Asymptotic curvature of Θ-metrics

Matthew B. Stenzel
Ohio State University, Newark Campus
1179 University Drive
Newark, OH 43055
stenzel@math.ohio-state.edu

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Abstract

We give an explicit and reasonably simple expression for the curvature tensor of a Θ-metric at boundary points, in terms of the metric tensor and invariants of the Θ-structure. We examine the behavior of the induced metric on level sets of a defining function near the boundary and describe the asymptotic behavior of its curvature tensor. Some applications of these results are given.

Key words: Θ-*metrics,* Θ-*curvatrue*
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1 Introduction

A Θ-metric on a compact manifold-with-boundary can be thought of as a Riemann metric on the interior having a certain type of singular behavior at the boundary. The singularity is measured by pairing the metric with an algebra of smooth vector fields, $\mathcal{V}_\Theta$, vanishing at the boundary in a way determined by a linear form Θ. This class of metrics, and the associated vector bundles which they are sections of, was introduced by Epstein, Melrose and Mendoza [5] to study the resolvent of the Laplacian of "Bergman-type" metrics on strictly pseudoconvex domains. They showed that a Kähler metric of the form $g = \partial\overline{\partial}\log\phi$, where ϕ is a strictly plurisuperharmonic defining function, lifts to a Θ-metric on the "square root" of the domain. For example the Cheng-Yau [4] and Bergman metrics lift to Θ-metrics.

It is well known that the curvature tensor of these Bergman-type metrics approaches a curvature tensor of constant holomorphic sectional curvature near the boundary [7], [4]. The main result of this paper is an expression for the dominant term near the boundary in the curvature tensor of an arbitrary Θ-metric, in terms of invariants attached to the Θ-structure at the boundary (theorem 5.1). We do not assume that there is any complex structure present, or if there is, that it has any particular relationship with the metric. Our results generalize those of Klembeck [7] and, we think, set them in the proper context. An advantage of our approach is that we are able to determine the structure of the curvature tensor of the induced metric on level sets of a defining function, in terms of the invariants mentioned above. As an application we show that for Bergman-type metrics, the curvature of the induced metric on level sets approaches the curvature tensor of a Berger sphere metric at the boundary, exactly as in the case of the complex hyperbolic ball.

We briefly summarize the contents and methods of this paper. In section 2 we introduce the basic ideas about Θ-structures. Let X be a smooth, compact manifold-with-boundary and let $\Theta \in \mathrm{C}^{\infty}(\partial X, T^*X)$ be a one-form which is non-vanishing when pulled back to ∂X. Θ determines a Lie algebra of vector fields, $\mathcal{V}_{\Theta}$, which lifts to the space of all smooth sections of a vector bundle ${}^{\Theta}TX$. At boundary points, the Lie algebra structure of $\mathcal{V}_{\Theta}$ lifts to a Lie algebra structure on the fiber, ${}^{\Theta}T_pX$. Sections of the dual bundle, ${}^{\Theta}T^*X$, correspond to sections of T^*X^o becoming singular in the appropriate way at the boundary. The usual exterior derivative lifts to an exterior derivative on sections of ${}^{\Theta}T^*X$, and at boundary points its action is determined by the Lie algebra structure on the fibers (lemma 2.1). This "freezing of coefficients" simplifies curvature computations at the boundary.

In section 3 we consider the familiar concepts of differential geometry in the Θ-setting. A C^k Θ-metric is a positive definite C^k section of $\otimes^2\, {}^{\Theta}T^*X$. A Θ-metric determines (and is determined by) a Riemann metric on X^o having a certain singular behavior at the boundary. We show that the Levi-Cevita connection of this Riemann metric lifts to a "Θ-connection" on ${}^{\Theta}TX$, and that its curvature lifts to a well-defined section of $\otimes^4\, {}^{\Theta}T^*X$. At boundary points, the lifted curvature tensor depends only on the metric tensor and the Lie algebra structure on the fibers ${}^{\Theta}T_pX$. In section 4 we define the invariants $\Omega \in C^k(\partial X, {}^{\Theta}\Lambda^2 X)$, $\xi \in C^k(\partial X, \mathrm{End}\,({}^{\Theta}TX))$, β^N and $\beta^n \in C^k(\partial X, {}^{\Theta}T^*X)$ that we need to express the Θ-curvature tensor at boundary points. These depend only on the metric tensor and the Θ-structure at boundary points. We then determine the Θ-connection matrix at boundary points, in terms of a frame compatible with β^n and β^N, by solving the Θ-analog of the first structure equation (lemma 4.1 and proposition 4.1).

In section 5 we prove our main result, theorem 5.1, which gives the $(0,4)$-curvature tensor and Ricci tensor of a Θ-metric at boundary points in terms of Ω, ξ, β^n, β^N and g. We show that the curvature tensor is Einstein at the boundary if and only if $\xi^2 = -4F^2 Id$ (where $F = |\beta^n|$). In section 5.1 we consider the Θ-analog of the induced metric on level sets of a defining function, and find the $(0,4)$ and Ricci curvature tensors. We show that if $\xi^2 = -4F^2 Id$, then a component of the covariant derivative of the Ricci tensor (with respect to its decomposition into irreducible components under the action of the orthogonal group) of the induced metric becomes small in norm near the boundary (proposition 5.1).

Section 6 gives some applications of these results. Section 6.1 recovers the well-known facts about Bergman-type metrics on strictly pseudoconvex domains, and gives analogous results for the induced metric on level sets of a defining function (corollaries 6.1 and 6.2). In section 6.2 we recover Klembeck's result on the Bergman metric. In section 6.3 we show that any manifold whose boundary is a contact manifold admits a complete "asymptotically Einstein" metric on its interior, in the sense that its Ricci tensor approaches, in the norm defined by the metric, a constant negative multiple of the metric near the boundary (theorem 6.1).

Our convention for wedge products is that if α, β are covectors, then $\alpha \wedge \beta = \frac{1}{2}(\alpha \otimes \beta - \beta \otimes \alpha)$.

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2 Θ-Structure

In this section we will review some facts about Θ-structures. For more details see [5]. Let X be a smooth, compact, N-dimensional ($N > 2$) manifold-with-boundary, let $\iota : \partial X \to X$ be the inclusion map, and let $\rho \geq 0$ be a smooth defining function for X.

A Θ-*structure* on X is a projective class of one-forms Θ,

$$\Theta \in C^{\infty}(\partial X; T^*X),$$

such that $\iota^*\Theta$ is non-vanishing. A necessary (and sufficient of ∂X is connected) condition for X to admit a Θ-structure is that the Euler characteristic of ∂X is zero (in particular if $\dim \partial X$ is odd then X always admits a Θ-structure). Associated with a Θ-structure is a subspace of $C^\infty(X,TX)$,

$$\mathcal{V}_\Theta := \left\{ V \in \rho C^\infty(X;TX) : \tilde{\Theta}(V) \in \rho^2 C^\infty(X) \right\},$$

where $\tilde{\Theta}$ is any extension of Θ to a smooth one-form on X. The space $\mathcal{V}_\Theta$ is a $C^\infty(X)$-module and a Lie algebra, independent of the choice of ρ, representative Θ, and extension $\tilde{\Theta}$. A *convenient frame* (for a given choice of ρ, $\tilde{\Theta}$) is a local frame $Y_1, \ldots, Y_N$ for TX such that, writing $N = 2n$ if N is even or $N = 2n+1$ otherwise,

$$d\rho(Y_N) = \tilde{\Theta}(Y_n) = 0, \qquad d\rho(Y_n) = \tilde{\Theta}(Y_N) = 1$$

and

$$Y_i \in \ker d\rho \cap \ker \tilde{\Theta} \quad \text{if } i \neq n, N.$$

A vector field V is in $\mathcal{V}_\Theta$ if and only if it can be written in terms of a convenient frame as

$$V = \rho b^n Y_n + \sum_{i \neq n,\, N} \rho b^i Y_i + \rho^2 b^N Y_N$$

for some smooth functions $b^1, \ldots, b^N$. For $p \in X$, let $\mathcal{I}_p$ be the subspace of $\mathcal{V}_\Theta$ consisting of those elements whose coefficients $b^1, \ldots, b^N$ all vanish at p for some (hence any) convenient frame. Let

$${}^\Theta TX := \bigsqcup_{p \in X} {}^\Theta T_p X, \quad \text{where } {}^\Theta T_p X := \mathcal{V}_\Theta \bmod \mathcal{I}_p.$$

${}^\Theta TX$ is a smooth vector bundle over X, with local trivialization given by the coefficients b^i. There is a natural "evaluation" map, $\iota_\Theta : {}^\Theta TX \longrightarrow TX$, obtained by choosing a representative element of $\mathcal{V}_\Theta$ and evaluating at p; it is an isomorphism over X^o and the zero map over ∂X. $\mathcal{V}_\Theta$ lifts to the space of all smooth section of ${}^\Theta TX$ in the sense that for every $V \in \mathcal{V}_\Theta$ there corresponds a unique $\widetilde{V} \in C^\infty(X, {}^\Theta TX)$ such that $\iota_\Theta \circ \widetilde{V} = V$. Of course, $\widetilde{V}(p)$ is simply $V \bmod \mathcal{I}_p$, which we will denote by $[V]_p$. The Lie algebra structure on $\mathcal{V}_\Theta$ lifts to a Lie bracket on sections of ${}^\Theta TX$, satisfying $\iota \circ [V,W] = [\iota \circ V, \iota \circ W]$.

When $p \in \partial X$, $\mathcal{I}_p$ is an ideal in the Lie algebra $\mathcal{V}_\Theta$. Thus the quotient, ${}^\Theta T_p X$, inherits a Lie algebra structure. For $p \in \partial X$, let

$$\begin{aligned} K_{1,p} &= \{V \in \mathcal{V}_\Theta : V = \rho W,\ W \in C^\infty(X;TX) \text{ and tangent to } \partial X\} \bmod \mathcal{I}_p \\ K_{2,p} &= \left\{V \in \mathcal{V}_\Theta : V = \rho^2 W,\ W \in C^\infty(X;TX)\right\} \bmod \mathcal{I}_p \end{aligned}$$

and

$$K_1 = \bigsqcup_{p \in \partial X} K_{1,p}, \quad K_2 = \bigsqcup_{p \in \partial X} K_{2,p}.$$

In terms of a convenient frame,

$$\begin{aligned} K_{1,p} &= \text{span } \left\{ [\rho Y_i]_p,\ i \neq n, N;\ [\rho^2 Y_N]_p \right\} \\ K_{2,p} &= \text{span } \left\{ [\rho^2 Y_N]_p \right\}. \end{aligned}$$

This shows that K_1 and K_2 are sub-bundles of ${}^\Theta TX$ over ∂X. K_2 is a trivial line bundle (see the remark following definition 4.1).

A calculation shows that

$$\left[{}^{\Theta}T_pX, {}^{\Theta}T_pX\right] = K_{1,p} \quad \text{and} \quad [K_{1,p}, K_{1,p}] \subseteq K_{2,p}$$

with equality holding if $K_{1,p}$ is not commutative. As a Lie algebra, ${}^{\Theta}T_pX$ is determined by its dimension and the dimension of the center of $K_{1,p}$ (which is the co-rank of $d\iota^*\Theta$).

Let ${}^{\Theta}T^*X$ be the dual bundle to ${}^{\Theta}TX$. If $Y_1, \ldots, Y_N$ is a convenient frame near $p \in \partial X$ and $\alpha^1, \ldots, \alpha^N$ is the dual frame, then $\iota_\Theta^*\rho^{-1}\alpha^1, \ldots, \iota_\Theta^*\rho^{-1}\alpha^{N-1}, \iota_\Theta^*\rho^{-2}\alpha^N$ extend smoothly up to the boundary as local sections of ${}^{\Theta}T^*X$ to give a local frame for ${}^{\Theta}T^*X$ near p (here ι_Θ^* is the transpose of the vector bundle map ι_Θ). Let ${}^{\Theta}\Lambda^m X$ be the m^{th} exterior power of ${}^{\Theta}T^*X$. Since the Lie algebra structure on $\mathcal{V}_\Theta$ lifts to a Lie bracket on sections of ${}^{\Theta}T^*X$, we can define an exterior derivative operator on sections of ${}^{\Theta}\Lambda^m X$ by the usual formula: for $p \in X$, $V_1, \ldots, V_{m+1} \in \mathcal{V}_\Theta$, $\beta \in C^1(X; {}^{\Theta}\Lambda^m X)$,

$$\begin{aligned}(d\beta)_p\left([V_1]_p, \ldots, [V_{m+1}]_p\right) \;=\; & \tfrac{1}{(m+1)!}\Big\{ \textstyle\sum_{i=1}^{m+1}(-1)^{i+1}V_i \cdot \Big(\beta(V_1, \ldots, \widehat{V_i}, \ldots, V_{m+1})\Big) \\ & + \textstyle\sum_{i<j}(-1)^{i+j}\beta([V_i, V_j], V_1, \ldots, \widehat{V_i}, \ldots, \widehat{V_j}, \ldots, V_{m+1})\Big\}\Big|_p .\end{aligned} \tag{1}$$

The value at p is independent of the choice of representatives V_i for the usual reasons at interior points, and at boundary points because $\mathcal{I}_p$ is then an ideal in $\mathcal{V}_\Theta$. Since ι_Θ preseves Lie bracket, we have $d\,\iota_\Theta^* = \iota_\Theta^* d$.

The following simple observation will be useful.

Lemma 2.1 *Let $\beta \in C^l(X; {}^{\Theta}\Lambda^m X)$ ($l > 0$) and $p \in \partial X$. Then $d\beta_p$ depends only on the value of β at p and the Lie algebra structure of ${}^{\Theta}T_pX$. If f is a C^1 function near p, then $d(f\beta)_p = f(p)d\beta_p$.*

Proof. The first sum in equation 1 vanishes at p because $V_i \in \rho C^\infty(X, TX)$. The second sum depends only on the value of β and $[V_i, V_j]$ at p. The latter is equal to the Lie bracket of the Θ-tangent vectors $[V_i]_p$, $[V_j]_p$ and is determined by the Lie algebra structure of ${}^{\Theta}T_pX$. The second statement follows in the same way. //

3 Θ-Geometry

Since some of the motivating examples of a Θ-metric (such as the Bergman metric of a strictly pseudoconvex domain (lifted to its square root), see section 6.2) are not smooth up to the boundary, we will work in the "C^k category" ($k \geq 2$). If ρ is a C^{k+1} defining function for X, then there is a positive function $f \in C^k(X)$ and a smooth defining function $\tilde{\rho}$ such that $\rho = f\tilde{\rho}$. Thus for example if T is a C^k section of TX, then $\rho^2 T$ is a C^k section of ${}^{\Theta}TX$. Equation 1 shows that d maps $C^k(X, {}^{\Theta}\Lambda^m X)$ into $C^{k-1}(X, {}^{\Theta}\Lambda^{m+1} X)$.

Definition 3.1 *A C^k Θ-metric on X is a symmetric, positive definite, C^k section of ${}^{\Theta}T^*X \otimes {}^{\Theta}T^*X$.*

Since ι_Θ canonically identifies ${}^{\Theta}TX$ and TX over X^o, we can think of a Θ-metric as a Riemann metric on X^o with a certain non-isotropic singularity at the boundary. We will briefly develop the idea of Θ-geometry, guided by the treatment of b-geometry in [8]. Let E, F be vector bundles over X. A Θ-*differential operator* of order m from sections of E to sections of F is a linear operator which can be expressed in terms of local trivializations of E and F as a polynomial of degree at most m in elements of $\mathcal{V}_\Theta$.

Definition 3.2 *A Θ-connection on a vector bundle E over X is a first order Θ-differential operator*

$${}^{\Theta}\nabla : C^k(X; E) \longrightarrow C^{k-1}(X; {}^{\Theta}T^*X \otimes E),$$

satisfying

$$^{\Theta}\nabla(f\mu) = \iota_{\Theta}^{*} df \otimes \mu + f\,^{\Theta}\nabla\mu$$

for all $f \in C^k(X)$*,* $\mu \in C^k(X;E)$*.*

Let g be a Θ-metric on X. The Levi-Cevita connection of $\left(X^o, (\iota_{\Theta}^{-1})^* g\right)$ lifts to a connection on $^{\Theta}TX|_{X^o}$. Pulling back by ι_{Θ}^*, we obtain an operator from $C^k(X^o, {}^{\Theta}TX)$ to $C^{k-1}(X^o, {}^{\Theta}T^*X \otimes {}^{\Theta}TX)$, which we denote by $^{\Theta}\nabla$.

Lemma 3.1 $^{\Theta}\nabla$ *extends to a* Θ*-connection on* $^{\Theta}TX$*.*

Proof. Let $\beta_1, \ldots, \beta_N$ be an orthonormal basis of sections of $^{\Theta}TX$ near $p \in \partial X$ and $\beta^1, \ldots, \beta^N$ the dual basis. We can define a matrix θ^i_j of C^{k-1} local sections of $^{\Theta}T^*X$ by solving the analog of the first structure equation,

$$\begin{array}{lcll} d\beta^i & = & -\sum_k \theta^i_k \wedge \beta^k, & \qquad i = 1, \ldots, N, \\ 0 & = & \theta^i_j + \theta^j_i. & \end{array} \tag{2}$$

We obtain

$$\theta^i_j = \sum -\frac{1}{2}(A^i_{kj} - A^k_{ji} + A^j_{ik})\beta^k$$

where the A^i_{jk} are defined by

$$d\beta^i = \sum_{j<k} A^i_{jk}\beta^j \wedge \beta^k$$

(and the A^i_{kj} are defined for $k \geq j$ by requiring that $A^i_{kj} = -A^i_{jk}$). Since ι_{Θ}^* is an isomorphism over the interior and commutes with d, the θ^i_j restricted to X^o are the lift of the connection one-forms associated with $(\iota_{\Theta}^{-1})^* g$ and the frame $\iota_{\Theta} \circ \beta_i$. Therefore $^{\Theta}\nabla\beta_i = \sum \theta^k_i \otimes \beta_k$ over X^o. If $\mu = \sum \mu^k \beta_k$ is local section of $^{\Theta}TX$ near $p \in \partial X$, then over the interior we have $^{\Theta}\nabla\mu = \sum_k \left(\iota_{\Theta}^* d\mu^k + \sum_j \mu^j \theta^k_j\right) \otimes \beta_k$. This extends to a C^{k-1} section near p, and shows that $^{\Theta}\nabla$ extends to a map $^{\Theta}\nabla : C^k(X, {}^{\Theta}TX) \to C^{k-1}(X, {}^{\Theta}T^*X \otimes {}^{\Theta}TX)$. It follows easily that $^{\Theta}\nabla$ is a Θ-connection. //

We can define a Θ-analog of the $(0,4)$-curvature tensor, $R^{\Theta} \in C^{k-2}(X, {}^{\Theta}\Lambda^2 X \otimes {}^{\Theta}\Lambda^2 X)$, by

$$\frac{1}{2}\sum_{j,k} R^i_{ljk}\beta^j \wedge \beta^k = d\theta^i_l + \sum_m \theta^i_m \wedge \theta^m_l \tag{3}$$

where we have written $R^{\Theta} = \sum R^i_{ljk}\beta^l \otimes \beta^j \otimes \beta^k \otimes \beta_i$. Over X^o, R^{Θ} is the lift of the curvature of $\left(X^o, (\iota_{\Theta}^{-1})^* g\right)$. It follows from lemma 2.1 and equations 2, 3 that when $p \in \partial X$, R^{Θ}_p depends only on the value of g at p and the Lie algebra structure, i.e., the Θ-structure, of $^{\Theta}T_pX$.

4 The Structure equations for a Θ-metric

A Θ-metric determines the following natural orthogonal decomposition of $^{\Theta}T_pX$ at the boundary:

$$^{\Theta}T_pX = K^{\perp}_{1,p} \oplus K_{1,p} \cap K^{\perp}_{2,p} \oplus K_{2,p}. \tag{4}$$

We will need an orthonormal Θ-frame adapted to this decomposition.

Definition 4.1 *A Θ-structure is oriented by a vector field T tangent to ∂X if $\Theta(T) > 0$ (for some representative of the projective class of one-forms determining the Θ-structure).*

We always can find such a vector field by choosing a Riemann metric on ∂X and identifying $\iota^*\Theta$ with a vector field. An orientation determines a natural trivialization of K_2 in the following way. Extend T to a vector field on X (still denoted by T). Since $\Theta(T) > 0$, $[\rho^2 T] \neq 0$ near ∂X. The unit length section corresponding to $[\rho^2 T]$ trivializes K_2 over ∂X, and depends only on the Θ-metric and the oriented Θ-structure. From now on we will fix ρ and assume that $d\rho(T) = 0$ and $\left|[\rho^2 T]\right| = 1$ near ∂X. An oriented Θ-structure and a Θ-metric determine a two-form, $\Omega \in C^k(\partial X; {}^{\Theta}\Lambda^2 X)$, by

$$\Omega_p(v_p, w_p)[\rho^2 T]_p := \mathrm{pr}_2([v_p, w_p]), \quad p \in \partial X,$$

where pr_2 is the orthogonal projection onto K_2 and the bracket is the pointwise Lie bracket of Θ-tangent vectors. We also obtain a skew-symmetric endomorphism, $\xi \in C^k(\partial X; \mathrm{End}\,({}^{\Theta}TX))$, by

$$g(v_p, \xi w_p) := \Omega_p(v_p, w_p).$$

These invariants will be used in our expression for the Θ-curvature tensor at boundary points.

We observe that $\iota_\Theta^* \rho^{-1} d\rho$ is a *canonical* section of ${}^{\Theta}T^*X$ over ∂X, independent of the choice of defining function (proof: if $\rho = f\tilde{\rho}$, then $\iota_\Theta^* \rho^{-1} d\rho = \iota_\Theta^* \tilde{\rho}^{-1} d\tilde{\rho} + \iota_\Theta^* f^{-1} df$, with $\iota_\Theta^* f^{-1} df \in \iota_\Theta^* C^k(X, \Lambda^1 X) \subset \rho C^k(X, {}^{\Theta}\Lambda^1 X)$). Denote by β^n the unit section corresponding to $\iota_\Theta^* \rho^{-1} d\rho$, and let β^N be the section of ${}^{\Theta}T^*X$ dual to $[\rho^2 T]$ by the metric. Since $d\rho(T) = 0$, β^n, β^N are orthogonal unit sections of ${}^{\Theta}T^*X$ in a neighborhood of ∂X. Over the boundary they depend only on the metric and oriented Θ-structure. We can now write the orthogonal decomposition 4 as

$${}^{\Theta}T_p X = (\ker \beta^n)^\perp \oplus \ker \beta^n \cap \ker \beta^N \oplus (\ker \beta^N)^\perp.$$

Fix a point $p_o \in \partial X$ and extend to a local orthonormal frame $\beta^1, \ldots, \beta^N$ for ${}^{\Theta}T^*X$ near p_o.

Lemma 4.1 *At boundary points near p_o,*

$$d\beta^N = -\frac{1}{2}\Omega \qquad \text{and} \qquad d\beta^i = -F\beta^n \wedge \beta^i \quad \text{if } i \neq N.$$

Here $F := |\iota_\Theta^* \rho^{-1} d\rho|$ depends only on the metric and Θ-structure over ∂X.

Proof. If $p \in \partial X$ and $v_p = [V]_p$, $w_p = [W]_p$ are in ${}^{\Theta}T_p X$, then

$$d(\beta^N)_p(v_p, w_p) = \frac{1}{2}\left\{V(g(\rho^2 T, W)) - W(g(\rho^2 T, V)) - g(\rho^2 T, [V, W])\right\}\Big|_p.$$

Since $\rho^2 T$, V, and W are C^k sections of ${}^{\Theta}TX$, the contraction with g is C^k up to the boundary. Since V, W vanish at the boundary,

$$-2d(\beta^N)_p(v_p, w_p) = g([\rho^2 T]_p, [v_p, w_p])_p.$$

This recovers the $K_{2,p}$ component of $[v_p, w_p]$ and shows that $-2d\beta^N = \Omega$.

Let $Y_N := T$ and extend to a C^k convenient frame for TX near p_o (with respect to ρ and some extension $\tilde{\Theta}$ such that $\tilde{\Theta}(T) = 1$). Let $\alpha^1, \ldots, \alpha^N$ be the dual frame. Then

$$\beta^i = \sum_{i \neq N} a^i_j \iota_\Theta^* \rho^{-1} \alpha^j + a^i_N \iota_\Theta^* \rho^{-2} \alpha^N$$

with $a^i_N = \beta^i([\rho^2 T]) = g(\beta^i, \beta^N) = \delta^i_N$. Therefore if $i \neq N$, we can write $\beta^i = \iota^*_\Theta \rho^{-1}\gamma^i$ where γ^i is C^k up to the boundary and

$$d\beta^i = -F\beta^n \wedge \beta^i + \iota^*_\Theta \rho^{-1} d\gamma^i.$$

Since $\iota^*_\Theta C^k(X, \Lambda^2 X) \subset \rho^2 C^k(X, {}^\Theta\Lambda^2 X)$, we have at boundary points $d\beta^i = -F\beta^n \wedge \beta^N$ as sections of ${}^\Theta\Lambda^2 X$. //

As in lemma 3.1, define local sections θ^i_j of ${}^\Theta T^* X$ satisfying the analog of the first structure equation by ${}^\Theta\nabla\beta_j = \sum \theta^i_j \otimes \beta_i$. Let $2\theta = \sum_{i,j=1}^N \theta^i_j \otimes \eta^j_i$, where η^j_i is the skew-symmetric matrix with 1 in row i, column j, -1 in row j, column i, and zero elsewhere. We write the first structure equation as

$$\sum_k \theta \wedge (\beta^k \otimes \beta_k) = \sum_{k=1}^{N-1} F\beta^n \wedge \beta^k \otimes \beta_k + \frac{1}{2}\Omega \otimes \beta_N. \tag{5}$$

Here the wedge product is

$$(\alpha \otimes \eta^j_i) \wedge (\gamma \otimes \beta_k) = \alpha \wedge \gamma \otimes \eta^j_i \beta_k$$

where $\eta^j_i \in so(N)$ acts on the frame $\beta_1, \ldots, \beta_N$ in the natural way: $\eta^j_i \beta_k = \delta_{kj}\beta_i - \delta_{ki}\beta_j$. Let ξ act on ${}^\Theta T^* X|_{\partial X}$ via the metric, so that $\xi\beta(v) = -\beta(\xi v)$.

Proposition 4.1 *The unique solution to equation 5 at boundary points near p_o is*

$$\theta = \frac{1}{2}\beta^N \otimes \xi - \frac{1}{2}\sum_{i=1}^N \xi\beta^i \otimes \eta^N_i - F\sum_{i=1}^{N-1} \beta^i \otimes \eta^n_i.$$

Proof. We have

$$\begin{aligned}\theta \wedge \left(\sum_{k=1}^N \beta^k \otimes \beta_k\right) &= \frac{1}{2}\sum_{k=1}^N \beta^N \wedge \beta^k \otimes \xi\beta_k + \frac{1}{2}\sum_{k=1}^N \beta^N \wedge \xi\beta^k \otimes \beta_k \\ &\quad + \frac{1}{2}\sum_{k=1}^N \xi\beta^k \wedge \beta^k \otimes \beta_N - F\sum_{k=1}^{N-1} \beta^k \wedge \beta^n \otimes \beta_k.\end{aligned}$$

Since $\sum_{k=1}^N \beta^k \otimes \xi\beta_k = -\sum_{k=1}^N \xi\beta^k \otimes \beta_k$,

$$\theta \wedge \left(\sum_{k=1}^N \beta^k \otimes \beta_k\right) = \frac{1}{2}\sum_{k=1}^N \xi\beta^k \wedge \beta^k \otimes \beta_N - F\sum_{k=1}^N \beta^k \wedge \beta^n \otimes \beta_k.$$

In this basis and with our wedge convention, $\Omega = \sum_{k=1}^N \xi\beta^k \wedge \beta^k$. The uniqueness follows in the usual way. //

5 The curvature tensor

Before we state the main result of this section we fix some notation. Given $(0,2)$-tensors h, k, we define a $(0,4)$-tensor $h \owedge k$ by

$$h \owedge k(x,y,z,w) = h(x,z)k(y,w) + h(y,w)k(x,z) - h(x,w)k(y,z) - h(y,z)k(x,w).$$

If h, k are both symmetric, then $h \owedge k$ is an algebraic curvature tensor; if ω is antisymmetric, then $4\omega \otimes \omega + \omega \owedge \omega$ is a curvature tensor (see [2], 1.110, and [3], §3). We write the symmetric product of tensors as $\alpha \circ \beta$, with the convention that $\alpha \circ \alpha = \alpha \otimes \alpha$ for $(0,1)$ tensors. If A is a $(1,1)$-tensor and h is a $(0,2)$-tensor, we let Ah denote the $(0,2)$-tensor defined by $Ah(V,W) = h(AV,W)$. Note that since ξ is skew-symmetric, $\Omega = -\xi g$ and $\xi^2 g$ is a symmetric $(0,2)$-tensor.

We will show below that at boundary points, $\xi\beta^N = 2F\beta^n$. It is not in general true that $\xi\beta^n = -2F\beta^N$ at boundary points. Put $\nu = \xi\beta^n + 2F\beta^N$. Then ν is defined on a neighborhood of ∂X, and over ∂X depends only on g and the oriented Θ-structure (see section 4). The main result of this section is the following expression for the curvature of a Θ-metric at the boundary in terms of g and invariants of the Θ-structure.

Theorem 5.1 *Let g be a C^k Θ-metric on X, $k \geq 2$. Then at boundary points the Θ-curvature tensor of g is*

$$-\frac{1}{8}\left(4F^2 g \owedge g + 4\Omega \otimes \Omega + \Omega \owedge \Omega\right) + Fg \owedge \beta^N \circ \nu - \frac{1}{4}\left(\xi^2 + 4F^2\right) g \owedge \beta^N \circ \beta^N.$$

The Ricci Θ-curvature tensor is

$$\frac{1}{2}\left(\xi^2 - 2F^2 N\right) g + F(N-1)\beta^N \circ \nu - \frac{1}{4}\mathrm{tr}\,\left(\xi^2 + 4F^2\right)\beta^N \circ \beta^N.$$

Proof. We first prove a technical lemma. Define ξ^i_j by $\xi\beta_j = \sum_i \xi^i_j \beta_i$, and note that $\xi\beta^j = -\sum_i \xi^j_i \beta^i$.

Lemma 5.1

1. *At boundary points, $\xi\beta_N = 2F\beta_n$, and $\xi\beta^N = 2F\beta^n$.*
2. *$[\eta^j_i, \eta^l_k] = \delta_{jk}\eta^l_i + \delta_{jl}\eta^i_k + \delta_{ik}\eta^j_l + \delta_{il}\eta^k_j$.*

Proof. Since $d\rho(Y_l) = \delta_{ln}$, we have $[\beta_l, \beta_N] = 2F\delta_{ln}\beta_N + \rho\mathcal{V}_\Theta$. Therefore at boundary points, $g(\beta_l, \xi\beta_N) = 2F\delta_{ln}$, and $\xi^l_N = 2F\delta_{ln}$. Since the ξ^i_j are skew-symmetric in i, j, $\xi\beta^N = 2F\beta^n$. The proof of 2 is a straightforward computation. //

Fix $p_o \in \partial X$ and choose a local orthonormal Θ-coframe $\beta^1, \ldots, \beta^N$ as in lemma 4.1. Using lemma 5.1.2, the second structure equation 3 can be written

$$\frac{1}{2}R^\Theta = d\theta + \theta \wedge \theta$$

where $R^\Theta = \sum_{i<l}\left(\sum_{j,k} R^i_{ljk}\right) \otimes \eta^l_i$. Using lemmas 2.1, 4.1, and 5.1.1, we have at boundary points

$$d\left(\xi\beta^i\right) = F\sum_{l=1}^{N-1} \xi^i_l \beta^n \wedge \beta^l + F\delta_{in}\Omega = -F\beta^n \wedge \xi\beta^i + F\delta_{in}\left(\Omega - 2F\beta^n \wedge \beta^N\right).$$

Therefore by proposition 4.1 we have, at boundary points,

$$d\theta = -\frac{1}{4}\Omega \otimes \left(\xi + 2F\eta^N_n\right) + \frac{F}{2}\sum_{i=1}^N \beta^n \wedge \xi\beta^i \otimes \eta^N_i + F^2\sum_{i=1}^N \beta^n \wedge \beta^i \otimes \eta^n_i + 2F^2\beta^n \wedge \beta^N \otimes \eta^N_n.$$

Let $A_k = \sum_{i=1}^N \xi_k^i \eta_i^N - 2F\eta_k^n$, $B = \xi - 2F\eta_n^N$. Then $2\theta = \beta^N \otimes B + \sum_{k=1}^N \beta^k \otimes A_k$, and

$$\begin{aligned}\theta \wedge \theta &= \frac{1}{4}\sum_{k=1}^N \beta^N \wedge \beta^k \otimes [B, A_k] + \frac{1}{8}\sum_{k,l=1}^N \beta^k \wedge \beta^l \otimes [A_k, A_l] \\ &= \frac{1}{4}\sum_{k=1}^N \beta^N \wedge \beta^k \otimes [B, A_k] + \frac{1}{8}\sum_{i,k=1}^N \xi\beta^i \wedge \xi\beta^k \otimes [\eta_i^N, \eta_k^N] \\ &\quad + \frac{F}{2}\sum_{i,k=1}^N \xi\beta^i \wedge \beta^k \otimes [\eta_i^N, \eta_k^n] + \frac{F^2}{2}\sum_{i,k=1}^N \beta^i \wedge \beta^k \otimes [\eta_i^n, \eta_k^n].\end{aligned}$$

Using lemma 5.1, and $\Omega = \sum_{k=1}^N \xi\beta^k \wedge \beta^k$, we obtain (at boundary points)

$$\begin{aligned}d\theta + \theta \wedge \theta &= -\frac{1}{8}\Big(2\Omega \otimes \xi + \sum_{i,j=1}^N (\xi\beta^i \wedge \xi\beta^j + 4F^2\beta^i \wedge \beta^j) \otimes \eta_i^j\Big) + 2F^2\beta^n \wedge \beta^N \otimes \eta_n^N \\ &\quad + \frac{F}{2}\sum_{j=1}^N (\xi\beta^j \wedge \beta^N \otimes \eta_j^n + \beta^j \wedge \xi\beta^n \otimes \eta_j^N) + \frac{1}{4}\sum_{k=1}^N \beta^N \wedge \beta^k \otimes [B, A_k].\end{aligned}$$

Again using lemma 5.1,

$$\begin{aligned}\frac{1}{4}\sum_{k=1}^N \beta^N \wedge \beta^k \otimes [B, A_k] &= -\frac{1}{4}\sum_{k=1}^N \beta^N \wedge \xi\beta^k \otimes ([\xi, \eta_k^N] - 2F\eta_k^n) \\ &\quad -\frac{F}{2}\sum_{k=1}^N \beta^N \wedge \beta^k \otimes ([\xi, \eta_k^n] + 2F\eta_k^N) - 2F^2\beta^n \wedge \beta^N \otimes \eta_n^N.\end{aligned}$$

Recall $\xi\beta^n = -2F\beta^N + \nu$. Then

$$\begin{aligned}d\theta + \theta \wedge \theta &= -\frac{1}{8}\Big(2\Omega \otimes \xi + \sum_{i,j=1}^N (\xi\beta^i \wedge \xi\beta^j + 4F^2\beta^i \wedge \beta^j) \otimes \eta_i^j\Big) \\ &\quad -\frac{F}{2}\sum_{k=1}^N \beta^N \wedge \beta^k \otimes [\xi, \eta_k^n] - \frac{1}{4}\sum_{k=1}^N \beta^N \wedge \xi\beta^k \otimes [\xi, \eta_k^N] + \frac{F}{2}\sum_{k=1}^N \nu \wedge \beta^k \otimes \eta_N^k.\end{aligned}$$

To obtain the $(0,4)$-curvature tensor, we identify $so(N)$ with ${}^\Theta\Lambda^2 X$ near p_o using the frame and the Θ-metric. Then η_i^j is identified with $2\beta^i \wedge \beta^j$ (the factor of 2 comes from our wedge convention), ξ is identified with Ω, and $[\xi, \eta_k^l]$ with $2\left(\xi\beta^k \wedge \beta^l + \beta^k \wedge \xi\beta^l\right)$. We have, for any $(0,1)$ tensors α, β, γ, δ,

$$4\alpha \wedge \beta \otimes \gamma \wedge \delta = \alpha \otimes \gamma \owedge \beta \otimes \delta, \tag{6}$$

so that

$$4F^2\sum_{i,j=1}^N \beta^i \wedge \beta^j \otimes \beta^i \wedge \beta^j = F^2\sum_{i,j=1}^N \beta^i \otimes \beta^i \owedge \beta^j \otimes \beta^j = F^2 g \owedge g$$

and

$$\sum_{i,j=1}^{N} \xi\beta^i \wedge \xi\beta^j \otimes \beta^i \wedge \beta^j = \frac{1}{4}\sum_{i,j=1}^{N} \xi\beta^i \otimes \beta^i \owedge \xi\beta^j \otimes \beta^j = \frac{1}{4}\Omega \owedge \Omega$$

(since ξ is skew-symmetric, $\sum_{i=1}^{N} \xi\beta^i \otimes \beta^i = \sum_{i=1}^{N} \xi\beta^i \wedge \beta^i$). Therefore the Θ-curvature tensor, evaluated at boundary points, is

$$\begin{aligned}
\frac{1}{2} R^{\Theta} &= -\frac{1}{16}\left(4F^2 g \owedge g + 4\Omega \otimes \Omega + \Omega \owedge \Omega\right) \\
&\quad + F \sum_{k=1}^{N} \left(\beta^k \wedge \beta^N \otimes \beta^k \wedge \nu + \beta^k \wedge \nu \otimes \beta^k \wedge \beta^N\right) \\
&\quad + \frac{1}{2}\sum_{k=1}^{N}\left(\xi\beta^k \wedge \beta^N \otimes \xi\beta^k \wedge \beta^N - 4F^2\beta^k \wedge \beta^N \otimes \beta^k \wedge \beta^N\right)
\end{aligned}$$

(a pair of terms have been cancelled using lemma 5.1.1 and the skew-symmetry of ξ). Using equation 6 we obtain the desired expression for the $(0,4)$-curvature.

An easy computation shows that if A is a $(1,1)$-tensor and h is a $(0,2)$-tensor, then the Ricci contraction of $Ag \owedge h$ is

$$\operatorname{Ric}(Ag \owedge h)(x,z) = (\operatorname{tr} h)\, Ag(x,z) + (\operatorname{tr} A)\, h(x,z) - h(Ax,z) - h(x,A^t z).$$

Using this and a few elementary computations we obtain:

$$\begin{aligned}
\operatorname{Ric}(g \owedge g) &= 2(N-1)g, \qquad \operatorname{Ric}(\Omega \otimes \Omega) = -\xi^2 g, \qquad \operatorname{Ric}(\Omega \owedge \Omega) = -2\xi^2 g \\
\operatorname{Ric}(g \owedge \beta^N \circ \nu) &= (N-2)\beta^N \circ \nu \quad (\text{by lemma 5.1.1, } \operatorname{tr}(\beta^N \circ \nu) = \nu(\beta_N) = 0) \\
\operatorname{Ric}(g \owedge \beta^N \circ \beta^N) &= g + (N-2)\beta^N \circ \beta^N \\
\operatorname{Ric}(\xi^2 g \owedge \beta^N \circ \beta^N) &= \xi^2 g + (\operatorname{tr}\xi^2)\beta^N \circ \beta^N - 2\operatorname{Sym}\left((\xi^2\beta^N) \otimes \beta^N\right)
\end{aligned}$$

where Sym is the symmetric part. Since $\xi^2\beta^N = 2F\nu - 4F^2\beta^N$,

$$\operatorname{Ric}(\xi^2 g \owedge \beta^N \circ \beta^N) = \xi^2 g + \left(\operatorname{tr}\xi^2 + 8F^2\right)\beta^N \circ \beta^N - 4F\beta^N \circ \nu.$$

This gives the desired expression for the Ricci tensor. //

Corollary 5.1 $\operatorname{Ric}^{\Theta} = cg$ *at boundary points for some function c on ∂X if and only if $\xi^2 = -4F^2 Id$ (in which case $c = -F^2(N+2)$).*

Proof. If $\xi^2 g = -4F^2 Id$, lemma 5.2.1 implies $\nu = 0$ and so by theorem 5.1, $\operatorname{Ric}^{\Theta} = -F^2(N+2)g$. Conversely, suppose $\operatorname{Ric}^{\Theta} = cg$ for some function c on ∂X. If $\nu \neq 0$, then $\nu(\beta_i) = \alpha \neq 0$ for some $i \neq n$, N. Evaluating $\operatorname{Ric}^{\Theta} = cg$ on (β_N, β_i) gives $-g(\beta_n, \xi\beta_i) = (N-1)\alpha$. On the other hand, $\alpha = \nu(\beta_i) = -g(\beta_n, \xi\beta_i)$. Since $N > 2$, we conclude that $\alpha = 0$ and $\nu = 0$. Now evaluating $\operatorname{Ric}^{\Theta} = cg$ on (β_n, β_n) gives $c = -F^2(N+2)$, and evaluating on (β_N, β_N) gives $\operatorname{tr}(\xi^2 + 4F^2) = 0$. Therefore $\xi^2 g = -4F^2 g$, and so $\xi^2 = -4F^2 Id$. //

5.1 Curvature of level sets of a defining function

We now examine the behavior near the boundary of the metric induced on level sets of a defining function by a Θ-metric. As above let (X,g) be a smooth manifold-with-boundary and C^k Θ-metric g ($k \geq 2$). Let $\rho \geq 0$

be a C^{k+1} defining function for X, and U a neighborhood of ∂X where $d\rho \neq 0$. Consider the hypersurfaces $M_\epsilon := \{\rho = \epsilon\}$, $\epsilon > 0$, with the metric g_ϵ induced by $(\iota_\Theta^{-1})^* g$. Let R_ϵ be the $(0,4)$-curvature tensor of (M_ϵ, g_ϵ), let pr be the orthogonal projection onto $\ker d\rho$ and let $R_\epsilon^\Theta = (\mathrm{pr} \circ \iota_\Theta)^* R_\epsilon$. Letting $\epsilon > 0$ vary we obtain a section

$$R_\rho^\Theta \in C^{k-2}(X^o \cap U, {}^\Theta\Lambda^2 X \otimes {}^\Theta\Lambda^2 X),$$

which we regard as the Θ-analog of the curvature tensor of M_ϵ. Let pr_1 denote the orthogonal projection onto $\beta_n^\perp$.

Theorem 5.2 *R_ρ^Θ extends to a C^{k-2} section of ${}^\Theta\Lambda^2 X \otimes {}^\Theta\Lambda^2 X$ over U. Over ∂X,*

$$R_\rho^\Theta = -\frac{1}{8}\left\{4\Omega \otimes \Omega + \Omega \owedge \Omega + 2\xi^2 g \owedge \beta^N \circ \beta^N - 4\beta^N \circ \nu \owedge \beta^N \circ \xi\beta^n\right\} \circ \mathrm{pr}_1$$

The Ricci contraction of R_ρ^Θ over ∂X is

$$\mathrm{Ric}_\rho^\Theta = \frac{1}{4}\left\{2\xi^2 g - \left(\mathrm{tr}\,(\xi^2 + \nu \circ \xi\beta^n) \circ \mathrm{pr}_1 - 4F^2\right)\beta^N \circ \beta^N + \nu \circ (\nu - 6F\beta^N)\right\} \circ \mathrm{pr}_1.$$

Proof. Lifting the Gauss curvature equation, we have over the interior

$$R_\rho^\Theta = \left\{R^\Theta + \frac{1}{2} l \owedge l\right\} \circ \mathrm{pr}_1,$$

where l is the symmetric $(0,2)$-tensor associated with ${}^\Theta\nabla\beta_n$. Since pr_1 extends to a smooth bundle map of ${}^\Theta TX$ over U (at the boundary pr_1 is orthogonal projection onto K_1), this clearly extends to $C^{k-2}(U, {}^\Theta\Lambda^2 X \otimes {}^\Theta\Lambda^2 X)$. By proposition 4.1 we have, at boundary points,

$${}^\Theta\nabla\beta_n = \frac{1}{2}\beta^N \otimes \xi\beta_n + \frac{1}{2}\xi\beta^n \otimes \beta_N - F \sum_{i \neq n,\, N} \beta^i \otimes \beta_i.$$

A computation gives

$$l \owedge l = \left\{F^2 g \owedge g - 2Fg \owedge \beta^N \circ \nu + 2F^2 g \owedge \beta^N \circ \beta^N + \beta^N \circ \nu \owedge \beta^N \circ \xi\beta^n\right\} \circ \mathrm{pr}_1.$$

The expression for R_ρ^Θ and Ric_ρ^Θ now follow from theorem 5.1 and some routine computations. //

Consider the case where $N = 2n$ is even and $\xi^2 = -4F^2 Id$ (see for example section 6). Then over ∂X,

$$\begin{array}{rcl} R_\rho^\Theta & = & -\frac{1}{8}\left(4\Omega \otimes \Omega + \Omega \owedge \Omega - 8F^2 g \owedge \beta^{2n} \circ \beta^{2n}\right) \circ \mathrm{pr}_1 \\ \mathrm{Ric}_\rho^\Theta & = & -2F^2\left(g - n\beta^{2n} \circ \beta^{2n}\right) \circ \mathrm{pr}_1. \end{array} \tag{7}$$

We would like to see whether the curvature tensor of (M_ϵ, g_ϵ) has any special properties in this case, at least asymptotically as $\epsilon \to 0^+$. Choosing a basis β_i, β_{i+n} $(i \neq n,\, 2n)$ for $\beta_n^\perp \cap \beta_{2n}^\perp$ such that $\xi\beta_i = -\beta_{i+n}$, we see that at boundary points, R_ρ^Θ restricted to K_1 is $4F^2$ times the curvature tensor of the Heisenberg group, $H^{2n-1} = \mathbf{R}_q^{n-1} \times \mathbf{R}_p^{n-1} \times \mathbf{R}_t$, with the metric

$$g_h = \left(\sum_{i=1}^{n-1} p_i dq_i - dt\right)^2 + \sum_{i=1}^{n-1}(dq_i)^2 + (dp_i)^2.$$

None of the irreducible components of the curvature tensor of H^{2n-1} with respect to the $O(2n-1)$ action on curvature tensors vanish. Therefore we consider the $(0,3)$-tensor $\nabla_\epsilon \mathrm{Ric}_\epsilon$ (where ∇_ϵ is the Levi-Cevita connection on M_ϵ, and assuming that g is C^k with $k \geq 3$). On any Riemannian manifold, the bundle $H \subset \otimes^3 T^*M$ where $\nabla\mathrm{Ric}$ takes its values decomposes into irreducible components under the action of the orthogonal group as $H = Q \oplus S \oplus A$ (see [2], chapter 16). For the Heisenberg group (H^{2n-1}, g_h), the Q and S components of $\nabla\mathrm{Ric}$ are zero. We will show that for (M_ϵ, g_ϵ), $\nabla_\epsilon\mathrm{Ric}_\epsilon$ has this special property "asymptotically."

Proposition 5.1 *Let $(\nabla_\epsilon \mathrm{Ric}_\epsilon)_{Q_\epsilon \oplus S_\epsilon}$ be the component of $\nabla_\epsilon\mathrm{Ric}_\epsilon$ in $Q_\epsilon \oplus S_\epsilon$. Then*

$$\lim_{\epsilon \to 0^+} \sup_{M_\epsilon} \left| (\nabla_\epsilon \mathrm{Ric}_\epsilon)_{Q_\epsilon \oplus S_\epsilon} \right|_{g_\epsilon} = 0.$$

Proof. We first show that the decomposition $H_\epsilon = Q_\epsilon \oplus S_\epsilon \oplus A_\epsilon$ lifts to the Θ-setting. For $\epsilon > 0$, ι_Θ^* gives an inner product preserving vector bundle isomorphism between T^*M_ϵ and $\beta^{n\perp}$ over M_ϵ. Letting ϵ vary, we define the correpsonding subbundles ${}^\Theta H = {}^\Theta Q \oplus {}^\Theta S \oplus {}^\Theta A \subset \otimes^3\, {}^\Theta T^* X$ over $X^o \cap U$. Since the fibers of H_ϵ, Q_ϵ, S_ϵ, A_ϵ and hence their Θ counterparts ${}^\Theta H$, etc., can be described in terms of the pointwise inner product g_ϵ, resp. g (see [2], 16.2), ${}^\Theta H$ and its orthogonal decomposition extends to the boundary.

It follows from the Gauss (connection) equation and the fact that ${}^\Theta\nabla$ is the lift of ∇ that for $\epsilon > 0$,

$$(\mathrm{pr} \circ \iota_\Theta)^* (\nabla_\epsilon \mathrm{Ric}_\epsilon) = {}^\Theta\nabla\, \mathrm{Ric}_\rho^\Theta \big|_{\otimes^3 \beta_n^\perp}.$$

Since ι_Θ preserves inner products and ${}^\Theta Q$, ${}^\Theta S$, etc., are the lift of Q_ϵ, S_ϵ, etc., we have for $\epsilon > 0$,

$$\left| (\nabla_\epsilon \mathrm{Ric}_\epsilon)_{Q_\epsilon \oplus S_\epsilon} \right|_{g_\epsilon} = \left| \left((\mathrm{pr} \circ \iota_\Theta)^* \nabla_\epsilon \mathrm{Ric}_\epsilon \right)_{{}^\Theta Q \oplus {}^\Theta S} \right|_g = \left| \left({}^\Theta\nabla\, \mathrm{Ric}_\rho^\Theta \right)_{{}^\Theta Q \oplus {}^\Theta S} \right|_g.$$

(In the last equality we are considering Ric_ρ^Θ as an element of $\otimes^3 \beta^{n\perp}$ (by restriction) and using the norm on $\otimes^3\beta^{n\perp}$ induced by g.) Since ${}^\Theta\nabla\,\mathrm{Ric}_\rho^\Theta$ extends continuously up to the boundary, it suffices to show that its component in ${}^\Theta Q \oplus {}^\Theta S$ vanishes at the boundary. Using equations 7 we can write

$$\mathrm{Ric}_\rho^\Theta = -2F^2 \left(g - \beta^n \circ \beta^n - n\beta^{2n} \circ \beta^{2n} \right) + \rho C^{k-3}(U, \otimes^3\, {}^\Theta T^* X).$$

Using lemma 2.1, we have at boundary points

$${}^\Theta\nabla\, \mathrm{Ric}_\rho^\Theta = 2nF^2\, {}^\Theta\nabla \left(\beta^n \circ \beta^n + n\beta^{2n} \circ \beta^{2n} \right).$$

Upon restricting to $\otimes^3 \beta_n^\perp$, the terms involving ${}^\Theta\nabla(\beta^n \circ \beta^n)$ will vanish. By proposition 4.1 we have at boundary points

$${}^\Theta\nabla\beta^{2n} = 2F\beta^{2n} \circ \beta^n - \frac{1}{2}\Omega.$$

From this it follows that if $X \in \beta_n^\perp$, $\left({}^\Theta\nabla\, \mathrm{Ric}^\Theta \right)(X, X, X) = 0$. This means that the component in ${}^\Theta Q \oplus {}^\Theta S$ is zero at boundary points (see [2], 16.4). //

6 Examples and Applications

6.1 Strictly pseudoconvex domains

Let $\mathcal{U}$ be a strictly pseodoconvex domain in a complex n-manifold with strictly plurisuperharmonic defining function ϕ, so that $\mathcal{U} = \{\phi > 0\}$, $\partial\mathcal{U} = \{\phi = 0\}$, and the symmetric two-tensor $h(u,v) := -dd^c\phi(u, Jv)$ is positive-definite (J is the complex structure operator). We will assume that $\phi \in C^{k+3}(\overline{\mathcal{U}})$, $k \geq 2$, and that $\partial\mathcal{U}$ is smooth. This implies that the oriented projective class of $d^c\phi\big|_{\partial\mathcal{U}}$ can be represented by a smooth one-form $\theta \in C^\infty(\partial\mathcal{U}, T^*\overline{\mathcal{U}})$. Let X be the "square root" of $\mathcal{U}$, that is, the smooth manifold-with-boundary formed by adjoining the square root of a smooth defining function to $C^\infty(\overline{\mathcal{U}})$ (see [5], §2). There exists a positive C^{k+2} function f such that $f\phi$ is a smooth defining function, so $\rho := \sqrt{\phi}$ lifts to a C^{k+2} defining function for X. The "identity" map determines a canonical C^∞ map

$$\iota_{1/2} : X \longrightarrow \overline{\mathcal{U}},$$

which is a C^∞ isomorphism over the interior and over the boundary (but not smoothly invertible up to the boundary). We obtain a Θ-structure on X by taking $\Theta = \iota_{1/2}^*\theta$. Consider the Kähler form $\omega_c := -2c^{-1}dd^c \log\phi$, parameterized by the positive number[1] c. The lift of ω_c to X is

$$\omega_{X,c} := \iota_{1/2}^*\omega_c = \frac{2}{c}\left\{2\rho^{-1}d\rho \wedge \rho^{-2}\iota_{1/2}^* d^c\phi - \rho^{-2}\iota_{1/2}^* dd^c\phi\right\},$$

and the associated (complete) Kähler metric $g_c(u,v) := \omega_c(u, Jv)$ lifts to

$$g_{X,c} := \iota_{1/2}^* g_c = \frac{2}{c}\left\{2\left(\rho^{-1}d\rho\right)^2 + \frac{1}{2}\left(\rho^{-2}\iota_{1/2}^* d^c\phi\right)^2 + \rho^{-2}\iota_{1/2}^* h\right\}.$$

Since $\iota_{1/2}^* d^c\phi$ is in the projective class of Θ at boundary points, this shows that g_c lifts to a C^k Θ-metric on X, and ω_c to a C^k section of ${}^\Theta\Lambda^2 X$. After determining the invariants Ω and ξ, we use theorem 5.1 to recover a well-known result about the curvature of g_c near $\partial\mathcal{U}$.

Proposition 6.1 *Over ∂X we have $\Omega = \sqrt{c}\,\iota_\Theta^*\omega_{X,c}$, $\xi^2 = -c\,Id$, $\nu = 0$ and $F = \sqrt{c}/2$.*

Proof. Let T be a vector field near ∂X with $\big|[\rho^2 T]\big|_{\iota_\Theta^* g_{X,c}} = 1$ and let β^{2n} be the Θ-covector field dual to $[\rho^2 T]$ by $\iota_\Theta^* g_{X,c}$ as in section 4. Using the expression for $g_{X,c}$ above we find

$$\beta^{2n} = \frac{1}{\sqrt{c}}\iota_\Theta^*\iota_{1/2}^* d^c \log\phi + \rho C^{k+1}(X, {}^\Theta T^* X)$$

and so

$$d\beta^{2n} = -\frac{\sqrt{c}}{2}\iota_\Theta^*\omega_{X,c} + \rho C^k(X, {}^\Theta\Lambda^2 X).$$

Using lemma 4.1 we obtain, at boundary points, $\Omega = \sqrt{c}\,\iota_\Theta^*\omega_{X,c}$. Let ${}^\Theta J$ denote the lift of the complex structure operator on $\mathcal{U}$ to an automorphism of ${}^\Theta TX$ over X^o. Then ${}^\Theta J^2 = -Id$, and $\iota_\Theta^* g_{X,c}(v,w) = \iota_\Theta^*\omega_{X,c}(v, {}^\Theta J w)$. It follows that ${}^\Theta J$ extends continuously up to ∂X, and $\xi = -\sqrt{c}\,{}^\Theta J$ over ∂X. Therefore $\xi^2 = -c\,Id$, $F = \sqrt{c}/2$, and $\nu = 0$. //

Lifting the curvature tensor of $g_{X,c}$ to ${}^\Theta\Lambda^2 X \otimes {}^\Theta\Lambda^2 X$ gives the Θ-curvature tensor of $\iota_\Theta^* g_{X,c}$. Applying theorem 5.1 we obtain

[1] The factor of 2 appears because of our wedge convention, which determines the relationship between exterior derivative and tensor products. The parameter $-c$ is the asymptotic value of the holomorphic sectional curvature of g_c.

Corollary 6.1 *The curvature R_c of g_c approaches the curvature tensor of constant holomorphic sectional curvature $-c$ at the boundary. I.e.,*

$$\lim_{\partial\mathcal{U}} \left| R_c + \frac{c}{8}\left(g_c \owedge g_c + 4\omega_c \otimes \omega_c + \omega_c \owedge \omega_c\right) \right|_{g_c} = 0$$

and the Ricci tensor Ric_c *satisfies*

$$\lim_{\partial\mathcal{U}} \left| \mathrm{Ric}_c + \frac{c}{2}(n+1) g_c \right|_{g_c} = 0.$$

Applying theorem 5.1 and proposition 5.1 we obtain the following results for the hypersurfaces $\mathcal{U}_\epsilon := \{\phi = \epsilon\}$, which give an analog of Klembeck's result [7] for these hypersurfaces. They show that for any strictly pseudoconvex domain, the curvature tensor of the induced metric $g_{c,\epsilon}$ behaves like a Berger sphere metric near the boundary.

Corollary 6.2 *Let $R_{c,\epsilon}$ be the $(0,4)$-curvature tensor and $\mathrm{Ric}_{c,\epsilon}$ the Ricci tensor of $(\mathcal{U}_\epsilon, g_{c,\epsilon})$, and let $\iota_\epsilon : \mathcal{U}_\epsilon \to \mathcal{U}$ be the inclusion map. Then*

1. $\lim_{\epsilon\to 0} \sup_{\mathcal{U}_\epsilon} \left| R_{c,\epsilon} + \frac{c}{8}\iota_\epsilon^* \left(4\omega_c \otimes \omega_c + \omega_c \owedge \omega_c - 2 g_c \owedge \left(c^{-1/2} d^c \log\phi\right)^2\right)\right|_{g_{c,\epsilon}} = 0$
2. $\lim_{\epsilon\to 0} \sup_{\mathcal{U}_\epsilon} \left| \mathrm{Ric}_{c,\epsilon} + \frac{c}{2}\left(g_{c,\epsilon} - n\,\iota_\epsilon^* \left(c^{-1/2} d^c \log\phi\right)^2\right)\right|_{g_{c,\epsilon}} = 0$
3. $\lim_{\epsilon\to 0} \sup_{\mathcal{U}_\epsilon} \left| (\nabla_{c,\epsilon} \mathrm{Ric}_{c,\epsilon})_{Q_{c,\epsilon}\oplus A_{c,\epsilon}} \right|_{g_{c,\epsilon}} = 0.$

In [2], §16.56, Besse asks for examples of Riemannian manifolds with $(\nabla \mathrm{Ric})_{Q\oplus A} = 0$ which are neither locally homogeneous, nor locally isometric to Riemannian products and have non-parallel Ricci tensor. Part 3 of corollary 6.2 shows that for any $\epsilon > 0$ and sufficiently small, $\partial\mathcal{U}$ admits a Riemann metric g with $|(\nabla \mathrm{Ric})_{Q\oplus A}|_g < \epsilon$.

6.2 The Bergman metric

Let $\mathcal{U}$ be a strictly pseudoconvex domain in $\mathbf{C}^n$, $n \geq 2$. Consider the Bergman metric

$$g_B = \sum \frac{\partial^2}{\partial z_i \partial \overline{z}_j} \log k_B \, dz_i \circ d\overline{z}_j,$$

where k_B is the Bergman kernel of $\mathcal{U}$ restricted to the diagonal. Using Fefferman's asymptotic expansion [6], this can be written in a neighborhood of $\partial\mathcal{U}$ as

$$g_B = -(n+1) \sum \frac{\partial^2}{\partial z_i \partial \overline{z}_j} \log\phi \, dz_i \circ d\overline{z}_j$$

with associated Kähler form

$$\omega_B = -\frac{n+1}{2} dd^c \log\phi.$$

Here ϕ has the form $\psi(\Phi + \tilde{\Phi}\psi^{n+1}\log\psi)^{-1/(n+1)}$, ψ being the Euclidean distance to $\partial\mathcal{U}$ and Φ, $\tilde{\Phi}$ are smooth up to the boundary with $\Phi > 0$. This shows that $\sqrt{\phi}$ lifts to a C^{2n+1} defining function for the square root of $\mathcal{U}$, and g_B to a C^{2n-1} Θ-metric. Corollary 6.1 shows that the curvature tensor of g_B approaches the curvature tensor of constant holomorphic sectional curvature $-4/(n+1)$ at the boundary, recovering the well-known result of Klembeck [7].

6.3 Manifolds with contact boundary

We will construct an "asymptotically Einstein" metric on the interior of any manifold whose boundary is a contact manifold. For simplicity we will work in the C^∞ setting.

Theorem 6.1 *Let X be any smooth, $2n$-dimensional manifold with contact boundary. For any $c > 0$ there exists a complete smooth metric g on X^o such that*

$$\lim_{\partial X} |\mathrm{Ric} + cg|_g = 0. \tag{8}$$

Proof. We may assume without loss of generality that $c = 2(n+1)$, since multiplying the metric by a constant $c > 0$ scales the Ricci tensor by c^{-1}. Fix an identification of a collar neighborhood of ∂X with $[0,1) \times \partial X$. We will construct a Θ-metric on $[0,1) \times \partial X$ whose Ricci tensor at boundary points satisfies $\mathrm{Ric} = -2(n+1)g$ (as a Θ-tensor). We then extend the metric in a smooth but otherwise arbitrary way to X. The restriction to X^o will be a Riemann metric whose Ricci tensor satisfies 8, and the boundary behavior of the metric will insure completeness.

To construct the metric on $[0,1) \times \partial X$ we need to recall some facts about contact geometry, following [1]. A *contact manifold* M is a $2n-1$ manifold equipped with a one-form θ such that $\theta \wedge (d\theta)^{n-1}$ is a volume form. A *contact metric structure* is a quadruple (J, T, θ, b) consisting of a field of automorphisms J of TM, a vector field T, a contact form θ, and a Riemann metric b such that:

1. $\theta(T) = 1$
2. $J^2 = -Id + \theta \otimes T$
3. $b(JX, JY) = b(X, Y) - \theta(X)\theta(Y)$ for all X, $Y \in TM$
4. If Φ is the two-form defined by $\Phi(X, Y) = b(X, JY)$, then $\Phi = d\theta$.

Item 3 implies that $\theta = b(T, \cdot)$, and 4 implies that $d\theta(T, \cdot) = 0$ (so that T is the characteristic vector field associated to the contact form). We will use the following fact (see [1], II.3): On any contact manifold (M, θ), there exists a contact metric structure (J, T, θ, b) (same θ).

Fix a contact metric structure on ∂X. Let p_2 be projection onto the second factor of $[0,1) \times \partial X$ and put $\widetilde{\theta} := p_2^* \theta$. We give $[0,1) \times \partial X$ the Θ-structure determined by the restriction of $\widetilde{\theta}$ to ∂X, and oriented by the vector field T. Let t be the coordinate on $[0,1)$, let $\widetilde{b}$ be the pull-back of the contact metric b and consider the Θ-metric

$$g := \left(t^{-1} dt\right)^2 + \left(t^{-2} \widetilde{\theta}\right)^2 + t^{-2} \left(\widetilde{b} - \widetilde{\theta}^2\right).$$

If we regard T as a vector field on $[0,1) \times \partial X$ annihilated by dt, then $\widetilde{b}(T, \cdot) = \widetilde{\theta}$ (this follows from item 3 above). Then $t^2 T$ has unit length, and by lemma 4.1,

$$\begin{aligned} \Omega = -2d\left(t^{-2}\widetilde{\theta}\right)\Big|_{\partial X} &= -2\left(-2t^{-1} dt \wedge t^{-2}\widetilde{\theta} + t^{-2} d\widetilde{\theta}\right)\Big|_{\partial X} \\ &= -2\left(-2t^{-2} dt \wedge t^{-2}\widetilde{\theta} + t^{-2} p_2^* \, d\Phi\right)\Big|_{\partial X} \end{aligned}$$

where Φ is the two-form in item 4. Therefore an extension of ξ to a continuous section of $\mathrm{End}\left({}^{\Theta}T([0,1) \times \partial X)\right)$ is given by $\tilde{\xi}$, where

$$g(U, \tilde{\xi} V) = -2\left(-2t^{-1} dt \wedge t^{-2}\widetilde{\theta} + t^{-2} p_2^* \, d\Phi\right).$$

Since $J^2 = -Id$ on $\ker dt \cap \ker \theta$ (by item 2 above) and $t^{-1}dt$, $t^{-2}\widetilde{\theta}$ are orthonormal, we see that $\tilde{\xi}^2 = -4Id$, and so $\xi^2 = -4Id$. By theorem 5.1, the Ricci tensor of g at boundary points, as a Θ-metric, is $-2(n+1)g$. Now extend g in a smooth but otherwise arbitrary way to X. For the Riemann metric determined by the restriction of g to X^o, we obtain equation 8. The metric is complete because the term $(t^{-1}dt)^2$ prevents geodesics from reaching the boundary in finite time. //

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