The Annals of Statistics
1976, Vol. 4, No. 1, 157–174

ASYMPTOTIC COMPARISON OF RANK TESTS FOR THE REGRESSION PROBLEM WHEN TIES ARE PRESENT

BY KONRAD BEHNEN

University of Freiburg

Without assumptions on the underlying distributions we prove the asymptotic normality of averaged scores rank statistics under all product distributions which are contiguous to the null hypothesis, and find a very simple form of the centering constants. In the one-sided two-sample and trend situations this enables us to show that monotonicity of the scores generating function is equivalent to the asymptotic unbiasedness of the corresponding averaged scores rank test. For such asymptotically unbiased tests we prove simple necessary and sufficient conditions for having bounds for their asymptotic relative efficiency under all contiguous alternatives of the model. As a by-product, we get the local asymptotic superiority of averaged scores rank tests to the associated tests with randomized ranks not only for shift but for general alternatives.

In addition we prove that every one-sided averaged scores rank test is asymptotically most powerful (asymptotically equivalent to likelihood ratio test) for a suitable nonparametric subclass of alternatives, provided the test and the associated subclass of alternatives are generated by a non-decreasing, square-integrable function defined on the unit interval.

1. Introduction. There is a substantial literature on the problem of ties with respect to rank tests (cf. the bibliography in Hájek and Šidák (1967)). However, most of the papers are concerned with special properties of special tests. The first paper dealing with more general properties in a rather general class of rank tests is by Vorličková (1970). The disadvantages of this paper are the heavy assumptions on the underlying distributions and the restrictive results gotten under alternatives caused by the methods applied. The same is true for a similar paper by Vorličková (1972) on the symmetry case. Especially, it is assumed that the underlying distributions are purely discrete with mass only at equidistant points. Moreover, the asymptotic normality and efficiency results under contiguous alternatives are proved only for special subclasses of such alternatives. The reason for the latter restriction is the application of Hájek and Šidák (1967), Lemma VI.1.4 (Le Cam's third lemma). Conover (1973) dropped the assumptions on the underlying distributions, but since his proofs follow the lines of Vorličková the restrictions on the alternatives still are there.

In this paper we shall prove asymptotic normality and power results under all product distributions which are contiguous to the null hypothesis, without any assumption on the underlying distributions besides the trivial one of not having

Received April 1973; revised May 1975.

AMS 1970 *subject classifications.* Primary 62G10; Secondary 62E20, 62G20.

Key words and phrases. Linear rank statistics, treatment of ties, averaged scores, randomized ranks, asymptotic normality, asymptotic optimality, contiguity, asymptotic relative efficiency, efficiency bounds.

one-point measures. As a consequence we are able to derive results on comparison of rank tests by means of asymptotic relative efficiency, which extend the results for the continuous case of Behnen (1972).

Let $X_{n1}, \cdots, X_{nn}$ be independent real valued random variables and let $F_{nj}(x) = \Pr(X_{nj} \leqq x)$ be the (arbitrary) cumulative distribution function of X_{nj}, $j = 1, \cdots, n$, $n \in \mathbb{N}$. As usual, the statistic

$$R_{ni} = \sum_{j=1}^{n} I_{[0,\infty)}(X_{ni} - X_{nj})$$

is called the rank of X_{ni} in the ordered sample

$$X_n^{(\ )} = (X_n^{(1)}, \cdots, X_n^{(n)}), \qquad X_n^{(1)} \leqq \cdots \leqq X_n^{(n)}, \qquad X_n^{(R_{ni})} = X_{ni}.$$

Following the notation of Hájek and Šidák (1967) and Vorličková (1970) we call the g_n groups of equal observations in $X_n^{(\ )}$ ties and denote the number of observations in the jth tie $(1 \leqq j \leqq g_n)$ by τ_{nj},

$$\tau_n = (\tau_{n1}, \cdots, \tau_{ng_n}), \qquad T_{nj} = \sum_{i=1}^{j} \tau_{ni}, \qquad j = 1, \cdots, g_n,$$

$T_{n0} = 0$. Thus

$$X_n^{(T_{nj-1}+1)} = \cdots = X_n^{(T_{nj})}, \qquad X_n^{(T_{nj})} < X_n^{(T_{nj}+1)}.$$

For convenience we will suppress the first subscript n throughout this paper, and limits are taken for $n \to \infty$ if not otherwise stated.

Randomization of ranks. From the definitions it is clear that exactly the observations in one tie get equal ranks. Therefore the randomization is done in the most convenient way: Let $U_1, U_2, \cdots$ be i.i.d. random variables (also independent from $(X_1, \cdots, X_n)$ for each $n \in \mathbb{N}$) each having a rectangular distribution on (0, 1). Put

$$R_i^* = \sum_{j=1}^{n} I_{[0,\infty)}((R_i - U_i) - (R_j - U_j)), \qquad R^* = (R_1^*, \cdots, R_n^*).$$

This implies

$$\{R_1^*, \cdots, R_n^*\} = \{1, \cdots, n\} \quad \text{a.e.},$$
$$R_i < R_j \Rightarrow R_i^* < R_j^*,$$
$$R_i^* \leqq R_i, \quad \forall\, i, j = 1, \cdots, n.$$

For the definition of scores let b be a square-integrable function defined on (0, 1) such that (for convenience)

$$(1.1) \qquad \int b\, d\lambda = 0, \qquad \int b^2\, d\lambda = 1,$$

where λ denotes Lebesgue-measure on (0, 1) throughout the paper. Then scores b_i are defined by

$$b_i = Eb(U_n^{(i)}), \qquad i = 1, \cdots, n,$$

or in some other way such that (cf. Hájek and Šidák (1967) or Vorličková (1970))

$$(1.2) \qquad \int (b_{[nu+1]} - b(u))^2 \lambda(du) \to 0, \qquad \sum_{i=1}^{n} b_i = 0.$$

Moreover, let regression coefficients c_i be given such that

(1.3) $$\sum_{i=1}^n c_i = 0\,, \qquad \sum_{i=1}^n c_i^2 = 1\,, \qquad \max_{1\leq i\leq n} c_i^2 \to 0\,.$$

With these assumptions and notations we shall consider *randomized linear rank statistics* $S_b{}^*$ of the form

(1.4) $$S_b{}^* = \sum_{i=1}^n c_i b_{R_i^*}\,,$$

and for given $\alpha \in (0, 1)$ *randomized ranks tests* $\varphi_b{}^*$ of the form

(1.5) $$\varphi_b{}^* = I_{\{S_b{}^* > k_n\}}\,,$$

with $k_n \to u_\alpha$, u_α the $(1 - \alpha)$-quantile of $N(0, 1)$.

Now we introduce another method of treating ties (cf. Hájek (1969)) which will be shown to be generally better than randomization. Nevertheless randomization is a convenient tool.

Averaged scores. In the general notation stated above average the scores b_i over the length of the ties, i.e.,

$$\bar{b}_i = b_i(\tau) = \frac{1}{\tau_k} \sum_{j=T_{k-1}+1}^{T_k} b_j \qquad \text{if} \quad T_{k-1} < i \leqq T_k\,.$$

Define *averaged scores rank statistics* $\bar{S}_b$ by

(1.6) $$\bar{S}_b = \sum_{i=1}^n c_i \bar{b}_{R_i}$$

and *averaged scores rank tests* $\bar{\varphi}_b$ by

(1.7) $$\bar{\varphi}_b = I_{\{\bar{S}_b > k_n (1/(n-1) \sum_{i=1}^n \bar{b}_i{}^2)^{\frac{1}{2}}\}}\,.$$

EXAMPLE. Consider the two-sample situation and the averaged scores versions of the normal scores test, the Wilcoxon test, and the rank median test, with centered and standardized scores generating functions $\Phi^{-1}(u)$, $(12)^{\frac{1}{2}}(u - \frac{1}{2})$, sign $(u - \frac{1}{2})$, respectively. Fix any point ϑ_0 of the hypothesis of randomness H_0. Then by Theorem 2.2.(c) and Remark 3.1 for each one of the three tests there exist sequences from the alternative K_0 (first sample stochastically larger than second sample) which are contiguous to ϑ_0 and for which the respective tests have optimal asymptotic power. On the other hand, Theorem 3.3 gives the range of the asymptotic relative efficiency (ARE) of two such tests for the alternatives from K_0 which are contiguous to ϑ_0. If ϑ_0 corresponds to a distribution which is not purely discrete, then the ARE of rank median to Wilcoxon and normal scores test ranges down to zero. If ϑ_0 corresponds to a distribution function F such that F^{-1} is strictly increasing in a neighborhood of $\frac{1}{2}$, then the ARE of Wilcoxon and normal scores to rank median test ranges down to zero, too. If ϑ_0 corresponds to a continuous distribution function, then the range of the ARE of normal scores to Wilcoxon test is the interval $[\pi/6, \infty]$. With a lot of computations one can prove the existence of a constant $c > 0$ such that the ARE of normal scores to Wilcoxon test is not less than c, uniformly in $\vartheta_0 \in H_0$. Any such constant is strictly less than $\pi/6$.

2. Statement of the general theorems. In this section we state the theorems concerning asymptotic normality under general contiguous alternatives and optimality of averaged scores rank tests. The proofs are given in Section 5. For the formulation of the statements we first have to introduce some general notation.

For each one-dimensional distribution function F and each λ-integrable function h defined on the interval (0, 1) we define a function h_F on (0, 1) by

$$(2.1)\qquad \begin{aligned} h_F(u) &= h(u) && \text{if } u \in (0,1) - \textstyle\bigcup_{x\in D(F)} (F(x-0), F(x)]\,, \\ &= \textstyle\int_{F(x-0)}^{F(x)} h\, d\lambda/(F(x) - F(x-0)) && \\ & && \text{if } u \in (F(x-0), F(x)]\,, \quad x \in D(F)\,, \end{aligned}$$

where $D(F)$ denotes the countable set of points where F is not continuous. For continuous F the function h_F is equal to h. For general F

$$(2.2)\qquad h_F = E_\lambda(h \mid \mathfrak{B}_F) \quad \lambda\text{-a.e.},$$

where $E_\lambda(h \mid \mathfrak{B}_F)$ denotes the conditional expectation of h with respect to λ and the sub-σ-algebra $\mathfrak{B}_F$ of the Borel-σ-algebra on (0, 1), which is generated by the intervals $(F(x-0), F(x)]$, $x \in D(F)$, and the Borel-subsets of the set $(0, 1) - \bigcup_{x\in D(F)} (F(x-0), F(x)]$.

In the sequel we denote the distribution of $(X_{n1}, \cdots, X_{nn})$ by $\times_{j=1}^n F_{nj}$, the product measure of distributions corresponding to the distribution functions F_{nj}. In case of $F_{nj} = F$ we use $F^{(n)}$ instead of $\times_{j=1}^n F_{nj}$. Moreover, we identify distribution functions with the corresponding probability measures, if convenient.

Let H_0 denote the hypothesis of randomness, i.e.,

$$H_0 = \{\{\times_{j=1}^n F_{nj}\} : F_{nj} = F \quad \forall j, n,\ F(x) - F(x-0) < 1 \quad \forall x \in \mathbb{R}\}\,.$$

As in Hájek and Šidák (1967) call a sequence of distributions $\{Q_n\}$ contiguous to the sequence $\{P_n\}$ if $P_n(A_n) \to 0$ implies $Q_n(A_n) \to 0$. Then define K_0 to be the set of all sequences of product distributions which are contiguous to some sequence from H_0,

$$K_0 = \{\{\times_{j=1}^n F_{nj}\} : \{\times_{j=1}^n F_{nj}\} \text{ contiguous to some } \{F^{(n)}\} \in H_0\}\,.$$

REMARK 2.1. Contiguity of $\vartheta \in K_0$ to $\vartheta_1 \in H_0$ and $\vartheta_2 \in H_0$ implies $\vartheta_1 = \vartheta_2$, since $F_1 \neq F_2$ implies the existence of measurable sets B_n such that $F_1^{(n)}(B_n) \to 0$, $F_2^{(n)}(B_n) \to 1$ (cf. Behnen and Neuhaus (1975), Proof of the Proposition) which contradicts contiguity of $\vartheta \in K_0$ to $\{F_1^{(n)}\}$ and $\{F_2^{(n)}\}$.

Now we are in the position to formulate the results. Throughout the section we assume (1.1), (1.2), and (1.3).

THEOREM 2.1. (a) $\{\times_1^n F_j\} \in H_0$ *implies* $\mathfrak{L}(S_b^*) \to N(0, 1)$.

(b) $\{\times_1^n F_j\} \in K_0$ *implies* $\mathfrak{L}(S_b^* - \delta_n(b, \{\times_1^n F_j\})) \to N(0, 1)$, *with*

$$(2.3)\qquad \delta_n(b, \{\times_1^n F_j\}) = \textstyle\sum_{i=1}^n c_i \int (\bar{b}_{nF} \circ F)\, dF_i\,,$$

where F is associated to $\{\times_1^n F_i\}$ according to the definition of K_0 (cf. Remark 2.1), *and where* $\{\bar{b}_n\}$ *is any sequence of Lebesgue-measurable functions defined on the unit interval and having the properties*

$$\int \bar{b}_n \, d\lambda = 0 , \qquad \sup_x |\bar{b}_n(x)| < \infty \quad \forall\, n \in \mathbb{N} , \tag{2.4}$$
$$\int (b - \bar{b}_n)^2 \, d\lambda \to 0 , \qquad \max_i c_i^2 \sup_x \bar{b}_n^4(x) \to 0 .$$

(c) *If* $b = b_F$ λ-a.e. *for* $\{F^{(n)}\} \in H_0$ *and if*

$$K_{bF} = \{\{\textstyle\times_1^n F_{id}\} : F_{id} \text{ has } F\text{-density } 1 + dc_i(\bar{b}_{nF} \circ F),\ n \in \mathbb{N},\ d > 0,\ \{\bar{b}_n\} \text{ any approximation (2.4)}\} ,$$

then the randomized rank test $\{\varphi_b{}^*\}$ *according to* (1.5) *is asymptotically most powerful for* H_0 *against* K_{bF} *at level* α. *Moreover, each sequence from* K_{bF} *is contiguous to* $\{F^{(n)}\}$.

REMARK 2.2. For any two choices of centering constants we have

$$\delta_n(b, \vartheta) - \delta_n'(b, \vartheta) \to 0 .$$

Moreover, contiguity and asymptotic normality imply

$$\limsup_n |\delta_n(b, \vartheta)| < \infty \qquad \forall\, \vartheta \in K_0 . \tag{2.5}$$

REMARK 2.3. Obviously, assumptions (1.1) and (1.3) ensure the existence of approximations (2.4) of b. In case of bounded b we may choose $\bar{b}_n$ equal to b. In general we could choose a recentered truncation, where the truncation points suitably depend on $\max_i |c_i|$.

THEOREM 2.2. (a) $\{\times_1^n F_j\} = \{F^{(n)}\} \in H_0$ *implies* $\mathfrak{L}(\bar{S}_b) \to N(0, \int b_F^2 \, d\lambda)$ *and* $\operatorname{Var}(\bar{S}_b \mid \tau) = 1/(n-1) \sum_{i=1}^n (\bar{b}_i)^2 \to \int b_F^2 \, d\lambda$ *in probability.*

(b) $\{\times_1^n F_j\} \in K_0$ *implies* $\mathfrak{L}(\bar{S}_b - \delta_n(b, \{\times_1^n F_j\})) \to N(0, \int b_F^2 \, d\lambda)$, *with* $\delta_n(b, \{\times_1^n F_j\})$ *and* F *defined as in Theorem* 2.1. (b).

(c) *Assume* b *to be* λ-a.e. *monotone. Then* $\{\bar{\varphi}_b\}$ *is an asymptotically most powerful level* α *test for* H_0 *against*

$$K_b \equiv \{\{\times_1^n F_j\} \in K_{bF} : F(x) - F(x - 0) < 1 \quad \forall\, x \in \mathbb{R}\} ,$$

where K_{bF} *is defined as in Theorem* 2.1. (c). *Moreover,* $K_b \subset K_0$.

REMARK 2.4. On one hand the monotonicity condition on b is assumed to ensure $\int b_F^2 \, d\lambda > 0$ on all of H_0, and thus to ensure that $\{\bar{\varphi}_b\}$ is asymptotically level α for H_0. On the other hand we know from Behnen (1972), Theorem 2.2, that in the special applications the monotonicity condition is necessary for having asymptotically unbiased tests, even under the restrictive assumption of continuous distributions. Therefore it seems natural to pose this condition in part (c) instead of restricting the hypothesis H_0 to F's with $\int b_F^2 \, d\lambda > 0$, which depends on b if b is not monotone λ-a.e.

REMARK 2.5. Part (b) is a generalization of Conover (1973), Theorem 8.3, to all contiguous alternatives K_0 without any parametrization and without any regularity assumptions, whereas the following corollary corresponds in a certain sense to the result of Conover.

COROLLARY 2.3. *Let h fulfill* $\int h\,d\lambda = 0$, $\int h^2\,d\lambda = 1$ *and define* K_{hF} *according to part* (c) *of Theorem* 2.1. *Then, for*

$$\{\times_1^n F_j\} = \{\times_1^n F_{jd}\} \in K_{hF},$$

$$\text{(a)} \qquad \delta_n(b, \{\times_1^n F_{jd}\}) \to d \int b_F h_F\,d\lambda,$$

$$\text{(b)} \qquad E\varphi_b^* \to \Phi(-u_\alpha + d \int b_F h_F\,d\lambda),$$

$$\text{(c)} \qquad E\bar{\varphi}_b \to \Phi(-u_\alpha + d \int b_F h_F\,d\lambda/(\int b_F^2\,d\lambda)^{\frac{1}{2}}),$$

where Φ *denotes the distribution function of* $N(0, 1)$.

3. Efficiency of two-sample tests. In this section we assume (1.1), (1.2),

$$\text{(3.1)} \qquad n = n_1 + n_2, \qquad c_{ni} = (n_1 n_2/n)^{\frac{1}{2}}/n_1 \quad \text{if } i = 1, \cdots, n_1,$$
$$= -(n_1 n_2/n)^{\frac{1}{2}}/n_2 \quad \text{if } i = n_1 + 1, \cdots, n,$$

and $n_1 \to \infty$, $n_2 \to \infty$, which means that condition (1.3) is fulfilled. As (one-sided) alternative we take

$$K_0' = \{\{\times_1^n F_j\} \in K_0 : F_1 = \cdots = F_{n_1}, F_{n_1+1} = \cdots = F_n, F_1 \leqq F_n, F_1 \neq F_n\},$$

i.e., we consider the alternative that the first sample is stochastically larger than the second sample, which is more general and more useful than shift alternatives.

REMARK 3.1. If b is λ-a.e. nondecreasing and if we take λ-a.e. nondecreasing approximations (2.4) in the definition of K_{bF} and K_b, then we have $K_b \subset K_0'$. Thus the optimality result of Theorem 2.2. (c) really is concerned with subclasses of K_0'. More generally we have $K_b \subset K_0'$ if $\int_0^t b\,d\lambda \leqq 0\ \forall\, t \in (0, 1)$, and $K_b \cap K_0' = \emptyset$ if $\int_0^t b\,d\lambda \geqq 0\ \forall\, t \in (0, 1)$, provided the approximations (2.4) in the definition of K_b are suitably chosen.

THEOREM 3.1. *The averaged scores rank test* $\{\bar{\varphi}_b\}$ *according to* (1.7) *is asymptotically unbiased on all of* K_0' *if and only if b is* λ-a.e. *nondecreasing.*

PROOF. Because of the asymptotic normality of S_b^* (Theorem 2.1) and $\bar{S}_b$ (Theorem 2.2) under H_0 and K_0' with the same centering constants $\delta_n(\cdot, \cdot)$ and because of $\int b_F^2\,d\lambda > 0\ \forall\, \{F^{(n)}\} \in H_0$ if b is λ-a.e. nondecreasing, the proof is immediate from Lemma 5.1, Lemma 5.3, and Behnen (1972), Theorem 2.2 (applied to $\{\varphi_b^*\}$). □

REMARK 3.2. For λ-a.e. nondecreasing b we have as a consequence

$$\text{(3.2)} \qquad \liminf_n \delta_n(b, \vartheta) \geqq 0 \quad \forall\, \vartheta \in K_0'.$$

In case of

$$E_{H_0}\varphi_{nj} \to \alpha, \qquad E_\vartheta \varphi_{nj} - \Phi(-u_\alpha + \delta_{nj}) \to 0, \qquad j = 1, 2,$$

we define asymptotic relative efficiency of $\{\varphi_{n1}\}$ relative to $\{\varphi_{n2}\}$ with respect to

ϑ by

$$\text{ARE}\,(\{\varphi_{n1}\}:\{\varphi_{n2}\}\,|\,\vartheta) = \liminf_n (\delta_{n1}^2/\delta_{n2}^2)$$
$$\text{iff}\quad 0 \leqq \liminf_n \delta_{n1}\,,\quad 0 < \liminf_n \delta_{n2}\,,\quad \limsup_n \delta_{n1} < \infty\,.$$

REMARK 3.3. This definition is consistent with the usual definition of Pitman-efficiency. Obviously any definition of ARE doesn't make sense without the assumption of unbiasedness, i.e., $\liminf_n \delta_{ni} \geqq 0$, $i = 1, 2$. Since we do not assume convergence of δ_{ni} we have to impose some other conditions on the behavior of the δ_{ni}.

THEOREM 3.2. *Let b be λ-a.e. nondecreasing. Then for each $\vartheta \in K_0'$ either*

$$\liminf_n E_\vartheta \bar\varphi_b = \liminf_n E_\vartheta \varphi_b^* = \alpha$$

or

$$\text{ARE}\,(\{\bar\varphi_b\}:\{\varphi_b^*\}\,|\,\vartheta) \geqq 1\,,$$

where equality iff *b is λ-a.e. constant over each interval $(F(x-0), F(x))$, $x \in D(F)$, where F is the distribution function such that ϑ is contiguous to $\{F^{(n)}\}$.*

REMARK 3.4. This theorem is an extension of Conover (1973), formula (10.8), to all contiguous alternatives K_0'. It should be mentioned that even in the restricted case Conover doesn't clarify the validity of his formula (10.8) (cf. Remark 3.3 or Hájek and Šidák (1967), Remark VII.2.1.a).

PROOF. Because of $\int b_F^2\, d\lambda > 0$ we get from Theorems 2.1 and 2.2 either the first statement or

$$\text{ARE}\,(\cdots) = \int b^2\, d\lambda / \int b_F^2\, d\lambda\,.$$

But formula (2.2) implies

$$\int b_F^2\, d\lambda = E_\lambda(E_\lambda(b\,|\,\mathfrak{B}_F))^2 \leqq E_\lambda E_\lambda(b^2\,|\,\mathfrak{B}_F) = \int b^2\, d\lambda\,,$$

with equality iff $(E_\lambda(b\,|\,\mathfrak{B}_F))^2 = E_\lambda(b^2\,|\,\mathfrak{B}_F)$ λ-a.e. Because of (2.1) this means

$$\left(\int_{F(x-0)}^{F(x)} b\, d\lambda/(F(x)-F(x-0))\right)^2 = \int_{F(x-0)}^{F(x)} b^2\, d\lambda/(F(x)-F(x-0)) \quad \forall\, x \in D(F)\,,$$

and this is equivalent to

$$b = \int_{F(x-0)}^{F(x)} b\, d\lambda/(F(x)-F(x-0))$$
$$\lambda\text{-a.e. on}\quad (F(x-0), F(x)) \qquad \forall\, x \in D(F)\,. \ \square$$

THEOREM 3.3. *Let b_1 and b_2 be λ-a.e. nondecreasing scores generating functions which fulfill* (1.1), (1.2), (3.1). *Then, for each F different from one-point measures and for each $c > 0$, the following two statements are equivalent:*

(i) *$b_{1F}/(\int b_{1F}^2\, d\lambda)^{\frac12} - c b_{2F}/(\int b_{2F}^2\, d\lambda)^{\frac12}$ is λ-a.e. nondecreasing.*

(ii) *ARE $(\{\bar\varphi_{b_1}\}:\{\bar\varphi_{b_2}\}\,|\,\vartheta) \geqq c^2$ for each $\vartheta \in K_0'$ which is contiguous to $\{F^{(n)}\}$ and which fulfills*

$$\liminf_n E_\vartheta \bar\varphi_{b_2} > \alpha\,.$$

REMARK 3.5. Continuity of F implies $b_{1F} = b_1$, $b_{2F} = b_2$, and $\int b_{1F}^2\, d\lambda = \int b_{2F}^2\, d\lambda = 1$. Thus Theorem 3.3 contains Theorem 2.3 of Behnen (1972) as a special case. (Notice the misprinted inequality sign in statement (II) of that theorem.)

REMARK 3.6. Theorem 2.2, formula (2.5), and formula (3.2) imply the existence of ARE $(\{\bar\varphi_{b_1}\}:\{\bar\varphi_{b_2}\}\mid\vartheta)$ for each $\vartheta\in K_0'$ with $\liminf_n E_\vartheta\bar\varphi_{b_2}>\alpha$.

PROOF. Given F and $c>0$ we define b_F to be the difference stated in (i). If b_F is not λ-a.e. nondecreasing we define similar to the proof of Theorem 2.2 in Behnen (1972) a b_0 such that

$$\int b_0\,d\lambda=0\,,\qquad \int b_0^2\,d\lambda=1\,,\qquad \int_0^t b_0\,d\lambda\leqq 0\qquad \forall\, t\in(0,1)\,,$$

and

$$\int b_F b_0\,d\lambda<0\,,\qquad b_0=b_{0F}\,,\qquad \sup_x|b_0(x)|\leqq M<\infty\,.$$

Therefore, with the definition

$$K_{b_0F}=\{\{\textstyle\times_1^n F_{jd}\}:F_{jd}\ \text{has } F\text{-density }\ 1+dc_j(b_0\circ F),\ n\in\mathbb{N},\ d>0\}\,,$$

we have (cf. Remark 3.1)

$$\emptyset\neq K_{b_0F}\subset K_0'\,,$$

and $\vartheta(d)=\{\times_1^n F_{jd}\}\in K_{b_0F}$ implies (cf. Corollary 2.3 and Remark 3.2)

$$\delta_n(b_i,\vartheta(d))\to d\int b_{iF}b_0\,d\lambda\geqq 0\,,\qquad i=1,2\,,$$

$$\delta_n(b_1,\vartheta(d))/(\textstyle\int b_{1F}^2\,d\lambda)^{\frac12}-c\delta_n(b_2,\vartheta(d))/(\int b_{2F}^2\,d\lambda)^{\frac12}\to d\int b_F b_0\,d\lambda<0\,.$$

Because of Theorem 2.2 and formula (2.5) this implies for each $\vartheta\in K_{b_0F}\subset K_0'$ the existence of

$$\text{ARE }(\{\bar\varphi_{b_1}\}:\{\bar\varphi_{b_2}\}\mid\vartheta)<c^2\qquad\text{and}\qquad \liminf_n E_\vartheta\bar\varphi_{b_2}>\alpha\,,$$

which contradicts (ii).

For the other direction of the proof we assume in a first step that b_F equals a constant $a\in\mathbb{R}$ λ-a.e. This implies

$$a=0\,,\qquad c=1\,,\qquad\text{and}\qquad b_{1F}/(\textstyle\int b_{1F}^2\,d\lambda)^{\frac12}=b_{2F}/(\int b_{2F}^2\,d\lambda)^{\frac12}\quad\lambda\text{-a.e.},$$

which proves (ii) because of Theorem 2.2.

To complete the proof we may now assume $0<\int b_F^2\,d\lambda$. Then we proceed the same way as in the corresponding part of the proof of Theorem 2.3 in Behnen (1972) using Theorem 2.2 and formula (3.2) here instead of Theorem 2.1 and Theorem 2.2 there, and taking into account the variances $\int b_{iF}^2\,d\lambda$ of the limiting distributions. □

COROLLARY 3.4. *Let b_1 and b_2 be λ-a.e. nondecreasing scores generating functions which fulfill* (1.1), (1.2), (3.1). *If for some $c_1>0$, $c_2>0$ the functions $b_1-c_1b_2$ and $b_2-c_2b_1$ are λ-a.e. nondecreasing, then for each $\vartheta\in K_0'$ either*

$$\liminf_n E_\vartheta\bar\varphi_{b_1}=\liminf_n E_\vartheta\bar\varphi_{b_2}=\alpha$$

or

$$(c_1c_2)^2\leqq \text{ARE }(\{\bar\varphi_{b_1}\}:\{\bar\varphi_{b_2}\}\mid\vartheta)\leqq(c_1c_2)^{-2}\,.$$

PROOF. From (2.1) and the assumptions on b_1 and b_2 we get for each F that $b_{1F} - c_1 b_{2F}$ and $b_{2F} - c_2 b_{1F}$ are λ-a.e. nondecreasing. Moreover, Theorem 3.1 implies $\liminf_n E_\vartheta \bar\varphi_{b_i} \geqq \alpha$, $i = 1, 2$. Assume $\liminf_n E_\vartheta \bar\varphi_{b_2} > \alpha$. Then Theorem 3.3 with $c_1(\int b_{2F}^2\, d\lambda)^{\frac12}/(\int b_{1F}^2\, d\lambda)^{\frac12}$ instead of c implies

$$\mathrm{ARE}\,(\{\bar\varphi_{b_1}\} : \{\bar\varphi_{b_2}\} \mid \vartheta) \geqq c_1^2 \int b_{2F}^2\, d\lambda / \int b_{1F}^2\, d\lambda > 0\,,$$

which implies $\liminf_n E_\vartheta \bar\varphi_{b_1} > \alpha$, too. Therefore we may again apply Theorem 3.3 in order to get

$$\mathrm{ARE}\,(\{\bar\varphi_{b_2}\} : \{\bar\varphi_{b_1}\} \mid \vartheta) \geqq c_2^2 \int b_{1F}^2\, d\lambda / \int b_{2F}^2\, d\lambda > 0\,,$$

or, because of Theorem 2.2 and formula (2.5),

$$c_2^{-2} \int b_{2F}^2\, d\lambda / \int b_{1F}^2\, d\lambda \geqq \mathrm{ARE}\,(\{\bar\varphi_{b_1}\} : \{\bar\varphi_{b_2}\} \mid \vartheta)\,.$$

Obviously, the same inequalities can be proved under the assumption $\liminf_n E_\vartheta \bar\varphi_{b_1} > \alpha$. Since $\int (b_{1F} - c_1 b_{2F})\, d\lambda = 0$ and $b_{1F} - c_1 b_{2F}$, $b_{1F} + c_1 b_{2F}$ are λ-a.e. nondecreasing we get from Behnen (1972), formulas (3.1.C), (3.4), (3.9.C), (3.12.C), the inequality

$$\begin{aligned} \int b_{1F}^2\, d\lambda &= c_1^2 \int b_{2F}^2\, d\lambda + \int (b_{1F} - c_1 b_{2F})(b_{1F} + c_1 b_{2F})\, d\lambda \\ &\geqq c_1^2 \int b_{2F}^2\, d\lambda\,, \end{aligned}$$

and similarly

$$\int b_{2F}^2\, d\lambda \geqq c_2^2 \int b_{1F}^2\, d\lambda\,.$$

This completes the proof. □

4. Efficiency of trend tests. To get the standardization assumptions of the c's, we take here for each $n \in \mathbb{N}$

$$c_j = (12/n)^{\frac12}(j/(n+1) - \tfrac12)((n+1)/(n-1))^{\frac12}\,, \qquad j = 1, \cdots, n\,.$$

As the (one sided) alternative we take

$$K_0' = \{\{\textstyle\times_1^n F_j\} \in K_0 : F_1 \geqq F_2 \geqq \cdots \geqq F_n,\ \text{at least one inequality different from equality}\}\,.$$

Similar to Section 3 we have to assume here that b is λ-a.e. increasing or, more generally, that b fulfills the condition

$$\int_0^t b\, d\lambda \leqq 0 \qquad \forall\, t \in (0, 1)\,.$$

This implies

$$K_b' \subset K_0'$$

NOTE. Compared with Section 3 we have here reversed signs *and* reversed inequalities in the definition of the c's and of K_0', respectively.

For this trend case we get by a slight modification of the proofs of Section 3 results which are completely similar to the ones stated in Section 3.

With respect to the proof of the counterpart of formula (3.2) one should note that $\delta_n(\cdot, \cdot)$ can be written in the following form:

$$\delta_n(b, \{\textstyle\times_1^n F_j\}) = \sum_{j=1}^{[n/2]} c_j \int (\bar b_{nF} \circ F)(dF_j - dF_{n+1-j})\,,$$

with $c_j \leqq 0$, $F_j \geqq F_{n+1-j}$, for all $j = 1, \cdots, [n/2]$.

5. Proofs of the theorems of Section 2. Let $\vartheta_0 = \{F^{(n)}\}$ and $\vartheta = \{\times_{j=1}^n F_{nj}\}$ be given, and let D denote a countable subset of the line which contains the points of discontinuity of F and F_{nj}, $j = 1, \cdots, n$, $n \in \mathbb{N}$,

$$\bigcup_{n=1}^\infty \bigcup_{j=1}^n D(F_{nj}) \cup D(F) \subset D\,, \qquad C(F_j) = \mathbb{R} - D(F_j)\,.$$

Given such countable set D, let H denote a (discrete) cumulative distribution function on $\mathbb{R}$ such that

$$h(x) \equiv H(x) - H(x-0) > 0 \quad \forall\, x \in D\,, \qquad \textstyle\sum_{x \in D} h(x) = 1\,.$$

With this notation we define a strictly monotone function $T\colon \mathbb{R} \to \mathbb{R}$ by

$$T(x) = x + H(x)$$

and random variables $X_j^* = X_j^*(X_j, U_j)$ by

$$X_j^* = T(X_j) - U_j h(X_j)\,, \qquad j = 1, \cdots, n\,. \tag{5.1}$$

These definitions give a suitable connection between the ranks, the randomized ranks, the given random variables, and the starred random variables, needed in the proofs of the main theorems.

LEMMA 5.1. *Under ϑ and under ϑ_0:*

(a) *$X_1^*, \cdots, X_n^*$ are independent random variables each of them having a continuous distribution.*

(b) *If F_j^* denotes the cumulative distribution function of X_j^* then*

$$\begin{aligned} F_j^*(y) &= F_j(T^{-1}(y)) && \text{if } y \in T(\mathbb{R})\,, \\ &= F_j(x-0) + \frac{F_j(x) - F_j(x-0)}{T(x) - T(x-0)}\,(y - T(x-0)) \\ &&& \text{if } y \in [T(x-0), T(x)]\,, \quad x \in D\,. \end{aligned} \tag{5.2}$$

(c) $X_i < X_j \Rightarrow X_i^* < X_j^*$;

$$X_j = \inf\{x \in \mathbb{R} : T(x) \geqq X_j^*\}\,, \qquad X^{(j)} = \inf\{x \in \mathbb{R} : T(x) \geqq X^{*(j)}\}\,.$$

(d) $R_i(X_1^*, \cdots, X_n^*) = R_i^*$ a.e. $\forall\, i = 1, \cdots, n$.

(e) $X_1, \cdots, X_n$ i.i.d. *implies the independence of R^* and $X^{(\,)}$ and consequently the independence of R^* and $\tau = \tau(X^{(\,)})$.*

(f) $R_i + 1 - \sum_{j=1}^n I_{\{0\}}(R_i - R_j) \leqq R_i^* \leqq R_i \;\forall\, i = 1, \cdots, n$.

(g) *Continuity of F_j for all $j = 1, \cdots, n$ implies $R^* = R$* a.e.

PROOF. (a) Independence is immediate, continuity is gotten from the strict monotonicity of T by the following conditional argument

$$\begin{aligned} 0 &\leqq P(X_j^* = x) = \textstyle\int P(T(y) - U_j h(y) = x)\, dF_j(y) \\ &= \textstyle\int I_{\{0\}}(h(y)) I_{\{x\}}(T(y))\, dF_j(y) = \int_{\mathbb{R}-D} I_{\{x\}}(T(y))\, dF_j(y) \\ &\leqq \textstyle\int_{C(F_j)} I_{\{T^{-1}(x)\}}(y)\, dF_j(y) = 0\,. \end{aligned}$$

(b) For all $x, y \in \mathbb{R}$ put $g(x, y) = \int_0^1 I_{(-\infty, y]}(T(x) - uh(x))\, du$. Then by the independence of X_j and U_j we have

$$F_j^*(y) = P(T(X_j) - U_j h(X_j) \leqq y) = \int g(x, y)\, dF_j(x)$$

and furthermore

$$\begin{aligned} g(x, y) &= 1 && \text{if } T(x) \leqq y, \\ &= 0 && \text{if } T(x) > y + h(x), \\ &= 1 - (T(x) - y)/h(x) && \text{if } y < T(x) \leqq y + h(x), \\ &= 1 && \text{if } T(x) \leqq y, \\ &= 0 && \text{if } T(x - 0) > y, \\ &= (y - T(x - 0))/(T(x) - T(x - 0)) && \text{if } T(x - 0) \leqq y < T(x). \end{aligned}$$

Thus we get from the strict monotonicity of T

$$\begin{aligned} F_j^*(y) &= P(T(X_j) \leqq y) \\ &\quad + \sum_{x \in D} I_{[T(x-0), T(x))}(y) \frac{F_j(x) - F_j(x - 0)}{T(x) - T(x - 0)} (y - T(x - 0)) \\ &= F_j(T^{-1}(y)) \qquad \text{if } y \in T(\mathbb{R}) \\ &= F_j(x - 0) + \frac{F_j(x) - F_j(x - 0)}{T(x) - T(x - 0)} (y - T(x - 0)) \\ &\qquad \text{if } y \in [T(x - 0), T(x)], \quad x \in D. \end{aligned}$$

(c) $$\begin{aligned} X_i < X_j \Rightarrow X_i^* &= T(X_i) - U_i h(X_i) \leqq T(X_i) < X_j + H(X_i) \\ &\leqq T(X_j) - h(X_j) \leqq T(X_j) - U_j h(X_j) = X_j^*. \end{aligned}$$

The second part is also an immediate consequence of the definition.

(d) On one side we have by definition

$$\begin{aligned} R_i^* &= \sum_{j=1}^n I_{(0,\infty)}(R_i - R_j) + \sum_{j=1}^n I_{\{0\}}(R_i - R_j) I_{[0,\infty)}(U_j - U_i) \\ &= \sum_{j=1}^n I_{(0,\infty)}(X_i - X_j) + \sum_{j=1}^n I_{\{0\}}(X_i - X_j) I_{[0,\infty)}(U_j - U_i), \end{aligned}$$

on the other side we get from part (c) a.e. the equality

$$\begin{aligned} R_i(X_1^*, \cdots, X_n^*) &= \sum_{j=1}^n I_{[0,\infty)}(X_i^* - X_j^*) \\ &= \sum_{j=1}^n I_{(0,\infty)}(X_i - X_j) + \sum_{j=1}^n I_{[0,\infty)}(X_i^* - X_j^*) I_{\{0\}}(X_i - X_j) \\ &= \sum_j I_{(0,\infty)}(X_i - X_j) + \sum_j I_{[0,\infty)}((U_j - U_i) h(X_i)) I_{\{0\}}(X_i - X_j) \\ &= \sum_j I_{(0,\infty)}(X_i - X_j) + \sum_j I_{\{0\}}(X_i - X_j) I_{[0,\infty)}(U_j - U_i), \end{aligned}$$

since $P(X_i = X_j, h(X_i) \leqq 0) = 0$ if $i \neq j$ by the independence of X_i and X_j.

(e) $X_1, \cdots, X_n$ i.i.d. implies $X_1^*, \cdots, X_n^*$ i.i.d. Therefore the independence of $R(X_1^*, \cdots, X_n^*)$ and $X^{*(\ )}$ (cf. Witting and Nölle (1970)) and part (d) imply the independence of R^* and $X^{*(\ )}$. Thus part (c) proves the statement (e).

(f) From the representation of R_i^* in the proof of part (d) we have

$$\begin{aligned} R_i^* &= R_i - \sum_{j=1}^n I_{\{0\}}(X_i - X_j) I_{(-\infty,0)}(U_j - U_i) \\ &\geqq R_i + 1 - \sum_{j=1}^n I_{\{0\}}(R_i - R_j). \end{aligned} \tag{5.3}$$

(g) This is an immediate consequence of (5.3). □

LEMMA 5.2. *For each $n \in \mathbb{N}$ let X_n and Y_n be n-dimensional random vectors defined on some measure space $(\Omega, \mathfrak{A})$. Let P and Q be probability measures on $(\Omega, \mathfrak{A})$ and let X_n and Y_n be independent with respect to P and Q for each $n \in \mathbb{N}$. Finally assume $\mathfrak{L}(Y_n \mid P) = \mathfrak{L}(Y_n \mid Q)\ \forall n \in \mathbb{N}$. Then, for each sequence $\{f_n\}$ of Borel-measurable functions f_n from $\mathbb{R}_{2n}$ to some $\mathbb{R}_{k_n}$, the following implication is true*:

$$\{\mathfrak{L}(X_n \mid Q)\} \text{ contiguous to } \{\mathfrak{L}(X_n \mid P)\} \text{ implies contiguity of } \{\mathfrak{L}(f_n(X_n, Y_n) \mid Q)\} \text{ to } \{\mathfrak{L}(f_n(X_n, Y_n) \mid P)\}.$$

PROOF. For each Borel-measurable subset $B_n \subset \mathbb{R}_{k_n}$ we get:

$$\begin{aligned} P^{f_n(X_n,Y_n)}(B_n) &= P(f_n(X_n, Y_n) \in B_n) = P((X_n, Y_n) \in f_n^{-1}(B_n)) \\ &= E_p I_{f_n^{-1}(B_n)}(X_n, Y_n) = \int (E_p I_{f_n^{-1}(B_n)}(x, Y_n)) P^{X_n}(dx), \end{aligned}$$

and a similar equation for Q instead of P.

Because of $\mathfrak{L}(Y_n \mid P) = \mathfrak{L}(Y_n \mid Q)$ this implies the following chain of implications which conclude the proof:

$$\begin{aligned} P^{f_n(X_n,Y_n)}(B_n) \to 0 &\Rightarrow P^{X_n}(\{x : E_P I_{f_n^{-1}(B_n)}(x, Y_n) > \varepsilon\}) \to 0 \qquad \forall \varepsilon > 0 \\ &\Rightarrow Q^{X_n}(\{x : E_P I_{f_n^{-1}(B_n)}(x, Y_n) > \varepsilon\}) \to 0 \qquad \forall \varepsilon > 0 \\ &\Rightarrow Q^{X_n}(\{x : E_Q I_{f_n^{-1}(B_n)}(x, Y_n) > \varepsilon\}) \to 0 \qquad \forall \varepsilon > 0 \\ &\Rightarrow \int (E_Q I_{f_n^{-1}(B_n)}(x, Y_n)) Q^{X_n}(dx) \to 0 \\ &\Rightarrow Q^{f_n(X_n,Y_n)}(B_n) \to 0 . \end{aligned}$$

□

LEMMA 5.3. *Let F^* and F_j^* denote the distribution functions of X_j^* under F and F_j, respectively. Then, with $\bar{b}_n$ according to* (2.4):

(a) $\{\times_1^n F_j\}$ *contiguous to* $\{F^{(n)}\}$ *implies contiguity of* $\{\times_1^n F_j^*\}$ *to* $\{F^{*(n)}\}$.

(b) $\int (\bar{b}_{nF} \circ F)\, dF_j = \int (\bar{b}_{nF} \circ F^*)\, dF_j^*$.

(c) $\{\times_1^n F_j\}$ *contiguous to* $\{F^{(n)}\}$ *implies*

$$nP_F(A_n) \to 0 \Rightarrow \textstyle\sum_{j=1}^n P_{F_j}(A_n) \to 0 , \tag{5.4}$$

and

$$\textstyle\sum_{j=1}^n c_j \int ((\bar{b}_n - \bar{b}_{nF}) \circ F^*)\, dF_j^* \to 0 . \tag{5.5}$$

PROOF. (a) This is an immediate consequence of Lemma 5.2 and the definition of the starred random variables.

(b) From (5.2) and (2.1) we get $\forall x \in \mathbb{R},\ \forall u \in (0, 1) : (\bar{b}_{nF} \circ F^*)(T(x) - uh(x)) = (\bar{b}_{nF} \circ F)(x)$. This implies $(\bar{b}_{nF} \circ F^*)(X_j^*) = (\bar{b}_{nF} \circ F)(X_j)$ and thus the equality.

(c)
$$\begin{aligned} nP_F(A_n) \to 0 &\Rightarrow P_F^{(n)}(\times_1^n A_n^c) = (1 - P_F(A_n))^n \to e^0 = 1 \\ &\Rightarrow \textstyle\prod_1^n P_{F_j}(A_n^c) = (\times_1^n P_{F_j})(\times_1^n A_n^c) \to 1 \quad \text{(by contiguity)} \\ &\Rightarrow 0 \leqq \textstyle\sum_1^n P_{F_j}(A_n) \leqq -\log \prod_1^n (1 - P_{F_j}(A_n)) \to 0 . \end{aligned}$$

This proves (5.4). For the proof of (5.5) we get from (5.1), (5.2), and (2.1), the following equality:

$$\begin{aligned}
\int (\bar{b}_n \circ F^*)\, dF_j^* &= \int\int (\bar{b}_n \circ F^*)(T(x) - uh(x))\, dF_j(x)\, d\lambda(u) \\
&= \int_{\mathbb{R}-D} (\bar{b}_n \circ F^*)(T(x))\, dF_j(x) \\
&\quad + \sum_{x \in D} (F_j(x) - F_j(x-0)) \int \bar{b}_n(F^*(T(x) - uh(x)))\, d\lambda(u) \\
&= \int_{C(F_j)C(F)} (\bar{b}_n \circ F)\, dF_j \\
&\quad + \sum_{x \in D(F_j)C(F)} (F_j(x) - F_j(x-0))\bar{b}_n(F(x)) \\
&\quad + \sum_{x \in D(F_j)D(F)} (F_j(x) - F_j(x-0)) \\
&\quad \times \int \bar{b}_n(F(x-0) + (F(x) - F(x-0))(1-u))\, d\lambda(u) \\
&= \int_{C(F)} (\bar{b}_n \circ F)\, dF_j + \int_{D(F_j)D(F)} (\bar{b}_{nF} \circ F)\, dF_j\,.
\end{aligned}$$

Similarly we get

$$\int (\bar{b}_{nF} \circ F^*)\, dF_j^* = \int_{C(F)} (\bar{b}_{nF} \circ F)\, dF_j + \int_{D(F_j)D(F)} (\bar{b}_{nF} \circ F)\, dF_j\,.$$

Therefore it suffices to show

$$\sum_j c_j \int_{C(F)} ((\bar{b}_n - \bar{b}_{nF}) \circ F)\, dF_j \to 0\,. \tag{5.6}$$

If we define

$$S(F) \equiv \{x \in \mathbb{R} : F(x-\varepsilon) < F(x) < F(x+\varepsilon) \quad \forall\, \varepsilon > 0\} \tag{5.7}$$

we have

$$F(C(F)S(F)) \subset (0, 1) - \bigcup_{x \in D(F)} [F(x-0), F(x)]\,, \tag{5.8}$$

and

$$\lambda((0, 1)F(C(F) - S(F))) = 0\,. \tag{5.9}$$

Thus by (2.1)

$$\int_{C(F)S(F)} ((\bar{b}_n - \bar{b}_{nF}) \circ F)\, dF_j = 0\,,$$

and by (5.4)

$$\sum_{j=1}^{n} P_{F_j}(C(F) - S(F)) \to 0\,, \tag{5.10}$$

since

$$\begin{aligned}
P_F(C(F) - S(F)) &= P(F^{-1}(U_1) \in C(F) - S(F)) \\
&\leqq P(U_1 \in F(\mathbb{R}), (F \circ F^{-1})(U_1) \in F(C(F) - S(F))) \\
&= P(U_1 \in F(C(F) - S(F))) = 0 \quad \text{by (5.9)}\,.
\end{aligned}$$

Therefore we get from (2.4)

$$\begin{aligned}
&\left|\sum_j c_j \int_{C(F)} ((\bar{b}_n - \bar{b}_{nF}) \circ F)\, dF_j\right| \\
&\qquad = \left|\sum_j c_j \int_{C(F)-S(F)} ((\bar{b}_n - \bar{b}_{nF}) \circ F)\, dF_j\right| \\
&\qquad \leqq 2 \max_j |c_j| \sup_x |\bar{b}_n(x)| \sum_j P_{F_j}(C(F) - S(F)) \to 0\,,
\end{aligned}$$

which concludes proof. □

LEMMA 5.4. (a) *Put* $F^{-1}(u) = \inf\{x \in \mathbb{R}: F(x) \geqq u\}$, $0 < u < 1$, *and* $\tilde{F}(x) = \lim_{t \downarrow 0} F(x-t) = F(x-0)$, $x \in \mathbb{R}$. *Then*

$$\begin{aligned}(F \circ F^{-1})(u) &= u && \text{if } u \in F(\mathbb{R}) - F(D(F))\,, \\ &= F(x) && \text{if } u \in (F(x-0), F(x)]\,, && x \in D(F)\,, \\ &= F(x) && \text{if } u = F(x-0) \notin F(\mathbb{R})\,, && x \in D(F)\,, \\ (\tilde{F} \circ F^{-1})u) &= u && \text{if } u \in F(\mathbb{R}) - F(D(F))\,, \\ &= F(x-0) && \text{if } u \in (F(x-0), F(x)]\,, && x \in D(F)\,, \\ &= F(x-0) && \text{if } u = F(x-0) \notin F(\mathbb{R})\,, && x \in D(F)\,.\end{aligned}$$

(b) *Given* $u \in C(F \circ F^{-1}) \cap C(\tilde{F} \circ F^{-1})$ *and* $j_n \in \{1, \cdots, n\}$ *such that* $j_n/n \to u$. *Then*

$$F(X_n^{(j_n)}) \to (F \circ F^{-1})(u) \quad \text{in } F^{(n)}\text{-probability},$$
$$\tilde{F}(X_n^{(j_n)}) \to (\tilde{F} \circ F^{-1})(u) \quad \text{in } F^{(n)}\text{-probability}.$$

(c) *For each* $u \in (0, 1)$ *define random variables* s_{nu} *and* t_{nu} *according to*

$$s_{nu} = T_{nk-1}\,, \qquad t_{nu} = T_{nk}\,, \qquad \text{if } \; T_{nk-1} \leqq nu < T_{nk}\,, \qquad k = 1, \cdots, g_n\,.$$

Then, for each $u \in C(F \circ F^{-1}) \cap C(\tilde{F} \circ F^{-1}) \cap (F(\mathbb{R}) - F(D(F)))$,

$$n^{-1}s_{nu} \to u \quad \text{in } F^{(n)}\text{-probability},$$
$$n^{-1}t_{nu} \to u \quad \text{in } F^{(n)}\text{-probability},$$

and for each $u \in (F(x-0), F(x))$, $x \in D(F)$,

$$n^{-1}s_{nu} \to F(x-0) \quad \text{in } F^{(n)}\text{-probability},$$
$$n^{-1}t_{nu} \to F(x) \qquad \text{in } F^{(n)}\text{-probability}.$$

(d) *Define* $\bar{b}_n(\cdot, \tau_n): (0, 1) \to \mathbb{R}$ *according to*

$$\bar{b}_n(u, \tau_n) = n\tau_{nk}^{-1} \textstyle\int_{T_{nk-1}/n}^{T_{nk}/n} b \, d\lambda \qquad \text{if } \; T_{nk-1} \leqq nu < T_{nk}\,, \qquad k = 1, \cdots, g_n\,.$$

Then, for λ*-almost all* $u \in (0, 1)$,

$$\bar{b}_n(u, \tau_n) \to b_F(u) \quad \text{in } F^{(n)}\text{-probability}.$$

PROOF. (a) Obvious.

(b) Immediate from $\mathfrak{L}(U_n^{(j_n)}) \to \mathfrak{L}(u)$ and $\mathfrak{L}(F^{-1}(U_n^{(j_n)}) = \mathfrak{L}(X_n^{(j_n)} \mid F^{(n)})$.

(c) From the definition of s_{nu} and t_{nu} we get

$$s_{nu} < [nu+1] \leqq t_{nu}\,, \qquad X_n^{(t_{nu})} = X_n^{([nu+1])}$$

and thus

$$t_{nu} = \textstyle\sum_{i=1}^n I_{(-\infty, X_n^{([nu+1])}]}(X_i)\,, \qquad s_{nu} = \sum_{i=1}^n I_{(-\infty, X_n^{([nu+1])})}(X_i)\,.$$

Therefore by Glivenko–Cantelli

$$t_{nu}/n - F(X_n^{([nu+1])}) \to 0 \quad \text{in } F^{(n)}\text{-probability},$$
$$s_{nu}/n - \tilde{F}(X_n^{([nu+1])}) \to 0 \quad \text{in } F^{(n)}\text{-probability}.$$

Application of part (a) and part (b) concludes the proof of part (c).

(d) Let $L(b)$ denote the Lebesgue set of b (e.g. Rudin (1970), page 158) and put

$$A = L(b) \cap C(F \circ F^{-1}) \cap C(\tilde{F} \circ F^{-1}) \cap (F(\mathbb{R}) - F(D(F))) .$$

From part (c) and (2.1) we get for each $u \in A$

$$\bar{b}_n(u, \tau_n) = (t_{nu} - s_{nu})^{-1} n \int_{s_{nu}/n}^{t_{nu}/n} b \, d\lambda \to b(u) = b_F(u) \quad \text{in } F^{(n)}\text{-probability}$$

and for each $u \in (F(x-0), F(x))$, $x \in D(F)$,

$$\bar{b}_n(u, \tau_n) \to (F(x) - F(x-0))^{-1} \int_{F(x-0)}^{F(x)} b \, d\lambda = b_F(u) \quad \text{in } F^{(n)}\text{-probability.}$$

Because of $\lambda(L(b)) = 1$ and because of the monotonicity of $F \circ F^{-1}$ and $\tilde{F} \circ F^{-1}$ we have

$$\lambda(A + \textstyle\sum_{x \in D(F)} (F(x-0), F(x))) = 1 ,$$

which concludes the proof. □

PROOF OF THEOREM 2.1. Given F and F_j define F^*, F_j^*, and X_j^* according to (5.1) and (5.2). Putting

$$T_{bF}^* = \textstyle\sum_1^n c_i b \circ F^* \circ X_i^*$$

we have *under the assumptions of part* (a):

$$ET_{bF}^* = \textstyle\sum c_i \int b \, d\lambda = 0 , \qquad \operatorname{Var} T_{bF}^* = \sum c_i^2 \int b^2 \, d\lambda = 1 .$$

Thus $\mathfrak{L}(T_{bF}^*) \to N(0, 1)$ by Lindeberg-Feller theorem, since $\max_i c_i^2 \to 0$ by assumption. Furthermore we get from Hájek and Šidák (1967), Theorem V.1.4.a and formula (10) in the proof of Theorem V.1.5.a, the statement

$$E(S_b^* - T_{bF}^*)^2 \to 0 . \tag{5.11}$$

For the proof of part (b) this and the contiguity assumption imply (cf. Lemma 5.3. (a))

$$S_b^* - T_{bF}^* \to 0 \quad \text{in } \textstyle\bigtimes_1^n F_j^*\text{-probability.} \tag{5.12}$$

Therefore and because of Lemma 5.3. (b), (c), it suffices to prove

$$\mathfrak{L}(T_{bF}^* - \textstyle\sum_j c_j \int (\bar{b}_n \circ F^*) \, dF_j^*) \to N(0, 1) .$$

Because of Lemma 5.3. (a), (1.3), (1.1), and (2.4) this is an immediate consequence of the Theorem and Remark 1 of Behnen and Neuhaus (1975).

(c) From part (a) we have that $\{\varphi_b^*\}$ is asymptotically level α for H_0. Also K_{bF} is well defined since (2.4) implies $1 + dc_j(\bar{b}_{nF} \circ F)$ to be an F-density for d small enough (or n large enough):

$$\begin{aligned}\int (\bar{b}_{nF} \circ F) \, dF &= \int (\bar{b}_{nF} \circ F \circ F^{-1}) \, d\lambda \\ &= \int_{F(\mathbb{R})} \bar{b}_{nF} \, d\lambda + \textstyle\sum_{x \in D(F)} \bar{b}_{nF}(F(x))(F(x) - F(x-0)) \\ &= \int_{F(\mathbb{R})} \bar{b}_n \, d\lambda + \textstyle\sum_{x \in D(F)} \int_{(F(x-0), F(x))} \bar{b}_n \, d\lambda = \int \bar{b}_n \, d\lambda = 0 .\end{aligned}$$

First we get by $\sum_j c_j^2 \bar{b}_{nF}^2(F^*(X_j^*)) \to \int b_F^2 \, d\lambda = \int b^2 \, d\lambda = 1$ in $F^{*(n)}$-probability and by Taylor-expansion

$$\log \prod_j (1 + dc_j \bar{b}_{nF} \circ F^* \circ X_j^*) + d^2/2 - dT^*_{bF} \to 0 \quad \text{in } F^{*(n)}\text{-probability},$$

since $\int (\bar{b}_{nF} - b)^2 \, d\lambda \to 0$. Secondly we have by (5.2) and (2.1) (cf. proof of Lemma 5.3. (b))

$$\bar{b}_{nF} \circ F^* \circ X_j^* = \bar{b}_{nF} \circ F \circ X_j \, .$$

This implies (cf. (5.11))

$$\log \prod_j (1 + dc_j \bar{b}_{nF} \circ F \circ X_j) + d^2/2 - dS_b^* \to 0 \quad \text{in } F^{*(n)}\text{-probability}.$$

Thus, part (a) and Hájek and Šidák (1967), Corollary VI.1.2, imply contiguity of K_{bF} to $\{F^{(n)}\}$. Therefore we get from part (a) and part (b) asymptotic power-equivalence of $\{\varphi_b^*\}$ and Neyman–Pearson test for $\{F^{(n)}\}$ against ϑ for each $\vartheta \in K_{bF}$, which concludes the proof. □

PROOF OF THEOREM 2.2. Let $\{F^{(n)}\} \in H_0$ be given. Then we get from Theorem 2.1 *under the assumption of part* (a):

$$\mathfrak{L}(S^*_{bF}) \to N(0, \textstyle\int b_F^2 \, d\lambda) \, ;$$

and *under the assumptions of part* (b):

$$\mathfrak{L}(S^*_{bF} - \delta_n(b_F, \{\textstyle\times_1^n F_j\})) \to N(0, \textstyle\int b_F^2 \, d\lambda) \, ;$$

and finally *under the assumptions of part* (c):

$$\{I_{\{S^*_{bF} > k_\alpha (\int b_F^2 \, d\lambda)^{\frac{1}{2}}\}}\}$$

is asymptotically most powerful for $\{F^{(n)}\}$ against K_{bF} at level α, since the monotonicity of b implies $\int b_F^2 \, d\lambda > 0$ for all F not being one-point measures. Since we may assume $\bar{b}_{Fn} = \bar{b}_{nF}$, which implies for $\delta_n(\cdot, \cdot)$ according to (2.3)

$$\delta_n(b_F, \{\textstyle\times_1^n F_j\}) = \delta_n(b, \{\textstyle\times_1^n F_j\}) \, ,$$

and since we have from Hájek (1969), page 130, under H_0

$$\operatorname{Var}(\bar{S}_b \mid \tau) = \frac{1}{n-1} \sum_{i=1}^n (\bar{b}_i)^2 \, ,$$

it suffices (by contiguity) to prove (for each $\{F^{(n)}\} \in H_0$)

$$\bar{S}_b - S^*_{bF} \to 0 \quad \text{in } F^{(n)}\text{-probability}, \tag{5.13}$$

and

$$\frac{1}{n-1} \sum_{i=1}^n (\bar{b}_i)^2 \to \int b_F^2 \, d\lambda \quad \text{in } F^{(n)}\text{-probability}. \tag{5.14}$$

For the proof we state

$$E(\bar{S}_b - S^*_{bF})^2 = \sum_i c_i^2 E(b_{R_i}(\tau) - b_{FR_i^*})^2 + \sum_{i \neq j} c_i c_j E(b_{R_i}(\tau) - b_{FR_i^*})(b_{R_j}(\tau) - b_{FR_j^*}) \, .$$

Then we get by Lemma 5.1 and (5.3) on one hand

$$E((b_{R_i}(\tau) - b_{FR_i^*})^2 \mid \tau) = \frac{1}{n} \sum_i (b_i(\tau) - b_{Fi})^2$$
$$= \int (b_{[nu+1]}(\tau) - b_{F[nu+1]})^2 \, d\lambda(u) ,$$

and on the other hand if $i \neq j$

$$E((b_{R_i}(\tau) - b_{FR_i^*})(b_{R_j}(\tau) - b_{FR_j^*}) \mid \tau)$$
$$= \frac{1}{n(n-1)} \sum_{i \neq j} (b_i(\tau) - b_{Fi})(b_j(\tau) - b_{Fj})$$
$$= -\frac{1}{n-1} \int (b_{[nu+1]}(\tau) - b_{F[nu+1]})^2 \, d\lambda(u) .$$

Because of $\sum_{i \neq j} c_i c_j = (\sum_i c_i)^2 - \sum_i c_i^2 = -1$ we have

$$E(\bar{S}_b - S_{b_F}^*) = \frac{n}{n-1} E \int (b_{[nu+1]}(\tau) - b_{F[nu+1]})^2 \, d\lambda(u)$$
$$\leqq 2 \frac{n}{n-1} (E \int (b_{[nu+1]}(\tau) - b_F(u))^2 \, d\lambda(u)$$
$$+ \int (b_{F[nu+1]} - b_F(u))^2 \, d\lambda(u)) .$$

But $\int (b_F(u) - b_{F[nu+1]})^2 \, d\lambda \to 0$ by assumption. Because of $1/(n-1) \sum_i (\bar{b}_i)^2 = n/(n-1) \int b_{[nu+1]}^2(\tau) \, d\lambda(u)$ it is therefore sufficient for the proof of (5.13) and (5.14) to prove

$$E \int (b_{[nu+1]}(\tau) - b_F(u))^2 \, d\lambda(u) \to 0 . \tag{5.15}$$

For this proof let $\bar{b}_n(\cdot, \tau)$ denote the (random) function defined in Lemma 5.4 (d). Then we have

$$E \int (b_{[nu+1]}(\tau) - b_F(u))^2 \, d\lambda(u)$$
$$\leqq 2E \int (b_{[nu+1]}(\tau) - \bar{b}_n(u, \tau))^2 \, d\lambda(u) + 2E \int (\bar{b}_n(u, \tau) - b_F(u))^2 \, d\lambda(u) ,$$

and

$$E \int (b_{[nu+1]}(\tau) - \bar{b}_n(u, \tau))^2 \, d\lambda(u)$$
$$= E \sum_{j=1}^{g} \int_{T_{j-1}/n}^{T_j/n} (\cdots)^2 \, d\lambda$$
$$= E \sum_{j=1}^{g} \left(\frac{n}{\tau_j}\right)^2 (\int_{T_{j-1}/n}^{T_j/n} (b_{[nv+1]} - b(v)) \, d\lambda(v))^2 \int_{T_{j-1}/n}^{T_j/n} d\lambda(u)$$
$$\leqq E \sum_{j=1}^{g} \int_{T_{j-1}/n}^{T_j/n} (b_{[nv+1]} - b(v))^2 \, d\lambda(v)$$
$$= \int (b_{[nv+1]} - b(v))^2 \, d\lambda(v) \to 0 \quad \text{by assumption} \quad (1.2) .$$

For the other term we get from Lemma 5.4 (d)

$$\bar{b}_n(U_1, \tau) \to b_F(U_1) \quad \text{in } \lambda \times F^{(n)}\text{-probability},$$

and thus

$$(\lambda \times P_F^{(n)})(A_{nk}) \to 0 \qquad \forall k \in \mathbb{N} ,$$

if we put

$$A_{nk} = \{(u, x) \in (0, 1) \times \mathbb{R}_n : (\bar{b}_n(u, \tau(x)) - b_F(u))^2 > k\} .$$

Therefore we get by the dominated convergence theorem and from the definition of $\bar{b}_n(\cdot, \tau)$ for each $k \in \mathbb{N}$:

$$\begin{aligned}
&\limsup_{n\to\infty} E \int (\bar{b}_n(u, \tau) - b_F(u))^2 \, d\lambda(u) \\
&\quad = \limsup_{n\to\infty} E(\bar{b}_n(U_1, \tau) - b_F(U_1))^2 \\
&\quad = \limsup_{n\to\infty} E(\bar{b}_n(U_1, \tau) - b_F(U_1))^2 I_{A_{nk}}(U_1, X) \\
&\quad \leqq 2 \limsup_{n\to\infty} E\bar{b}_n^2(U_1, \tau) I_{A_{nk}}(U_1, X) \\
&\quad \leqq 2 \limsup_n E\left(\sum_{j=1}^{g} I_{[T_{j-1}, T_j)}(nU_1) \left(\frac{n}{\tau_j}\right) \left(\int_{T_{j-1}/n}^{T_j/n} b^2 \, d\lambda\right) I_{A_{nk}}(U_1, X)\right) \\
&\quad \leqq 2 \limsup_n E\left(\sum_{j=1}^{g} I_{[T_{j-1}, T_j)}(nU_1) k I_{A_{nk}}(U_1, X)\right) \\
&\qquad + 2 \limsup_n E\left(\sum_{j=1}^{g} I_{[T_{j-1}, T_j)}(nU_1) \left(\frac{n}{\tau_j}\right) \int_{T_{j-1}/n}^{T_j/n} b^2 I_{(k,\infty)}(b^2) \, d\lambda\right) \\
&\quad \leqq 2k \limsup_n (\lambda \times P_F^{(n)})(A_{nk}) + 2 \limsup_n E\left(\sum_{j=1}^{g} \int_{T_{j-1}/n}^{T_j/n} b^2 I_{(k,\infty)}(b^2) \, d\lambda\right) \\
&\quad = 2 \int b^2 I_{(k,\infty)}(b^2) \, d\lambda \, .
\end{aligned}$$

The last expression converges to zero as k tends to infinity, which concludes the proof. □

PROOF OF COROLLARY 2.3. This is obvious from Theorem 2.1 and Theorem 2.2, since for $\delta_n(b, \vartheta(d))$ according to (2.3) we get (similar to the proof of Lemma 5.3. (b))

$$\begin{aligned}
\delta_n(b, \vartheta(d)) &= \sum_i c_i \int (\bar{b}_{nF} \circ F)(1 + dc_i(\bar{h}_{nF} \circ F)) \, dF \\
&= d \int \bar{b}_{nF} \bar{h}_{nF} \, d\lambda \to d \int b_F h_F \, d\lambda \, .
\end{aligned}$$
□

REFERENCES

[1] BEHNEN, K. (1972). A characterization of certain rank-order tests with bounds for the asymptotic relative efficiency. *Ann. Math. Statist.* **43** 1839–1851.

[2] BEHNEN, K. and NEUHAUS, G. (1975). A central limit theorem under contiguous alternatives. *Ann. Statist.* **3** 1349–1353.

[3] CONOVER, W. J. (1973). Rank tests for one sample, two samples, and k samples without the assumption of a continuous distribution function. *Ann. Statist.* **1** 1105–1125.

[4] HÁJEK, J. (1969). *A Course in Nonparametric Statistics.* Holden-Day, San Francisco.

[5] HÁJEK, J. and ŠIDÁK, Z. (1967). *Theory of Rank Tests.* Academic Press, New York.

[6] RUDIN, W. (1970). *Real and Complex Analysis.* McGraw-Hill, London.

[7] VORLIČKOVÁ, D. (1970). Asymptotic properties of rank tests under discrete distributions. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **14** 275–289.

[8] VORLIČKOVÁ, D. (1972). Asymptotic properties of rank tests of symmetry under discrete distributions. *Ann. Math. Statist.* **43** 2013–2018.

[9] WITTING, H. and NÖLLE, G. (1970). *Angewandte Mathematische Statistik.* Teubner, Stuttgart.

SB MATHEMATIK
UNIVERSITÄT BREMEN
28 BREMEN 33
POSTFACH
WEST GERMANY