Asymptotic Bounds for Bipartite Ramsey Numbers

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Abstract

The bipartite Ramsey number $b(m, n)$ is the smallest positive integer r such that every (red, green) coloring of the edges of $K_{r,r}$ contains either a red $K_{m,m}$ or a green $K_{n,n}$. We obtain asymptotic bounds for $b(m, n)$ for $m \geq 2$ fixed and $n \to \infty$.

1 Introduction

Recent exact results for bipartite Ramsey numbers [4] have rekindled interest in this subject. The bipartite Ramsey number $b(m, n)$ is the smallest integer r such that every (red, green) coloring of the edges of $K_{r,r}$ contains either a red $K_{m,m}$ or a green $K_{n,n}$. In early work on the subject [1], Beineke and Schwenk proved that $b(2, 2) = 5$ and $b(3, 3) = 17$. In [4] Hattingh and Henning prove that $b(2, 3) = 9$ and $b(2, 4) = 14$. The following variation was considered by Beineke and Schwenk [1] and also by Irving [5]: for $1 \leq m \leq n$, the bipartite Ramsey number $R(m, n)$ is the smallest integer r such that every (red, green) coloring of the edges of $K_{r,r}$ contains a monochromatic $K_{m,n}$. Irving found that $R(2, n) \leq 4n - 3$, with equality if n is odd and there is Hadamard matrix of order $2(n-1)$. The bound $R(m, n) \leq 2^m(n-1)+1$ was proved by Thomason in [7]. Note that $b(m, m) = R(m, m)$. In this note, we obtain asymptotic bounds for $b(m, n)$ with m fixed and $n \to \infty$.

2 The Main Result

Theorem 1. *Let $m \geq 2$ be fixed. Then there are constants A and B such that*

$$A\left(\frac{n}{\log n}\right)^{(m+1)/2} < b(m, n) < B\left(\frac{n}{\log n}\right)^{m}, \qquad n \to \infty.$$

Specifically, these bounds hold with

$$A = (1-\epsilon)m^{-1/(m-1)} \left(\frac{m-1}{m^2}\right)^{(m+1)/2}$$

and

$$B = (1+\epsilon)\left(\frac{1}{m-1}\right)^{m-1},$$

where $\epsilon > 0$ *is arbitrary.*

Proof. The upper bound is based on well-known results for the Zarankiewicz function. Let $z(r,s)$ denote the maximum number of edges that a subgraph of $K_{r,r}$ can have if it does not contain $K_{s,s}$ as a subgraph. We use the bound

$$z(r;s) < \left(\frac{s-1}{r}\right)^{1/s} r(r-s+1) + (s-1)r, \tag{1}$$

which is found in [2] and elsewhere. To prove $b(m,n) \leq r$ it suffices to show that $z(r;m) + z(r;n) < r^2$. Take $\epsilon > 0$ and set $r = c(n/\log n)^m$ where $c = (m-1)^{-(m-1)}(1+\epsilon)$. Then

$$\begin{aligned} \frac{z(r;m)}{r^2} &< \left(\frac{m-1}{r}\right)^{1/m}\left(1-\frac{m-1}{r}\right) + \frac{m-1}{r} \\ &= \left(\frac{m-1}{c}\right)^{1/m} \frac{\log n}{n} + O\left(\left(\frac{\log n}{n}\right)^m\right). \end{aligned} \tag{2}$$

To bound $z(r;n)/r^2$, we begin with the evident asymptotic formula

$$\left(\frac{n-1}{r}\right)^{1/n} = \left(\frac{(n-1)(\log n)^m}{cn^m}\right)^{1/n} = 1 - \frac{(m-1)\log n}{n} + O\left(\frac{\log\log n}{n}\right).$$

Hence

$$\begin{aligned} \frac{z(r;n)}{r^2} &< \left(\frac{n-1}{r}\right)^{1/n}\left(1-\frac{n-1}{r}\right) + \frac{n-1}{r} \\ &= 1 - \frac{(m-1)\log n}{n} + O\left(\frac{\log\log n}{n}\right). \end{aligned} \tag{3}$$

Adding (2) and (3) we obtain

$$\begin{aligned} \frac{z(r;m)+z(r;n)}{r^2} &= 1 - \left(m-1-\left(\frac{m-1}{c}\right)^{1/m}\right)\frac{\log n}{n} + O\left(\frac{\log\log n}{n}\right) \\ &= 1-(m-1)\left(1-\frac{1}{(1+\epsilon)^{1/m}}\right)\frac{\log n}{n} + O\left(\frac{\log\log n}{n}\right), \end{aligned}$$

so $(z(r;m)+z(r;n))/r^2<1$ for all sufficiently large n, completing the proof.

To prove the lower bound, we use the Lovász Local Lemma in the manner pioneered by Spencer [6]. Consider a random coloring of the edges of $K_{r,r}$ in which, independently, each edge is colored red with probability p. For each set S of $2m$ vertices, m from each vertex class of the $K_{r,r}$, let R_S denote the event in which each edge of the $K_{m,m}$ spanned by S is red. Similarly, for each set T consisting of n vertices from each color class, let G_T denote the event in which each edge of the $K_{n,n}$ spanned by T is green. Then $\mathbb{P}(R_S)=p^{m^2}$ for each of the $\binom{r}{m}^2$ choices of S, and we simply write $\mathbb{P}(R)$ for the common value. In the same way, $\mathbb{P}(G)=(1-p)^{n^2}$ for each of $\binom{r}{n}^2$ possible $G=G_T$ events. Let S be a fixed choice of m vertices from each class. Then N_{RR} denotes the number of events $R_{S'}$ such that R_S and $R_{S'}$ are dependent, that is the bipartite graphs spanned by S and S' share at least one edge. Similarly, let N_{RG} denote the number of events G_T such that R_S and G_T are dependent. In the same way, for fixed a fixed choice T of n vertices from each class, we define the dependence numbers N_{GR} and N_{GG}. By the Local Lemma, the probability that a random coloring has neither a red $K_{m,m}$ or a green $K_{n,n}$ is positive provided there exist positive numbers x_R and x_G such that

$$\begin{aligned} 1 &> x_R\,\mathbb{P}(R), && (4)\\ 1 &> x_G\,\mathbb{P}(G), && (5)\\ \log x_R &> x_R N_{RR}\,\mathbb{P}(R)+x_G N_{RG}\,\mathbb{P}(G), && (6)\\ \log x_G &> x_R N_{GR}\,\mathbb{P}(R)+x_G N_{GG}\,\mathbb{P}(G). && (7)\end{aligned}$$

With positive constants c_1 through c_4 to be chosen, set

$$\begin{aligned} p &= c_1 r^{-2/(m+1)},\\ n &= c_2 r^{2/(m+1)}\log r,\\ x_R &= c_3,\\ x_G &= \exp\left(c_4 r^{2/(m+1)}(\log r)^2\right).\end{aligned}$$

To prove that there are choices of the constants $c_1,\ldots,c_4$ for which (4) through (7) hold, we begin by noting the following bounds:

$$N_{RR}\le m^2\binom{r}{m-1}^2<r^{2(m-1)},$$

$$N_{GR}\le n^2\binom{r}{m-1}^2<n^2r^{2(m-1)},$$

$$N_{RG},N_{GG}\le\binom{r}{n}^2<\left(\frac{e\,r}{n}\right)^{2n}.$$

We have

$$N_{RR}\mathbb{P}(R)<r^{2(m-1)}\left(c_1r^{-2/(m+1)}\right)^{m^2}=c_1^{m^2}r^{-2/(m+1)}=o(1),\qquad r\to\infty, \tag{8}$$

independent of the choice of c_1. Also $\log N_{RG} < 2n \log r = 2c_2\, r^{2/(m+1)}(\log r)^2$ and

$$\mathbb{P}(G) = (1-p)^{n^2} \le \exp(-pn^2) = \exp\left(-c_1 c_2^2 r^{2/(m+1)}(\log r)^2\right),$$

so $x_G N_{RG}\, \mathbb{P}(G) \le \exp\left((c_4 + 2c_2 - c_1 c_2^2) r^{2/(m+1)}(\log r)^2\right)$. Hence $x_G N_{RG}\, \mathbb{P}(G) = o(1)$ and $x_G N_{GG}\, \mathbb{P}(G) = o(1)$. provided we choose c_1, c_2 and c_4 so that

$$c_4 < c_1 c_2^2 - 2c_2. \tag{9}$$

Note that (4) is automatically fulfilled, and also $x_G N_{RG}\, \mathbb{P}(G) = o(1)$ implies (5). In view of (8) and $x_G N_{RG}\, \mathbb{P}(G) = o(1)$, which is implied by (9), condition (6) holds for all sufficiently large r if we choose

$$c_3 > 1. \tag{10}$$

Finally, since

$$\begin{aligned} x_R N_{GR}\, \mathbb{P}(R) &\le c_3 (c_2\, r^{2/(m+1)} \log r)^2 r^{2(m-1)} (c_1 r^{-2/(m+1)})^{m^2} \\ &= c_1^{m^2} c_2^2\, c_3\, r^{2/(m+1)} (\log r)^2, \end{aligned}$$

we see that (7) holds provided the constants $c_1, \ldots, c_4$ are chosen so that

$$c_4 > c_1^{m^2} c_2^2\, c_3. \tag{11}$$

To satisfy (9), (10), and (11), and at the same time find a near optimal (minimum) choice for c_2, we begin by considering the case of equality in (7)-(9). Set $c_3 = 1$ and

$$c_1^{m^2} c_2^2 = c_4 = c_1 c_2^2 - 2c_2.$$

Since both c_1 and c_2 are positive, c_1 must satisfy $0 < c_1 < 1$. To minimize $c_2 = 1/(c_1 - c_1^{m^2})$ we choose $c_1 = m^{-2/(m^2-1)}$. To satisfy (7)-(9) and still make a nearly optimal choice of c_2, set

$$c_1 = m^{-2/(m^2-1)}, \qquad c_2 = \frac{2(1+\epsilon)}{c_1 - (1+\epsilon)c_1^{m^2}}, \qquad c_3 = 1+\epsilon,$$

where ϵ is positive and small enough that $c_1 - (1+\epsilon)c_1^{m^2} > 0$. Then $c_1^{m^2} c_2^2 c_3 < c_1 c_2^2 - 2c_2$, which is equivalent to $c_2(c_1 - c_3 c_1^{m^2}) > 2$, is satisfied and there is a suitable choice of c_4 so that $c_1^{m^2} c_2^2 c_3 < c_4 < c_1 c_2^2 - 2c_2$. A routine computation shows that this justifies the lower bound statement with

$$A = (1-\epsilon) m^{-1/(m-1)} \left(\frac{m-1}{m^2}\right)^{(m+1)/2},$$

where $\epsilon > 0$ is arbitrary. □

3 Open Questions

Our knowledge of $b(2,n)$ closely parallels that of $r(C_4, K_n)$. Concerning the latter, Erdős conjectured at the 1983 ICM in Warsaw that $r(C_4, K_n) = o(n^{2-\epsilon})$ for some $\epsilon > 0$ [3, p. 19].

Open Question 1. *Prove or disprove that* $b(2,n) = o(n^{2-\epsilon})$ *for some* $\epsilon > 0$*.*

Also, very little is known about the diagonal case. A well-known question in classical Ramsey theory concerning the asymptotic behavior of $r(n)$ [3, p. 10] has the following counterpart for bipartite Ramsey numbers.

Open Question 2. *Determine the value of*

$$\lim_{n\to\infty} b(n,n)^{1/n},$$

if it exists.

From [4] and [7] it is known that $\sqrt{2}e^{-1}n2^{n/2} < b(n,n) \leq 2^n(n-1)+1$, so if the limit exists, it is between $\sqrt{2}$ and 2.

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