the existence and approximation of solutions for a discrete system. Stability criteria are derived for difference equations of Volterra type degenerate Kernels in paper [2]. The main objective in papers [3, 5] is to extend some of the main results in asymptotic theory to difference systems of Volterra type. In the paper [6], topological methods were used to study stability in the first approximation of some nonlinear Volterra difference equations. In papers [7, 8], weighted norms are used to find sufficient conditions under which discrete Volterra equations have unique solutions. The problem of asymptotic equivalence in difference equations has been considered for example in papers [3, 4, 9, 10, 11]. In these papers block dichotomy was used to study relations between the solution of a linear difference system and perturbed difference system associated with the linear system.

This paper is divided into two sections not including Introduction and Resolvent. In the first section (Section 3) we consider linear system of difference equations and give sufficient condition for this equation has the solution which tends to a constant vector. In Section 4, using the resolvent kernel we provide a criterion for the asymptotic equivalence between the unperturbed linear and perturbed nonlinear Volterra systems.

Let

$$Z = \{0, 1, 2, \ldots\}, \quad N(n_0) = \{n_0 + 1, n_0 + 2, \ldots\}, \quad n_0 \in Z,$$

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R^k – the k-dimensional real Euclidean space with norm

$$|x| = \sum_{i=1}^{k} |x_i|, \quad x = (x_1, \dots, x_k),$$

M^k – the space of all $k \times k$ metrics $A = (a_{ij})$ with norm $|\circ|$ given by

$$|A| = \sum_{i=1}^{k} \sum_{j=1}^{k} |a_{ij}|.$$

The identity matrix is defined by E.

2. Resolvent

Consider a system of linear equations

$$y(n) = f(n) + \sum_{s=n+1}^{\infty} K(n,s)y(s) \tag{1}$$

where $f, y\ Z \to R^k, K(n,s)$ is from M^k.

Let us assume that a unique solution y of system (1) exists for all finite n. Let us find the solution y as a function of f and auxiliary $k \times k$ matrix $R(n,j), n \leq j < \infty$ referred to as resolvent.

Let

$$\begin{aligned} K_1(n,s) &= K(n,s), \\ K_q(n,s) &= \sum_{r=n+1}^{s-1} K(n,r)K_{q-1}(r,s) \end{aligned}$$

and

$$R(n,s) = \sum_{q=1}^{\infty} K_q(n,s). \tag{2}$$

The $k \times k$ matrix $R(n,s)$ is called the resolvent kernel associated with the kernel $K(n,s)$.

It is now easy to conclude that the resolvent $R(n,s)$ satisfies the relations

$$R(n,j) = K(n,j) + \sum_{s=n+1}^{j-1} R(n,s)K(s,j) \tag{3}$$

and

$$R(n,j) = K(n,j) + \sum_{s=n+1}^{j-1} K(n,s)R(s,j), \tag{3'}$$

for $j \geq n$, where $\sum_{s=k}^{l} u(s) \equiv 0$ for $l < k$.

In terms of the resolvent matrix $R(n, s)$ of (2) (analogously as in integral equations) the solution of (1) can be written as

$$y(n) = f(n) + \sum_{s=n+1}^{\infty} R(n, s) f(s). \tag{4}$$

From (1), multiplying by $R(n, j)$ and summing with respect to j between $n+1$ and ∞, we obtain

$$\sum_{j=n+1}^{\infty} R(n, j)(y(j) - f(j)) = \sum_{j=n+1}^{\infty} \Big(\sum_{s=n+1}^{j-1} R(n, s) K(s, j) \Big) y(j).$$

Then, by virtue of (3') and (1) we obtain the desired form (4) of the solution of the system (1).

3. Asymptotic properties

Asymptotic properties of the Volterra discrete system (1) is discussed in this part.

Lemma 3.1. *Suppose that*

1° *the functions $f(n)$ and $K(n, s)$ are defined for $n \geq n_0, s \geq n_0$,*

2° $\overline{\lim_{n\to\infty}} |f(n)| = M < \infty$,

3° $\overline{\lim_{n\to\infty}} \sum_{s=n_0}^{\infty} |K(n, s)| = \mu < 1$, $\lim_{n\to\infty} \sum_{s=n_0}^{n_1} |K(n, s)| = 0$ *for each* $n_1 \geq n_0$,

4° *the equation*

$$y(n) = f(n) + \sum_{s=n_0}^{\infty} K(n, s) y(s) \quad (n \geq n_0) \tag{5}$$

has a solution $\overline{y}(n)$ such that $|\overline{y}(n)| \leq L$ for $n \geq n_0$.

Then the following inequality holds

$$\overline{\lim_{n\to\infty}} |\overline{y}(n)| \leq \frac{M}{1 - \mu}.$$

Proof. Agarwal [1] gave sufficient conditions for the existence of the solution of equation (5). For the given $\varepsilon \in (0, 1-\mu)$ we choose $n_1 \geq n_0$ and $n_2 \geq n_1$ so that

$$\sum_{s=n_0}^{n_1} |K(n,s)| \leq \varepsilon, \quad \sum_{s=n_0}^{\infty} |K(n,s)| \leq \mu + \varepsilon,$$
$$|f(n)| \leq M + \varepsilon \quad \text{and} \quad |\overline{y}(n)| \leq L_1 + \varepsilon$$

for $n \geq n_2$ where $L_1 = \overline{\lim_{n\to\infty}} |\overline{y}(n)|$. We obtain

$$\begin{aligned} |\overline{y}(n)| &\leq M + \varepsilon + L \sum_{s=n_0}^{n_1} |K(n,s)| + (L_1 + \varepsilon) \sum_{s=n_1+1}^{\infty} |K(n,s)| \\ &\leq M + \varepsilon + \varepsilon L + (L_1 + \varepsilon)(\mu + \varepsilon). \end{aligned}$$

Hence

$$L_1 \leq \frac{M + \varepsilon(1 + L + \mu + \varepsilon)}{1 - \mu - \varepsilon}.$$

□

Theorem 3.1. *Let f, F and ψ be defined for $n \in N(n_0)$ and let $N(n,s)$ be defined for $s \geq n \geq n_0$. Suppose that for $s \geq n \geq n_0$*

1° $\sum\limits_{l=n+1}^{s-1} |N(n,l)|\,|N(l,s)|^{\alpha} \leq \lambda |N(n,s)|^{\alpha}$, *with some $\alpha \in [0,1]$ and fixed $\lambda < 1$,*

2° $|N(n,s)| \leq F(s)$, *$F(s)$ uniformly bounded for $s \geq n \geq n_0$,*

3° $\sum\limits_{s=n+1}^{\infty} |\psi(s)| < \infty$, *$\psi(n)$ is uniformly bounded, for $n \geq n_0$,*

4° $\lim\limits_{n\to\infty} \sup\limits_{n_0 \leq l \leq n+1} \sum\limits_{s=n+1}^{\infty} F^{1-\alpha}(s) |N(l,s)|^{\alpha} = 0$,

5°_{a} $\overline{\lim\limits_{n\to\infty}} |f(n)| = M < \infty$ *or*

5°_{b} $\lim\limits_{n\to\infty} f(n) = s \quad (|s| < \infty)$.

Then the equation

$$y(n) = f(n) + \sum_{s=n+1}^{\infty} K_0(n,s) y(s) \tag{6}$$

where $K_0(n,s) = N(n,s) + \psi(s)$ for $s \geq n \geq n_0$ has for large $n (n \geq n_0)$ exactly one solution $\overline{y}(n)$ bounded for $n \to \infty$. We have $\overline{\lim_{n\to\infty}} |\overline{y}(n)| = M$ in case 5°_{a}, respectively $\lim_{n\to\infty} \overline{y}(n) = s$ in case 5°_{b}.

Proof. We choose a number a satisfying the condition $\lambda < a < 1$. Then, with some $n_1 \ge n_0$ there exists $\sum\limits_{l=n+1}^{s-1} F^{1-\alpha}(l)|N(n,l)|^\alpha$ for $s \ge n \ge n_1$ and we have $\sum\limits_{l=n+1}^{\infty} |K_0(n,l)| \le a$ for $n \ge n_1$.

Let

$$K_q(n,s) = \sum_{l=n+1}^{s-1} K_0(n,l) K_{q-1}(l,s) \tag{7}$$

for $s \ge n \ge n_0,\ q = 1, 2, \dots$.

We will prove by induction the inequality

$$|K_q(n,s)| \le a^q F^{1-\alpha}(s)|N(n,s)|^\alpha + q a^{q-1}\psi_1(n,s) + a^q|\psi(s)| \tag{8}$$

for $s \ge n \ge n_1$ and $q = 0, 1, 2, \dots$, where

$$\psi_1(n,s) = F^{1-\alpha}(s) \sum_{l=n+1}^{s-1} |N(l,s)|^\alpha |\psi(l)|.$$

We immediately verify that (8) is true for $q = 0$. Suppose now that it is true for the index $q-1$ ($q \ge 1$). Then, observing that $\psi_1(n,s)$ is a decreasing function of the variable n for $s \ge n \ge n_1$, we have

$$\begin{aligned}
|K_q(n,s)| \ \le\ & \sum_{l=n+1}^{s-1} |N(n,l) + \psi(l)| \cdot \{a^{q-1} F^{1-\alpha}(s)|N(l,s)|^\alpha \\
& \qquad + (q-1)a^{q-2}\psi_1(l,s) + a^{q-1}|\psi(s)|\} \\
\le\ & a^{q-1}|\psi(s)| \sum_{l=n+1}^{s-1} |N(n,l) + \psi(l)| \\
& + (q-1)a^{q-2} \sum_{l=n+1}^{s-1} |N(n,l) + \psi(l)|\psi_1(l,s) \\
& + a^{q-1} \sum_{l=n+1}^{s-1} |N(n,l) + \psi(l)| F^{1-\alpha}(s)|N(l,s)|^\alpha \\
\le\ & a^q|\psi(s)| + (q-1)a^{q-2}\psi_1(n,s) \sum_{l=n+1}^{s-1} (F^{1-\alpha}(s)|N(n,l)|^\alpha + |\psi(l)|) \\
& + a^{q-1} F^{1-\alpha}(s) \sum_{l=n+1}^{s-1} |N(n,l)|\,|N(l,s)|^\alpha \\
& + a^{q-1} F^{1-\alpha}(s) \sum_{l=n+1}^{s-1} |N(n,l)|^\alpha |\psi(l)|
\end{aligned}$$

$$\begin{aligned}
&\le a^q|\psi(s)| + (q-1)a^{q-1}\psi_1(n,s) \\
&\qquad + a^{q-1}F^{1-\alpha}(s)\lambda|N(n,s)|^\alpha + a^{q-1}\psi_1(n,s) \\
&= a^q|\psi(s)| + (q-1)a^{q-1}\psi_1(n,s) \\
&\qquad + a^{q-1}F^{1-\alpha}(s)\lambda|N(n,s)|^\alpha + a^{q-1}\psi_1(n,s) \\
&= a^q|\psi(s)| + qa^{q-1}\psi_1(n,s) + a^qF^{1-\alpha}(s)|N(n,s)|^\alpha, \quad (\lambda < a < 1).
\end{aligned}$$

This proves (8). Therefore, the series $\sum_{q=0}^{\infty} K_q(n,s)$ is uniformly convergent for $s \ge n \ge n_1$. Taking $\sum_{q=0}^{\infty} K_q(n,s) = R(n,s)$ we obtain from (8) for $s \ge n \ge n_1$

$$\begin{aligned}
|R(n,s)| &\le \sum_{q=0}^{\infty}(a^q|\psi(s)| + qa^{q-1}\psi_1(n,s) + a^qF^{1-\alpha}(s)|N(n,s)|^\alpha) \\
&\le \frac{1}{1-a}|\psi(s)| + \frac{1}{(1-a)^2}\psi_1(n,s) + \frac{1}{1-a}F^{1-\alpha}(s)|N(n,s)|^\alpha.
\end{aligned}$$

We have

$$\begin{aligned}
\overline{\lim_{n\to\infty}}\sum_{s=n+1}^{\infty}\psi_1(n,s) &= \overline{\lim_{n\to\infty}}\sum_{s=n+1}^{\infty}F^{1-\alpha}(s)\sum_{i=n+1}^{s-1}|N(i,s)|^\alpha|\psi(i)| \\
&= \overline{\lim_{n\to\infty}}\sum_{i=n+1}^{\infty}|\psi(i)|\sum_{s=i+1}^{\infty}F^{1-\alpha}(s)|N(i,s)|^\alpha \\
&\le \lim_{n\to\infty}\sum_{i=n+1}^{\infty}|\psi(i)| \cdot \lim_{n\to\infty}\sum_{s=n+1}^{\infty}F^{1-\alpha}(s)|N(n,s)|^\alpha = 0.
\end{aligned}$$

We choose $n_2 \ge n_1$ such that the functions

$$\sum_{s=n+1}^{\infty}F^{1-\alpha}(s)|N(n,s)|^\alpha, \qquad \sum_{s=n+1}^{\infty}\psi_1(n,s)$$

and $f(n)$ are bounded for $n \ge n_2$ and we find that the $\sum_{s=n+1}^{\infty}|R(n,s)|$ is convergent and uniformly bounded for $n \ge n_2$. Then the functions

$$I(n) = \sum_{s=n+1}^{\infty}R(n,s)f(s) \quad \text{and} \quad \overline{y}(n) = f(n) + I(n)$$

remain bounded for $n \ge n_2$.

We will prove the uniform convergence of $\sum\limits_{s=n+1}^{\infty} R(t_0,s)f(s)$ for $t_1 \le t_0 \le t_2,\ t_1 \ge n_2$. We have for $n \ge n_3,\ t_1 \le t_0 \le t_2$

$$\begin{aligned}
&\sum_{s=n+1}^{\infty} F^{1-\alpha}(s) \sum_{i=t_0+1}^{s-1} |N(i,s)|^{\alpha}|\psi(i)| = \\
&\quad = \sum_{s=n+1}^{\infty}\sum_{i=t_0+1}^{n} F^{1-\alpha}(s)|N(i,s)|^{\alpha}|\psi(i)| + \sum_{s=n+1}^{\infty}\sum_{i=n+1}^{s-1} F^{1-\alpha}(s)|N(i,s)|^{\alpha}|\psi(i)| \\
&\quad \le \sum_{i=t_0+1}^{n} |\psi(i)| \sum_{s=n+1}^{\infty} F^{1-\alpha}(s)|N(i,s)|^{\alpha} + \sum_{i=n+1}^{\infty} |\psi(i)| \sum_{s=i+1}^{\infty} F^{1-\alpha}(s)|N(i,s)|^{\alpha} \\
&\quad \le \varepsilon A + \varepsilon^2,
\end{aligned}$$

where $A = \sum\limits_{i=t_0+1}^{\infty} |\psi(i)|$.

We have for $n \ge n_2$

$$\begin{aligned}
\sum_{s=n+1}^{\infty} |R(t_0,s)f(s)| \quad \le \quad & N_0\Big\{\frac{1}{1-a}\sum_{s=n+1}^{\infty}|\psi(s)| \\
& + \frac{1}{(1-a)^2}\sum_{s=n+1}^{\infty} F^{1-\alpha}(s)\cdot \sum_{i=t_0+1}^{s-1}|N(i,s)|^{\alpha}|\psi(i)| \\
& + \frac{1}{1-a}\sum_{s=n+1}^{\infty} F^{1-\alpha}(s)|N(t_0,s)|^{\alpha}\Big\},
\end{aligned}$$

where $N_0 = \sup\limits_{n \ge n_2} |f(n)|$.

For given $\varepsilon > 0$ we choose $n_3 \ge t_2$ so that for $n \ge n_3$ we get

$$\sup_{t_1 \le u \le n+1} \sum_{s=n+1}^{\infty} F^{1-\alpha}(s)|N(u,s)|^{\alpha} \le \varepsilon \quad \text{and} \quad \sum_{s=n+1}^{\infty} |\psi(s)| \le \varepsilon.$$

Finally, for $n \ge n_3,\ t_1 \le t_0 \le t_2$ we obtain

$$\sum_{s=n+1}^{\infty} |R(t_0,s)f(s)| \le \frac{\varepsilon N_0}{1-a} + \frac{\varepsilon A N_0}{(1-a)^2} + \frac{\varepsilon^2 N_0}{(1-a)^2} + \frac{\varepsilon N_0}{1-a}.$$

It follows that $I(n)$ exists in every finite interval $[a,b]$ $(n_2 \le a < b < \infty)$. Since the $\sum\limits_{i=n+1}^{\infty} |K_0(n,s)| \sum\limits_{i=s+1}^{\infty} |R(s,i)|$ converges for $n \ge n_2$, we have for $n \ge n_2$

$$\sum_{s=n+1}^{\infty} K_0(n,s)I(s) = \sum_{s=n+1}^{\infty} K_0(n,s) \sum_{i=s+1}^{\infty} R(s,i)f(i)$$

$$
\begin{aligned}
&= \sum_{i=n+1}^{\infty}\Big(\sum_{s=n+1}^{i-1} K_0(n,s)R(s,i)\Big)f(i)\\
&= \sum_{i=n+1}^{\infty}\Big(\sum_{s=n+1}^{i-1} K_0(n,s)\sum_{q=0}^{\infty}K_q(s,i)\Big)f(i)\\
&= \sum_{i=n+1}^{\infty}\sum_{q=0}^{\infty}\sum_{s=n+1}^{i-1} K_0(n,s)K_q(s,i)f(i)\\
&= \sum_{i=n+1}^{\infty}\sum_{q=0}^{\infty}K_{q+1}(n,i)f(i)
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{s=n+1}^{\infty} K_0(n,s)\overline{y}(s) &= \sum_{s=n+1}^{\infty} K_0(n,s)(f(s)+I(s))\\
&= \sum_{s=n+1}^{\infty} K_0(n,s)f(s) + \sum_{s=n+1}^{\infty} K_0(n,s)I(s)\\
&= \sum_{s=n+1}^{\infty} K_0(n,s)f(s) + \sum_{i=n+1}^{\infty}\sum_{q=0}^{\infty}K_{q+1}(n,i)f(i)\\
&= \sum_{s=n+1}^{\infty} K_0(n,s)f(s) + \sum_{i=n+1}^{\infty}\Big(\sum_{q=0}^{\infty}K_q(n,i)-K_0(n,i)\Big)f(i)\\
&= \sum_{i=n+1}^{\infty}\sum_{q=0}^{\infty}K_q(n,i)f(i) = \sum_{i=n+1}^{\infty}R(n,i)f(i) = I(n).
\end{aligned}
$$

Hence it follows that $\overline{y}(n)$ satisfies (1) for $n \geq n_2$. Next, by (5) the equality

$$
\lim_{q\to\infty}\sum_{s=n+1}^{\infty}|K_q(n,s)| = 0 \tag{9}
$$

holds for $n \geq n_2$. Indeed, from (8) and assumptions of Theorem we have

$$
\lim_{q\to\infty}\sum_{s=n+1}^{\infty}\{a^qF^{1-\alpha}(s)|N(n,s)|^{\alpha} + qa^{q-1}\psi_1(n,s) + a^q|\psi(s)|\} = 0.
$$

Every solution $y(n)$ of (6) for $f(n)=0$ satisfies the relation

$$
y(n) = \sum_{s=n+1}^{\infty} K_0(n,s)y(s).
$$

Indeed, let $y(n)$ be solution of the equation (6). Now substituting for $y(s)$ relation

$$y(s) = \sum_{l=s+1}^{\infty} K_0(s,l)y(l)$$

we get from (7)

$$\begin{aligned} y(n) &= \sum_{s=n+1}^{\infty} K_0(n,s) \sum_{l=s+1}^{\infty} K_0(s,l)y(l) \\ &= \sum_{l=n+1}^{\infty} \Big(\sum_{s=n+1}^{l-1} K_0(n,s)K_0(s,l) \Big) y(l) = \sum_{l=n+1}^{\infty} K_1(n,l)y(l). \end{aligned}$$

Substituting the last equality into (6) we obtain

$$y(n) = \sum_{l=n+1}^{\infty} K_2(n,l)y(l).$$

Repeating the above procedure $(q-1)$ times we have

$$y(n) = \sum_{l=n}^{\infty} K_q(n,l)y(l). \tag{10}$$

Our next objective is to show that equation (6) has a unique solution. Suppose (for contradiction) that there are two solutions y_1, y_2, $y_1 \neq y_2$ bounded for $n \to \infty$. Subtracting we get

$$u(n) = \sum_{l=n+1}^{\infty} K_0(n,l)u(l); \quad u(l) = y_1(l) - y_2(l). \tag{11}$$

From (10), we see that

$$u(n) = \sum_{l=n+1}^{\infty} K_q(n,l)u(l). \tag{12}$$

Hence, by the boundedness of the function u and the condition (9) we have $u(n) = 0$ for all $n \geq n_2$.

We infer hence that in the general case there exists for $n \geq n_2$ exactly one solution of (6) bounded for $n \to \infty$.

We have

$$\overline{y}(n) - f(n) = \sum_{s=n+1}^{\infty} K_0(n,s)\overline{y}(s) \to 0$$

as $n \to \infty$ by (3) and (4).

It follows that $\overline{\lim_{n\to\infty}} |\overline{y}(n)| = M$ in case (5_{a}) and $\lim_{n\to\infty} \overline{y}(n) = s$ in case (5_{b}). $\square$

Now we consider the scalar situation.

Theorem 3.2. *Suppose that*

1° *the function* $g^p(n)$ *has property* $\lim_{n\to\infty} g^p(n) = 0$, $\sum_{s=n+1}^{\infty} |\Delta g^p(s)| \le K|g^p(n)|$, $K \ge 1$, *uniformly for* $p \in (0,1]$, $g(n) \ne 0$, $|g(n)|$ *is monotone and* $|\Delta g(n)|$ *uniformly bounded for* $n \ge n_0$,

2° $\varphi(n), f(n)$ *and* $\psi(n)$ *are bounded on* $N(n_0)$,

2°_{a} $\overline{\lim_{n\to\infty}} |f(n)| = M < \infty$,

2°_{b} $\lim_{n\to\infty} f(n) = s \quad (|s| < \infty)$,

3° $\sum_{n=n_0}^{\infty} |\psi(n)| < \infty$,

4° $\lim_{n\to\infty} \varphi(n) = 0$.

Let $K(n,s) = \frac{\Delta g(s)}{g(n)}\varphi(s) + \psi(s)$ *and* $N(n,s) = \frac{\Delta g(s)}{g(n)}\varphi(s)$ *for* $n \ge n_0,\ s \ge n_0$. *Then, in the case of* $\lim_{n\to\infty} g(n) = 0$ *the equation*

$$y(n) = f(n) + \sum_{s=n+1}^{\infty} K(n,s)y(s), \tag{13}$$

has for large $(n \ge n_0)$ *exactly one solution* $\overline{y}(n)$ *bounded for* $n \to \infty$.

We have $\overline{\lim_{n\to\infty}} |\overline{y}(n)| = M$ *in case* 2°_{a}, *resp.* $\lim \overline{y}(n) = s$ *in case* 2°_{b}.

Proof. In the case of $\lim_{n\to\infty} g(n) = 0$ we choose a fixed $\alpha \in (0,1)$ and for given $\varepsilon > 0$ a small enough $\delta > 0$ such that the inequality $K\delta \le \varepsilon < 1$ is true. Next, we choose $n_2 \ge n_0$ such that $|\varphi(n)| \le \delta$ for $n \ge n_2$ and

$$\sum_{l=n+1}^{\infty} |g(l)|^{p-1} |\Delta g(l)| \le K|g(n)|^p$$

is satisfied for $n \ge n_2$ and every $p \in (0,1]$.

We obtain by 1° and 4° for $s \ge n \ge n_2$

$$\begin{aligned}
\sum_{l=n+1}^{s-1} |N(n,l)|\,|N(l,s)|^\alpha &= \sum_{l=n+1}^{s-1} \Big|\frac{\Delta g(l)}{g(n)}\varphi(l)\Big|\,\Big|\frac{\Delta g(s)}{g(l)}\varphi(s)\Big|^\alpha \\
&= \frac{|\Delta g(s)\varphi(s)|}{|g(n)|} \sum_{l=n+1}^{s-1} |\Delta g(l)|\,|\varphi(l)|\,|g(l)|^{-\alpha} \\
&\le \delta \frac{|\Delta g(s)\varphi(s)|^\alpha}{|g(n)|} \sum_{l=n+1}^{s-1} |\Delta g(l)|\,|g(l)|^{-\alpha} \\
&\le \frac{|\Delta g(s)\varphi(s)|^\alpha}{|g(n)|} \delta K |g(n)|^{1-\alpha} = \delta K |N(n,s)|^\alpha.
\end{aligned}$$

The inequality in hypothesis 1° of Theorem 3.1 is satisfied with $\lambda = K\delta$.

Next, we state that hypothesis 2° of Theorem 3.1 is satisfied for $F(s) = \left|\frac{\Delta g(s)}{g(s)}\varphi(s)\right|$ for $s \geq n_2$. We shall show that hypothesis 4° of Theorem 3.1 is also satisfied.

We have

$$\begin{aligned}
&\sup_{n_0 \leq l \leq n+1} \sum_{s=n+1}^{\infty} F^{1-\alpha}(s)|N(l,s)|^{\alpha} = \\
&\quad = \sup_{n_0 \leq l \leq n+1} \sum_{s=n+1}^{\infty} F^{1-\alpha}(s)\left|\frac{\Delta g(s)}{g(l)}\varphi(s)\right|^{\alpha} \\
&\quad = \sup_{n_0 \leq l \leq n+1} \frac{1}{|g(l)|^{\alpha}} \sum_{s=n+1}^{\infty} \left|\frac{\Delta g(s)}{g(l)}\varphi(s)\right|^{1-\alpha} |\Delta g(s)\varphi(s)|^{\alpha} \\
&\quad = \sup_{n_0 \leq l \leq n+1} \frac{1}{|g(l)|^{\alpha}} \sum_{s=n+1}^{\infty} |\Delta g(s)\varphi(s)|\,|g(s)|^{\alpha-1} \\
&\quad \leq \sup_{n_0 \leq l \leq n+1} \frac{\delta}{|g(l)|^{\alpha}} K|g(n)|^{\alpha} \leq \varepsilon \quad \text{for} \quad n \geq n_2.
\end{aligned}$$

To prove this part of Theorem 3.2 we now use Theorem 3.1. □

Remark 3.1. *System (1) can be extended in the form*

$$y(n) = f(n) + \sum_{s=n}^{\infty} K(n,s)y(s). \tag{$*$}$$

Let det$(E - K(n,n)) \neq 0$ *for all* $n \geq n_0$, *then*

$$y(n) = h(n) + \sum_{s=n+1}^{\infty} \overline{K}(n,s)y(s)$$

where

$$\begin{aligned}
h(n) &= (E - K(n,n))f(n), \\
\overline{K}(n,s) &= (E - K(n,n))^{-1}K(n,s).
\end{aligned}$$

4. Asymptotic equivalence

In this section we are going to get some asymptotic formulae which relate the solutions $y(n)$ of the system

$$y(n) = f(n) + \sum_{s=0}^{n-1} K(n,s)y(s) \tag{14}$$

and solutions $x(n)$ of the system

$$x(n) = f(n) + \sum_{s=0}^{n-1} K(n,s)[x(s) + g(s,x(s))]. \tag{15}$$

In particular, we will show that

$$\lim_{n\to\infty} |x(n) - y(n)| = 0.$$

Our results complete those concerning various asymptotic relationships between (14) and (15) that have been obtained recently, [3, 4, 9, 10].

The resolvent kernel associated with the kernel $K(n,s)$ is defined to be the (unique) solution of the system (see Section 2, Resolvent)

$$R(n,s) = K(n,s) + \sum_{q=s+1}^{n-1} K(n,q)R(q,s), \quad n > s \tag{16}$$

and

$$R(n,s) = K(n,s) + \sum_{q=s+1}^{n-1} R(n,q)K(q,s), \quad n > s. \tag{17}$$

In terms of the resolvent matrix $R(n,s)$ of (14) the system (15) equivalent to the system

$$x(n) = y(n) + \sum_{s=0}^{n-1} R(n,s)g(s,x(s)), \tag{18}$$

where $y(n)$ is the solution of the linear system (14) given by

$$y(n) = f(n) + \sum_{s=0}^{n-1} R(n,s)f(s). \tag{19}$$

Let $S(o) \equiv S$ be the set of all sequences $\{z(n)\}_{n\geq 0}$ of k-dimensional vectors and let $BS(o) \equiv BS$ be the space of all bounded sequences equipped with the norm $|z| = \sup\limits_{n\geq 0} |z(n)|$.

Theorem 4.1. *Let the resolvent kernel $R(n,s)$ satisfy the following conditions:*

1° *there exist constants $p > 1$ and $B > 0$ such that*

$$\Big(\sum_{s=0}^{n} |R(n,s)|^p\Big)^{\frac{1}{p}} \leq B, \quad n \in N,\ p > 1, \tag{20}$$

2° *for each fixed* $m > 0$

$$\lim_{n\to\infty} \sum_{s=0}^{m} |R(n,s)|^p = 0. \tag{21}$$

Let $g(n,x)$ *be defined for* $n \geq 0$, $|x| < \infty$ *and continuous for each* x, *and let there exist a function* $\lambda(n) \geq 0$, $\lambda \in l_q(0,\infty)$ *where* $p+q = pq$ *such that for all* $n \geq 0$, $|x| < \infty$

$$|g(n,x)| \leq \lambda(n)(1+|x|). \tag{22}$$

Then, given a solution $y \in BS$ *of system (14), there exists a solution* $x \in BS$ *of the system (15) such that*

$$\lim_{n\to\infty} (x(n) - y(n)) = 0. \tag{23}$$

And conversely, given a solution $u \in BS$ *of the system (15), there exists a solution* $v \in BS$ *of the system (14) such that*

$$\lim_{n\to\infty} (u(n) - v(n)) = 0. \tag{23'}$$

As can be seen from (19), a sufficient condition for $y \in BS$ is that $f \in BS$, $f \in l_q(0,\infty)$ and $R(n,s)$ satisfy (20).

Proof of Theorem 4.1. The proof is divided into four parts.

I. Assuming the existence of a solution $y \in BS$ of (14), we prove that there exists a solution x of (15) for $n \geq 0$. We make use of the Volterra equation (18) equivalent to (15). Sufficient conditions under which equation (18) has unique solution is given in [1].

II. Next we show that $x \in BS$. Let $0 < \varepsilon < 1$, since $\lambda \in l_q(0,\infty)$, choose a number $n_* > 0$ so large that

$$\Big(\sum_{s=n_*}^{n} \lambda^q(s)\Big)^{\frac{1}{q}} \leq \frac{\varepsilon}{B} \quad (n_* \leq n < \infty,\ 1 < q < \infty). \tag{24}$$

Since $x(n)$ is defined on $\langle 0,\infty)$, there exists a constant $M = M(n_*) > 0$ so that $M = \sup\limits_{0\leq n\leq n_*} |x(n)|$.

Choose a number $P > 0$ so that

$$|y| + \sum_{s=0}^{n-1} |R(n,s)|(1+|x(s)|)\lambda(s) \leq \tag{25}$$

$$
\begin{aligned}
&\leq |y| + \sum_{s=0}^{n-1} |R(n,s)|\lambda(s)(1+M) \\
&\leq |y| + (1+M)\Big(\sum_{s=0}^{n-1} |R(n,s)|^p\Big)^{\frac{1}{p}} \Big(\sum_{s=0}^{n-1} \lambda^q(s)\Big)^{\frac{1}{q}} \leq (1-\varepsilon)P.
\end{aligned}
$$

We assert that $|x(n)| < P$ for all $n \geq 0$. If not, there exists a $n_1 \geq n_* + 2$ such that $|x(n)| < P$ for $0 \leq n < n_1$ and $|x(n_1)| = P$. But from (18) and assumption of Theorem we obtain

$$
\begin{aligned}
P = |x(n_1)| \quad &\leq \quad |y| + \sum_{s=0}^{n_*} |R(n_1,s)|\lambda(s)(1+|x(s)|) \\
&\qquad + \sum_{s=n_*+1}^{n_1-1} |R(n_1,s)|\lambda(s)(1+|x(s)|) \\
&\leq \quad |y| + (1+M)B(|\lambda|_q + (1+P)B\Big(\sum_{s=n_*+1}^{n_1-1} \lambda^q(s)\Big)^{\frac{1}{q}}.
\end{aligned}
$$

Applying (24) and (25) yields

$$
P \leq |y| + (1+M)B|\lambda|_q + (1+P)\varepsilon < (1-\varepsilon)P + \varepsilon P = P,
$$

what is a contradiction. Thus $|x(n)| < P$ for all $n \geq 0$.

III. We show that $\lim_{n\to\infty}(x(n) - y(n)) = 0$, where $y \in BS$ is a solution of (14) and $x \in BS$ is the solution of (15), the existence of which was established in (I) and (II).

Let $\sup_{0\leq n<\infty} |x(n)| = M_0$ and let $\varepsilon > 0$ be given. Choose $m > 0$ so large that

$$
\Big(\sum_{s=m}^{n} \lambda^q(s)\Big)^{\frac{1}{q}} < \frac{\varepsilon}{2B(1+M_0)} \quad \text{for} \quad n \geq m. \tag{26}
$$

By (21) choose $m_1 > m$ so that

$$
\Big(\sum_{s=0}^{m} |R(n,s)|^p\Big)^{\frac{1}{p}} < \frac{\varepsilon}{2(1+M_0)|\lambda|_q} \quad (n \geq m_1). \tag{27}
$$

Then, from (19), (20), (22), the Hölder inequality and (26), (21) we obtain successively

$$
|x(n) - y(n)| \quad \leq \quad \sum_{s=0}^{m} |R(n,s)|\lambda(s)(1+|x(s)|) +
$$

$$
\begin{aligned}
&+\sum_{s=m+1}^{n}|R(n,s)|\lambda(s)(1+|x(s)|)\\
\leq\ & (1+M)|\lambda|_q\Big(\sum_{s=0}^{m}|R(n,s)|^p\Big)^{\frac{1}{p}}+(1+M)B\Big(\sum_{s=m+1}^{n}\lambda^q(s)\Big)^{\frac{1}{q}}\\
<\ & \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon
\end{aligned}
$$

for $n > m_1$. Since $\varepsilon > 0$ is arbitrary, this completes the proof.

IV. Let $u \in BS$ be a solution of (15). We will show that there exists a solution $v \in BS$ of (14) such that

$$\lim_{n\to\infty}(u(n)-v(n))=0.$$

Let $v(n) = u(n) - \sum_{s=0}^{n-1} R(n,s)g(s,u(s))$. The $v(n)$ is a solution of (14). Using assumptions of Theorem and the Hölder inequality one has

$$
\begin{aligned}
|v(n)| &\leq |u(n)|+\sum_{s=0}^{n-1}|R(n,s)|\,|g(s,u(s))|\\
&\leq |u|+\sum_{s=0}^{n-1}|R(n,s)|\lambda(s)(1+|u|)\\
&= |u|+(1+|u|)\sum_{s=0}^{n-1}|R(n,s)|\lambda(s)\\
&\leq |u|+(1+|u|)B|\lambda|_q<\infty.
\end{aligned}
$$

Hence $v \in BS$.

Define m and m_1 as in (26), (27) with M_0 replaced by $|u|$. Then as in (III) one obtains

$$
\begin{aligned}
|u(n)-v(n)| &\leq \sum_{s=0}^{m}|R(n,s)|\lambda(s)(1+|u|)+\sum_{s=m+1}^{n-1}|R(n,s)|\lambda(s)(1+|u|)\\
&< \varepsilon\frac{(1+|u|)|\lambda|_q}{2(1+|u|)|\lambda|_q}+\varepsilon\frac{B(1+|u|)}{2B(1+|u|)}=\varepsilon \quad \text{for} \quad n\geq m_1.
\end{aligned}
$$

Since $\varepsilon > 0$ is arbitrary,

$$\lim_{n\to\infty}(u(n)-v(n))=0.$$

This completes the proof. □

Corollary 4.1. *Let the resolvent Kernel $R(n,s)$ satisfy (20), (21) with $p = 1$. Let $g(n,x)$ be continuous in (n,x) for $n \in \langle 0,\infty)$, $|x| < \infty$ and let there exists a function $\lambda \in BS$ such that $\lambda(n) \geq 0$, $0 \leq n < \infty$, $\lim_{n\to\infty} \lambda(n) = 0$ and such that (22) is satisfied. Then the systems (14)-(15) are asymptotically equivalent.*

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