ASYMPTOTIC DELSARTE CLIQUES IN DISTANCE-REGULAR GRAPHS

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Abstract. We give a new bound on the parameter λ (number of common neighbors of a pair of adjacent vertices) in a distance-regular graph G, improving and generalizing a bound for strongly regular graphs implicit in Spielman (1996). The new bound is one of the ingredients of recent progress on the complexity of testing isomorphism of strongly regular graphs (Babai, Chen, Sun, Teng, Wilmes 2013). The proof is based on a remarkable clique geometry found by Metsch (1991). We also give a simplified proof of the following asymptotic consequence of Metsch's result: if $k\mu = o(\lambda^2)$ then each edge of G belongs to a unique maximal clique of size $\sim \lambda$, and all other cliques have size $o(\lambda)$. Here k denotes the degree and μ the number of common neighbors of a pair of vertices at distance 2. We point out that Metsch's cliques are "asymptotically Delsarte" when $k\mu = o(\lambda^2)$, so families of distance-regular graphs with parameters satisfying $k\mu = o(\lambda^2)$ are "asymptotically Delsarte-geometric."

1. Introduction

A graph is called *amply regular* with parameters (n, k, λ, μ) if it is k-regular on n vertices, any two adjacent vertices have exactly λ common neighbors, and any two vertices at distance two from each other have exactly μ common neighbors. Amply regular graphs have been well-studied, as they generalize distance-regular graphs while preserving many of their properties [6, Section 1.1]. Our first result gives a new bound on λ.

In fact, our bound applies more generally to "sub-amply regular" graphs. We say a graph is *sub-amply regular* when it satisfies the weaker condition that any two vertices at distance two from each other have *at most* μ common neighbors.

Theorem 1.1. *Let G be a sub-amply regular graph with parameters (n, k, λ, μ) which is not a disjoint union of cliques. Then*

$$\lambda + 1 < \max\left\{4\sqrt{2n}, \frac{6}{\sqrt{13}-1}\sqrt{k(\mu-1)}\right\}. \tag{1}$$

Even in the very special case of strongly regular graphs, this result considerably improves the previously known bound for λ (Spielman [15]) (see Sec. 2.2). The new bound was used in [2] to improve Spielman's $\exp(\widetilde{O}(n^{1/3}))$ bound on the (worst-case) complexity of testing isomorphism of strongly regular graphs to $\exp(\widetilde{O}(n^{1/5}))$ where the $\widetilde{O}$ notation hides polylogarithmic ($(\log n)^C$) factors. This application was the key motivation of the present paper.

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1.1. **Clique geometry.** We say that a collection $\mathcal{C}$ of cliques of a graph is a *clique geometry* if (i) all cliques in $\mathcal{C}$ are maximal and (ii) every pair of adjacent vertices of G belongs to a unique member of $\mathcal{C}$. We shall refer to the members of $\mathcal{C}$ as *special cliques*.

Our result relies on the remarkable clique geometry appearing in sub-amply regular graphs under certain constraints on the parameters, discovered by Metsch [12] (Theorem 1.2). We observe in particular that Metsch's constraints are met when $k\mu/\lambda^2$ is small; furthermore, in this case, the special cliques have nearly uniform order. (The *order* of a clique is the number of its vertices.)

Sub-amply regular graphs G with $\mu \leq 1$ trivially have a (unique) clique geometry. When $\mu = 0$, G is a disjoint union of cliques of order $\lambda + 2 = 1 + k$. When $\mu = 1$, the common neighbors of two adjacent vertices form a clique. When $\mu = 2$ and $k < (1/2)\lambda(\lambda+3)$, Brouwer and Neumaier showed that again the common neighbors of any pair of adjacent vertices form a clique [7]. In such graphs, every edge lies in a unique maximal clique, and every maximal clique has order exactly $\lambda + 2$. A clique geometry exists under much more general conditions, as proved by Metsch [12].

Theorem 1.2 (Metsch [13, Result 2.1])**.** *Let G be a sub-amply regular graph with parameters (n, k, λ, μ), and let t be an integer such that*

$$\lambda > (2t - 1)(\mu - 1) - 1, \text{ and}$$

$$k < (t + 1)(\lambda + 1) - \frac{1}{2}t(t + 1)(\mu - 1).$$

Then the maximal cliques of order at least $\lambda + 2 - (t - 1)(\mu - 1)$ form a clique geometry, and each vertex belongs to at most t special cliques.

Remark 1.3. We note that special cliques of Theorem 1.2 can be easily recognized by the degree of the vertices in the common neighborhood of a pair of adjacent vertices. In particular, if u and v are two adjacent vertices of G, then a common neighbor w of u and v lies in the special clique containing u and v iff in the subgraph of G induced on the common neighborhood of u and v, the degree of w is at least $\lambda - (t - 1)(\mu - 1) - 1$.

Corollary 1.4. *Let G be a sub-amply regular graph with parameters (n, k, λ, μ) such that*

$$(\lambda + 1)^2 > (3k + \lambda + 1)(\mu - 1). \tag{2}$$

Then the maximal cliques of order at least $\lambda + 2 - (\lceil (3/2)k/(\lambda + 1)\rceil - 1)(\mu - 1)$ form a clique geometry.

The corollary is obtained from Theorem 1.2 by setting $t = \lceil 3k/(2(\lambda + 1))\rceil$. □

The starting point of our work was Spielman's 1996 paper [15] in which he derived asymptotic consequences of Neumaier's 1979 classification of strongly regular graphs [14], including a bound on the parameter λ. Our bound (1) applies more generally to sub-amply regular graphs (and hence does not require Neumaier's classification), and improves Spielman's bound for $k > n^{5/8}$.

We prove our bound (1) in Section 2. In Section 2.2, we compare Spielman's bound to our own. Then, in Section 2.3, we explain the connection to graph isomorphism testing in some detail.

The asymptotic viewpoint makes the results considerably more transparent. In Section 3, we give a short self-contained proof of Theorem 1.5 (below), an asymptotic corollary to Metsch's theorem.

To interpret asymptotic statements such as "Let G be an amply regular graph with $k\mu = o(\lambda^2)$," we think of our graph G as belonging to some infinite family for which the asymptotic relation holds. All hidden constants are absolute, and all limits are uniform as the number of vertices $n \to \infty$. We use common notation for asymptotic relations, including writing $f \sim g$ (asymptotic equality) for functions f and g for which $\lim_{n\to\infty}(f(n)/g(n)) = 1$. We write $f(n) \gtrsim g(n)$ if $f(n) \sim \max\{f(n), g(n)\}$.

Theorem 1.5. *Let G be a sub-amply regular graph with parameters (n, k, λ, μ) such that $k\mu = o(\lambda^2)$. Then (for n sufficiently large) every pair of adjacent vertices belongs to a unique maximal clique of order $\sim \lambda$, and all other maximal cliques in G have order $o(\lambda)$.*

So the large maximal cliques form a clique geometry.

The key lemma used in the proof, Lemma 3.2, is used in the recent characterization of primitive coherent configurations with more than $\exp(n^{1/3+\varepsilon})$ automorphisms by Sun and Wilmes [16].

1.2. **Asymptotic Delsarte geometry.** Let s denote the least eigenvalue of (the adjacency matrix of) the graph G (so $s < 0$). The following bound on the order of cliques in distance-regular graphs was established by Delsarte.

Lemma 1.6 (Delsarte [8])**.** *If G is a distance-regular graph then no clique in G has order greater than $1 + k/|s|$.*

Any clique achieving this order is called a *Delsarte clique* [9].

We call a graph G *Delsarte-geometric* if G is distance-regular and it has a clique geometry in which all special cliques are Delsarte. This concept was introduced by Godsil [10] who called such graphs "geometric." Johnson and Hamming graphs are examples of Delsarte-geometric graphs.

Godsil [10] gave the following sufficient condition for a distance-regular graph to be Delsarte-geometric.

An *m-claw* in a graph is an induced $K_{1,m}$ subgraph.

Theorem 1.7 (Godsil [10])**.** *Let G be a distance-regular graph with least eigenvalue s. If there are no m-claws in G with $m > |s|$ and*

$$\lambda + 1 > (2|s| - 1)(\mu - 1) \tag{3}$$

then G is Delsarte-geometric.

It would seem desirable to replace the structural assumption (bound on claw size) in Godsil's theorem by a reasonable assumption involving the parameters of the graph only since this would allow broader applicability of the result. Bang and Koolen make a step in this direction, removing the structural assumption but strengthening the constraint on the parameters.

Theorem 1.8 (Bang, Koolen [11])**.** *If $\lambda > \lfloor s \rfloor^2 \mu$ for a distance-regular graph G with least eigenvalue s then G is Delsarte-geometric.*

Note that for large $|s|$, the Bang–Koolen constraint $s^2\mu \lesssim \lambda$ requires essentially a factor of $|s|/2$ larger λ than does Godsil's constraint (3) which for large $|s|$ and μ requires $2|s|\mu \lesssim \lambda$.

We point out that already an increase by a factor that goes to infinity arbitrarily slowly compared to Godsil's contraint, $|s|\mu = o(\lambda)$, suffices for an *asymptotic*

Delsarte geometry, i. e., a clique geometry where the order of the special cliques is $\sim k/|s|$.

Theorem 1.9. *Let G be a distance-regular graph with satisfying $|s|\mu = o(\lambda)$. Then G is asymptotically Delsarte-geometric.*

Theorem 1.9 is proved in Section 4.

2. Bounding λ

In this section, we derive our bound on λ (Theorem 1.1) from Corollary 1.4. In the special case of strongly regular graphs, we compare our bound with Spielman's and indicate the significance of the improvement to bounding the complexity of testing isomorphism of strongly regular graphs.

2.1. Proof of the bound.

Lemma 2.1. *Let $\mathcal{C}$ be a geometric collection of cliques in a graph G on n vertices such that every vertex is in at least $r \geq 2$ and at most R cliques, and each clique has order at least ℓ. Then*

$$\ell \leq \frac{R}{\sqrt{r(r-1)}}\sqrt{n}.$$

Proof. Let $m = |\mathcal{C}|$ and let N be the number of vertex–clique incidences. Then $\ell m \leq N \leq nR$. Let T be the number of triples (p, ℓ_1, ℓ_2) where $p \in \mathcal{P}$, $\ell_1, \ell_2 \in \mathcal{L}$, $\ell_1 \neq \ell_2$, and p is incident with both ℓ_1 and ℓ_2. Then $T = \sum_p \deg(p)(\deg(p)-1) \geq nr(r-1)$. On the other hand, by the intersection assumption, $T \leq m(m-1) < m^2$. Comparing,

$$nr(r-1) < m^2 \leq \left(\frac{nR}{\ell}\right)^2 . \qquad \square$$

Proof of Theorem 1.1. Case 1. Suppose $(3k+\lambda+1)(\mu-1) < (\lambda+1)^2$. Then by Corollary 1.4, every edge lies in a special clique of order at least $\lambda+2-(3/2)k(\mu-1)/(\lambda+1) > (1/2)(\lambda+1)$ and at most $\lambda+2$. The number of special cliques containing a given vertex is at most $2k/(\lambda+1)$, and at least $k/(\lambda+1)$. Since each vertex lies in at least two special cliques, Lemma 2.1 gives $\lambda+1 < 4\sqrt{2n}$.

Case 2. Otherwise, $(\lambda+1)^2 \leq (3k+\lambda+1)(\mu-1)$. Set $\delta = (\sqrt{13}-1)/6$.

Case 2a. Suppose $\mu - 1 \geq \delta(\lambda+1)$. Then

$$\lambda + 1 \leq (1/\delta)(\mu-1) < (1/\delta)\sqrt{k(\mu-1)}. \tag{4}$$

Case 2b. Otherwise, $\mu - 1 < \delta(\lambda+1)$, and we have

$$(1-\delta)(\lambda+1)^2 < 3k(\mu-1),$$

which is equivalent to Eq. (4) by our choice of δ. The Theorem follows by combining Eq. (4) with Case 1. $\square$

2.2. Spielman's bound for strongly regular graphs.

A strongly regular graph with parameters (n, k, λ, μ) is a k-regular graph on n vertices such that any two adjacent vertices have exactly λ common neighbors, and any two distinct nonadjacent vertices have exactly μ common neighbors. Hence, strongly regular graphs are subamply regular, and indeed distance-regular if connected. We will state Neumaier's classification of strongly regular graphs [14], along with its asymptotic consequences to the parameters of strongly regular graphs, derived by Spielman [15].

A *partial geometry* $\mathfrak{X} = (\mathcal{P}, \mathcal{L})$ with parameters (R, K, α), where $R, K \geq 2$, is a geometric 1-design with parameters R, K with the property that for every nonincident pair $(p, \ell) \in \mathcal{P} \times \mathcal{L}$, there are exactly α lines containing p that intersect ℓ. Examples of partial geometries include *Steiner 2-designs*, which are the partial geometries with $\alpha = K$, and *transversal designs*, which are the partial geometries with $\alpha = K - 1$. The *dual* of a partial geometry $(\mathcal{P}, \mathcal{L})$ with parameters (R, K, α) is the incidence structure $(\mathcal{L}, \mathcal{P})$. It is a partial geometry with parameters (K, R, α). The *line-graph* of a partial geometry is the point-graph of its dual.

Every line-graph (or point-graph) of a partial geometry is strongly regular, and the geometric strongly regular graphs are point-graphs (hence line-graphs) of partial geometries. Other examples of strongly regular graphs include disjoint unions of cliques of equal order and the complements of such graphs; we call these two types *trivial*; and *conference graphs*, which have parameters $(n, (n-1)/2, (n-5)/4), (n-1)/4)$. We now state Neumaier's classification.

Theorem 2.2 (Neumaier [14]). *A strongly regular graph G with eigenvalues $k \geq r > s$ is of one of one the following types: (i) trivial; (ii) the line-graph of a Steiner 2-design or the line-graph of a transversal design; (iii) a conference graph; or (iv) G satisfies the inequality*

$$r \leq \max\left\{2(-s-1)(\mu+1+s)+s, \frac{s(s+1)(\mu+1)}{2} - 1\right\} \tag{5}$$

Inequality (5) is called the "claw bound."

The following asymptotic consequences of Neumaier's classification are implicit in Spielman's paper on testing isomorphism of strongly regular graphs [15].

Theorem 2.3 (Spielman [15]). *Let G be a nontrivial strongly regular graph with parameters (n, k, λ, μ) and non-principal eigenvalues $r > s$. If $k = o(n)$, then*

(a) $\mu = o(k)$.

Suppose in addition that inequality (5) *(the claw bound) holds. Then*

(b) $r \lesssim k^{2/3}(\mu+1)^{1/3}$;
(c) $\lambda \lesssim k^{2/3}(\mu+1)^{1/3}$;
(d) $\lambda = o(k)$;
(e) $\mu \sim k^2/n$.

Spielman explicitly states (a) and (d). For the reader's convenience, we give an organized presentation of a proof of the full statement of Theorem 2.3 in Appendix A.

Our Theorem 1.1 improves inequality (c) for $k \geq n^{5/8}$. We summarize the combination of our bound and Spielman's bound (c) over the full range of possible degrees k. Let

$$h(n,k) = \min\left\{\left(\frac{k}{n}\right)^{4/3}, \max\left\{\left(\frac{k}{n}\right)^{3/2}, \left(\frac{1}{n}\right)^{1/2}\right\}\right\}.$$

We assume $k \leq (n-1)/2$ (otherwise we can take the complement of G).

Theorem 2.4. *Let G be a strongly regular graph with parameters (n, k, λ, μ) satisfying Eq.* (5) *(the claw bound). Then*

$$\frac{\lambda}{n} = O(h(n,k)).$$

The function $h(n,k)$ evaluates to

- $(k/n)^{4/3}$ for $k \leq n^{5/8}$
- $n^{-1/2}$ for $n^{5/8} \leq k \leq n^{2/3}$
- $(k/n)^{3/2}$ for $k \geq n^{2/3}$.

Note that the function $h(n,k)$ is continuous so up to constant factors the transition is continuous around the boundaries of the intervals above.

2.3. **Connection to graph isomorphism testing.** The key motivation for our main result comes from its application to the complexity of graph isomorphism testing (GI). While strong theoretical evidence suggests that this problem is not NP-complete, the worst-case bound of $\exp(\widetilde{O}(\sqrt{n}))$, established three decades ago [4, 17, 3], continues to be unchallenged.

Strongly regular graphs have long been recognized as a difficult although probably not complete class for GI; there has been slightly more progress on the complexity of testing their isomorphism. The first bound for strongly regular graphs was $\exp(\widetilde{O}(n^{1/2}))$ [1] (1980), followed by $\exp(\widetilde{O}(n^{1/3}))$ [15] (Spielman, 1996) and $\exp(\widetilde{O}(n^{1/5}))$ [2] (2013). The two main components of the recent result are an $\exp(n^{O(\mu+\log n)})$ bound and an $\exp(n^{\widetilde{O}(1+\lambda/\mu)})$ bound. While under Neumaier's claw bound, the value of μ is asymptotically determined by n and k ($\mu \sim k^2/n$, see Theorem 2.3 (e)), the value of λ can vary widely, thus the significance of an improved bound on λ that contributed to reducing the exponent of the exponent to 1/5.

3. Proof of clique structure

We now give a simple proof of Theorem 1.5. We use the following lemma, based on a lemma of Spielman for strongly regular graphs [15, Lemma 17].

If u is a vertex of a graph G, we write $N(u)$ for the neighborhood of u, i. e., the set of vertices adjacent to u, and write $N^+(u) = N(u) \cup \{u\}$.

Lemma 3.1. *Let G be a graph on k vertices which is regular of degree λ and such that any pair of nonadjacent vertices have at most $\mu - 1$ common neighbors. Then for any vertex u, there are at most $(k-\lambda-1)(\mu-1)$ ordered pairs of nonadjacent vertices in $N(u)$.*

Proof. Let X be the number of ordered pairs of nonadjacent vertices in $N(u)$, and let K be the number of ordered pairs of *adjacent* vertices in $N(u)$, so $K + X = \lambda(\lambda-1)$. Let P be the number of ordered pairs (x,y) of vertices such that (u,x,y) induces a path of length two. For every neighbor x of u, and every neighbor $y \neq u$ of x, the pair (x,y) is counted in either K or P, so $K+P = \lambda(\lambda-1)$ and so $P = X$. On the other hand, there are $k-\lambda-1$ vertices not adjacent to u, each of which has at most $\mu-1$ common neighbors with u, and so $X = P \leq (k-\lambda-1)(\mu-1)$. □

Lemma 3.2. *Let G be a graph on k vertices which is regular of degree λ and such that any pair of nonadjacent vertices have at most $\mu-1$ common neighbors. Suppose that $k\mu = o(\lambda^2)$. Then there is a partition of $V(G)$ into maximal cliques of order $\sim \lambda$, and all other maximal cliques of G have order $o(\lambda)$.*

Proof. Fix a vertex u and consider the induced subgraph H of G on $N^+(u)$. Suppose x and y are distinct non-adjacent vertices of H. They have at most $\mu-1$ common neighbors in H, so there are at least $\lambda - \mu$ vertices in $H \setminus \{x,y\}$ which are not common neighbors of x and y. Hence, at least one of x and y has codegree at least $\kappa := (\lambda - \mu)/2$ in H (i. e., degree at most $\lambda - \kappa$). Let D be the set of vertices in W of codegree at least κ, and let $C = W \setminus D$. It follows that C is a clique, and clearly $u \in C$.

Now by Lemma 3.1, $|D|\kappa < (k - \lambda - 1)(\mu - 1) = o(\lambda^2)$, and so $|D| = o(\lambda)$. In particular, $C \sim \lambda$, and every element of D has at least one non-neighbor in C. Hence, C is a maximal clique, and every element not in C, having at least one non-neighbor in C, has at most μ neighbors in C. Thus, any maximal clique which contains u as well as a vertex not in C has order at most $|D| + \mu = o(\lambda)$. □

Theorem 1.5 then follows immediately by applying Lemma 3.2 to the graphs induced by G on $N(u)$ for $u \in V$. □

4. Asymptotic Delsarte cliques

We finally give a prove of Theorem 1.9.

Suppose that G is distance-regular with intersection numbers b_i, c_i, where for any pair u, v of vertices u at distance i, the number of neighbors of u at distance $i+1$ from v is b_i and the number of neighbors of u at distance $i-1$ from v is c_i (cf. [6, Chap. 4.1]). (Note that every distance-regular graph is sub-amply regular with parameters $\lambda = b_0 - b_1 - 1$ and $\mu = c_2$.)

Lemma 4.1. *Let G be a distance-regular graph with least eigenvalue s. Then*

$$\lambda + \frac{k}{\lambda} > \frac{k}{|s|}.$$

Proof. Let $\{u_0, u_1, \ldots, u_d\}$ be the standard sequence of polynomials for G (see, e.g., [6, Section 4.1B]). It is well known that $u_0(x) = 1$, $u_1(x) = x/k$, and

$$c_1 u_0(x) + a_1 u_1(x) + b_1 u_2(x) = x u_1(x) \tag{6}$$

(cf. eq. (13) in [6, Section 4.1B]). Furthermore, if θ_i is the ith greatest eigenvalue of G, then the sequence $\{u_0(\theta_i), u_1(\theta_i), \ldots, u_d(\theta_i)\}$ has exactly i sign changes [6, Corollary 4.1.2]. In particular, the sequence $\{u_0(s), u_1(s), \ldots, u_d(s)\}$ is alternating, and so $u_2(s) > 0$. Hence, from Eq. (6),

$$\lambda - s = \frac{k}{-s} + \frac{k^2}{-s} u_2(s) > \frac{k}{-s}.$$

So, if $\lambda \leq k/|s|$, then $\lambda + k/\lambda > \lambda - s > k/|s|$. Thus, in any case, $\lambda + k/\lambda > k/|s|$. □

We note that Lemma 4.1 is a slight improvement over Lemma 3.2 of [11] which states $\lambda + |s| > k/|s|$. The method of proof is virtually identical.

Proof of Theorem 1.9. Since $|s|\mu = o(\lambda)$, by Lemma 4.1, we have

$$k\mu < |s|\mu \left(\lambda + \frac{k}{\lambda} \right) = o(\lambda^2 + k).$$

We therefore have $k\mu = o(\lambda^2)$, so by Theorem 1.5, G has a clique geometry $\mathcal{C}$ with special cliques of order $\sim \lambda$. By Lemma 1.6, we have $\lambda \lesssim 1 + k/|s|$. But since $\lambda \gtrsim k/|s|$ by Lemma 4.1, it follows that $k/|s|$ is unbounded and the special cliques have order $\sim k/|s|$. □

5. Conclusion

We have derived a new bound on the parameter λ of sub-amply regular graphs, and hence for distance-regular graphs. In the particular case of strongly regular graphs, the improved bound contributed to the improved complexity estimate for testing isomorphism of strongly regular graphs [2]. Our proof relies on Metsch's clique geometry when $k\mu = o(\lambda^2)$.

Examples of this clique structure arise in geometric strongly regular graphs, in in particular in point-graphs of partial geometries, including Steiner designs and their duals.

We are not aware of infinite families of sub-amply regular graphs satisfying $k\mu = o(\lambda^2)$ which are not in fact point-graphs of geometric 1-designs. If such families do not exist, this would considerably strengthen the conclusion of Theorem 1.5.

In fact, we are not aware of even a single non-geometric sub-amply regular graph satisfying inequality (2).

We note that if any examples of non-geometric strongly regular graphs satisfying inequality (2) exist, they will be rather large. No example has fewer than 1500 vertices; this was verified by checking all feasible parameters of strongly regular graphs in the table compiled by Andries Brouwer [5].

Appendix A. Spielman's bounds on the parameters of strongly regular graphs

In this section we prove Theorem 2.3.

Let G be a strongly regular graph with parameters (n, k, λ, μ). Let $k \geq r \geq s$ denote the eigenvalues of G. We use the following standard observations (cf. [6, Ch. 1.3]).

Proposition A.1. (i) $(n-k-1)\mu = k(k-\lambda-1)$

(ii) $-rs = k - \mu$

(iii) $r + s = \lambda - \mu$

(iv) *If G is not a conference graph, then r and s are integers.*

Proof of Theorem 2.3. From Prop. A.1 (i) and $k = o(n)$, it follows that $\mu = o(k)$, proving item (a) of the Theorem. Then Proposition A.1 (ii) gives $k \sim -rs$.

For any strongly regular graph, $s \leq -1$ (see, e.g., [6, Corollary 3.5.4]). Therefore $2(-s-1)(\mu+1+s) + s \leq s^2(\mu+1)$, and so, assuming the claw bound, we have

$$r \leq s^2(\mu+1). \tag{7}$$

Combining this with $k \sim -rs$ gives $k \lesssim -s^3(\mu+1)$, hence

$$-s \gtrsim \left(\frac{k}{\mu+1}\right)^{1/3}.$$

Combining this last inequality with $k \sim -rs$, we obtain

$$r \lesssim k^{2/3}(\mu+1)^{1/3},$$

proving item (b) of the Theorem.

Now $\mu = o(k^{2/3}\mu^{1/3})$ by (a), and so $\lambda \lesssim k^{2/3}(\mu+1)^{1/3}$ follows from (b) and Prop. A.1 (iii), proving item (c) of the Theorem. But then $\lambda = o(k)$ follows by items (a) and (c), proving item (d). Finally, item (e) follows from item (d) by Prop. A.1 (i). □

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