Commun. Math. Phys. 132, 73–101 (1990)

Communications in Mathematical Physics

Asymptotic Completeness for N-Body Short-Range Quantum Systems: A New Proof

Gian Michele Graf

Institut für Theoretische Physik, ETH-Hönggerberg, CH-8093 Zürich, Switzerland

Dedicated to Res Jost and Arthur Wightman

Abstract. We give an alternative geometrical proof of asymptotic completeness for an arbitrary number of quantum particles interacting through short-range pair potentials. It relies on an estimate showing that the intercluster motion concentrates asymptotically on classical trajectories.

Table of Contents

1. Introduction

The first task of quantum scattering theory is to give a classification of the possible large time behaviours of Schrödinger orbits $e^{-itH}\psi$. In this paper we study this problem for an arbitrary number of particles interacting via short range interactions. In the intuitive picture of the scattering process, this system is well described at large times by a number of bound clusters which do not feel each other. This statement is called asymptotic completeness. For $N=2,3$ it was proved by several authors (see [2, Sect. 5.7] for a review), and in particular using geometric ideas by Enss [6–8]. For arbitrary N the proof is due to Sigal and Soffer [22].

Our main intermediate result is a propagation estimate showing that asymptotically $2p_a \approx x_a/t$ on that part of configuration space, which corresponds to a given cluster decomposition a. Here we have used the notation of [22] which is reviewed at the end of this section. Such a property in fact is typical for the clusters of a moving freely. This result is not new, since it was derived earlier in [23] (with the only modification that the partition of unity used there is in phase space).

Somewhat similar results relying on a spectral rather than on a geometrical break-up were derived in [9]. New, however, is our approach for proving this property, where a boosted Hamiltonian

$$K(t)=(p-\nabla\varphi(x,t))^2+V$$

is the main propagation observable. Boosted Hamiltonians have been used in [1, 3] to establish exponential bounds on bound state wave functions of N-body systems. The difference here is that the function φ is no longer imaginary, but real. This is related to the analogous change of the phase of the wave function, when passing from an energetically forbidden to an energetically allowed region. The propagation estimate we mentioned is based neither on channel expansions, nor on local decay, nor on the Mourre estimate. Yet, the latter becomes crucial when proving asymptotic completeness.

The hypothesis on the pair potentials V_{ij} as real multiplication operators on $L^2(X^{(ij)})$ are grouped according to their different role in the theory.

Decay Assumption.

$$\|F(|y|>R)\,V_{ij}(y)\,(p^2+1)^{-1}\|\leqq \text{const}\,R^{-\mu_1}, \tag{1.1}$$

$$\|F(|y|>R)\,\nabla V_{ij}(y)\,(p^2+1)^{-1}\|\leqq \text{const}\,R^{-(1+\mu_2)} \tag{1.2}$$

for $R>R_0$ and some R_0, $\mu:=\min(\mu_1,\mu_2)>0$.

Short-Range Assumption.

$$\mu_1>1\,. \tag{1.3}$$

Compactness Assumption.

$$V_{ij}(p^2+1)^{-1},\,(p^2+1)^{-1}y\cdot\nabla V_{ij}(y)\,(p^2+1)^{-1}\ \textit{are compact}. \tag{1.4}$$

The Hamiltonian of the N-particle system is

$$H=p^2+V:=p^2+\sum_{(ij)}V_{ij}(x^{(ij)}) \tag{1.5}$$

on $L^2(X)$, where X is the N-body configuration space with center of mass motion removed.

We now define the wave operators

$$\Omega_a:=s-\lim_{t\to+\infty}e^{itH}e^{-itH_a}P^a \tag{1.6}$$

for all cluster decompositions a with $\#(a)\geqq 2$. Here $P^a=\mathbf{1}\otimes P^a$ with respect to $L^2(X)=L^2(X_a)\otimes L^2(X^a)$ is the bound state projection for H^a. States in the range of Ω_a are asymptotic to bound non-interacting clusters of a.

Theorem 1.1. *Assume the pair potentials V_{ij} are infinitesimally small with respect to p^2 on $L^2(X^{(ij)})$, and satisfy* (1.1)–(1.3). *Then the wave operators* (1.6) *exist, their ranges are closed, mutually orthogonal, and satisfy*

$$\bigoplus_{\#(a)\geqq 2}\operatorname{Ran}\Omega_a\subset\operatorname{Ran}(1-P)\,.$$

These facts are rather well known [15, 12, 14, 19, 20]. Nevertheless we include a proof, because the ones we are aware of make assumptions on the potentials in terms of L^p-spaces. Our aim, however, is to prove

Theorem 1.2. *The quantum N-body system* (1.5) *satisfying* (1.1)–(1.4) *is asymptotically complete, i.e.*

$$\bigoplus_{\#(a)\geqq 2} \operatorname{Ran}\Omega_a = \operatorname{Ran}(1-P). \tag{1.7}$$

The proof we present is self-contained, except for the Mourre estimate [16, 17, 10, 2]. By the same techniques some related results can be proved:

- Inclusion of 3- (or more) body potentials.
- Inclusion of permutation symmetry [21].
- Propagation theorem [22, 5] for a time-dependent propagation set.
- Asymptotic clustering for Coulomb-type potentials [23].

We now introduce a number of notions and notations pertaining to N-body systems. The physical space is $\mathbb{R}^\nu$, $\nu \geqq 1$. The configuration space of N mass points $m_i > 0$ in the center of mass (CM-) frame is the real vector space

$$X := \left\{ x = (x^1, \dots, x^N) \,\middle|\, x^i \in \mathbb{R}^\nu, \sum_{i=1}^N m_i x^i = 0 \right\}$$

equipped with the metric

$$x \cdot y := 2 \sum_{i=1}^N m_i x^i \cdot y^i ,$$

where $x^i \cdot y^i$ is the scalar product on $\mathbb{R}^\nu$. We will also use the notation $x^2 = x \cdot x$ and $|x| = (x \cdot x)^{1/2}$. The Hilbert space of the quantum mechanical N-body system is $L^2(X)$, where the volume element of X is defined by the metric.

Clusters are nonempty subsets $C \subset \{1, \dots, N\}$. The subspace

$$X^C := \{ x \in X \,|\, x^i = 0 \text{ if } i \notin C \}$$

represents the configuration space ofthe cluster C in its own CM-frame. Evidently $X^{C_1} \perp X^{C_2}$ if $C_1 \cap C_2 = \emptyset$. A cluster decomposition $a = (C_1, \dots, C_{\#(a)})$ is a partition of $\{1, \dots, N\}$ into clusters. We thereby set $\#(a)$ to be the number of clusters $C \in a$. We define

$$X^a := \bigoplus_{C \in \boxed{a}} X^C = \left\{ x \in X \,\middle|\, \sum_{i \in \boxed{C}} m_i x^i = 0 \text{ for } C \in a \right\}. \tag{1.8}$$

The meaning of this notation [17] is that variables within boxes are kept fixed, i.e. that the sums are ranging over the other variables only. We also write (ij) for the cluster decomposition $(ij) = (1) \dots \widehat{(i)} \dots \widehat{(j)} \dots (N)(ij)$, where $\hat{}$ indicates omission. Then $X^{(ij)}$ is the configuration space for the relative coordinate ofthe pair i, j. The orthogonal complement of X^a is

$$X_a := \{ x \in X \,|\, x^i = x^j \text{ if } i, j \in C \text{ for some } C \in a \}.$$

We denote the orthogonal projection onto X^a, X_a by $\mathbf{1}^a$, $\mathbf{1}_a$ respectively, and set the shorter notations $x^a := \mathbf{1}^a x$, $x_a := \mathbf{1}_a x$. The splitting $X = X_a \oplus X^a$ induces the factorization

$$L^2(X) = L^2(X_a) \otimes L^2(X^a), \tag{1.9}$$

whereas (1.8) induces

$$L^2(X^a) = \bigotimes_{C \in \boxed{a}} L^2(X^C). \tag{1.10}$$

The cluster decompositions are partially ordered by $a \subset b$, expressing that each cluster of a is a subset of a cluster of b. By $a \cup b$ we mean the smallest cluster c with $a \subset c$, $b \subset c$, whose existence and uniqueness is readily verified. The relation $a \subset b$ implies $\mathbf{1}_a \geqq \mathbf{1}_b$ (in the sense of orthogonal projections and hence of quadratic forms), as well as $x^b{}_a = x_a{}^b =: x_a^b$.

The operators describing the particle velocities are the components of $2p = (2/i)\nabla$, where ∇ is the (contravariant) gradient. The operator

$$H^a := (p^a)^2 + \sum_{(ij) \subset \boxed{a}} V_{ij}(x^{(ij)}),$$

on $L^2(X^a)$ then describes the internal motion of a system of non-interacting clusters. With respect to the factorization (1.10) it has the structure

$$H^a = \sum_{C \in \boxed{a}} \mathbf{1} \otimes \ldots \otimes H^C \otimes \ldots \otimes \mathbf{1}, \tag{1.11}$$

where H^C is the Hamiltonian of the cluster C in its own CM-frame. By P^a we denote the bound state projection of H^a. In particular, the Hamiltonian (1.5) is $H = H^{(1 \ldots N)}$, and $P := P^{(1 \ldots N)}$ is its bound state projection.

The free intercluster motion of the non-interacting clusters is included in the description when considering

$$H_a := (p_a)^2 \otimes \mathbf{1} + \mathbf{1} \otimes H^a \tag{1.12}$$

on (1.9). Alternatively, it can be written as $H_a = H - I_a$, where

$$I_a := \sum_{(ij) \not\subset \boxed{a}} V_{ij}(x^{(ij)})$$

are the intercluster interactions. Under the assumptions used in this paper the Hamiltonians above are selfadjoint on the domain $D((p^a)^2)$, respectively $D(p^2)$ of the kinetic energy.

We use the notation $F(x \in A)$ for the (sharp) characteristic function of $A \subset X$.

2. The Partition of Unity and the Vector Field

In this section we will construct a partition of unity as characteristic functions of a partition of the N-body configuration space X. Like for other partitions of unity indexed by cluster decompositions, each member has a support designed to "kill" intercluster interactions, but in addition the normal to its boundary is at any point orthogonal to the motion within those clusters, which are compatible with this point. Related to this is the construction of a vector field, whose derivative (as a matrix) has suitable positivity properties.

The seminorm $((x^a)^2 + (x^b)^2)^{1/2}$ is a norm on $X^{a \cup b}$, hence

$$10q \cdot (x^{a \cup b})^2 < (x^a)^2 + (x^b)^2 \tag{2.1}$$

for all a, b and $x^{a \cup b} \neq 0$, provided $q > 0$ is small enough. We also require $5q \leqq 1$ and set

$$q^a := \begin{cases} q^{\#(a)-1} & \text{if } a \neq (1) \ldots N \\ 0 & \text{if } a = (1) \ldots (N), \end{cases}$$

and $q_b^a := q^a - q^b$. The separate definition if $a=(1)\dots(N)$ is not essential, but it eases the notation due to $q_{(1)\dots(N)}^a = q^a$, in imitation of $x_{(1)\dots(N)}^a = x^a$. We will frequently use (2.1) in the form of

Lemma 2.1. *Assume that $a \not\subset b$ and $b \not\subset a$, and let $c \subset a$, $c \subset b$. If $(x_c^{a\cup b})^2 \geqq (1/2) q_a^{a\cup b}$ and $(x_c^a)^2 \leqq 2q^a$, then*

$$(x_c^b)^2 > 2q_c^b. \tag{2.2}$$

Proof. Notice that if $\#(f) > \#(g)$, then $q \cdot q^g \geqq q^f$. Moreover, like in set theory, $a \subset b \Leftrightarrow a \cup b = b$, hence under our assumptions $a\cup b \supsetneqq b$ and $a \cup b \supsetneqq a$. By (2.1)

$$\begin{aligned}(x_c^b)^2 &> 10q \cdot (1/2)(q^{a\cup b} - q^a) - 2q^a \geqq 5q \cdot q^{a\cup b} - q^a - 2q^a \\ &= 2q \cdot q^{a\cup b} + 3\underbrace{(q \cdot q^{a\cup b} - q^a)}_{\geqq 0} \geqq 2q^b \geqq 2(q^b - q^c),\end{aligned}$$

where we used $5q \leqq 1$ in the second step. □

For the rest of this section we will use this lemma in the weaker version with factors 1/2 and 2 dropped.

We can now define our partition of unity $\{J_a\}_a$, which is not smooth and which is indexed by all cluster decompositions of $\{1, \dots, N\}$, including the trivial one $(1\dots N)$:

$$J_a(x) := \left[\prod_{f \supsetneqq \boxed{a}} F((x_a^f)^2 > q_a^f)\right]\left[\prod_{g \subsetneqq \boxed{a}} F((x_g^a)^2 \leqq q_g^a)\right]. \tag{2.3}$$

Related ideas in the construction of the partition of unity were used in [22].

Lemma 2.2.

$$\sum_a J_a(x) = 1. \tag{2.4}$$

Proof. We define

$$\Sigma_a := \{x \in X \mid (x_a^f)^2 > q_a^f,\ \forall f \supsetneqq a, (x_g^a)^2 \leqq q_g^a,\ \forall g \subsetneqq a\},$$

and prove (2.4) by showing

(a) For each $x \in X$ there is at least one a with $x \in \Sigma_a$,
(b) For each $x \in X$ there is at most one a with $x \in \Sigma_a$.

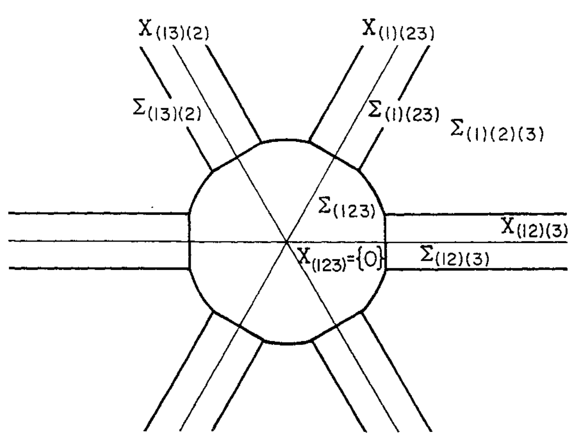

Fig. 1. The configuration space X for $N=3$, with subspaces X_a and subsets Σ_a

Proof of (a). Let $x \in X$ be given. Choose a maximal a such that $(x^a)^2 - q^a \leqq (x^b)^2 - q^b$ for all cluster decompositions b (i.e. an a such that no $a' \supsetneqq a$ enjoys this property). We claim $x \in \Sigma_a$. Indeed, for $f \supsetneqq a$, $(x_a^f)^2 - q_a^f = (x^f)^2 - q^f - ((x^a)^2 - q^a) > 0$ by the maximality of a. Similarly, $(x_g^a)^2 - q_g^a \leqq 0$ for $g \subsetneqq a$.

Proof of (b). Let $a \neq b$. We have to show $\Sigma_a \cap \Sigma_b = \emptyset$. We distinguish between i) $a \subset b$ or $b \subset a$, and ii) $a \not\subset b$ and $b \not\subset a$.

i) It is enough to consider $a \subset b$. $x \in \Sigma_a$ implies $(x_a^b)^2 > q_a^b$, and $x \in \Sigma_b$ implies $(x_a^b)^2 \leqq q_a^b$. Hence $\Sigma_a \cap \Sigma_b = \emptyset$.

ii) Here $a \cup b \supsetneqq a$. Thus for $x \in \Sigma_a$ we have $(x^{a \cup b})^2 \geqq (x_a^{a \cup b})^2 > q_a^{a \cup b}$ and $(x^a)^2 \leqq q^a$. This proves $(x^b)^2 > q^b$ by (2.2), which implies $x \notin \Sigma_b$. □

Another property we will use is

Lemma 2.3. *For $x \in \Sigma_a$ and $(ij) \not\subset a$*

$$(x^{(ij)})^2 > q^{(ij)} = q^{N-2}. \tag{2.5}$$

Proof. Since $a \cup (ij) \supsetneqq a$, we see that $(x^{a \cup (ij)})^2 \geqq (x_a^{a \cup (ij)})^2 > q_a^{a \cup (ij)}$, as well as $(x^a)^2 \leqq q^a$. If $a \not\subset (ij)$, we get (2.5) from (2.2); otherwise $a = (1) \dots (N)$ and (2.5) is included in the definition of Σ_a. □

We now define the basic vector field on X:

$$W(x) := \sum_a J_a(x) x_a . \tag{2.6}$$

One of the reasons for our choice of the partition of unity lies in

Theorem 2.4. *As a distribution, the derivative $W_*(x)$ is symmetric and satisfies*

$$W_*(x) \geqq \sum_a J_a(x) \mathbf{1}_a . \tag{2.7}$$

Proof. In the following we consider products of distributions. It is not difficult to see that they are always well defined, since their singular supports are manifolds which intersect transversally. Evidently

$$W_*(x) = \sum_a x_a \otimes \nabla J_a(x) + \sum_a J_a(x) \mathbf{1}_a ,$$

since $(x_a)_* = \mathbf{1}_a$. We thus have to show that the first term on the right-hand side is positive semidefinite. Applying the product rule to (2.3), we obtain

$$\nabla J_a = \sum_{b \supsetneqq \boxed{a}} J_{a\partial b} + \sum_{b \subsetneqq \boxed{a}} J_{a\partial b} ,$$

where

$$J_{a\partial b} = \Bigg[\prod_{\substack{f \supsetneqq \boxed{a} \\ f \neq \boxed{b}}} F((x_a^f)^2 \geqq q_a^f) \Bigg] \Bigg[\prod_{g \subsetneqq \boxed{a}} F((x_g^a)^2 \leqq q_g^a) \Bigg] \nabla F((x_a^b)^2 \geqq q_a^b) \tag{2.8}$$

for $b \supsetneqq a$, and similarly for $b \subsetneqq a$ (for the sake of interpretation, $J_{a\partial b}$ is a term related to the boundary between Σ_a and Σ_b). It will be enough to show

$$x_a \otimes J_{a\partial b} + x_b \otimes J_{b\partial a} \geqq 0 \tag{2.9}$$

for all a, b with $b \supsetneqq a$, since then

$$\sum_a x_a \otimes \nabla J_a(x) = \sum_{b \supsetneqq a} x_a \otimes J_{a\partial b} + \sum_{a \supsetneqq b} x_a \otimes J_{a\partial b}$$
$$= \sum_{b \supsetneqq a} x_a \otimes J_{a\partial b} + \sum_{b \supsetneqq a} x_b \otimes J_{b\partial a} \geqq 0\,,$$

where we interchanged the names of the variables in the second step. We now prove (2.9) by performing some replacements in (2.8), so to make $J_{a\partial b}$ comparable with $J_{b\partial a}$.

[1] For $g \subsetneqq b$ with $g \not\subset a$, $g \not\supset a$ we have either i) $a \cup g = b$ or ii) $a \cup g \neq b$. In case i) we have

$$F((x^a)^2 \leqq q^a)\, \nabla F((x_a^b)^2 \geqq q_a^b) = F((x_g^b)^2 \leqq q_g^b)\, F((x^a)^2 \leqq q^a)\, \nabla F((x_a^b)^2 \geqq q_a^b)\,, \qquad (1\,\mathrm{i})$$

while in case ii) we have

$$F((x^a)^2 \leqq q^a)\, F(x_a^{a\cup g})^2 \geqq q_a^{a\cup g})\, \nabla F((x_a^b)^2 \geqq q_a^b)$$
$$= F((x_g^b)^2 \leqq q_g^b)\, F((x^a)^2 \leqq q^a)\, F(x_a^{a\cup g})^2 \geqq q_a^{a\cup g})\, \nabla F((x_a^b)^2 \geqq q_a^b)\,. \qquad (1\,\mathrm{ii})$$

In fact in both cases we have on the support of the left-hand side $(x_a^b)^2 = q_a^b$, as well as $(x^{a\cup g})^2 \geqq (x_a^{a\cup g})^2 \geqq q_a^{a\cup g}$ and $(x^a)^2 \leqq q^a$, hence $(x^g)^2 > q^g$ by (2.2). Since $(x_g^b)^2 + (x^g)^2 = (x^b)^2 = (x_a^b)^2 + (x^a)^2$, we obtain $(x_g^b)^2 \leqq q_a^b + q^a - q^g = q_g^b$.

[2] For $g \subsetneqq a$

$$F((x_g^a)^2 \leqq q_g^a)\, \nabla F((x_a^b)^2 \geqq q_a^b) = F((x_g^b)^2 \leqq q_g^b)\, \nabla F((x_a^b)^2 \geqq q_a^b)\,, \qquad (2)$$

since for $(x_a^b)^2 = q_a^b$, $(x_g^a)^2 \leqq q_g^a$ iff $(x_g^b)^2 \leqq q_g^b$, due to $(x_g^b)^2 = (x_a^b)^2 + (x_g^a)^2$.

[3] For $f \supsetneqq a$ with $f \not\subset b$, $f \not\supset b$

$$F((x_a^f)^2 \geqq q_a^f)\, F((x_a^{b\cup f})^2 \geqq q_a^{b\cup f})\, \nabla F((x_a^b)^2 \geqq q_a^b)$$
$$= F((x_a^{b\cup f})^2 \geqq q_a^{b\cup f})\, \nabla F((x_a^b)^2 \geqq q_a^b)\,, \qquad (3)$$

since for x in the support of the right-hand side $(x_a^{b\cup f})^2 \geqq q_a^{b\cup f} > q_b^{b\cup f}$, $(x_a^b)^2 = q_a^b \leqq q^b$, and hence $(x_a^f)^2 \geqq q_a^f$ by (2.2).

[4] For $f \supsetneqq b$

$$F((x_a^f)^2 \geqq q_a^f)\, \nabla F((x_a^b)^2 \geqq q_a^b) = F((x_b^f)^2 \geqq q_b^f)\, \nabla F((x_a^b)^2 \geqq q_a^b)\,, \qquad (4)$$

since for $(x_a^b)^2 = q_a^b$, $(x_a^f)^2 \geqq q_a^f$ iff $(x_b^f)^2 \geqq q_b^f$, due to $(x_a^f)^2 = (x_b^f)^2 + (x_a^b)^2$.

[5] Finally, for $b \supsetneqq f \supsetneqq a$

$$F((x_a^f)^2 \geqq q_a^f)\, \nabla F((x_a^b)^2 \geqq q_a^b) = F((x_f^b)^2 \leqq q_f^b)\, \nabla F((x_a^b)^2 \geqq q_a^b)\,, \qquad (5)$$

since for $(x_a^b)^2 = q_a^b$, $(x_a^f)^2 \geqq q_a^f$ iff $(x_f^b)^2 \leqq q_f^b$, due to $(x_a^b)^2 = (x_f^b)^2 + (x_a^f)^2$.

Performing successively the replacements (1) up to (5) in (2.8), we get

$$J_{a\partial b} = \left[\prod_{f \supsetneqq \boxed{b}} F((x_b^f)^2 \geqq q_b^f)\right]\left[\prod_{\substack{g \subsetneqq \boxed{b} \\ g \neq \boxed{a}}} F((x_g^b)^2 \leqq q_g^b)\right] \nabla F((x_a^b)^2 \geqq q_a^b)\,.$$

This is, up to a reversal of the inequality in the argument of ∇F, equal to $J_{b\partial a}$. Disregarding the common nonnegative factors, we obtain for the left-hand side of

(2.9)

$$x_a \otimes \nabla F((x_a^b)^2 \geqq q_a^b) + x_b \otimes \nabla F((x_a^b)^2 \leqq q_a^b) = 2\delta((x_a^b)^2 = q_a^b) \cdot (x_a \otimes x_a^b - x_b \otimes x_a^b)$$
$$= 2\delta((x_a^b)^2 = q_a^b) \cdot x_a^b \otimes x_a^b \geqq 0,$$

since $x_a - x_b = x^b - x^a = x_a^b$. □

We conclude this section by proving some further properties of the vector field $W(x)$.

Lemma 2.5.

$$W(x)^{(ij)} = 0 \quad \text{for} \quad (x^{(ij)})^2 < q^{N-2}, \tag{2.10}$$

$$\|x - W(x)\|_\infty < \infty, \tag{2.11}$$

$$\|(x - W(x)) \cdot x\|_\infty < \infty. \tag{2.12}$$

Here $\|\cdot\|_\infty$ *for vector fields is the norm on* $L^\infty(X, X)$.

Proof. Let $(x^{(ij)})^2 < q^{N-2}$ and $(ij) \not\subset a$. We then have $x \notin \overline{\Sigma_a} = \operatorname{supp} J_a$ by (2.5), and hence

$$W(x) = \sum_{a \supset \boxed{(ij)}} J_a(x) x_a .$$

Thus (2.10) follows from $x_a^{(ij)} = 0$ for $a \supset (ij)$. We compute

$$x - W(x) = \sum_a J_a(x)(x - x_a) = \sum_a J_a(x) x^a ,$$

where we used (2.4), and

$$(x - W(x)) \cdot x = \sum_a J_a(x) x^a \cdot x = \sum_a J_a(x)(x^a)^2 .$$

Thus (2.11), (2.12) follow from $(x^a)^2 \leqq q^a$ on $\overline{\Sigma_a} = \operatorname{supp} J_a$. □

3. Smearing Them Out

We will need the partition of unity and the vector field to be smooth. This will be obtained by taking convolutions with a smooth φ. We list the required properties of the mollifier:

$$\begin{aligned} &\varphi \in C_0^\infty(X), && \varphi \geqq 0, \\ &\int \varphi(x)\, dx = 1, && \int x\varphi(x)\, dx = 0, \\ &\operatorname{supp} \varphi \subset \{x \in X \mid |x| \leqq \sigma\}, && \sigma > 0. \end{aligned} \tag{3.1}$$

We will refer to σ as the sharpness. The fact that the smeared quantities will have a slightly larger support than the original ones motivates the following definition: for a set $A \subset X$ we define

$$A^\sigma = \{x \in X \mid \operatorname{dist}(x, A) \leqq \sigma\}.$$

We now define smooth partitions of unity, $\tilde{j}_a$ and j_a

$$\tilde{j}_a := J_a * \varphi\,, \tag{3.2}$$

$$j_a := \frac{\tilde{j}_a}{\left(\sum_b \tilde{j}_b^2\right)^{1/2}} \tag{3.3}$$

[see (2.3) for the definition of J_a], and state

Lemma 3.1. *$\tilde{j}_a, j_a$ are smooth with bounded derivatives, and*

$$\sum_a \tilde{j}_a = 1\,, \tag{3.4}$$

$$\sum_a j_a^2 = 1\,. \tag{3.5}$$

Furthermore, for $y \in \operatorname{supp} \tilde{j}_a = \operatorname{supp} j_a$, and $(ij) \not\subset a$

$$|y^{(ij)}| \geqq \tfrac{1}{2} q^{\frac{N-2}{2}}\,, \tag{3.6}$$

provided the sharpness $\sigma > 0$ is small enough.

Proof. $\sum_a \tilde{j}_a = \left(\sum_a J_a\right) * \varphi = 1 * \varphi = 1$ by (2.4); (3.5) follows from (3.3) and from the fact that the denominator there never vanishes by (3.4).

We impose $\sigma \leqq (1/2) q^{\frac{N-2}{2}}$. For $y \in \operatorname{supp} \tilde{j}_a \subset \Sigma_a^\sigma$ there is an $x \in \overline{\Sigma_a}$ with $|y - x| \leqq \sigma$. Hence

$$|y^{(ij)}| \geqq |x^{(ij)}| - |y^{(ij)} - x^{(ij)}| \geqq q^{\frac{N-2}{2}} - \sigma \geqq \tfrac{1}{2} q^{\frac{N-2}{2}}$$

by (2.5). □

The next two lemmas will be needed in the proof of two further important properties of our partitions of unity.

Lemma 3.2. *Given $k_0 > 1$ and provided the sharpness $\sigma > 0$ is small enough,*

$$\Sigma_a^\sigma \cap (\Sigma_b^\sigma \cdot k^{-1}) = \emptyset \tag{3.7}$$

for $b \not\subset a$ and $k \geqq k_0$. (By this we mean that σ depends on k_0 but not on b, a, k.)

Proof. By taking $\sigma > 0$ small enough, we can impose

$$k_0((q_a^b)^{1/2} - \sigma) > (q_a^b)^{1/2} + \sigma$$

for all $a \subsetneqq b$; and

$$(q^b)^{1/2} + \sigma < \sqrt{2}(q^b)^{1/2}\,,$$

$$(q_a^{a\cup b})^{1/2} - \sigma > \frac{1}{\sqrt{2}} (q_a^{a\cup b})^{1/2}$$

for all $b \neq (1)\dots(N)$ and all a with $a \cup b \supsetneqq a$.

From $b \not\subset a$ it follows $b \neq (1) \ldots (N)$. We distinguish between i) $a \subset b$, and ii) $a \not\subset b$.

i)
$$x \in \Sigma_b^\sigma \cdot k^{-1} \Rightarrow |kx_a^b| \leqq (q_a^b)^{1/2} + \sigma\,;$$
$$x \in \Sigma_a^\sigma \Rightarrow |x_a^b| \geqq (q_a^b)^{1/2} - \sigma$$
$$\Rightarrow |kx_a^b| \geqq k((q_a^b)^{1/2} - \sigma) > (q_a^b)^{1/2} + \sigma\,.$$

Hence $\Sigma_a^\sigma \cap (\Sigma_b^\sigma \cdot k^{-1}) = \emptyset$.

ii) In this case we also have $a \neq (1) \ldots (N)$. Let $x \in \Sigma_a^\sigma$. Since $a \cup b \supsetneqq a$,

$$|x^{a \cup b}| \geqq |x_a^{a \cup b}| \geqq (q_a^{a \cup b})^{1/2} - \sigma \geqq \frac{1}{\sqrt{2}} (q_a^{a \cup b})^{1/2}\,,$$

and $|x^a| \leqq (q^a)^{1/2} + \sigma \leqq \sqrt{2}(q^a)^{1/2}$. Together with (2.2) this proves

$$|kx^b| > |x^b| > \sqrt{2}(q^b)^{1/2} > (q^b)^{1/2} + \sigma\,,$$

thus $x \notin \Sigma_b^\sigma \cdot k^{-1}$. □

Lemma 3.3. *Given $k > 1$ and provided the sharpness $\sigma > 0$ is small enough,*

$$J_a(x) = \Bigg[\prod_{f \supsetneqq \boxed{a}} F((x_a^f)^2 > q_a^f) \Bigg] \Bigg[\prod_{\boxed{b} \subset g \subsetneqq \boxed{a}} F((x_g^a)^2 \leqq q_g^a) \Bigg], \tag{3.8}$$

for $b \subset a$ and $x \in (\Sigma_b^\sigma \cdot k^{-1})^\sigma$.

Proof. By taking $\sigma > 0$ small enough we can impose

$$k^{-1}((q_g^b)^{1/2} + \sigma) + \sigma \leqq (q_g^b)^{1/2}$$

for all $g \subsetneqq b$. We may assume $b \neq (1) \ldots (N)$, since otherwise (3.8) coincides with (2.3).

$\boxed{1}$ For $g \subsetneqq a$ with $b \not\subset g$, $b \not\supset g$ we have either i) $b \cup g = a$ or ii) $b \cup g \subsetneqq a$. For $x \in (\Sigma_b^\sigma \cdot k^{-1})^\sigma$ we have in case i)

$$F((x^a)^2 \leqq q^a)\, F((x_g^a)^2 \leqq q_g^a) = F((x^a)^2 \leqq q^a)\,, \tag{1 i}$$

while in case ii)

$$F((x^a)^2 \leqq q^a)\, F((x_{b \cup g}^a)^2 \leqq q_{b \cup g}^a)\, F((x_g^a)^2 \leqq q_g^a) = F((x^a)^2 \leqq q^a)\, F((x_{b \cup g}^a)^2 \leqq q_{b \cup g}^a)\,. \tag{1 ii}$$

Assume the contrary, i.e. that there is an $x \in (\Sigma_b^\sigma \cdot k^{-1})^\sigma$ in the support of the right-hand side of (1 i) or of (1 ii), such that $(x_g^a)^2 > q_g^a$. In case i) this is equivalent to $(x_g^{b \cup g})^2 > q_g^{b \cup g}$, but also in case ii) we have $(x_g^{b \cup g})^2 = (x_g^a)^2 - (x_{b \cup g}^a)^2 > q_g^a - q_{b \cup g}^a = q_g^{b \cup g}$, which in both cases implies $(x^{b \cup g})^2 \geqq (x_g^{b \cup g})^2 > q_g^{b \cup g}$.

On the other hand $(x^g)^2 = (x^a)^2 - (x_g^a)^2 \leqq q^a - q_g^a = q^g$, and hence $(x^b)^2 > q^b$ by (2.2). This contradicts $x \in (\Sigma_b^\sigma \cdot k^{-1})^\sigma$, since for $y \in \Sigma_b^\sigma \cdot k^{-1}$ we have $|y^b| \leqq k^{-1}((q^b)^{1/2} + \sigma)$, and thus $|x^b| \leqq k^{-1}((q^b)^{1/2} + \sigma) + \sigma \leqq (q^b)^{1/2}$.

$\boxed{2}$ For $g \subsetneqq a$ with $g \subsetneqq b$ and for $x \in (\Sigma_b^\sigma \cdot k^{-1})^\sigma$,

$$F((x_b^a)^2 \leqq q_b^a)\, F((x_g^a)^2 \leqq q_g^a) = F((x_b^a)^2 \leqq q_b^a)\,. \tag{2}$$

In fact, for $y \in \Sigma_b^\sigma \cdot k^{-1}$ we have $|y_g^b| \leqq k^{-1}((q_g^b)^{1/2} + \sigma)$, hence $|x_g^b| \leqq k^{-1}((q_g^b)^{1/2} + \sigma) + \sigma \leqq (q_g^b)^{1/2}$. If we restrict x further to the support of the right-hand side of (2), we get $(x_g^a)^2 = (x_b^a)^2 + (x_g^b)^2 \leqq q_b^a + q_g^b = q_g^a$.

Performing successively the replacements (1) and (2) in (2.3) we arrive at (3.8). □

Lemma 3.4. *Given $k_0 > 1$ and provided the sharpness $\sigma > 0$ is small enough,*

$$j_a(y)\tilde{j}_b(ky) = 0 \quad \text{for} \quad b \not\subset a, \tag{3.9}$$

$$j_a(y) = j_a(y) \sum_{b \subset \boxed{a}} \tilde{j}_b(ky), \tag{3.10}$$

$$\nabla\tilde{j}_a(y) = \nabla\tilde{j}_a(y) \sum_{b \subset \boxed{a}} j_b(ky)^2, \tag{3.11}$$

$$\nabla j_a(y) = \nabla j_a(y) \sum_{b \subset \boxed{a}} j_b(ky)^2 \tag{3.12}$$

for $k \geqq k_0$.

Proof. Equations (3.10), (3.11), (3.12) would hold by (3.4), (3.5), if the sums extended to all b. We have to show that (3.9) (and similar terms) vanish for $b \not\subset a$:

$$\operatorname{supp} j_a, \operatorname{supp} \nabla\tilde{j}_a, \operatorname{supp} \nabla j_a \subset \Sigma_a^\sigma,$$

$$\operatorname{supp} \tilde{j}_b(k\,\cdot), \operatorname{supp} j_b(k\,\cdot) \subset \Sigma_b^\sigma \cdot k^{-1}.$$

But $\Sigma_a^\sigma \cap (\Sigma_b^\sigma \cdot k^{-1}) = \emptyset$ by (3.7). □

Lemma 3.5. *Given $k > 1$ and provided the sharpness $\sigma > 0$ is small enough,*

$$j_b(ky)\, \nabla\tilde{j}_a(y) = j_b(ky)\, \nabla_b \tilde{j}_a(y), \tag{3.13}$$

$$j_b(ky)\, \nabla j_a(y) = j_b(ky)\, \nabla_b j_a(y) \tag{3.14}$$

for $b \subset a$.

Proof. For $y \in \operatorname{supp} j_b(k\,\cdot) \subset \Sigma_b^\sigma \cdot k^{-1}$ we have

$$\tilde{j}_a(y) = \int J_a(y-z)\, \varphi(z)\, dz,$$

where $y - z \in (\Sigma_b^\sigma \cdot k^{-1})^\sigma$ for z in the support of the integrand. Then $\nabla\tilde{j}_a(y) = \nabla_b \tilde{j}_a(y)$, since on $(\Sigma_b^\sigma \cdot k^{-1})^\sigma$ (which is the closure of its interior) $J_a(x)$ is a function of x_b alone by (3.8). We now prove (3.14). Let $c \not\supset b$. Since $\operatorname{supp} \tilde{j}_c \subset \Sigma_c^\sigma$, we have $\tilde{j}_c(y) = 0$ for $y \in \Sigma_b^\sigma \cdot k^{-1}$ by (3.7). Thus

$$j_a(y) = \frac{\tilde{j}_a(y)}{\left(\sum_{c \supset \boxed{b}} \tilde{j}_c(y)^2\right)^{1/2}},$$

and (3.14) then follows from (3.13), which also holds for a replaced by $c \supset b$. □

We now turn to the smoothing of the vector field W. We define

$$w := W * \varphi \tag{3.15}$$

[see (2.6) for the definition of W].

Lemma 3.6. *Given $k_0 > 1$ and provided the sharpness $\sigma > 0$ is small enough,*

$$j_a(y)\, w(ky)_a = j_a(y)\, k y_a \tag{3.16}$$

for $k \geqq k_0$.

Proof. We first prove

$$W(ky)_a = ky_a \tag{3.17}$$

for $y \in \Sigma_a^{2\sigma}$ and $k \geqq k_0$. By definition $W(ky) = \sum_b J_b(ky) ky_b$, but for $b \not\subset a$, $\operatorname{supp} J_b(k\cdot) \subset \Sigma_b \cdot k^{-1} \subset \Sigma_b^{2\sigma} \cdot k^{-1}$ is disjoint from $\Sigma_a^{2\sigma}$ by (3.7), hence

$$W(ky) = \sum_{b \subset \boxed{a}} J_b(ky) ky_b$$

and (3.17) holds because $y_{b,a} = y_a$ for $b \subset a$ and $J_b(ky) = 0$ for $b \not\subset a$. We can now prove (3.16): for $y \in \operatorname{supp} j_a \subset \Sigma_a^\sigma$

$$w(ky) = \int W(ky - z)\varphi(z)\,dz = \int W(k(y - zk^{-1}))\varphi(z)\,dz\,,$$

where $y - zk^{-1} \in \Sigma_a^{\sigma + \sigma k^{-1}} \subset \Sigma_a^{2\sigma}$ for z in the support of the integrand. Therefore

$$w(ky)_a = \int (k(y - zk^{-1}))_a \varphi(z)\,dz = ky_a\,,$$

by (3.17), (3.1). □

Finally, we carry over to w the properties of W derived in Sect. 2.

Lemma 3.7. *For sufficiently small sharpness $\sigma > 0$, w has bounded derivatives (of degree $\geqq 1$), w_* is symmetric, and*

$$w(y)^{(ij)} = 0 \quad \text{for} \quad |y^{(ij)}| \leqq \frac{1}{2} q^{\frac{N-2}{2}}\,, \tag{3.18}$$

$$w_*(y) \geqq \sum_a \tilde{j}_a(y) \mathbf{1}_a\,, \tag{3.19}$$

$$\|w(y) - y\|_\infty < \infty\,, \tag{3.20}$$

$$\|w(y) - w_*(y)y\|_\infty < \infty\,. \tag{3.21}$$

Proof. We impose $\sigma < (1/2) q^{\frac{N-2}{2}}$.

$$w(y)^{(ij)} = \int W(y - z)^{(ij)} \varphi(z)\,dz\,,$$

where $|(y - z)^{(ij)}| \leqq (1/2) q^{\frac{N-2}{2}} + \sigma < q^{\frac{N-2}{2}}$ for z in the support of the integrand. Thus (3.18) follows from (2.10).

$w_* = W_* * \varphi$ is symmetric since W_* is, and (3.19) follows from (2.7) by $\varphi \geqq 0$.

We have $y * \varphi = y$ by (3.1); $\|w(y) - y\|_\infty = \|(W - y) * \varphi\|_\infty \leqq \|W - y\|_\infty < \infty$ follows from (2.11).

From $w = W * \varphi = (W - y) * \varphi + y$ we see that w has bounded derivatives, because y has, and by (2.11).

By explicit computation, $((y - W)\cdot y) * \varphi = ((y - W) * \varphi)\cdot y - (y - W) * (y\varphi)$. This proves

$$[((y - W) * \varphi)\cdot y]_* = ((y - W)\cdot y) * \varphi_* + (y - W) * (y\varphi)_* \in L^\infty(X, X) \tag{3.22}$$

by (2.11), (2.12). Using again $y * \varphi = y$, we get for any tangent vector X

$$\begin{aligned}[((y - W) * \varphi)\cdot y]_*(X) &= ((y - w)\cdot y)_*(X) = (y - w)\cdot X + y\cdot(\mathbf{1} - w_*(y))(X) \\ &= [y - w + (1 - w_*(y))y]\cdot X\,,\end{aligned}$$

where we used that $w_*(y)$ is symmetric. By (3.20), (3.22) we conclude $\|(1-w_*(y))y\|_\infty<\infty$ and $\|w(y)-w_*(y)y\|_\infty \leqq \|w(y)-y\|_\infty+\|(1-w_*(y))y\|_\infty <\infty$. □

4. Propagation Estimates

The purpose of this section is to show that, roughly speaking, the intercluster motion concentrates asymptotically on classical trajectories. Throughout this section we will assume that the pair interactions $V_{ij}(y)$ on $L^2(X^{(ij)})$ are infinitesimally small with respect to p^2 and satisfy (1.1) and (1.2). The Hamiltonian H is of course (1.5), and the basic propagation observable is

$$K(t):=(p-v(x,t))^2+V(x), \tag{4.1}$$

where the vector field $v(x,t)$ will be specified later as a mild modification of $x/2t$. The starting point of our propagation estimates are the Heisenberg equation of motion. We thus define

$$D\cdot:=i[H,\cdot]+\frac{\partial\cdot}{\partial t}. \tag{4.2}$$

A straightforward computation leads to

$$\begin{aligned} DK(t)= & -2(p-v)(v_*+v_*^t)(p-v)-2\left(p\cdot\left(v_*v+\frac{1}{2}\frac{\partial v}{\partial t}\right)+\left(v_*v+\frac{1}{2}\frac{\partial v}{\partial t}\right)\cdot p\right) \\ & +4v\cdot\left(v_*v+\frac{1}{2}\frac{\partial v}{\partial t}\right)+\Delta(\nabla\cdot v)+2v\cdot\nabla V, \end{aligned} \tag{4.3}$$

where v_* is the x-derivative of v, and v_*^t its transpose. Since $K(t)$ is not bounded one can either derive propagation estimates holding on a time invariant dense set only, or introduce some cutoffs, thus replacing $K(t)$ by a bounded operator. We will follow the latter alternative and borrow the maximal velocity bound of [23]:

Theorem 4.1. *Let $\Omega\subset\mathbb{R}$ be bounded and measurable, and $\lambda>0$ large enough. Then*

$$\int_1^\infty \|F(\lambda\leqq|x/t|\leqq 2\lambda)e^{-itH}E_\Omega(H)\psi\|^2\frac{dt}{t}\leqq \text{const}\,\|\psi\|^2 \tag{4.4}$$

for all $\psi\in L^2(X)$, where $E_\Omega(H)$ is the spectral projection for H associated with Ω and the constant depends on Ω,λ.

Proof. We take $h\in C^\infty(\mathbb{R})$ with $h'\geqq 0$, $(h')^{1/2}\in C_0^\infty(\mathbb{R})$, and

$$h'(y)\geqq 1 \quad \text{for} \quad 1\leqq y\leqq 4, \tag{4.5}$$

$$\operatorname{supp} h'\subset\{y\in\mathbb{R}\,|\,\tfrac{1}{2}\leqq y\leqq 9\}. \tag{4.6}$$

For the propagation observable $\Phi=-h((x/\lambda t)^2)$ we compute

$$\frac{d}{dt}(\psi_t,E_\Omega\Phi E_\Omega\psi_t)=(\psi_t,E_\Omega(D\Phi)E_\Omega\psi_t), \tag{4.7}$$

$$D\Phi=\frac{2}{t}\left(\frac{x}{\lambda t}\right)^2h'-\frac{2}{\lambda t}\left(h'\frac{x}{\lambda t}\cdot p+p\cdot\frac{x}{\lambda t}h'\right),$$

where $\psi_t = e^{-itH}\psi$, $E_\Omega = E_\Omega(H)$ and $h' = h'((x/\lambda t)^2)$. We have $(x/\lambda t)^2 h' \geqq (1/2)h'$ by (4.6), and

$$E_\Omega h' \frac{x}{\lambda t} \cdot p E_\Omega = E_\Omega (h')^{1/2} \frac{x}{\lambda t} (h')^{1/2} \cdot p E_\Omega$$

$$= E_\Omega (h')^{1/2} \chi(H) \frac{x}{\lambda t} (h')^{1/2} \cdot p E_\Omega + O(t^{-1})$$

$$= E_\Omega (h')^{1/2} \chi(H) p \cdot \frac{x}{\lambda t} (h')^{1/2} E_\Omega + O(t^{-1}),$$

where we took $\chi \in C_0^\infty(\mathbb{R})$ with $\chi(x) = 1$ for $x \in \Omega$, and we used $[(h')^{1/2}, \chi(H)] = O(t^{-1})$, which is proved at the very end of this section. This shows

$$E_\Omega \left(h' \frac{x}{\lambda t} \cdot p + p \cdot \frac{x}{\lambda t} h' \right) E_\Omega \leqq 2 \|\chi(H) p\| \cdot 3 E_\Omega h' E_\Omega + O(t^{-1}),$$

since $|x/\lambda t| \leqq 3$ for $x \in \operatorname{supp} h'((\cdot/\lambda t)^2)$, and thus

$$E_\Omega (D\Phi) E_\Omega \geqq \frac{1}{t} \left(1 - \frac{12}{\lambda} \|\chi(H) p\| \right) E_\Omega h' E_\Omega + O(t^{-2}) \geqq \frac{1}{2t} E_\Omega h' E_\Omega + O(t^{-2})$$

provided $\lambda > 0$ is large enough. By integrating (4.7) we conclude

$$\int_1^\infty (\psi_t, E_\Omega h' E_\Omega \psi_t) \frac{dt}{2t} \leqq 2 \sup_{t \geqq 1} |(\psi_t, E_\Omega \Phi E_\Omega \psi_t)| + \text{const} \|\psi\|^2 \leqq \text{const} \|\psi\|^2,$$

and (4.4) then follows by (4.5). □

In the sequel f, g will denote two fixed functions with $f, g \in C_0^\infty(X)$, $0 \leqq f, g \leqq 1$, $f(x) = 0$ for $|x| \geqq 2$, $f(x) = 1$ for $|x| \leqq 1$, $\operatorname{supp} g \subset \{x \in X \mid 1 \leqq |x| \leqq 2\}$, and $g = 1$ on $\operatorname{supp} \nabla f$.

We also define the vector field v in (4.1) by

$$v(x,t) := t^{-\delta} w\left(\frac{x}{2t^{1-\delta}} \right),$$

with $\delta > 0$, which will be adjusted later [see (3.15) for the definition of w].

Proviso. From now on the clause "provided the sharpness $\sigma > 0$ is small enough" will be omitted in the propositions, although it is assumed whenever quantities depending on σ are involved.

Theorem 4.2. *Let $\delta > 0$ be small enough, $\Omega \subset \mathbb{R}$ bounded and measurable, and $\lambda > 0$ large enough. Then*

$$\int_1^\infty (\psi_t, f(p - v)(v_* + v_*^t)(p - v) f \psi_t)\, dt \leqq \text{const} \|\psi\|^2 \tag{4.8}$$

for all $\psi = E_\Omega(H)\psi$, where $\psi_t = e^{-itH}\psi$ and $f = f(x/\lambda t)$.

Proof.

$$\frac{d}{dt}(\psi_t, fKf\psi_t) = (\psi_t, D(fKf)\psi_t) = (\psi_t, [f(DK)f + (Df)Kf + fK(Df)]\psi_t). \tag{4.9}$$

We first show that the terms in (4.9) arising from

$$Df=\frac{2}{\lambda t}(\nabla f)\left(\frac{x}{\lambda t}\right)\cdot\left(p-\frac{x}{2t}\right)-\frac{i}{(\lambda t)^2}(\Delta f)\left(\frac{x}{\lambda t}\right)$$

are integrable in time. Since $g\nabla f=\nabla f$ and since $K(t)$ is a local operator, we have $fK(Df)=gfK(Df)g$, where $g=g(x/\lambda t)$. Then, setting $E_\Omega=E_\Omega(H)$,

$$\begin{aligned}E_\Omega fK(Df)E_\Omega&=E_\Omega gfK(Df)\,g(H+i)^{-1}E_\Omega(H+i)\\&=E_\Omega gfK(Df)(H+i)^{-1}gE_\Omega(H+i)+\underbrace{E_\Omega gf}_{=O(1)}\underbrace{K(Df)(H+i)^{-1}}_{=O(t^{-1})}\underbrace{[H,g]E_\Omega}_{=O(t^{-1})}\\&=(H-i)\,E_\Omega g\underbrace{(H-i)^{-1}fK}_{=O(1)}\underbrace{(Df)(H+i)^{-1}}_{=O(t^{-1})}gE_\Omega(H+i)+O(t^{-2}),\end{aligned}\tag{4.10}$$

where the $O(1)$ estimates will be proved below. This proves

$$\begin{aligned}\int_1^\infty|(\psi_t,E_\Omega fK(Df)\,E_\Omega\psi_t)|\,dt&\leqq\text{const}\int_1^\infty\|gE_\Omega(H+i)\psi_t\|^2\frac{dt}{t}+\text{const}\|\psi\|^2\\&\leqq\text{const}\|E_\Omega(H+i)\psi\|^2+\text{const}\|\psi\|^2\leqq\text{const}\|\psi\|^2\end{aligned}$$

by (4.4). We estimate

$$|v(x,t)|=t^{-\delta}\left|w\left(\frac{x}{2t^{1-\delta}}\right)\right|\leqq t^{-\delta}\left(\text{const}+\left|\frac{x}{2t^{1-\delta}}\right|\right)\leqq\text{const}\,t^{-\delta}+\left|\frac{x}{2t}\right|$$

by (3.20), proving fv, fv^2, $(\nabla f)\cdot v=O(1)$,

$$v_*(x,t)=t^{-\delta}w_*\left(\frac{x}{2t^{1-\delta}}\right)\frac{1}{2t^{1-\delta}}=\frac{1}{2t}w_*\left(\frac{x}{2t^{1-\delta}}\right).$$

Since w has bounded derivatives, we get $K=H-2p\cdot v+v^2+O(t^{-1})$; by $fp\cdot v=p\cdot fv+O(t^{-1})$ we then see $(H-i)^{-1}fK=O(1)$, as announced. Next we show that the terms on the right-hand side of (4.3), up to the first one, lead to integrable contributions to (4.9),

$$\begin{aligned}\frac{\partial v}{\partial t}&=-\delta t^{-(1+\delta)}w\left(\frac{x}{2t^{1-\delta}}\right)-(1-\delta)t^{-(1+\delta)}w_*\left(\frac{x}{2t^{1-\delta}}\right)\frac{x}{2t^{1-\delta}}\\&=-t^{-(1+\delta)}[w_*(y)y+\delta(w(y)-w_*(y)y)]|_{y=x/2t^{1-\delta}},\end{aligned}$$

$$v_*v+\frac{1}{2}\frac{\partial v}{\partial t}=\frac{1}{2}t^{-(1+\delta)}[w_*(y)(w(y)-y)-\delta(w(y)-w_*(y)y)]|_{y=x/2t^{1-\delta}},$$

$$\Delta(\nabla\cdot v)=\left(\frac{1}{2t^{1-\delta}}\right)^3t^{-\delta}[\Delta(\nabla\cdot w)]|_{y=x/2t^{1-\delta}}.$$

By (3.20), (3.21) we thus obtain $\left\|v_*v+\frac{1}{2}\frac{\partial v}{\partial t}\right\|_\infty=O(t^{-(1+\delta)})$, and $\|\Delta(\nabla\cdot v)\|_\infty=O(t^{-(3-2\delta)})$. Finally,

$$v\cdot\nabla V_{ij}=v^{(ij)}\cdot\nabla^{(ij)}V_{ij}(x^{(ij)})=t^{-\delta}w\left(\frac{x}{2t^{1-\delta}}\right)^{(ij)}F(|x^{(ij)}|\geqq\text{const}\,t^{1-\delta})\cdot\nabla V_{ij}(x^{(ij)})$$

by (3.18). $fE_\Omega = ((p^{(ij)})^2+1)^{-1}B$, where $B = f((p^{(ij)})^2+1)E_\Omega + [(p^{(ij)})^2, f]E_\Omega = O(1)$. Thus

$$\begin{aligned} E_\Omega f v\cdot \nabla V_{ij} f E_\Omega &= E_\Omega f v^{(ij)} F(|x^{(ij)}| \geqq \mathrm{const}\, t^{1-\delta}) \cdot \nabla V_{ij}(x^{(ij)})((p^{(ij)})^2+1)^{-1}B \\ &= O(t^{-(1-\delta)(1+\mu_2)}) \end{aligned}$$

by (1.2). The claimed integrability thus holds, provided $\min(1+\delta,\ 3-2\delta,\ (1-\delta)(1+\mu_2)) > 1$. Integrating (4.9) we then obtain

$$\begin{aligned} 2\int_1^\infty (\psi_t, f(p-v)(v_* + v_*^t)(p-v)f\psi_t)\,dt &\leqq 2\sup_{t\geqq 1} |(\psi_t, E_\Omega f K f E_\Omega \psi_t)| + \mathrm{const}\,\|\psi\|^2 \\ &\leqq \mathrm{const}\,\|\psi\|^2 , \end{aligned}$$

since $\sup_{t\geqq 1}\|E_\Omega f(x/\lambda t)K(t)f(x/\lambda t)E_\Omega\| < \infty$, and $v_* = (1/2)(v_* + v_*^t) \geqq 0$ by (3.19). □

Theorem 4.3. *Let $\Omega \subset \mathbb{R}$ be bounded and measurable, $\lambda > 0$ large enough, and $v > 0$. Then*

$$\int_1^\infty \left(\psi_t, f j_a \left(p - \frac{x}{2t}\right)_a^2 j_a f \psi_t\right)\frac{dt}{t} \leqq \mathrm{const}\,\|\psi\|^2 \tag{4.11}$$

for all $\psi = E_\Omega(H)\psi$, where $\psi_t = e^{-itH}\psi$, $f = f(x/\lambda t)$ and $j_a = j_a(x/vt)$.

Proof. [See (3.3) for the definition of j_a],

$$f j_a \left(p-\frac{x}{2t}\right)_a^2 j_a f = f\left(p-\frac{x}{2t}\right)_a j_a^2 \left(p-\frac{x}{2t}\right)_a f - \frac{1}{(vt)^2} f(\Delta_a j_a) j_a f . \tag{4.12}$$

We claim that for large t

$$j_a\left(\frac{x}{vt}\right)\frac{x_a}{2t} = j_a\left(\frac{x}{vt}\right) v(x,t)_a , \tag{4.13}$$

$$\frac{1}{t} j_a^2 \mathbf{1}_a \leqq 2v_* . \tag{4.14}$$

In fact

$$j_a\left(\frac{x}{vt}\right) v(x,t)_a = j_a\left(\frac{x}{vt}\right) t^{-\delta} w\left(\frac{vt^\delta}{2}\frac{x}{vt}\right)_a = j_a\left(\frac{x}{vt}\right) t^{-\delta}\frac{x_a}{2t^{1-\delta}} = j_a\left(\frac{x}{vt}\right)\frac{x_a}{2t},$$

where we took $k_0 > 1$ and used (3.16), since $vt^\delta/2 \geqq k_0$ for large t. By (3.19),

$$v_*(x,t) = \frac{1}{2t} w_*\left(\frac{x}{2t^{1-\delta}}\right) \geqq \frac{1}{2t}\sum_b \tilde{j}_b\left(\frac{x}{2t^{1-\delta}}\right)\mathbf{1}_b .$$

Thus

$$\begin{aligned} 2v_* \geqq 2v_* j_a^2 &\geqq \frac{1}{t}\sum_b \tilde{j}_b\left(\frac{x}{2t^{1-\delta}}\right)\mathbf{1}_b j_a^2\left(\frac{x}{vt}\right) = \frac{1}{t}\sum_{b\subset \boxed{a}} \tilde{j}_b\left(\frac{x}{2t^{1-\delta}}\right)\mathbf{1}_b j_a^2\left(\frac{x}{vt}\right) \\ &\geqq \frac{1}{t}\sum_{b\subset \boxed{a}} \tilde{j}_b\left(\frac{x}{2t^{1-\delta}}\right)\mathbf{1}_a j_a^2\left(\frac{x}{vt}\right) = \frac{1}{t}\mathbf{1}_a j_a^2 , \end{aligned}$$

where we used successively $v_* \geqq 0$, (3.9), $\mathbf{1}_b \geqq \mathbf{1}_a$ for $b \subset a$, and (3.10). Using (4.12), (4.13), and (4.14) we obtain

$$\frac{1}{t} f j_a \left(p - \frac{x}{2t}\right)_a^2 j_a f = \frac{1}{t} f (p-v) j_a^2 \mathbf{1}_a (p-v) f + O(t^{-3})$$
$$\leqq f(p-v)\, 2v_* (p-v) f + O(t^{-3}).$$

Thus (4.11) follows from (4.8). □

As a first illustration of the importance of the boundaries of the partition of unity, we improve (4.11).

Theorem 4.4. *Let $\Omega \subset \mathbb{R}$ be bounded and measurable, $\lambda > 0$ large enough, and $v > 0$. Then*

$$\int_1^\infty \left(\psi_t, f j_a \left|p - \frac{x}{2t}\right|_a j_a f \psi_t\right) \frac{dt}{t} \leqq \text{const} \|\psi\|^2 \tag{4.15}$$

for all $\psi = E_\Omega(H)\psi$, where $\psi_t = e^{-itH}\psi$, $f = f(x/\lambda t)$, $j_a = j_a(x/vt)$, and $\left|p - \frac{x}{2t}\right|_a = \left[\left(p - \frac{x}{2t}\right)_a^2\right]^{1/2}$.

Proof. Since certain commutators of $\left|p - \frac{x}{2t}\right|_a$ are not well behaved, due to the singularity of $\sqrt{s}$ at $s=0$, we resort to the regularization of [23],

$$A_a(t) := \left(p - \frac{x}{2t}\right)_a^2 + t^{-2\beta} \tag{4.16}$$

for $\beta > 0$. We compute

$$D(f j_a A_a^{1/2} j_a f) = f D(j_a A_a^{1/2} j_a) f + (Df) j_a A_a^{1/2} j_a f + f j_a A_a^{1/2} j_a (Df), \tag{4.17}$$

and again we first prove that the terms arising from Df are integrable:

$$f j_a A_a^{1/2} j_a (Df) = f j_a A_a^{1/2} j_a (Df) g^2 = g f j_a A_a^{1/2} j_a (Df) g + [f j_a A_a^{1/2} j_a (Df), g] g,$$

where $g = g(x/\lambda t)$, with

$$E_\Omega [f j_a A_a^{1/2} j_a (Df), g] g E_\Omega = E_\Omega f j_a [A_a^{1/2}, g] j_a \underbrace{(Df) g E_\Omega}_{=O(t^{-1})}. \tag{4.18}$$

The estimate of $[A_a^{1/2}, g]$ is deferred after the conclusion of this proof [see (4.24)]. The result is (for $0 < \beta < 1$)

$$\|E_\Omega f j_a [A_a^{1/2}, g]\| = \|[A_a^{1/2}, g] j_a f E_\Omega\| \leqq O(t^{-(1-\beta)}) \|j_a f E_\Omega\|$$
$$+ O(t^{-1}) \underbrace{\|A_a^{1/2} j_a f E_\Omega\|}_{O(1)}$$
$$= O(t^{-(1-\beta)}). \tag{4.19}$$

Hence (4.18) is $O(t^{-(2-\beta)})$ and

$$\int_1^\infty |(\psi_t, E_\Omega f j_a A_a^{1/2} j_a (Df) E_\Omega \psi_t)|\, dt \leqq \text{const}\, \|\psi\|^2$$

can be shown by following the pattern (4.10). We now turn to the first term on the right-hand side of (4.17),

$$D(j_a A_a^{1/2} j_a) = j_a (D A_a^{1/2}) j_a + (D j_a) A_a^{1/2} j_a + j_a A_a^{1/2} (D j_a), \tag{4.20}$$

and we are going to prove the integrability of the contributions arising from

$$D j_a = (2/vt)(\nabla j_a)(x/vt) \cdot (p - x/2t) - i/(vt)^2 (\Delta j_a)(x/vt):$$

$$\begin{aligned}(D j_a) f E_\Omega &= \frac{2}{vt} \sum_{b \subset \boxed{a}} j_b^2 (\nabla_b j_a) \cdot \left(p - \frac{x}{2t}\right)_b f E_\Omega + O(t^{-2}) \\ &= \frac{2}{vt} \sum_{b \subset \boxed{a}} j_b (\nabla_b j_a) \cdot \left(p - \frac{x}{2t}\right)_b j_b f E_\Omega + O(t^{-2}),\end{aligned}$$

where we took $0 < v_0 < v$, $j_b = j_b(x/v_0 t)$, and used (3.12), (3.14) with $k = v/v_0$. We thus obtain

$$\begin{aligned}&|(\psi_t, E_\Omega f j_a A_a^{1/2} (D j_a) f E_\Omega \psi_t)| \\ &\leqq \frac{\text{const}}{t} \sum_{b \subset \boxed{a}} \|A_a^{1/2} j_a f E_\Omega \psi_t\| \left\| \left(p - \frac{x}{2t}\right)_b j_b f E_\Omega \psi_t \right\| + O(t^{-2}) \|\psi\|^2,\end{aligned}$$

since $A_a^{1/2} j_a f E_\Omega = O(1)$. The term in this sum is a product of two functions in $L^2([1,\infty), dt/t)$ by (4.11), (4.16), showing that

$$\int_1^\infty |(\psi_t, E_\Omega f j_a A_a^{1/2} (D j_a) f E_\Omega \psi_t)|\, dt \leqq \text{const}\, \|\psi\|^2 .$$

We are now left with the first term on the right-hand side of (4.20):

$$D A_a^{1/2} = D_a A_a^{1/2} + i[I_a, A_a^{1/2}], \tag{4.21}$$

where D_a is defined by (4.2) with H replaced by H_a. The contribution arising from the last term in (4.21), $E_\Omega f j_a [I_a, A_a^{1/2}] j_a f E_\Omega$, is integrable in time (for $2\beta < \mu$) by a computation deferred after the end of this proof [see (4.34)]. The first term on the right-hand side of (4.21) can be computed using $e^{itH_a}(D_a A_a^{1/2}) e^{-itH_a} = \frac{d}{dt}(e^{itH_a} A_a^{1/2} e^{-itH_a})$ and $e^{itH_a} A_a^{1/2} e^{-itH_a} = \left(\frac{x_a^2}{4t^2} + t^{-2\beta}\right)^{1/2}$. The result is

$$\begin{aligned}D_a A_a^{1/2} &= -\frac{1}{t} A_a^{-1/2} \left[\left(p - \frac{x}{2t}\right)_a^2 + \beta t^{-2\beta}\right] \\ &= -\frac{1}{t} A_a^{1/2} + (1-\beta) t^{-(1+2\beta)} A_a^{-1/2},\end{aligned} \tag{4.22}$$

where the contribution arising from the last term is integrable due to $\|A_a^{-1/2}\| \leqq t^\beta$.

Collecting (4.17), (4.20), (4.21), (4.22), and knowing that all contributions, except the one arising from the first term on the right-hand side of (4.22), are integrable in time for $0<\beta<\min(\mu/2,1)$, we obtain

$$\int_1^\infty (\psi_t, fj_a A_a^{1/2} j_a f\psi_t)\frac{dt}{t} \leqq 2\sup_{t\geqq 1}|(\psi_t, E_\Omega fj_a A_a^{1/2} j_a f E_\Omega \psi_t)| + \text{const}\,\|\psi\|^2$$
$$\leqq \text{const}\,\|\psi\|^2. \tag{4.23}$$

This is equivalent to (4.15) by $\left|p-\frac{x}{2t}\right|_a \leqq A_a^{1/2} \leqq \left|p-\frac{x}{2t}\right|_a + t^{-\beta}$. □

The estimates (4.11) and (4.15) can be summarized by replacing $\left|p-\frac{x}{2t}\right|_a$ in (4.15) by $\left|p-\frac{x}{2t}\right|_a^\varepsilon$ with $1\leqq\varepsilon\leqq 2$. This can be extended to $0<\varepsilon\leqq 2$ by considering the propagation observable $fj_a A_a^{\varepsilon/2} j_a f$, and by using the techniques and the results obtained so far. However, we will not use this extension.

We now fill the technical gaps left open in the previous proof.

Lemma 4.5. i) *Let $h\in C^\infty(X)$ with bounded derivatives. Then, as $t\to\infty$,*

$$\|[A_a(t)^{1/2}, h(x/t)]\varphi\| \leqq O(t^{-(1-\beta)} + t^{-2(1-\beta)})\,\|\varphi\| + O(t^{-1})\,\|A_a(t)^{1/2}\varphi\|, \tag{4.24}$$

and for $0<\alpha<1/2$,

$$[A_a(t)^\alpha, h(x/t)] = O(t^{-(1-\beta)}). \tag{4.25}$$

ii) *Let $[A_a(t), B(t)]$ be bounded for fixed t. Then*

$$\|[A_a(t)^{1/2}, B(t)]\| \leqq O(t^{2\beta})\,\|[A_a(t), B(t)]\|. \tag{4.26}$$

iii)

$$[A_a(t)^{1/4}, p(H_a+i)^{-1}] = O(t^{-(1-\beta)}). \tag{4.27}$$

Proof. We drop the subscript a and use

$$A^\alpha\psi = C\int_0^\infty d\omega\, \omega^{\alpha-1}(\omega+A)^{-1}A\psi$$

for $\psi\in D(A)$ and $0<\alpha\leqq 1/2$, where $C^{-1}=\int_0^\infty dx x^{\alpha-1}(x+1)^{-1}$. Then

$$[A^\alpha, B] = C\int_0^\infty d\omega\,\omega^{\alpha-1}[(\omega+A)^{-1}A, B]$$
$$= C\int_0^\infty d\omega\,\omega^{\alpha-1}\omega(\omega+A)^{-1}[A,B](\omega+A)^{-1}, \tag{4.28}$$

since $(\omega+A)^{-1}A = 1-\omega(\omega+A)^{-1}$. Notice that

$$\int_0^\infty d\omega\,\omega^{\alpha-1}\underbrace{\|\omega(\omega+A)^{-1}\|}_{\leqq 1}\;\underbrace{\|(\omega+A)^{-1}\|}_{\leqq \min(\omega^{-1}, t^{2\beta})}$$
$$\leqq \int_0^1 d\omega\,\omega^{\alpha-1}t^{2\beta} + \int_1^\infty d\omega\,\omega^{\alpha-2} = O(t^{2\beta}), \tag{4.29}$$

which proves (4.26). Setting $B=h(x/t)$ in (4.28), and using

$$i[A,h]=\frac{2}{t}\nabla h\cdot\left(p-\frac{x}{2t}\right)-\frac{i}{t^2}\Delta h$$

(the subindex a is dropped on the right-hand side too), we obtain

$$\|[A^\alpha,h]\varphi\|\leqq\frac{\text{const}}{t}\int_0^\infty d\omega\,\omega^{\alpha-1}\|\omega(\omega+A)^{-1}\|\,\|A^{1/2}(\omega+A)^{-1}\varphi\|$$

$$+\frac{\text{const}}{t^2}\int_0^\infty d\omega\,\omega^{\alpha-1}\|\omega(\omega+A)^{-1}\|\,\|(\omega+A)^{-1}\|\,\|\varphi\|,\qquad(4.30)$$

where we used $\left\|\left(p-\frac{x}{2t}\right)\psi\right\|\leqq\|A^{1/2}\psi\|$. The last term is $O(t^{-(2-2\beta)})$ by (4.29), while for the first one we use $\|A^{1/2}(\omega+A)^{-1}\|\leqq\|(\omega+A)^{-1/2}\|\leqq t^\beta$, as well as

$$\|A^{1/2-\varepsilon}(\omega+A)^{-1}A^\varepsilon\varphi\|\leqq\|(\omega+A)^{-(1/2+\varepsilon)}A^\varepsilon\varphi\|\leqq\omega^{-(1/2+\varepsilon)}\|A^\varepsilon\varphi\|$$

for any $\varepsilon\leqq 1/2$. This leads to the estimate

$$\|[A^\alpha,h]\varphi\|\leqq\frac{\text{const}}{t}\left[\int_0^1 d\omega\,\omega^{\alpha-1}t^\beta\|\varphi\|+\int_1^\infty d\omega\,\omega^{\alpha-1/2}\omega^{-(1+\varepsilon)}\|A^\varepsilon\varphi\|\right]$$

$$+O(t^{-2(1-\beta)})\,\|\varphi\|\,.$$

If $\alpha=1/2$ the second integral is finite for $\varepsilon=1/2$; if $\alpha<1/2$ it is finite even for $\varepsilon=0$. In the latter case we can use the symmetric expression for $i[A,h]$, which leaves us without the $O(t^{-2(1-\beta)})$ term. We are now left with iii), and we first show

$$[A^{1/4},p]=[A^{1/4},p_a]=O(t^{-(1-\beta)})\,.\qquad(4.31)$$

This follows from $i[A,p_a]=i\left[\left(p-\frac{x}{2t}\right)_a^2,p_a\right]=\frac{1}{t}\left(p-\frac{x}{2t}\right)_a$ by setting $B=p_a$ in (4.28), which is then estimated like the first term on the right-hand side of (4.30), i.e. by (4.31). Thus

$$[A^{1/4},p(H_a+i)^{-1}]=[A^{1/4},p]\,(H_a+i)^{-1}+p[A^{1/4},(H_a+i)^{-1}]=O(t^{-(1-\beta)})$$

by (4.31) and because of

$$\begin{aligned}p[A^{1/4},(H_a+i)^{-1}]&=-p(H_a+i)^{-1}\,[A^{1/4},H_a]\,(H_a+i)^{-1}\\&=-p(H_a+i)^{-1}\,[A^{1/4},p^2]\,(H_a+i)^{-1}\\&=-p(H_a+i)^{-1}\,[A^{1/4},p]\,p(H_a+i)^{-1}-p(H_a+i)^{-1}\,p[A^{1/4},p]\,(H_a+i)^{-1}\\&=O(t^{-(1-\beta)})\,.\quad\square\end{aligned}$$

Lemma 4.6.

$$\tilde{j}_aI_a(p^2+1)^{-1}=O(t^{-\mu_1})\,,\qquad(4.32)$$

$$\tilde{j}_a(\nabla I_a)(p^2+1)^{-1}=O(t^{-(1+\mu_2)})\,,\qquad(4.33)$$

where $\tilde{j}_a=\tilde{j}_a(x/vt)$, $v>0$.

Proof. For $(ij)\not\subset a$ we have

$$\tilde{j}_a V_{ij}(x^{(ij)})(p^2+1)^{-1}$$
$$=\tilde{j}_a F(|x^{(ij)}|\geq \text{const}\, t)\, V_{ij}(x^{(ij)})((p^{(ij)})^2+1)^{-1}((p^{(ij)})^2+1)(p^2+1)^{-1}=O(t^{-\mu_1})$$

by (3.6), (1.1). The proof of (4.33) is similar. □

Lemma 4.7.

$$(p^2+1)^{-1} f j_a [I_a, A_a^{1/2}] j_a f (p^2+1)^{-1}=O(t^{-(1+\mu-2\beta)}). \tag{4.34}$$

Proof. (See Theorem 4.4 for notation.) By

$$\begin{aligned}[(p^2+1)^{-1} f j_a I_a j_a f (p^2+1)^{-1}, A_a^{1/2}] &= (p^2+1)^{-1} f j_a [I_a, A_a^{1/2}] j_a f (p^2+1)^{-1}\\ &\quad + [(p^2+1)^{-1} f j_a, A_a^{1/2}]\, I_a j_a f (p^2+1)^{-1}\\ &\quad + (p^2+1)^{-1} f j_a I_a [j_a f (p^2+1)^{-1}, A_a^{1/2}],\end{aligned} \tag{4.35}$$

we have to show that the left-hand side as well as the last two terms on the right-hand side are $O(t^{-(1+\mu-2\beta)})$. We compute

$$[(p^2+1)^{-1} f j_a, A_a]=(p^2+1)^{-1}[f j_a, A_a]+[(p^2+1)^{-1}, A_a] f j_a,$$

$$i[f j_a, A_a]=-\frac{2}{t}\left(p-\frac{x}{2t}\right)\cdot \nabla_a(f(\cdot/\lambda) j_a(\cdot/v))-\frac{i}{t^2}\Delta_a(f(\cdot/\lambda) j_a(\cdot/v)),$$

$$i[(p^2+1)^{-1}, A_a]=\frac{1}{t}(p^2+1)^{-1}\left(\left(p-\frac{x}{2t}\right)\cdot p_a+p_a\cdot\left(p-\frac{x}{2t}\right)\right)(p^2+1)^{-1},$$

proving $[(p^2+1)^{-1} f j_a, A_a]=O(t^{-1})$. Moreover

$$(p^2+1)^{-1} f j_a i[I_a, A_a] j_a f (p^2+1)^{-1}$$
$$=-(p^2+1)^{-1} f j_a\left(\left(p-\frac{x}{2t}\right)\cdot \nabla I_a+\nabla I_a\cdot\left(p-\frac{x}{2t}\right)\right) j_a f (p^2+1)^{-1}=O(t^{-(1+\mu_2)}),$$

by (4.33). Using also (4.32), it follows from (4.35) with $A_a^{1/2}$ replaced by A_a that

$$[(p^2+1)^{-1} f j_a I_a j_a f (p^2+1)^{-1}, A_a]=O(t^{-(1+\mu)}).$$

Hence, applying (4.26) to (4.35), we obtain (4.34). □

We conclude this section by some commutator estimates, the last of them has already been used in the proof of Theorem 4.1:

Lemma 4.8. *Let $\chi\in C_0^\infty(\mathbb{R})$ or $\chi(x)=(x+i)^{-1}$, and $h\in C^\infty(X)$ with bounded derivatives. Then*

$$\tilde{j}_a\chi(H)-\chi(H_a)\tilde{j}_a=O(t^{-\mu'}), \tag{4.36}$$

$$j_a\chi(H)-\chi(H_a) j_a=O(t^{-\mu'}), \tag{4.37}$$

$$h\chi(H)-\chi(H)h=O(t^{-1}), \tag{4.38}$$

where $\mu'=\min(\mu_1, 1)$, $\tilde{j}_a=\tilde{j}_a(x/vt)$, and $h=h(x/t)$.

Proof. We only prove (4.36).

$$\tilde{j}_a(H+i)^{-1}-(H_a+i)^{-1}\tilde{j}_a=(H_a+i)^{-1}(H_a\tilde{j}_a-\tilde{j}_a H)(H+i)^{-1}=O(t^{-\mu'})$$

by $(\tilde{j}_a H - H_a \tilde{j}_a)(H+i)^{-1} = ([\tilde{j}_a, p^2] + \tilde{j}_a I_a)(H+i)^{-1} = O(t^{-\mu'})$, due to (4.32). For $\chi \in C_0^\infty(\mathbb{R})$ we set $\chi_1(x) = \chi(x)(x+i) \in C_0^\infty(\mathbb{R})$, and compute

$$
\begin{aligned}
&(\tilde{j}_a \chi_1(H) - \chi_1(H_a)\tilde{j}_a)(H+i)^{-1} \\
&= \int_{-\infty}^{\infty} dr \hat{\chi}_1(r) e^{-irH_a}(e^{irH_a}\tilde{j}_a e^{-irH} - \tilde{j}_a)(H+i)^{-1} \\
&= (-i) \int_{-\infty}^{\infty} dr \hat{\chi}_1(r) \int_0^r ds e^{i(s-r)H_a}(\tilde{j}_a H - H_a \tilde{j}_a)(H+i)^{-1} e^{-isH} \\
&= O(t^{-\mu'}).
\end{aligned}
$$

Then

$$
\begin{aligned}
\tilde{j}_a \chi(H) - \chi(H_a)\tilde{j}_a &= (\tilde{j}_a \chi_1(H) - \chi_1(H_a)\tilde{j}_a)(H+i)^{-1} \\
&\quad + \chi_1(H_a)(\tilde{j}_a (H+i)^{-1} - (H_a+i)^{-1}\tilde{j}_a) = O(t^{-\mu'}). \quad \square
\end{aligned}
$$

5. Existence of Deift-Simon Wave Operators

We now focus on short range potentials, i.e. we assume in addition to the hypothesis of the preceding section, that (1.1) holds with $\mu_1 > 1$. In this section we will establish the existence of the Deift-Simon wave operators for velocity $v > 0$, defined as

$$W_a := s - \lim_{t \to +\infty} e^{itH_a} \tilde{j}_a \left(\frac{x}{vt}\right) e^{-itH}, \tag{5.1}$$

where a is any cluster decomposition [see (3.2) for the definition of $\tilde{j}_a$]. We caution the reader that this result alone does not imply asymptotic completeness, since (unlike in [4, 22]) a is running over all cluster decompositions, including the trivial one $(1 \ldots N)$. We start by proving that the convergence above occurs when suitable cutoffs are added.

Lemma 5.1. *Let $v > 0$, $\Omega, \Omega' \subset \mathbb{R}$ bounded and measurable, and $\lambda > 0$ large enough. Then*

$$s - \lim_{t \to +\infty} e^{itH_a} E_{\Omega'}(H_a) f \tilde{j}_a f E_\Omega(H) e^{-itH} \tag{5.2}$$

exists, where $f = f(x/\lambda t)$, and $\tilde{j}_a = \tilde{j}_a(x/vt)$.

Proof. We set $E_\Omega = E_\Omega(H)$, $E_{\Omega'} = E_{\Omega'}(H_a)$ and we denote by $\tilde{W}_a(t)$ the operator whose limit is taken in (5.2). Since we are going to use Cook's method, we compute

$$
\begin{aligned}
\frac{d\tilde{W}_a}{dt} &= e^{itH_a} E_{\Omega'} \left[i(H_a f \tilde{j}_a f - f \tilde{j}_a f H) + \frac{\partial}{\partial t}(f \tilde{j}_a f) \right] E_\Omega e^{-itH} \\
&= e^{itH_a} E_{\Omega'} D(f \tilde{j}_a f) E_\Omega e^{-itH} - i e^{itH_a} E_{\Omega'} f \tilde{j}_a f I_a E_\Omega e^{-itH},
\end{aligned} \tag{5.3}
$$

where the last term is integrable in norm by (4.32) (this is the only place where we use $\mu_1 > 1$). We now turn to the first term on the right-hand side of (5.3)

$$D(f \tilde{j}_a f) = f(D\tilde{j}_a) f + (Df) \tilde{j}_a f + f \tilde{j}_a (Df), \tag{5.4}$$

and consider the contributions arising from Df:

$$E_{\Omega'} f\tilde{j}_a(Df)E_\Omega = E_{\Omega'} g f\tilde{j}_a(Df) g E_\Omega$$
$$= E_{\Omega'} g \underbrace{f\tilde{j}_a(Df)(H+i)^{-1}}_{=O(t^{-1})} g(H+i)E_\Omega + O(t^{-2}),$$

with $g = g(x/\lambda t)$ as given in the foregoing section. Then

$$|(\varphi, E_{\Omega'} f\tilde{j}_a(Df)E_\Omega \psi)| \leqq \frac{\text{const}}{t} \|gE_{\Omega'}\varphi\| \, \|g(H+i)E_\Omega\psi\| + O(t^{-2})\|\varphi\| \, \|\psi\|.$$

Hence, as $t_1, t_2 \to +\infty$,

$$\int_{t_1}^{t_2} |(\varphi, e^{itH_a} E_{\Omega'} f\tilde{j}_a(Df)E_\Omega e^{-itH}\psi)| \, dt$$
$$\leqq \text{const}\left[\int_1^\infty \|gE_{\Omega'} e^{-itH_a}\varphi\|^2 \frac{dt}{t}\right]^{1/2} \left[\int_{t_1}^{t_2} \|g(H+i)E_\Omega e^{-itH}\psi\|^2 \frac{dt}{t}\right]^{1/2} + o(1)\|\varphi\| \, \|\psi\|$$
$$\leqq \text{const}\|\varphi\| \cdot o(1) + o(1)\|\varphi\| \, \|\psi\| = o(1)\|\varphi\| \tag{5.5}$$

by (4.4), which applies to H_a as well. Equation (5.5) holds for fixed ψ uniformly in $\varphi \in L^2(X)$. We are now left with the first term on the right-hand side of (5.4).

$$f(D\tilde{j}_a)f = \frac{2}{vt} f(\nabla\tilde{j}_a) \cdot \left(p - \frac{x}{2t}\right) f + O(t^{-2})$$
$$= \frac{2}{vt} \sum_{b \subset \boxed{a}} f j_b^2(\nabla_b \tilde{j}_a) \cdot \left(p - \frac{x}{2t}\right)_b f + O(t^{-2})$$
$$= \frac{2}{vt} \sum_{b \subset \boxed{a}} f j_b(\nabla\tilde{j}_a) \cdot \left(p - \frac{x}{2t}\right)_b j_b f + O(t^{-2}),$$

where we took $0 < v_0 < v$, $j_b = j_b(x/v_0 t)$, and used (3.11), (3.13) with $k = v/v_0$. We write

$$(\nabla\tilde{j}_a) \cdot \left(p - \frac{x}{2t}\right)_b = A_b(t)^{1/4} B(t) A_b(t)^{1/4} \tag{5.6}$$

[see (4.16) for the definition of $A_b(t)$] with

$$B(t) = A_b(t)^{-1/4} (\nabla\tilde{j}_a) A_b(t)^{1/4} \cdot \underbrace{A_b(t)^{-1/4}\left(p - \frac{x}{2t}\right)_b A_b(t)^{-1/4}}_{=O(1)} = O(1),$$

since $A_b(t)^{-1/4}(\nabla\tilde{j}_a)A_b(t)^{1/4} = \nabla\tilde{j}_a + A_b(t)^{-1/4} O(t^{-(1-\beta)}) = O(1)$, where we used (4.25) (with $\alpha = 1/4$ and $\beta \leqq 2/3$), and $A_b(t)^{-1/4} = O(t^{\beta/2})$. This implies

$$|(\varphi, E_{\Omega'} f(D\tilde{j}_a) f E_\Omega \psi)|$$
$$\leqq \frac{\text{const}}{t} \sum_{b \subset \boxed{a}} \|A_b(t)^{1/4} j_b f E_{\Omega'}\varphi\| \, \|A_b(t)^{1/4} j_b f E_\Omega \psi\| + O(t^{-2})\|\varphi\| \, \|\psi\|,$$

from where we obtain as $t_1, t_2 \to +\infty$

$$\int_{t_1}^{t_2} |(\varphi, e^{itH_a} E_{\Omega'} f(D\tilde{j}_a) f E_\Omega e^{-itH}\psi)| \, dt = o(1)\|\varphi\|$$

by (4.23), with the uniformity of (5.5). We conclude that for given $\psi \in L^2(X)$

$$|(\varphi, (\tilde{W}_a(t_2) - \tilde{W}_a(t_1))\psi)| \leqq \int_{t_1}^{t_2} \left| \left(\varphi, \frac{d\tilde{W}_a}{dt}\psi \right) \right| dt = o(1)\, \|\varphi\|,$$

as $t_1, t_2 \to +\infty$, so we see that $\tilde{W}_a(t)\psi$ is Cauchy as $t \to +\infty$. □

Now the task is to remove the cutoffs from (5.2). We use a result of [13, 18]:

Lemma 5.2. *If* $\psi \in D(p) \cap D(x)$, *then* $e^{-itH}\psi \in D(p) \cap D(x)$, *and*

$$\|xe^{-itH}\psi\| \leqq \text{const}(1+|t|)(\|p\psi\| + \|x\psi\| + \|\psi\|). \tag{5.7}$$

Proof. By the relative boundedness of H and p^2 with respect to each other, we estimate $\sup_{t\in\mathbb{R}} \|p^2 e^{-itH}\psi\| \leqq \text{const}(\|p^2\psi\| + \|\psi\|)$ for $\psi \in D(p^2)$, and by interpolation,

$$\sup_{t\in\mathbb{R}} \|pe^{-itH}\psi\| \leqq \text{const}(\|p\psi\| + \|\psi\|) \tag{5.8}$$

for $\psi \in D(p)$. We regularize x by $u^\varepsilon(x) = x/(1+\varepsilon x^2)$. We have $u^\varepsilon_* \xrightarrow{s} 1$, $\Delta u^\varepsilon \xrightarrow{s} 0$ as $\varepsilon \searrow 0$, where u_* is the derivative of u. Let $\varphi, \psi \in D(p^2)$. Integrating $d(e^{-itH}\varphi, u^\varepsilon e^{-itH}\psi)/dt = (e^{-itH}\varphi, (2u^\varepsilon_* p - i\Delta u^\varepsilon)e^{-itH}\psi)$ we get

$$u^\varepsilon e^{-itH}\psi = e^{-itH}u^\varepsilon\psi + \int_0^t d\tau\, e^{-i(t-\tau)H}(2u^\varepsilon_* p - i\Delta u^\varepsilon)e^{-i\tau H}\psi,$$

since by the continuity of the integrand, the integral can be carried inside the scalar product. This then extends to $\psi \in D(p)$ by (5.8). Furthermore, the integrand is uniformly bounded in ε and τ. If in addition $\psi \in D(x)$, we obtain as $\varepsilon \searrow 0$

$$xe^{-itH}\psi = e^{-itH}x\psi + 2\int_0^t d\tau\, e^{-i(t-\tau)H}pe^{-i\tau H}\psi$$

by dominated convergence and by the closedness of x. From this (5.7) readily follows. □

Theorem 5.3. *For any* $v > 0$, *the limits* (5.1) *exist for all cluster decompositions* a.

Proof. We set $W_a(t) = e^{itH_a}\tilde{j}_a(x/vt)e^{-itH}$. It is enough to show that $W_a(t)\psi$ is Cauchy as $t \to +\infty$ for $\psi \in D(p) \cap D(x)$. Given $\varepsilon > 0$,

$$\begin{aligned}\sup_{t\geqq 1} \|(1-f(x/\lambda t)^2)e^{-itH}\psi\| &\leqq \sup_{t\geqq 1} \|F(|x/t| \geqq \lambda)e^{-itH}\psi\| \\ &\leqq \sup_{t\geqq 1} \lambda^{-1}\|(x/t)e^{-itH}\psi\| \\ &\leqq \text{const}\,\lambda^{-1}(\|p\psi\| + \|x\psi\| + \|\psi\|) \leqq \varepsilon \end{aligned} \tag{5.9}$$

for $\lambda > 0$ large enough. We also take a bounded measurable $\Omega \subset \mathbb{R}$ with

$$\|(1 - E_\Omega(H))\psi\| \leqq \varepsilon. \tag{5.10}$$

Finally, we take a bounded measurable $\Omega' \supset \Omega$ and a $\chi \in C_0^\infty(\mathbb{R})$ with $\chi(x) = 1$ for $x \in \Omega$, and $\chi(x) = 0$ for $x \notin \Omega'$. Then, setting $E_{\Omega'} = E_{\Omega'}(H_a)$, $E_\Omega = E_\Omega(H)$

$$\begin{aligned} &\|E_{\Omega'}W_a(t_2)\psi - E_{\Omega'}W_a(t_1)\psi\| \\ &\leqq \|E_{\Omega'}e^{it_2H_a}\tilde{j}_a f^2 e^{-it_2H}\psi - E_{\Omega'}e^{it_1H_a}\tilde{j}_a f^2 e^{-it_1H}\psi\| + 2\sup_{t\geqq 1}\|(1-f^2)e^{-itH}\psi\| \\ &\leqq \|e^{it_2H_a}E_{\Omega'}f\tilde{j}_a f E_\Omega e^{-it_2H}\psi - e^{it_1H_a}E_{\Omega'}f\tilde{j}_a f E_\Omega e^{-it_1H}\psi\| + 2\|(1-E_\Omega)\psi\| + 2\varepsilon \\ &\leqq \varepsilon + 2\varepsilon + 2\varepsilon = 5\varepsilon \end{aligned}$$

for t_1, t_2 large, by (5.9), (5.10), (5.2). Now (5.1) follows from this and from

$$\begin{aligned}&\|(1-E_{\Omega'})W_a(t_2)\psi-(1-E_{\Omega'})W_a(t_1)\psi\|\\&\leqq\|(1-\chi(H_a))W_a(t_2)E_\Omega\psi-(1-\chi(H_a))W_a(t_1)E_\Omega\psi\|+2\|(1-E_\Omega)\psi\|\\&\leqq\varepsilon+2\varepsilon=3\varepsilon\end{aligned}$$

for large t_1, t_2. This follows from (5.10) and from

$$(1-\chi(H_a))W_a(t)E_\Omega=e^{itH_a}(\tilde{J}_a\chi(H)-\chi(H_a)\tilde{J}_a)e^{-itH}E_\Omega\xrightarrow[t\to+\infty]{}0\,,$$

where we used $(1-\chi(H))E_\Omega=0$ and (4.36). □

Proof of Theorem 1.1. In the foregoing proofs in this section the roles of H and H_a are interchangeable. This implies the existence of

$$s-\lim_{t\to+\infty} e^{itH}\tilde{J}_a\left(\frac{x}{vt}\right)e^{-itH_a} \tag{5.11}$$

for any $v>0$. Next we notice that the finite linear combinations of eigenstates of H^a are dense in $\operatorname{Ran}P^a$. It is therefore enough to prove (1.6) assuming that P^a is the orthogonal projection on such an eigenstate: $H^aP^a=EP^a$. Moreover it suffices to prove the existence of the so modified limit on states in the dense set $D=\bigcup_{v>0} D_v$,

$$D_v=\operatorname{Ran}\prod_{f\supsetneqq \boxed{a}} F(|2p_a^f|\geqq v((q_a^f)^{1/2}+\sigma))\,.$$

We then remark

$$F(|x_a^f/t|\leqq v((q_a^f)^{1/2}+\sigma))e^{-it(p_a)^2}F(|2p_a^f|\geqq v((q_a^f)^{1/2}+\sigma))\xrightarrow{s}0\,,$$

$$F(|x_g^a/t|\geqq v((q_g^a)^{1/2}-\sigma))\xrightarrow{s}0$$

as $t\to+\infty$ for $f\supsetneqq a$, $g\subsetneqq a$, where the first limit is an immediate consequence of [19], Theorem IX.31. For $\psi\in D_v$, $v>0$ this leads to

$$\begin{aligned}&e^{-itH_a}P^a\psi\\&=\left[\prod_{f\supsetneqq\boxed{a}}F(|x_a^f/vt|\geqq(q_a^f)^{1/2}+\sigma)\right]\left[\prod_{g\subsetneqq\boxed{a}}F(|x_g^a/vt|\leqq(q_g^a)^{1/2}-\sigma)\right]e^{-itH_a}P^a\psi+o(1)\\&=\tilde{J}_a\left(\frac{x}{vt}\right)e^{-itH_a}P^a\psi+o(1)\end{aligned}$$

as $t\to+\infty$, where we used $e^{-itH_a}P^a=e^{-itE}e^{-it(p_a)^2}P^a$ and the commutation relations arising from (1.9), as well as $\tilde{J}_a=1$ on the set given by the characteristic function above. The existence of (5.11) then implies that of (1.6), from which in turn the remaining claims follow. □

6. Asymptotic Completeness

The whole analysis done up to now holds also at eigenvalue and at threshold energies. Also compactness has never been used. This will no longer be so in this section. We assume (1.1), (1.2), (1.4) and, as far as the proof of Theorem 1.2 is concerned, $\mu_1 > 1$. We recall that under these assumptions the Mourre estimate holds [2]. We will use it to prove

Theorem 6.1. *Let $\Omega \subset \mathbb{R}$ be a compact set, which does not contain any eigenvalues or thresholds. Then*

$$W_{(1 \dots N)} E_\Omega(H) = 0, \tag{6.1}$$

provided $v > 0$ in the definition (5.1) *of $W_{(1 \dots N)}$ is small enough.*

Proof. We remark that $\mu_1 > 1$ is not required for the existence of $W_{(1 \dots N)}$, as can be seen from (5.3) by $I_{(1 \dots N)} = 0$. We now drop the subscript $(1 \dots N)$. By a covering argument, it suffices to show that for any $E \in \mathbb{R}$, which is neither an eigenvalue nor a threshold of H, there is an open interval $\Delta \ni E$ such that

$$W E_\Delta(H) = 0 \tag{6.2}$$

for small enough $v > 0$. For such an $E \in \mathbb{R}$ the Mourre estimate reads

$$E_{\Delta_0}(H)\, i[H, A]\, E_{\Delta_0}(H) \geqq \theta E_{\Delta_0}(H) \tag{6.3}$$

for some $\theta > 0$ and some open interval $\Delta_0 \ni E$. Here $A = (p \cdot x + x \cdot p)/2$. We take $\Delta \ni E$ to be an open interval with $\bar{\Delta} \subset \Delta_0$, and compute

$$\frac{d}{dt}\left(\psi_t, E_\Delta \tilde{j} \frac{A}{t} \tilde{j} E_\Delta \psi_t\right) = \left(\psi_t, E_\Delta \left[\tilde{j}\left(D\frac{A}{t}\right)\tilde{j} + (D\tilde{j})\frac{A}{t}\tilde{j} + \tilde{j}\frac{A}{t}(D\tilde{j})\right] E_\Delta \psi_t\right), \tag{6.4}$$

where $\psi_t = e^{-itH}\psi$, $E_\Delta = E_\Delta(H)$, and $\tilde{j} = \tilde{j}_{(1 \dots N)}(x/vt)$. We want to prove that the contributions arising from $D\tilde{j}$ are integrable in time. Remark that as $t \to +\infty$,

$$\frac{A}{t}\tilde{j} E_\Delta = \frac{x \cdot p}{t}\tilde{j} E_\Delta + O(t^{-1}) = \frac{x}{t}\tilde{j} \cdot p E_\Delta + O(t^{-1}) = O(1),$$

since $\operatorname{supp}\tilde{j} \subset \Sigma^\sigma_{(1 \dots N)}$ is bounded,

$$\begin{aligned}
E_\Delta \tilde{j}\frac{A}{t}(D\tilde{j}) E_\Delta &= E_\Delta f \tilde{j}\frac{A}{t}(D\tilde{j}) f E_\Delta \\
&= \frac{2}{vt} E_\Delta f \tilde{j}\frac{A}{t}(\nabla\tilde{j}) \cdot \left(p - \frac{x}{2t}\right) f E_\Delta + O(t^{-2}) \\
&= \frac{2}{vt}\sum_b E_\Delta f \tilde{j}\frac{A}{t} j_b^2 (\nabla_b \tilde{j}) \cdot \left(p - \frac{x}{2t}\right)_b f E_\Delta + O(t^{-2}) \\
&= \frac{2}{vt}\sum_b E_\Delta f j_b p\left(\frac{x}{t} \otimes \tilde{j}\nabla\tilde{j}\right)\left(p - \frac{x}{2t}\right)_b j_b f E_\Delta + O(t^{-2}),
\end{aligned} \tag{6.5}$$

where we inserted $f = f(x/\lambda t)$ due to $f = 1$ on $\operatorname{supp}\tilde{j}(\cdot/vt)$ for $\lambda > 0$ large enough, we applied the by now familiar trick (3.11), (3.13) with $j_b = j_b(x/v_0 t)$, $0 < v_0 < v$, and we used $\tilde{j} A j_b = j_b A \tilde{j} + O(1)$. We further discuss the term in the sum (6.5). It is equal to

$$\begin{aligned}
&(H - i) E_\Delta f j_b (H_b - i)^{-1} p\left(\frac{x}{t} \otimes \tilde{j}\nabla\tilde{j}\right) \cdot \left(p - \frac{x}{2t}\right)_b j_b f E_\Delta + O(t^{-\mu'}) \\
&= (H - i) E_\Delta f j_b (H_b - i)^{-1} p A_b(t)^{1/4} O(1) A_b(t)^{1/4} j_b f E_\Delta + O(t^{-\mu'}) \\
&= (H - i) E_\Delta f j_b A_b(t)^{1/4} (H_b - i)^{-1} p\, O(1) A_b(t)^{1/4} j_b f E_\Delta + O(t^{-\mu''}),
\end{aligned}$$

where $\mu''=\min(\mu',1-\beta)>0$. Here we used (4.37), (4.38), we applied (5.6) with $\nabla\tilde{j}_a$ replaced by $(x/t)\otimes\tilde{j}\nabla\tilde{j}$, and finally we used (4.27). We then conclude by (4.23) that

$$\int_{t_1}^{t_2}\left|\left(\psi_t, E_\Delta\tilde{j}\frac{A}{t}(D\tilde{j})E_\Delta\psi_t\right)\right|dt \xrightarrow[t_1,t_2\to+\infty]{} 0$$

for given $\psi\in L^2(X)$. Next we consider

$$E_\Delta\tilde{j}\left(D\frac{A}{t}\right)\tilde{j}E_\Delta=\frac{1}{t}E_\Delta\tilde{j}\left(i[H,A]-\frac{A}{t}\right)\tilde{j}E_\Delta,$$

and

$$\begin{aligned}E_\Delta\tilde{j}i[H,A]\tilde{j}E_\Delta&=E_\Delta\tilde{j}\chi(H)\,i[H,A]\,\chi(H)\tilde{j}E_\Delta+O(t^{-1})\\&\geqq\theta E_\Delta\tilde{j}\chi(H)^2\tilde{j}E_\Delta+O(t^{-1})\\&=\theta E_\Delta\tilde{j}^2E_\Delta+O(t^{-1}),\end{aligned}$$

where we took $\chi\in C_0^\infty(\mathbb{R})$ with $\chi(x)=1$ for $x\in\Delta$, $\chi(x)=0$ for $x\notin\Delta_0$, and we used (4.38), (6.3). We also have

$$E_\Delta\tilde{j}\frac{A}{t}\tilde{j}E_\Delta\leqq\text{const}\,vE_\Delta\tilde{j}^2E_\Delta+O(t^{-1}),$$

where the constant is independent of $v>0$, since by the boundedness of $\operatorname{supp}\tilde{j}$

$$\begin{aligned}\left|\left(\varphi,E_\Delta\tilde{j}\frac{A}{t}\tilde{j}E_\Delta\varphi\right)\right|&\leqq\left|\left(\varphi,E_\Delta\tilde{j}p\cdot\frac{x}{t}\tilde{j}\chi(H)E_\Delta\varphi\right)\right|+O(t^{-1})\,\|\varphi\|^2\\&\leqq v\left|\left(\varphi,E_\Delta\tilde{j}p\chi(H)\cdot\frac{x}{vt}\tilde{j}E_\Delta\varphi\right)\right|+O(t^{-1})\|\varphi\|^2\\&\leqq\text{const}\,v\|\tilde{j}E_\Delta\varphi\|^2+O(t^{-1})\,\|\varphi\|^2.\end{aligned}\tag{6.6}$$

This implies

$$E_\Delta\tilde{j}\left(D\frac{A}{t}\right)\tilde{j}E_\Delta\geqq\frac{1}{t}(\theta-\text{const}\,v)E_\Delta\tilde{j}^2E_\Delta+O(t^{-2}),$$

where we take $v>0$ such that $\theta-\text{const}\,v=:\delta>0$. Integrating (6.4) from t_1 to $t_2>t_1$ we thus obtain

$$\left(\psi_t,E_\Delta\tilde{j}\frac{A}{t}\tilde{j}E_\Delta\psi_t\right)\Bigg|_{t_1}^{t_2}\geqq\delta\int_{t_1}^{t_2}\|\tilde{j}E_\Delta\psi_t\|^2\frac{dt}{t}+o(1)$$

as $t_1,t_2\to+\infty$. Given $\varepsilon>0$ we have $\|\tilde{j}E_\Delta\psi_t\|^2=\|e^{itH}\tilde{j}e^{-itH}E_\Delta\psi\|^2\geqq\|WE_\Delta\psi\|^2-\varepsilon$ for t large enough by (5.1). This, together with (6.6) shows

$$O(1)\geqq\delta\int_{t_1}^{t_2}(\|WE_\Delta\psi\|^2-\varepsilon)\frac{dt}{t}=\delta(\|WE_\Delta\psi\|^2-\varepsilon)\log\frac{t_2}{t_1}.$$

Taking $t_1=t$, $t_2=t^2$, this proves $\|WE_\Delta\psi\|^2\leqq\varepsilon$, and hence (6.2). □

We remark that one can easily prove (6.2) by using the minimal velocity bounds of [23, 24], which however require the additional hypothesis that $y\cdot\nabla(y\cdot\nabla V_{ij})(p^2+1)^{-1}$ is bounded on $L^2(X^{(ij)})$.

We are now ready for the

Proof of Theorem 1.2. The proof goes by induction, i.e. we assume that completeness holds for M bodies with $M<N$. Equation (1.7) trivially holds for $N=1$, since both sides are $\{0\}$. So let $N>1$. Given $\psi\in\operatorname{Ran}(1-P)$ and $\varepsilon>0$, we can find a compact set $\Omega\subset\mathbb{R}$ which does not contain any eigenvalues or thresholds, such that

$$\|(1-E_\Omega(H))\psi\|\leqq\varepsilon. \tag{6.7}$$

This follows from the fact that the set of eigenvalues and thresholds is countable and closed, which is a consequence of the Mourre estimate. We then take a $v>0$ such that (6.1) holds. Thus

$$\begin{aligned} e^{-itH}E_\Omega\psi &= \sum_a \tilde{j}_a\left(\frac{x}{vt}\right)e^{-itH}E_\Omega\psi \\ &= \sum_{\#(a)\geqq 2} e^{-itH_a}W_aE_\Omega\psi+o(1) \\ &= \sum_{\#(a)\geqq 2} e^{-itH_a}P^aW_aE_\Omega\psi+\sum_{\#(a)\geqq 2} e^{-itH_a}(1-P^a)W_aE_\Omega\psi+o(1) \end{aligned}$$

as $t\to+\infty$ by (3.4), (5.1), (6.1). We now apply the completeness result to H^C in (1.12), (1.11), with the result

$$\bigoplus_{b\subsetneqq \boxed{a}} \operatorname{Ran}\Omega(H_a,H_b)=\operatorname{Ran}(1-P^a),$$

where $\Omega(H_a,H_b)=s-\lim\limits_{t\to+\infty} e^{itH_a}e^{-itH_b}P^b$. Hence there are $\psi_b^a\in L^2(X)$ for $b\subsetneqq a$, such that

$$(1-P^a)W_aE_\Omega\psi=\sum_{b\subsetneqq \boxed{a}}\Omega(H_a,H_b)\psi_b^a,$$

which implies

$$e^{-itH_a}(1-P^a)W_aE_\Omega\psi=\sum_{b\subsetneqq \boxed{a}} e^{-itH_b}P^b\psi_b^a+o(1)$$

as $t\to+\infty$, and

$$E_\Omega\psi=\sum_{\#(a)\geqq 2}\Omega^aW_aE_\Omega\psi+\sum_{\#(a)\geqq 2}\sum_{b\subsetneqq \boxed{a}}\Omega^b\psi_b^a\in\bigoplus_{\#(a)\geqq 2}\operatorname{Ran}\Omega^a.$$

Since the latter is a closed subspace, $\psi\in\bigoplus_{\#(a)\geqq 2}\operatorname{Ran}\Omega^a$ follows by (6.7). □

Acknowledgements. I am indebted to W. Hunziker and to I. M. Sigal for their interest and for many remarks which helped to improve this work. I also thank E. Skibsted who provided the proof of Lemma 2.2 (a), in replacement of a more involved one.

References

1. Combes, J.M., Thomas, L.: Asymptotic behavior of eigenfunctions for multiparticle Schrödinger operators. Commun. Math. Phys. **34**, 251–270 (1973)
2. Cycon, H.L., Froese, R.G., Kirsch, W., Simon, B.: Schrödinger operators. Berlin, Heidelberg, New York: Springer 1987

3. Deift, P., Hunziker, W., Simon, B., Vock, E.: Pointwise bounds on eigenfunctions and wave packets in N-body quantum systems. IV. Commun. Math. Phys. **64**, 1–34 (1978)
4. Deift, P., Simon, B.: A time-dependent approach to the completeness of multiparticle quantum systems. Commun. Pure Appl. Math. **30**, 573–583 (1977)
5. Dereziński, J.: A new proof of the propagation theorem for N-body quantum systems. Commun. Math. Phys. **122**, 203–231 (1989)
6. Enss, V.: Asymptotic completeness for quantum-mechanical potential scattering. I. Short-range potentials. Commun. Math. Phys. **61**, 285–291 (1978)
7. Enss, V.: Asymptotic completeness for quantum-mechanical potential scattering. II. Singular and long-range potentials. Ann Phys. **119**, 117–132 (1979)
8. Enss, V.: Completeness of three-body quantum scattering. In: Dynamics and processes, Blanchard, P., Streit, L. (eds.). Lecture Notes in Mathematics, vol. 1031, pp. 62–88. Berlin, Heidelberg, New York: Springer 1983
9. Enss, V.: Introduction to asymptotic observables for multi-particle quantum scattering. In: Schrödinger operators, Aarhus 1985. Balslev, E. (ed.). Lecture Notes in Mathematics, vol. 1218, pp. 61–92. Berlin, Heidelberg, New York: Springer 1986
10. Froese, R.G., Herbst, I.: A new proof of the Mourre estimate. Duke Math. J. **49**, 1075–1085 (1982)
11. Graf, G.M.: Phase space analysis of the charge transfer model. Helv. Phys. Acta **63**, 107–138 (1990)
12. Hack, M.N.: Wave operators in multichannel scattering. Nuovo Cim. Ser. X **13**, 231–236 (1959)
13. Hunziker, W.: On the space-time behavior of Schrödinger wavefunctions. J. Math. Phys. **7**, 300–304 (1966)
14. Hunziker, W.: Time dependent scattering theory for singular potentials. Helv. Phys. Acta **40**, 1052–1062 (1967)
15. Jauch, J.M.: Theory of the scattering operator. I, II. Helv. Phys. Acta **31**, 127–158, 661–684 (1958)
16. Mourre, E.: Absence of singular continuous spectrum for certain selfadjoint operators. Commun. Math. Phys. **78**, 391–408 (1981)
17. Perry, P., Sigal, I.M., Simon, B.: Spectral analysis of N-body Schrödinger operators. Ann. Math. **114**, 519–567 (1981)
18. Radin, C., Simon, B.: Invariant domains for the time-dependent Schrödinger equation. J. Diff. Eq. **29**, 289–296 (1978)
19. Reed, M., Simon, B.: Methods of modern mathematical physics, Vol. I–IV. New York: Academic Press 1972–79
20. Sigal, I.M.: Scattering theory for Many-Body quantum mechanical systems. Lecture Notes in Mathematics, Vol. 1011. Berlin, Heidelberg, New York: Springer 1983
21. Sigal, I.M., Sigalov, A.G.: Description of the spectrum of the energy operator of quantum mechanical systems that is invariant with respect to permutations of identical particles. Theor. Math. Phys. **5**, 990–1005 (1970)
22. Sigal, I.M., Soffer, A.: The N-particle scattering problem: asymptotic completeness for short-range systems. Ann. Math. **126**, 35–108 (1987)
23. Sigal, I.M., Soffer, A.: Long-range many-body scattering. Asymptotic clustering for Coulomb-type potentials. Invent. Math. **99**, 115–143 (1990)
24. Signal, I.M., Soffer, A.: Local decay and propagation estimates for time-dependent and time-independent Hamiltonians, Princeton University preprint (1988), to appear in Acta Math.

Communicated by B. Simon

Received November 29, 1989; in revised form March 12, 1990