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Asymptotic Completeness for Quantum Mechanical Potential Scattering*

I. Short Range Potentials

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Abstract. A new (geometrical) proof is given for the asymptotic completeness of the wave operators and the absence of a singular continuous spectrum of the Hamiltonian for potentials which decrease faster than in the Coulomb case, the space dimension is arbitrary.

Introduction and Results

Quantum mechanical potential scattering is completely under control for most potentials of interest. As long as the potential vanishes at infinity fast enough to exclude the Coulomb potential the completeness of the ordinary wave operators can be proved using eigenfunction expansions (see e.g. [1] and references given therein) or other methods in special cases.

Instead of using these rather abstract methods we give a new proof for the completeness of the wave operators and the absence of a singular continuous spectrum in the Hamiltonian which follows the intuition of how a scattering particle behaves in space and time. That this "geometrical" approach to the completeness problem is the natural one was pointed out to me years ago by R. Haag. This point of view has also recently been advocated by Deift and Simon [8], Simon [7].

The main idea of our proof is as follows: a state from the continuous spectral subspace of the Hamiltonian is known [6] to leave in the time-mean any finite region of space. Such a far out localized state is decomposed into its outgoing components where $\mathbf{x} \cdot \mathbf{p}$ is positive (up to a tail) and the remaining incoming components. The outgoing components cannot interact any more, i.e. $(\Omega_{-} - \mathbb{1})$ is small on them, thus they lie in the range of the outgoing wave operator Ω_{-}. Similarly the incoming components cannot have interacted in the past. This is used to

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show that the incoming components become orthogonal to the whole state, consequently the latter cannot be orthogonal to the range of Ω_-.

We do not assume any symmetry for the potential nor shall we need information about the point spectrum of H.

We investigate the following system: The free time evolution of a particle of mass 1 is generated by the selfadjoint operator on $\mathscr{H} = L^2(\mathbb{R}^\nu)$,

$$H_0 = \tfrac{1}{2}\sum_1^\nu p_i^2. \tag{1}$$

The potential V is a bounded perturbation of H_0 with H_0—bound smaller than 1[3] (it may be velocity dependent):

$$\| V\Psi \| \leqq a \| H_0 \Psi \| + b \| \Psi \| \quad \forall \Psi \in \mathscr{D}(H_0), a < 1. \tag{2}$$

$H = H_0 + V$ is a selfadjoint operator with $\mathscr{D}(H) = \mathscr{D}(H_0) \subset \mathscr{D}(V)$. Let $F(\cdot)$ denote the projection onto L^2 of that subset of $\mathbf{x}$-space which is specified inside the round brackets. The falloff of the potential at infinity is fast enough such that

$$\| V(H_0 + \mathbb{1})^{-1} F(|\mathbf{x}| \geqq R) \| = : h(R) \in L^1([0, \infty), dR). \tag{3}$$

Wave operators are defined as usual [3]:

$$\Omega_\mp = \text{s} - \lim_{t \to \pm\infty} \exp(iHt) \exp(-iH_0 t). \tag{4}$$

Condition (3) avoids long range potentials for which only modified wave operators exist.

Theorem. *The range of $\Omega_\pm$ is the subspace of $\mathscr{H}$ corresponding to the continuous spectrum of H.*

As the range of $\Omega_\pm$ must be contained in the absolutely continuous spectral subspace for H, the singular continuous spectrum of H must be empty.

More singular potentials can be treated but we wish to avoid technicalities which might hide the simple ideas of proof.

Asymptotic Properties of States with Continuous Energy Spectrum

Ruelle [6], Amrein and Georgescu [2] showed that for a particle in a potential the bound states and the scattering states can be characterized "geometrically", namely by their behavior in space and time. Whenever the dynamics is such that $F(|\mathbf{x}| < R)(H + i\mathbb{1})^{-1}$ is a compact operator for all $R < \infty$ it was shown in [6] that any state Ψ from the continuous spectral subspace of the Hamiltonian will leave any finite region of space in the time mean:

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T d\tau \| F(|\mathbf{x}| < R) \exp(-iH\tau)\Psi \| = 0 \quad \forall R < \infty. \tag{5}$$

We will show for our more restricted class of interactions that this state is a scattering state in the traditional sense namely that it belongs to the range of the wave operators. ($F(|\mathbf{x}| < R)(H + i\mathbb{1})^{-1}$ is compact because $F(|\mathbf{x}| < R)(H_0 + i\mathbb{1})^{-1}$ is compact and $(H_0 + i\mathbb{1})(H + i\mathbb{1})^{-1}$ is bounded by (2).)

Let Ψ be a vector from the domain of H which belongs to the continuous spectral subspace of H. Setting $R=n$ it follows from (5) that for any n

$$\int_{-n}^{n} dt \| F(|\mathbf{x}|<n)\exp(-iH(t+\tau))(H+i\mathbb{1})\Psi \|$$

goes to 0 in the mean as a function of τ. Therefore we can find a sequence τ_n such that as $n \to \infty$

$$\| F(|\mathbf{x}|<13n)\exp(-iH\tau_n)\Psi \| \to 0, \tag{6}$$

$$\int_{-n}^{n} dt \| F(|\mathbf{x}|<n)\exp(-iH(t+\tau_n))(H+i\mathbb{1})\Psi \| \to 0. \tag{7}$$

$$\Psi_n := \exp(-iH\tau_n)\Psi \tag{8}$$

represents a sequence of states localized farther and farther away from the scatterer. Their kinetic energy distributions approximate their total energy distribution:

Lemma 1. *Let $\tilde{\varphi}$ be the Fourier transform of an $L^1(\mathbb{R}, dt)$-function, then*

$$\lim_{n\to\infty} \| (\tilde{\varphi}(H) - \tilde{\varphi}(H_0))\Psi_n \| = 0. \tag{9}$$

Remark. The lemma is true for potentials vanishing arbitrarily slowly at infinity.

Proof. The expression (9) equals

$$\left\| \int_{-\infty}^{\infty} dt\varphi(t)\{\exp(-iHt) - \exp(-iH_0t)\}\Psi_n \right\|$$

$$\leqq \int_{-n}^{n} dt|\varphi(t)| \, \| (\exp(iH_0t)\exp(-iHt) - \mathbb{1})\Psi_n \| + 2 \int_{|t|>n} dt|\varphi(t)|.$$

The first term is bounded by the L^1-norm of φ times

$$\sup_{|t|\leqq n} \| (\exp(iH_0t)\exp(-iHt) - \mathbb{1})\Psi_n \|$$

$$\leqq \int_{-n}^{n} dt \| V\exp(-iHt)\Psi_n \|$$

$$\leqq \int_{-n}^{n} dt\{ \| V(H+i\mathbb{1})^{-1} \| \, \| F(|\mathbf{x}|<n)\exp(-iHt)(H+i\mathbb{1})\Psi_n \|$$

$$+ \| V(H+i\mathbb{1})^{-1}F(|\mathbf{x}|>n) \| \, \| (H+i\mathbb{1})\Psi_n \| \}.$$

The first summand vanishes for $n \to \infty$ due to (7), the other one because $\| V(H+i\mathbb{1})^{-1}F(|\mathbf{x}|>n) \| < 2h(n)$ (see (3)) for large enough n and $n \cdot h(n) \to 0$.

Asymptotic Decomposition of States

From now on we will assume that Ψ is taken from the dense subset of the continuous spectral subspace for H, which contains those states whose energy is finite and bounded away from 0. For suitably chosen numbers $0<a<b<\infty$ the energy support of Ψ and Ψ_n is contained in the interval $[73a^2, (b-a)^2/4]$. Choose a function $\varphi \in \mathscr{D}(\mathbb{R})$ such that $\tilde{\varphi}(\omega) = 1$ for $73a^2 \leqq \omega \leqq (b-a)^2/4$ and $\tilde{\varphi}(\omega) = 0$ if $\omega < 72a^2$ or $\omega > (b-a)^2/2$, then $\tilde{\varphi}(H)\Psi_n = \Psi_n$.

$$\Phi_n := \tilde{\varphi}(H_0)\Psi_n \tag{10}$$

is an approximation of Ψ_n: $\|\Phi_n - \Psi_n\| \to 0$ by Lemma 1, which has momentum support in $12a \leqq |\mathbf{p}| \leqq b - a$. It is crucial for the estimates below that a and b can be chosen independently of n.

Let $\{B_n, C_{ni}\}$ be a disjoint decomposition of $\mathbb{R}^\nu$ into the ball $B_n = \{\mathbf{x} \in \mathbb{R}^\nu \mid |\mathbf{x}| \leqq n\}$ and finitely many truncated cones $C_{ni} \subset \{\mathbf{x} \in \mathbb{R}^\nu \mid |\mathbf{x}| > n, \mathbf{x}\cdot\mathbf{e}_i \geqq |\mathbf{x}|/2\}$ for a suitable set of unit vectors $\mathbf{e}_i$ which are independent of n. In addition to the projectors $F(\cdot)$ we shall need smooth multiplication operators in $\mathbf{x}$-space, $F_0(C)$, which are obtained by convoluting the characteristic function of $C \subset \mathbb{R}^\nu$ with a fixed function $\zeta \in \mathscr{S}(\mathbb{R}^\nu)$. Thereby $\operatorname{supp} \tilde{\zeta}(\mathbf{p}) \subset \{\mathbf{p} \in \mathbb{R}^\nu \mid |\mathbf{p}| \leqq a\}$, $\tilde{\zeta}(0) = 1$, and $\{F_0(B_n), F_0(C_{ni})\}$ is a resolution of the identity. The estimate (6) implies both $\|F_0(B_{12n})\Phi_n\| \to 0$, and

$$\|\Psi_n - \sum_i F_0(C_{12n,i})\Phi_n\| \to 0. \tag{11}$$

Each $F_0(C_{12n,i})\Phi_n$ which is essentially localized far away from the scatterer for large n, will be decomposed further into its "outgoing" and "incoming" components by splitting it in momentum space such that $\mathbf{p}\cdot\mathbf{x} \gtrsim 0$ or $\lesssim 0$ for $\mathbf{x} \in C_{ni}$. Let $\chi_i(\mathbf{p}) \in \mathscr{D}(\mathbb{R}^\nu)$ be such that $\chi_i(\mathbf{p}) = 0$ for $\mathbf{p}\cdot\mathbf{e}_i < -a$, and $\chi_i(\mathbf{p}) + \chi_i(-\mathbf{p}) = 1$ for $|\mathbf{p}| < b$. Then

$$\Phi_n = [\chi_i(\mathbf{p}) + \chi_i(-\mathbf{p})]\Phi_n \quad \forall i, n. \tag{12}$$

$$\Phi_n(i, \mathrm{out/in}) = F_0(C_{12n,i})\chi_i(\pm\mathbf{p})\Phi_n \tag{13}$$

are the outgoing/incoming components of Φ_n. Before we show that $\Phi_n(i, \mathrm{out/in})$ evolves almost freely in the future/past we have to prove a technical estimate on the space-time behavior of $\exp(-iH_0t)\Phi_n(i, \mathrm{out/in})$.

Localization of $\exp(-iH_0t)\Phi_n(i, \mathrm{out/in})$

We have constructed $\Phi_n(i, \mathrm{out})$ such that it is mainly localized far away from the scatterer and $\mathbf{p}$ points away from the origin. One expects that for later times the distance from the scatterer increases linearly in time, in fact the tail decreases faster than any inverse power of $|\mathbf{x}|$ and t. The following estimate will be used later.

Lemma 2.

$$\lim_{n\to\infty} \int_0^\infty dt \, \|F(|\mathbf{x}| \leqq n + at)\exp(-iH_0t)(H_0 + \mathbb{1})\Phi_n(i, \mathrm{out})\| = 0, \tag{14}$$

$$\lim_{n\to\infty} \int_{-\infty}^0 dt \, \|F(|\mathbf{x}| \leqq n - at)\exp(-iH_0t)(H_0 + \mathbb{1})\Phi_n(i, \mathrm{in})\| = 0 \tag{15}$$

for all $\Phi_n(i, \text{out/in})$ *as constructed above.*

Proof. The ball $|\mathbf{x}| \leqq n + at$ is contained in all half spaces $\mathbf{x}\cdot\mathbf{f} \leqq (n+at)$. Let us decompose $\Phi_n(i, \text{out})$ further and show that for each term the estimate analogous to (14) is true for a suitable half space. This reduces the problem to one dimension.

Split χ_i into finitely many summands; e.g. $\xi_j \in \mathscr{D}(\mathbb{R}^\nu)$, $\chi_i(\mathbf{p}) = \sum \xi_j(\mathbf{p})$. Simple trigonometry shows that the ξ_j can be chosen such that

$$\operatorname{supp} \xi_j(\mathbf{p}) \subset \{\mathbf{p} \in \mathbb{R}^\nu \,|\, \mathbf{p}\cdot\mathbf{f}_j \geqq 2a\} \tag{16}$$

for a suitably chosen set of unit vectors $\mathbf{f}_j$ which make with $\mathbf{e}_i$ an angle of $20°$ ($\sin 10° > 1/6$). Furthermore $\mathbf{x}\cdot\mathbf{f}_j > 2n$ $\forall \mathbf{x} \in C_{12n,i}$; $\forall j$. The lemma follows if we show that for all i, j:

$$\lim_{n\to\infty} \int_0^\infty dt \,\| F(\mathbf{x}\cdot\mathbf{f}_j < n + at)\exp(-iH_0 t)(H_0 + \mathbb{1})F_0(C_{12n,i})\xi_j(\mathbf{p})\Phi_n \| = 0 . \tag{17}$$

To simplify notation pick a coordinate system such that the x_1-axis is parallel to the $\mathbf{f}_j$ under consideration. With $F_m = (H_0 + \mathbb{1})F_0(C_{12n,i} \cap \{2n + m \leqq x_1 \leqq 2n + m + 1\})$ we have a further decomposition which obeys

$$(H_0 + \mathbb{1})F_0(C_{12n,i}) = \sum_0^\infty F_m . \tag{18}$$

For any set $A : F_0(A)\Phi_n$ has momentum support in $|\mathbf{p}| \leqq b$, therefore $(H_0 + \mathbb{1})$ can be implemented by multiplication in $\mathbf{p}$-space with a fixed function from $\mathscr{D}(\mathbb{R}^\nu)$, consequently the localization is not seriously extended. A shift to the left by $2n + m$ along the x_1-axis transforms $F_m \xi_j(\mathbf{p})\Phi_n$ into a state well localized near $x_1 = 0$, the momentum support is contained in $a \leqq p_1 \leqq b$. Denote by $\varphi(\mathbf{x})$ the wave function of any such state. Then $\exp[-i(p_2^2 + \ldots + p_\nu^2)/2]$ commutes with $F(x_1 < n + at)$, and so drops out of (17).

Next we modify and extend a well known asymptotic expression for $(\exp(-iH_0 t)\varphi)(\mathbf{x})$, see e.g. Theorem IX.31 and its proof in [5].

$$\begin{aligned}(W_t\varphi)(\mathbf{x}) := {} & (2\pi i t)^{-1/2}\exp(ix_1^2/2t) \\ & \cdot \int (1 + iy_1^2/2t + (iy_1^2/2t)^2/2)\exp(-ix_1 y_1/t)\varphi(\mathbf{y}) .\end{aligned} \tag{19}$$

The support of $\tilde{\varphi}(\mathbf{p})$ implies that $(W_t\varphi)(\mathbf{x}) = 0$ if $x_1 < at$. An analogous calculation as in [5] and Taylor's formula yield

$$\begin{aligned}&\|(1+t)(-x_1 + 2at)^2[(\exp(-ip_1^2 t/2)\varphi)(\mathbf{x}) - (W_t\varphi)(\mathbf{x})]\| \\ &= \|(1+t)t^2(-i\partial/\partial y_1 + 2a)^2\{[\exp(iy_1^2/2t) - (1 + iy_1^2/2t + (iy_1^2/2t)^2/2]\varphi(\mathbf{y})\}\| \\ &\leqq \|P(|y_1|)[|\varphi(\mathbf{y})| + |\partial/\partial y_1 \varphi(\mathbf{y})| + |\partial^2/\partial y_1^2 \varphi(\mathbf{y})|]\| ,\end{aligned} \tag{20}$$

where $P(\cdot)$ is a polynomial of 6th order with coefficients depending on a only. Due to the momentum restrictions of $\tilde{\varphi}(\mathbf{p})$ and the good localization near $x_1 = 0$ which is uniform for all normalized φ under consideration, a straightforward estimate shows that (20) is bounded by a constant A depending only on a, b, ζ and the function implementing $(H_0 + \mathbb{1})$. The remark following (19) and (20) imply for $z > 0$

$$\| F(x_1 < -z + at)\exp(-iH_0 t)\varphi \| \leqq A[(1+t)(z+at)^2]^{-1}. \tag{21}$$

Replacing φ by $F_m \xi_j(\mathbf{P})\Phi_n$ and $z = n + m$ in (21):

$$\begin{aligned} &\| F(x_1 < n + at)\exp(-iH_0 t)(H_0 + \mathbb{1})F_0(C_{12n,i})\xi_j(\mathbf{P})\Phi_n \| \\ &\leqq A \sum_{m=0}^{\infty} [(1+t)(m+n+at)^2]^{-1} \leqq A[(1+t)(n-1+at)]^{-1}. \end{aligned} \tag{22}$$

Thus (17) holds and (14) follows. The same proof with a few signs reversed gives (15). ■

Completeness of $\Omega_{\mp}$

We first estimate the interaction on asymptotic states.

Lemma 3.

$$\lim_{n\to\infty} \| (\Omega_- - \mathbb{1})\Phi_n(i, \text{out}) \| = 0, \tag{23}$$

$$\lim_{n\to\infty} \| (\Omega_+ - \mathbb{1})\Phi_n(i, \text{in}) \| = 0. \tag{24}$$

Proof.

$$\begin{aligned} &\| (\Omega_- - \mathbb{1})\Phi_n(i, \text{out}) \| \leqq \int_0^\infty dt \| V \exp(-iH_0 t)\Phi_n(i, \text{out}) \| \\ &\leqq \| V(H_0 + \mathbb{1})^{-1} \| \int_0^\infty dt \| F(|x| \leqq n + at)\exp(-iH_0 t)(H_0 + \mathbb{1})\Phi_n(i, \text{out}) \| \\ &+ \| (H_0 + \mathbb{1})\Phi_n(i, \text{out}) \| \int_0^\infty dt \| V(H_0 + \mathbb{1})^{-1} F(|\mathbf{x}| \geqq n + at) \|. \end{aligned}$$

The first term vanishes by Lemma 2, the second by (3). For (24) analogously. ■

The existence of $\Omega_{\mp}$ on $\mathscr{H}$ is well known and easy to prove, e.g. [4]. Also it follows immediately from our Lemma 3 and the fact that for t_n large enough $\exp(\mp iH_0 t_n)\Phi$ is of the type $\Phi_n(i, \text{out/in})$ for a total set of states Φ.

Property (24) implies that $\Phi_n(i, \text{in})$ and Ψ_n (see (8)) become orthogonal:

$$\begin{aligned} &|(\Phi_n(i, \text{in}), \Psi_n)| \\ &\leqq \| (\mathbb{1} - \Omega_+)\Phi_n(i, \text{in}) \| + |(\exp(iH_0 \tau_n)\Phi_n(i, \text{in}), \Omega_+^* \Psi)|, \end{aligned} \tag{25}$$

The last summand is bounded by

$$\| F(|\mathbf{x}| \leqq n + a\tau_n)\exp(iH_0 \tau_n)\Phi_n(i, \text{in}) \| + \| F(|\mathbf{x}| \geqq n + a\tau_n)\Omega_+^* \Psi \|.$$

The second term obviously decreases, and similarly for the first from the estimates leading to (15) in Lemma 2.

Now assume that Ψ (and every Ψ_n) is a state with continuous energy spectrum orthogonal to the range of Ω_-. Substracting $\sum_i \Phi_n(i, \text{in})$ from Ψ_n we obtain (by (11, 12)) asymptotically $\sum_i \Phi_n(i, \text{out})$ which lies in the range of Ω_- by (23). As $\| \sum_i \Phi_n(i, \text{in}) \|$ is bounded uniformly in n this is a contradiction. Interchanging the roles of $\Phi_n(\text{in})$ and $\Phi_n(\text{out})$ one shows that Ψ must lie in the range of Ω_+ as well. This concludes the proof of our theorem.

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