# Asymptotic Diagonalizations of a Linear Ordinary Differential System 

Feipeng Xie<br>Western Michigan University

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# ASYMPTOTIC DIAGONALIZATIONS OF A LINEAR ORDINARY DIFFERENTIAL SYSTEM 

by<br>Feipeng Xie

A Dissertation<br>Submitted to the<br>Faculty of The Graduate College<br>in partial fulfillment of the requirements for the<br>Degree of Doctor of Philosophy<br>Department of Mathematics and Statistics

Western Michigan University
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# ASYMPTOTIC DIAGONALIZATIONS OF A LINEAR ORDINARY DIFFERENTIAL SYSTEM 

Feipeng Xie, Ph.D.<br>Western Michigan University, 1995

Consider the following system:

$$
\frac{d y}{d t}=A(\dot{t}) y, \quad t \in I_{0}=\left[t_{0},+\infty\right)
$$

where $A(t)=\Lambda(t)+R(t)$ with $\Lambda(t)$ a diagonal matrix and $t_{0}$ is a finite number. In this dissertation, we will discuss the asymptotic solution of the system for the following two cases:

1. The real parts of the diagonal elements of $\Lambda(t)$ are separated from each other for $t \in I_{0}$ by a fixed number and matrix $R(t) \in L^{p}\left(I_{0}\right)$, where $p \geq 1$.
2. The real parts of some eigenvalues of $A(t)$ have same limit as $t$ goes to infinite.

Harris and Lutz (1977) proved that, in Case 1, the asymptotic solution of the system can be obtained by performing certain transformations repeatedly $k$ times ( $k$ satisfies $2^{k} \geq p$ ). In Chapter II of this dissertation, the result of Harris and Lutz is generalized and the asymptotic solutions are given directly.

The asymptotic solution of the system in Case 2 is discussed in Chapter III and some known results are extended. In Chapter IV the main result of Chapter III is used to find the deficiency index of a certain fourth order ordinary self-adjoint differential operator.

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I dedicate this dissertation to my dear wife, Mei Yi. We have been struggling together with all the difficulties in these years.

Feipeng Xie

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## CHAPTER I

## INTRODUCTION

The result of N. Levinson [31] in 1948, known as the Levinson Theorem, plays an important role in the study of the asymptotic behavior of solution of a linear system of differential equations, but as it is mentioned in the preface of [14] the initial applications of the theorem were only of a very straightforward kind. In order to find the asymptotic solutions of the linear systems of differential equations for which the Levinson Theorem can not be used directly, we have seen many extensions of this important theorem in last forty years (e.g. R. Bellman [1], [2], K. Chiba and T. Kimura [3], E.A. Coddington and N. Levinson [4], W.A. Coppel [5], A. Devinatz [6], A. Devinatz and J. Kaplan [9], M.S.P. Eastham [10], H. Gingold [17], H. Gingold, P.F. Hsieh and Y. Sibuya [18], W.A. Harris, Jr. and D.A. Lutz [21], [22], [23], W.A. Harris, Jr. and Y. Sibuya [24], P. Hartman [25], P. Hartman and A. Wintner [26], [27], P.F. Hsieh and F. Xie [28]). In this Chapter we will give a brief introduction of the development of this theory. A good source for the general theory can be found in the book by Eastham [14].

Consider the linear system of differential equations

$$
\begin{equation*}
\frac{d y}{d t}=A(t) y \tag{1.1}
\end{equation*}
$$

where y is an n -dimensional vector and $A(t)$ is an $n \times n$ matrix continuous on $I_{0}=\left[t_{0}, \infty\right),\left(t_{0}:\right.$ finite $)$. In order to state the Levinson theorem we need :

Assumption 1.1. The matrix $A(t)$ is in the form

$$
A(t)=\Lambda(t)+R(t)
$$

where $\Lambda(t)=\operatorname{diag}\left\{\lambda_{1}(t), \lambda_{2}(t), \cdots, \lambda_{n}(t)\right\}$, with $\lambda_{j}(t),(j=1,2, \cdots, n)$, continuous on $I_{0}$.
Assumption 1.2. Let $D_{j k}(t)$ be the real part of $\lambda_{j}(t)-\lambda_{k}(t)$, i.e. $D_{j k}(t)=$ $\mathcal{R}\left(\lambda_{j}(t)-\lambda_{k}(t)\right), \quad(j, k=1,2, \cdots, n)$. For each fixed j , the set of positive integers $\{1,2, \cdots, n\}$ is the union of two disjoint subsets $P_{j 1}$ and $P_{j 2}$, where
(i) $k \in P_{j 1}$ if

$$
\lim _{t \rightarrow+\infty} \int_{t_{0}}^{t} D_{j k}(\tau) d \tau=-\infty, \quad \int_{s}^{t} D_{j k}(\tau) d \tau<K, \quad t_{0} \leq s \leq t
$$

for some positive number K ;
(ii) $k \in P_{j 2}$ if

$$
\int_{s}^{t} D_{j k}(\tau) d \tau<K, \quad t_{0} \leq t \leq s
$$

for some positive number K .
Assumption 1.3. The matrix $R(t)$ is an $n \times n$ matrix satisfying

$$
R(t) \in L^{1}\left(I_{0}\right)
$$

A version of the Levinson theorem can be stated as the following.
Theorem L [31]. Under the Assumptions 1.1, 1.2 and 1.3, there exists an $n \times n$ matrix $Q(t)$ such that

1. the derivative $Q^{\prime}(t)$ exists, and the entries of $Q(t)$ and $Q^{\prime}(t)$ are continuous in $t$ on the interval $I_{0}$,
2. $\lim _{t \rightarrow+\infty} Q(t)=0$,
3. the transformation:

$$
y=\left[I_{n}+Q(t)\right] z
$$

changes system (1.1) into

$$
\frac{d z}{d t}=\Lambda(t) z
$$

on the interval $I_{0}$, where $I_{n}$ is the $n \times n$ identity matrix.
Levinson also considered the following system:

$$
\begin{equation*}
\frac{d y}{d t}=[A+V(t)+R(t)] y \tag{1.2}
\end{equation*}
$$

where A is a constant matrix, $V(t)$ and $R(t)$ are $n \times n$ continuous matrices.
Assumption 1.4. The entries of matrix $V^{\prime}(t)$ are continuous for $t \geq t_{0}, V(t) \rightarrow 0$ as $t \rightarrow \infty$, and $\left|V^{\prime}(t)\right| \in L^{1}\left(I_{0}\right)$.

Theorem LE [31]. Assume that: (i) A is a constant matrix with distinct eigenvalues; (ii) $V(t)$ satisfies Assumption 1.4; (iii) $R^{\prime}(t)$ is continuous for $t \geq t_{0}$, $R(t) \in L^{1}\left(I_{0}\right)$; and (iv) the eigenvalues of the matrix $A+V(t)$ satisfy Assumption 1.2. Then there exists an $n \times n$ matrix $Q(t)$ such that

1. the derivative $Q^{\prime}(t)$ exists, and the entries of $Q(t)$ and $Q^{\prime}(t)$ are continuous in $t$ on the interval $I_{0}$,
2. $\lim _{t \rightarrow+\infty} Q(t)=0$,
3. the transformation:

$$
y=P\left[I_{n}+Q(t)\right] z
$$

changes system (1.2) into

$$
\frac{d z}{d t}=\Lambda(t) z
$$

on the interval $I_{0}$, where $P$ is a constant matrix such that $P^{-1} A P$ is a diagonal matrix and $\Lambda(t)$ is a diagonal matrix with diagonal elements the eigenvalues of $A+V(t)$.

Assumption 1.2 and 1.3 are not independent, but related each other. If $D_{j k}(t),(j, k=1,2, \cdots, n)$, satisfy some stronger condition, then $R(t)$ does not need to be in $L^{1}\left(I_{0}\right)$, but in $L^{p}\left(I_{0}\right)$ for some $p>1$.

An important extension of Theorem $L$ for system (1.1) is the HartmanWintner theorem [27]. In order to state this theorem we need another two assumptions.

Assumption 1.5. There exists a positive constant $\delta$ such that for each pair of indices j and $\mathrm{k},(j \neq k)$,

$$
\left|D_{j k}(t)\right| \geq \delta>0, \quad \text { for } t \in I_{0}
$$

where $D_{j k}(t),(j, k=1,2, \cdots, n)$, are defined in assumption 1.2.
Assumption 1.6. $R(t) \in L^{p}\left(I_{0}\right), \quad$ for some constant $p \geq 1$.
A version of the Hartman-Wintner Theorem can be given as follows:
Theorem HW [27]. Under the Assumptions 1.1, 1.5 and 1.6 with $1 \leq p \leq 2$, there exists an $n \times n$ matrix $Q(t)$ such that

1. the derivative $Q^{\prime}(t)$ exists, and the entries of $Q(t)$ and $Q^{\prime}(t)$ are continuous in $t$ on the interval $I_{0}$,
2. $\lim _{t \rightarrow+\infty} Q(t)=0$,
3. the transformation:

$$
y=\left[I_{n}+Q(t)\right] z
$$

changes system (1.1) into

$$
\frac{d z}{d t}=[\Lambda(t)+\operatorname{diag}\{R(t)\}] z
$$

on the interval $I_{0}$, where $I_{n}$ is the $n \times n$ identity matrix.
Hartman-Wintner Theorem has been further generalized by Harris and Lutz [23].

Theorem HL [23]. Under the Assumptions 1.1, 1.5 and 1.6 (for any $p \geq 1$ ), there exists a sequence of $n \times n$ matrices $Q_{i}(t),(i=0,1, \cdots, k-1)$, such that

1. the derivatives $Q_{i}^{\prime}(t)$ exist, and the entries of $Q_{i}(t)$ and $Q_{i}^{\prime}(t)$ are continuous in $t$ on the interval $I_{0}$ for all $i,(i=0,1, \cdots, k-1)$,
2. $\lim _{t \rightarrow+\infty} Q_{i}(t)=0,(i=0,1, \cdots, k-1)$,
3. the transformation:

$$
y=\prod_{i=0}^{k-1}\left[I+Q_{i}(t)\right] z_{k}
$$

changes system (1.1) into

$$
\frac{d z_{k}}{d t}=\Lambda_{k}(t) z_{k}
$$

on the interval $\Lambda_{0}$, where $\Lambda_{k}(t)$ is a diagonal matrix which depends on all transformation matrices $Q_{i}(t), \quad(i=0,1, \cdots, k-1)$, and $k$ satisfies $2^{k} \geq p$.

The inconvenience of the application of Theorem HL is that in order to find $\Lambda_{k}(t)$ one has to find $Q_{i}(t), \quad(i=0,1, \cdots, k-1)$. In the next Chapter, a different approach will be taken, and a new result Theorem 2.1.1 will be introduced, where like the results of Theorem $L$ and Theorem HW, the asymptotic solutions will be given directly. It is worth to point out that the application range of Theorem 2.1.1 strictly includes the application range of Theorem HL.

Now let us consider another case, if

$$
\lim _{t \rightarrow+\infty} D_{j k}(t)=0, \quad \text { for some pair of indices } \mathrm{j} \text { and } \mathrm{k},(j \neq k)
$$

then Assumption 1.5 is not satisfied. In this case, if $D_{j k}(t),(j, k=1,2, \cdots, n)$, still satisfy some condition which is a little stronger than Assumption 1.2, then $R(t)$ needs only to partially satisfy Assumption 1.3.

Let

$$
\begin{equation*}
R(t)=\left\{r_{j k}(t)\right\}_{j, k=1}^{n} \tag{1.3}
\end{equation*}
$$

and

$$
\begin{equation*}
r_{1}(t)=\max _{j>k}\left|r_{j k}(t)\right|, \quad r_{2}(t)=\max _{j<k}\left|r_{j k}(t)\right| \tag{1.4}
\end{equation*}
$$

Assumption 1.7. There exist two positive constants $\delta$ and K and a constant $\alpha$, $(0 \leq \alpha<1)$, independent of $t$, such that for each pair of indices j and k , $(j, k=1,2, \ldots, n ; k \neq j)$, either

$$
\begin{equation*}
\exp \left\{\int_{t}^{s} D_{j k}(\tau) d \tau\right\} \leq K \exp \left\{-\delta\left(s^{1-\alpha}-t^{1-\alpha}\right)\right\}, \text { for all } s \geq t \tag{1.5}
\end{equation*}
$$

or

$$
\begin{equation*}
\exp \left\{\int_{t}^{s} D_{j k}(\tau) d \tau\right\} \leq K \exp \left\{-\delta\left(t^{1-\alpha}-s^{1-\alpha}\right)\right\}, \text { for all } s \leq t \tag{1.6}
\end{equation*}
$$

where $D_{j k}(t)=\mathcal{R}\left(\lambda_{j}(t)-\lambda_{k}(t)\right), \quad(j, k=1,2, \cdots, n)$, are defined in Assumption 1.2 .

Assumption 1.8. The matrix $R(t)$ is continuous and satisfies:

$$
\begin{equation*}
|R(t)|=o\left(t^{-\alpha}\right), \quad \text { as } \quad t \rightarrow+\infty \tag{1.7}
\end{equation*}
$$

and either

$$
\begin{equation*}
\int_{t_{0}}^{+\infty} t^{-\alpha} \max _{s \geq t}\left[s^{\alpha} r_{1}(s)\right] d t<+\infty \tag{1.8}
\end{equation*}
$$

or

$$
\begin{equation*}
\int_{t_{0}}^{+\infty} t^{-\alpha} \max _{s \geq t}\left[s^{\alpha} r_{2}(s)\right] d t<+\infty \tag{1.9}
\end{equation*}
$$

In [28] we proved the following theorem:
Theorem HX [28]. Under Assumptions 1.1, 1.7 and 1.8, there exists an $n \times n$ matrix $Q(t)$ such that

1. the derivative $Q^{\prime}(t)$ exists, and the entries of $Q(t)$ and $Q^{\prime}(t)$ are continuous in $t$ on the interval $I_{0}$,
2. $\lim _{t \rightarrow+\infty} Q(t)=0$,
3. the transformation:

$$
\begin{equation*}
y=\left[I_{n}+Q(t)\right] z \tag{1.10}
\end{equation*}
$$

changes systems (1.1) into

$$
\begin{equation*}
\frac{d z}{d t}=[\Lambda(t)+\operatorname{diag}\{R(t)\}] z \tag{1.11}
\end{equation*}
$$

on the interval $I_{0}$, where $I_{n}$ is the $n \times n$ identity matrix.

In Chapter III, Theorem HX will be generalized. There we will consider the case $D_{j k}(t)=O(1 / t)$, for some $j, k \in\{1,2, \cdots, n\}$.

Asymptotic integration formulas for linear systems of differential equations with multiple eigenvalues for $\lim _{t \rightarrow+\infty} A(t)$ have important application in the deficiency index problem for self-adjoint ordinary differential operators. In Chapter IV, we will apply the main result of Chapter III to the deficiency index problem for certain fourth order differential operators.

## CHAPTER II

## A GENERALIZATION OF THEOREM HW

### 2.1 Introduction and Main Theorems

In Chapter I, we learned that for the system:

$$
\begin{equation*}
\frac{d y}{d t}=A(t) y, \quad t \in I_{0}=\left(t_{0},+\infty\right) \tag{2.1}
\end{equation*}
$$

if $A(t)$ satisfies Assumptions $1.1,1.5$ and 1.6 (with $p \geq 1$ ), then the asymptotic solution of (2.1) can be obtained via a sequence of linear transformations as stated in Theorem HL. In practice, however, it is sometimes very difficult to find all the transformation matrices $Q_{i}(t)$ or their suitable approximations, $(i=0,1, \cdots, k-$ 1), on which the asymptotic integration formula depends. In this Chapter we will take a different approach and give another generalization of Theorem HW from Chapter I. This will provide the asymptotic integration formula directly.

Assumption 2.1.1. The matrix $A(t)$ is in the form

$$
A(t)=\Lambda(t)+P(t)+R(t)
$$

where $\Lambda(t)=\operatorname{diag}\left\{\lambda_{1}(t), \lambda_{2}(t), \cdots, \lambda_{n}(t)\right\}$ with $\lambda_{j}(t), \quad(j=1,2, \cdots, n)$, continuous on $I_{0}$, and $R(t) \in L^{1}\left(I_{0}\right)$.

Assumption 2.1.2. There exist two positive constants $K$ and $\delta$ such that for each pair of indices j and $\mathrm{k},(j, k=1,2, \ldots, n ; k \neq j)$, we have

$$
\begin{equation*}
\exp \left\{\int_{t}^{s} D_{j k}(\tau) d \tau\right\} \leq K \exp \{-\delta(s-t)\}, \text { for } s \geq t \text { and } j>k \tag{2.2}
\end{equation*}
$$

and

$$
\begin{equation*}
\exp \left\{\int_{t}^{s} D_{j k}(\tau) d \tau\right\} \leq K \exp \{-\delta(t-s)\}, \text { for } s \leq t \text { and } j<k \tag{2.3}
\end{equation*}
$$

where $D_{j k}(t)=\mathcal{R}\left(\lambda_{j}(t)-\lambda_{k}(t)\right), \quad(j, k=1,2, \cdots, n)$, are defined in Assumption 1.2.

Assumption 2.1.3. The matrix $P(t)$ satisfies:

$$
\|P(t)\| \leq p(t), \text { for some } p(t) \in L^{m}\left(I_{0}\right), \quad m \geq 1
$$

Theorem 2.1.1. Under Assumptions 2.1.2, 2.1.3 and 2.1.1 with $R(t) \equiv 0$, there exists a linear transformation:

$$
\begin{equation*}
y=\left[I_{n}+T(t)\right] z \tag{2.4}
\end{equation*}
$$

with $T(t)=\left\{T_{j k}(t)\right\}$ and $T_{j j}(t)=0$, for all $j$, such that

1. for every pair of indices $j$ and $k,(j \neq k)$, the derivative $T_{j k}^{\prime}(t)$ exists, and both $T_{j k}(t)$ and $T_{j k}^{\prime}(t)$ are continuous on the interval $I_{0} ;$
2. $\lim _{t \rightarrow+\infty} T_{j k}(t)=0, \quad(j \neq k) ;$
3. the transformation (2.4) reduces the system

$$
\begin{equation*}
\frac{d y}{d t}=[\Lambda(t)+P(t)] y \tag{2.5}
\end{equation*}
$$

into

$$
\begin{equation*}
\frac{d z_{j}}{d t}=\left[\lambda_{j}(t)+p_{j j}(t)+\sum_{h \neq j} p_{j h}(t) T_{h j, r}(t)+d_{j}(t)\right] z_{j}, \quad(j=1,2, \cdots, n) \tag{2.6}
\end{equation*}
$$

where $p_{i j}(t),(i, j=1,2, \cdots, n)$, denote the entries of the $n \times n$ matrix $P(t)$; $d_{j}(t)=\sum_{h \neq j} p_{j h}(t) T_{h j}(t)-\sum_{h \neq j} p_{j h}(t) T_{h j, r}(t) \in L^{1}\left(I_{0}\right)$, for all $j$, and $T_{j l, r}(t)$ which are approximations of $T_{j l}, j, l=1,2, \cdots, n, j \neq l$, are defined as follows:

$$
\left\{\begin{align*}
T_{j l, k+1}(t) & =\int_{\tau j l}^{t} \exp \left\{\int_{s}^{t}\left(\lambda_{j}(\tau)-\lambda_{l}(\tau)\right) d \tau\right\} U_{j l}\left(s, T_{k}(s)\right) d s, \text { with }  \tag{2.7}\\
T_{j l, 0}(t)= & 0, \text { and } \\
U_{j l}\left(t, T_{k}(t)\right) & =p_{j l}(t)+\sum_{h \neq l} p_{j h}(t) T_{h l, k}(t) \\
& \quad-T_{j l, k}(t)\left[p_{l l}(t)+\sum_{h \neq l} p_{l h}(t) T_{h l, k}(t)\right]
\end{align*}\right.
$$

with $T_{k}$ to be approximations of $T(t), k=0,1,2, \cdots, r-1$, and $r$ to be the smallest non-negative integer such that $r \geq m-2$, and $\tau_{j l}$ to be defined as the follows:

$$
\tau_{j l}= \begin{cases}t_{0}, & \text { if } j<l \\ +\infty, & \text { if } j>l\end{cases}
$$

In order to obtain a more general theorem, we need to first establish:
Lemma 2.1.1. If there exists an $n \times n$ transformation matrix $G(t)$ such that: (i) the derivative $G^{\prime}(t)$ exists, and the entries of $G(t)$ and $G^{\prime}(t)$ are continuous in $t$ on the interval $I_{0}$, (ii) $\lim _{t \rightarrow+\infty} G(t)=0$, (iii) the transformation: $z=\left[I_{n}+G(t)\right] w$ changes systems $z^{\prime}=[\Lambda(t)+P(t)] z$ into

$$
\frac{d w}{d t}=\tilde{\Lambda}(t) w
$$

where $\tilde{\Lambda}(t)$ is a diagonal matrix satisfying Assumption 1.2, then under Assumption 2.1.1, there exists an $n \times n$ transformation matrix $Q(t)$ such that:

1. the derivative $Q^{\prime}(t)$ exists, and the entries of $Q(t)$ and $Q^{\prime}(t)$ are continuous in $t$ on the interval $I_{0}$,
2. $\lim _{t \rightarrow+\infty} Q(t)=0$,
3. the transformation:

$$
\begin{equation*}
y=\left[I_{n}+Q(t)\right] z \tag{2.8}
\end{equation*}
$$

changes system (2.1) into

$$
\begin{equation*}
\frac{d z}{d t}=[\Lambda(t)+P(t)] z \tag{2.9}
\end{equation*}
$$

on the interval $I_{0}$.

The proofs of the Theorem 2.1.1 and Lemma 2.1.1 will be given in Section 2.3 and Section 2.4, respectively. Combine Lemma 2.1.1 and Theorem 2.1.1, we directly have:

Theorem 2.1.2. Under Assumptions 2.1.1, 2.1.2, and 2.1.3, there exist $n \times n$ matrices $Q(t)$ and $T(t)=\left\{T_{j k}(t)\right\}$ with $T_{j j}(t)=0$, for all $j$, such that

1. the derivatives $Q^{\prime}(t)$ and $\dot{T}^{\prime}(t)$ exist, and the entries of $Q(t), T(t), Q^{\prime}(t)$ and $T^{\prime}(t)$ are all continuous in $t$ on the interval $I_{0}$,
2. $\lim _{t \rightarrow+\infty} Q(t)=0$, and $\lim _{t \rightarrow+\infty} T(t)=0$,
3. the transformation:

$$
\begin{equation*}
y=\left[I_{n}+Q(t)\right]\left[I_{n}+T(t)\right] z \tag{2.10}
\end{equation*}
$$

changes systems (2.1) into

$$
\begin{equation*}
\frac{d z_{j}}{d t}=\left[\lambda_{j}(t)+p_{j j}(t)+\sum_{h \neq j} p_{j h}(t) T_{h j, r}(t)+d_{j}(t)\right] z_{j}, j=1, \cdots, n \tag{2.11}
\end{equation*}
$$

where $p_{i j}(t),(i, j=1,2, \cdots, n)$, denote the entries of $n \times n$ matrix $P(t)$; $d_{j}(t) \in L^{1}\left(I_{0}\right)$, for all $j$, and $T_{h j, r}$ are defined as in (2.7).

Remark for Assumption 2.1.2. If $D_{j k}(t)$ satisfy Assumption 1.5, i.e. $\left|D_{j k}(t)\right| \geq$ $\delta>0$, then Assumption 2.1.2 follows. On the other hand Assumption 2.1.2 gives the separation of real parts of $\lambda_{j}(t)$ in the average sense. For instance, if $\lambda_{1}(t)=1$
and $\lambda_{2}(t)=\sin (t)$, then Assumption 1.5 is not satisfied, but Assumption 2.1.2 is satisfied, since

$$
\exp \left\{\int_{t}^{s} D_{21}(\tau) d \tau\right\} \leq e^{2} e^{-(s-t)}, \text { for } s \geq t
$$

and

$$
\exp \left\{\int_{t}^{s} D_{12}(\tau) d \tau\right\} \leq e^{2} e^{-(t-s)}, \text { for } s \leq t
$$

### 2.2 Examples

Three examples will now be given to show that the above new Theorems are often much simpler to apply than the classical results and have a wider range of application.

Example 1. Consider the differential equation:

$$
\begin{equation*}
x^{\prime \prime}-(1+\phi(t)) x=0, \quad \phi(t) \in L^{m}\left(t \geq t_{0}\right) \tag{2.12}
\end{equation*}
$$

It can be changed into the following equivalent $2 \times 2$ system of differential equations:

$$
y^{\prime}=\left\{\left[\begin{array}{cc}
-1 & 0 \\
0 & 1
\end{array}\right]+\frac{1}{2}\left[\begin{array}{cc}
-\phi & -\phi \\
\phi & \phi
\end{array}\right]\right\} y
$$

by means of

$$
\left[\begin{array}{c}
x \\
x^{\prime}
\end{array}\right]=\left[\begin{array}{cc}
1 & 1 \\
-1 & 1
\end{array}\right] y
$$

For $m=3$, the asymptotic solutions of differential equation (2.12) have been given by Bellman [2]. The method which Bellman used is complicated. In fact Bellman himself wrote that his method is in practice difficult to use because the algebraic complexities become overwhelming for larger $\mathrm{m}([2], \mathrm{p} .133)$.

The above differential equation with $m=3$ has also been studied by Harris and Lutz as an application of Theorem HL [23]. The treatment there is not simple
either, and becomes more difficult as $m$ gets larger. By comparison to the two approachs above, Theorem 2.1.1 is easy to apply, as will now be shown.

Case 1. If $\phi(t) \in L^{3}\left(I_{0}\right)$, by Theorem 2.1.1 with $r=1$, we have linear transformation $y_{1}=z_{1}+T_{12} z_{2}$ and $y_{2}=z_{2}+T_{21} z_{1}$, such that

$$
\begin{aligned}
& \frac{d z_{1}}{d t}=\frac{1}{2}\left[-2-\phi(t)-\phi(t) T_{21,1}(t)+d_{1}(t)\right] z_{1} \\
& \frac{d z_{2}}{d t}=\frac{1}{2}\left[2+\phi(t)+\phi(t) T_{12,1}(t)+d_{2}(t)\right] z_{2}
\end{aligned}
$$

where $d_{1}(t), d_{2}(t) \in L^{1}\left(I_{0}\right)$,

$$
\begin{equation*}
T_{21,1}(t)=\frac{1}{2} \int_{+\infty}^{t} e^{-2(s-t)} \phi(s) d s \tag{2.13}
\end{equation*}
$$

and

$$
\begin{equation*}
T_{12,1}(t)=-\frac{1}{2} \int_{t_{0}}^{t} e^{-2(t-s)} \phi(s) d s \tag{2.14}
\end{equation*}
$$

Since

$$
\left[\begin{array}{c}
x \\
x^{\prime}
\end{array}\right]=\left[\begin{array}{cc}
1 & 1 \\
-1 & 1
\end{array}\right]\left[\begin{array}{cc}
1 & o(1) \\
o(1) & 1
\end{array}\right]\left[\begin{array}{cc}
z_{1} & 0 \\
0 & z_{2}
\end{array}\right]
$$

we have

$$
x_{1}=(1+o(1)) \exp \left\{\int_{t_{0}}^{t}\left[-1-\frac{1}{2} \phi(s)-\frac{1}{4} \phi(s) \int_{+\infty}^{s} e^{-2(\tau-s)} \phi(\tau) d \tau\right] d s\right\},
$$

and

$$
x_{2}=(1+o(1)) \exp \left\{\int_{t_{0}}^{t}\left[1+\frac{1}{2} \phi(s)-\frac{1}{4} \phi(s) \int_{t_{0}}^{s} e^{-2(s-\tau)} \phi(\tau) d \tau\right] d s\right\}
$$

Case 2. If $\phi(t) \in L^{4}\left(I_{0}\right)$, by Theorem 2.1.1 with $r=2$, we have linear transformation $y_{1}=z_{1}+T_{12} z_{2}$ and $y_{2}=z_{2}+T_{21} z_{1}$, such that

$$
\begin{aligned}
& \frac{d z_{1}}{d t}=\frac{1}{2}\left[-2-\phi(t)-\phi(t) T_{21,2}(t)+d_{1}(t)\right] z_{1} \\
& \frac{d z_{2}}{d t}=\frac{1}{2}\left[2+\phi(t)+\phi(t) T_{12,2}(t)+d_{2}(t)\right] z_{2}
\end{aligned}
$$

where $d_{1}(t), d_{2}(t) \in L^{1}\left(I_{0}\right)$,

$$
\begin{array}{r}
T_{21,2}(t)=\frac{1}{2} \int_{+\infty}^{t} e^{-2(s-t)}\left[\phi(s)+2 \phi(s) T_{21,1}(s)\right] d s \\
+\frac{1}{2} \int_{+\infty}^{t} e^{-2(s-t)} \phi(s) T_{21,1}^{2}(s) d s \tag{2.16}
\end{array}
$$

and

$$
\begin{array}{r}
T_{12,2}(t)=-\frac{1}{2} \int_{t_{0}}^{t} e^{-2(t-s)}\left[\phi(s)+2 \phi(s) T_{12,1}(s)\right] d s \\
-\frac{1}{2} \int_{t_{0}}^{t} e^{-2(t-s)} \phi(s) T_{12,1}^{2}(s) d s \tag{2.18}
\end{array}
$$

Note that since

$$
\phi(t) \int_{+\infty}^{t} e^{-2(s-t)} \phi(s) T_{21,1}^{2}(s) d s \text { and } \phi(t) \int_{t_{0}}^{t} e^{-2(t-s)} \phi(s) T_{12,1}^{2}(s) d s \in L^{1}\left(I_{0}\right)
$$

we have
$x_{1}=(1+o(1)) \exp \left\{\int_{t_{0}}^{t}\left[-1-\frac{1}{2} \phi(s)-\frac{1}{4} \phi(s) \int_{+\infty}^{s} e^{-2(\tau-s)}\left[\phi(\tau)+2 \phi(\tau) T_{21,1}(\tau)\right] d \tau\right] d s\right\}$,
and
$x_{2}=(1+o(1)) \exp \left\{\int_{t_{0}}^{t}\left[1+\frac{1}{2} \phi(s)-\frac{1}{4} \phi(s) \int_{t_{0}}^{s} e^{-2(s-\tau)}\left[\phi(\tau)+2 \phi(\tau) T_{12,1}(\tau)\right] d \tau\right] d s\right\}$,
where $T_{21,1}(t)$ and $T_{12,1}(t)$ are defined in (2.13) and (2.14), respectively.
Case 3. In general, if $\phi \in L^{m}\left(I_{0}\right)$, for $m>2$, by Theorem 2.1.1, with $r$ taken to be the smallest integer such that $r \geq m-2$, we have the following asymptotic solution:

$$
x_{1}=(1+o(1)) \exp \left\{\int_{t_{0}}^{t}\left[-1-\frac{1}{2} \phi(s)-\frac{1}{2} \phi(s) T_{21, r}(s)\right] d s\right\}
$$

and

$$
x_{2}=(1+o(1)) \exp \left\{\int_{t_{0}}^{t}\left[1+\frac{1}{2} \phi(s)+\frac{1}{2} \phi(s) T_{12, r}(s)\right] d s\right\},
$$

where $T_{21, r}(t)$ and $T_{12, r}(t)$ are defined as follows:

$$
T_{21, k+1}(t)=\frac{1}{2} \int_{+\infty}^{t} e^{-2(s-t)}\left[\phi(s)+2 \phi(s) T_{21, k}(s)+\phi(s) T_{21, k}^{2}(s)\right] d s
$$

$$
T_{12, k+1}(t)=-\frac{1}{2} \int_{t_{0}}^{t} e^{-2(t-s)}\left[\phi(s)+2 \phi(s) T_{12, k}(s)+\phi(s) T_{12, k}^{2}(s)\right] d s
$$

$k=1,2, \cdots, r-1$, and $T_{21,1}(t)$ and $T_{12,1}(t)$ are defined in (2.13) and (2.14), respectively.

Example 2. Consider the following system of differential equations:

$$
\left[\begin{array}{l}
y_{1}  \tag{2.19}\\
y_{2}
\end{array}\right]^{\prime}=\left[\begin{array}{cc}
\sin t & \phi_{1}(t) \\
\phi_{2}(t) & 1
\end{array}\right]\left[\begin{array}{l}
y_{1} \\
y_{2}
\end{array}\right], t \geq 0
$$

where $\phi_{1}(t), \phi_{2}(t) \in L^{m}\left(I_{0}\right)$, for some $m>1$. As the diagonal elements coincide at $t=\left(2 k+\frac{1}{2}\right) \pi,(\mathrm{k}:$ non-negative integers $)$, Theorem HL is not applicable, but Assumption 2.1.2 is satisfied for $\lambda_{1}=\sin t$, and $\lambda_{2}=1$, so the asymptotic solution of (2.19) can be obtained by Theorem 2.1.1.

Case 1. If $\phi_{1}(t), \phi_{2}(t) \in L^{2}\left(I_{0}\right)$, by Theorem 2.1.1 with $r=0$, we know that the asymptotic solution of (2.19) is:

$$
\left[\begin{array}{l}
y_{1}  \tag{2.20}\\
y_{2}
\end{array}\right]=\left[\begin{array}{cc}
1 & o(1) \\
o(1) & 1
\end{array}\right]\left[\begin{array}{cc}
(1+o(1)) e^{-\cos t+1} & 0 \\
0 & (1+o(1)) e^{t}
\end{array}\right]
$$

Case 2. If $\phi_{1}(t), \phi_{2}(t) \in L^{3}\left(I_{0}\right)$, by Theorem 2.1.1 with $r=1$, we have linear transformation $y_{1}=z_{1}+T_{12,1} z_{2}$ and $y_{2}=z_{2}+T_{21,1} z_{1}$ so that

$$
\begin{aligned}
& \frac{d z_{1}}{d t}=\left[\sin t+\phi_{1}(t) T_{21,1}(t)+d_{1}(t)\right] z_{1} \\
& \frac{d z_{2}}{d t}=\left[1+\phi_{2}(t) T_{12,1}(t)+d_{2}(t)\right] z_{2}
\end{aligned}
$$

where $d_{1}(t), \quad d_{2}(t) \in L^{1}\left(I_{0}\right)$,

$$
\begin{equation*}
T_{21,1}(t)=\int_{+\infty}^{t} e^{\int_{s}^{t}(1-\sin \tau) d \tau} \phi_{2}(s) d s \tag{2.21}
\end{equation*}
$$

and

$$
\begin{equation*}
T_{12,1}(t)=\int_{0}^{t} e^{\int_{1}^{t}(\sin r-1) d \tau} \phi_{1}(s) d s \tag{2.22}
\end{equation*}
$$

Thus the asymptotic solution of (2.19) is:

$$
\left[\begin{array}{l}
y_{1}  \tag{2.23}\\
y_{2}
\end{array}\right]=\left[\begin{array}{cc}
1 & o(1) \\
o(1) & 1
\end{array}\right]\left[\begin{array}{cc}
(1+o(1)) w_{1} & 0 \\
0 & (1+o(1)) w_{2}
\end{array}\right]
$$

where $w_{1}(t), w_{2}(t)$ are:

$$
w_{1}(t)=\exp \left\{\int_{0}^{t} \sin s+\phi_{1}(s) T_{21,1}(s) d s\right\}
$$

and

$$
w_{2}(t)=\exp \left\{\int_{0}^{t} 1+\phi_{2}(s) T_{12,1}(s) d s\right\}
$$

Example 3. Consider the following system of differential equations

$$
\left[\begin{array}{l}
y_{1}  \tag{2.24}\\
y_{2}
\end{array}\right]^{\prime}=\left[\begin{array}{cc}
-1 & t^{-\frac{2}{5}} \sin \left(t+t^{-\frac{1}{5}}\right) \\
t^{-\frac{3}{7}} \cos \left(t+t^{-\frac{4}{5}}\right) & 1
\end{array}\right]\left[\begin{array}{l}
y_{1} \\
y_{2}
\end{array}\right], t \geq 1
$$

Since

$$
\sin \left(t+t^{-\frac{4}{5}}\right)-\sin t=O\left(t^{-\frac{4}{5}}\right), \text { and } \cos \left(t+t^{-\frac{4}{5}}\right)-\cos t=O\left(t^{-\frac{4}{5}}\right)
$$

we can rewrite (2.24) as

$$
\left[\begin{array}{l}
y_{1}  \tag{2.25}\\
y_{2}
\end{array}\right]^{\prime}=\left\{\left[\begin{array}{cc}
-1 & t^{-\frac{3}{5}} \sin t \\
t^{-\frac{3}{7}} \cos t & 1
\end{array}\right]+R(t)\right\}\left[\begin{array}{l}
y_{1} \\
y_{2}
\end{array}\right], t \geq 1
$$

where the matrix $R(t)$ is $2 \times 2$ and in $L^{1}\left(I_{0}\right)$.

Since $t^{-\frac{2}{5}} \sin t$ and $t^{-\frac{3}{7}} \cos t \in L^{3}\left(I_{0}\right)$, by Theorem 2.1.2 with $r=1$, there exist $Q(t)$ and $T(t)$ with $\lim _{t \rightarrow \infty} Q(t)=0$ and $\lim _{t \rightarrow \infty} T(t)=0$ such that the linear transformation $y=\left[I_{2}+Q(t)\right]\left[I_{2}+T(t)\right] z$ changes (2.24) into:

$$
\begin{align*}
& \frac{d z_{1}}{d t}=\left[-1+\left(t^{-\frac{2}{5}} \sin t\right) T_{21,1}(t)+d_{1}(t)\right] z_{1}  \tag{2.26}\\
& \frac{d z_{2}}{d t}=\left[1+\left(t^{-\frac{3}{7}} \cos t\right) T_{12,1}(t)+d_{2}(t)\right] z_{2} \tag{2.27}
\end{align*}
$$

where

$$
\begin{equation*}
T_{21,1}(t)=\int_{+\infty}^{t} e^{-2(s-t)} s^{-\frac{3}{7}} \cos s d s=\frac{1}{5} t^{-\frac{3}{7}}(\sin t-2 \cos t)+r_{21}(t) \tag{2.28}
\end{equation*}
$$

with $r_{21}(t) \in L^{1}\left(I_{0}\right)$, and

$$
\begin{equation*}
T_{12,1}(t)=\int_{1}^{t} e^{-2(t-s)} s^{-\frac{2}{5}} \sin s d s=\frac{1}{5} t^{-\frac{2}{5}}(2 \sin t-\cos t)+r_{12}(t) \tag{2.29}
\end{equation*}
$$

with $r_{12}(t) \in L^{1}\left(I_{0}\right)$. So the asymptotic solution of (2.24) is:

$$
\left[\begin{array}{l}
y_{1}  \tag{2.30}\\
y_{2}
\end{array}\right]=\left[\begin{array}{cc}
1+o(1) & o(1) \\
o(1) & 1+o(1)
\end{array}\right]\left[\begin{array}{cc}
(1+o(1)) w_{1} & 0 \\
0 & (1+o(1)) w_{2}
\end{array}\right]
$$

where $w_{1}(t), w_{2}(t)$ are:

$$
\begin{gathered}
w_{1}(t)=\exp \left\{\int_{1}^{t}\left[-1+s^{-\frac{2}{5}} \sin t \cdot \frac{1}{5} s^{-\frac{3}{7}}(\sin s-2 \cos s)\right] d s\right\} \\
=\left(C_{1}+o(1)\right) \exp \left\{-t+\frac{7}{12} t^{\frac{6}{35}}\right\}
\end{gathered}
$$

and

$$
\begin{gathered}
w_{2}(t)=\exp \left\{\int_{1}^{t}\left[1+s^{-\frac{3}{7}} \cos s \cdot \frac{1}{5} s^{-\frac{2}{5}}(2 \sin s-\cos s)\right] d s\right\} \\
=\left(C_{2}+o(1)\right) \exp \left\{t-\frac{7}{35} t^{\frac{6}{35}}\right\}
\end{gathered}
$$

with $C_{1}$ and $C_{2}$ to be two constants.

### 2.3 The Proof of Theorem 2.1.1

The proof of Theorem 2.1.1 is based on a result obtained by Y. Sibuya [33]. Suppose that the system (2.1) is in the form

$$
\begin{equation*}
\frac{d y_{j}}{d t}=A_{j}(t) y_{j}+\sum_{k=1}^{\sigma} B_{j k}(t) y_{k}, \quad(j=1,2, \cdots, \sigma), \tag{2.31}
\end{equation*}
$$

where $y_{j}$ are $n_{j}$-column vectors, $A_{j}(t)$ and $B_{j k}(t)$ are $n_{j} \times n_{j}$ and $n_{j} \times n_{k}$ matrices, respectively, where $n_{1}+n_{2}+\cdot+n_{\sigma}=n$. Let $G_{j}(t, s)$ be the $n_{j} \times n_{j}$ matrix such that

$$
\left\{\begin{array}{l}
G_{j}^{\prime}(t, s)=A_{j}(t) G_{j}(t, s), \\
G_{j}(s, s)=I_{n_{j}}, \quad(j=1,2, \cdots, \sigma)
\end{array}\right.
$$

for $t, s \in I_{0}$. Assume the following:
Assumption 2.3.1. There exist two positive constants $K$ and $\delta$ such that for any $n_{j} \times n_{k}$ matrix $C_{j k}$, we have

$$
\left\|G_{j}(t, s) C_{j k} G_{k}(t, s)^{-1}\right\| \leq K e^{+\delta(t-s)}\left\|C_{j k}\right\|, \quad \text { for } \quad t \leq s, j>k
$$

and

$$
\left\|G_{j}(t, s) C_{j k} G_{k}(t, s)^{-1}\right\| \leq K e^{-\delta(t-s)}\left\|C_{j k}\right\|, \quad \text { for } \quad t \geq s, \quad j<k
$$

for $t, s \in I_{0}$. Here $\|\cdot\|$ denote the Euclidian norm.
Assumption 2.3.2. There exists a function $f(t)$ such that

$$
\left\|B_{j k}(t)\right\| \leq f(t), \quad(j, k=1,2, \cdots, \sigma)
$$

and

$$
\sup _{p \geq t}(1+p-t)^{-1} \int_{t}^{p} f(\tau) d \tau \rightarrow 0, \text { as } t \rightarrow+\infty
$$

Y. Sibuya [33] proved the following:

Theorem S [33]. Under the Assumptions 2.3.1 and 2.3.2, there exists a linear transformation:

$$
\begin{equation*}
y_{j}=z_{j}+\sum_{k \neq j} T_{j k}(t) z_{k}, \quad(j=1,2, \cdots, \sigma) \tag{2.32}
\end{equation*}
$$

with $n_{j} \times n_{k}$ matrices $T_{j k}(t)$, such that

1. for every pair of indices $j$ and $k,(j \neq k)$, the derivative $T_{j k}^{\prime}(t)$ exists, and the entries of $T_{j k}(t)$ and $T_{j k}^{\prime}(t)$ are continuous on the interval $I_{0} ;$
2. $\lim _{t \rightarrow+\infty} T_{j k}(t)=0, \quad(j \neq k) ;$
3. the transformation (2.32) reduces the system (2.31) to

$$
\begin{equation*}
\frac{d z_{j}}{d t}=\left[A_{j}(t)+B_{j j}(t)+\sum_{h \neq j} B_{j h}(t) T_{h j}(t)\right] z_{j}, \quad(j=1,2, \cdots, \sigma) \tag{2.33}
\end{equation*}
$$

From the proof of this theorem, we know

$$
T_{j l}(t)=\int_{\tau_{j l}}^{t} \exp \left\{\int_{s}^{t} A_{j}(\tau) d \tau\right\} U_{j l}(s, T(s)) \exp \left\{-\int_{s}^{t} A_{l}(\tau) d \tau\right\} d s, \quad j \neq l
$$

where

$$
\tau_{j l} \doteq \begin{cases}+\infty, & \text { if } j>l \\ t_{0}, & \text { if } j<l\end{cases}
$$

and

$$
U_{j l}(t, T)=B_{j l}(t)+\sum_{h \neq l} B_{j h}(t) T_{h l}(t)-T_{j l}(t)\left[B_{l l}(t)+\sum_{h \neq l} B_{l h}(t) T_{h l}(t)\right], \quad j \neq l
$$

for $t \in I_{0}$.
If all $A_{j}(t), \quad(j=1,2, \cdots, \sigma)$, are $1 \times 1$ matrices (i.e. $\left.\sigma=n\right)$, and $A_{j}(t)=\lambda_{j}(t)$, for all j , then we have:

Theorem $\mathbf{S}^{\prime}$. Under the Assumptions 2.3.1 and 2.3.2, there exists a linear transformation:

$$
\begin{equation*}
y_{j}=z_{j}+\sum_{k \neq j} T_{j k}(t) z_{k}, \quad(j=1,2, \cdots, n) \tag{2.34}
\end{equation*}
$$

with scalars $T_{j k}(t)$ such that

1. for every pair of indices $j$ and $k,(j \neq k)$, the derivative $T_{j k}^{\prime}(t)$ exists, and both $T_{j k}(t)$ and $T_{j k}^{\prime}(t)$ are continuous on the interval $I_{0} ;$
2. $\lim _{t \rightarrow+\infty} T_{j k}(t)=0, \quad(j \neq k) ;$
3. the transformation (2.34) reduces the system (2.31) to

$$
\frac{d z_{j}}{d t}=\left[\lambda_{j}(t)+b_{j j}(t)+\sum_{h \neq j} b_{j h}(t) T_{h j}(t)\right] z_{j}, \quad(j=1,2, \cdots, n)
$$

where $b_{i j}(t)$ denotes the $1 \times 1$ matrix $B_{i j}(t),(i, j=1,2, \cdots, n)$.
In this case, we have

$$
T_{j l}(t)=\int_{\tau_{j l}}^{t} \exp \left\{\int_{s}^{t}\left(\lambda_{j}(\tau)-\lambda_{l}(\tau)\right) d \tau\right\} U_{j l}(s, T(s)) d s, \quad j \neq l
$$

where

$$
\tau_{j l}= \begin{cases}+\infty, & \text { if } j>l \\ t_{0}, & \text { if } j<l\end{cases}
$$

and

$$
U_{j l}(t, T)=b_{j l}(t)+\sum_{h \neq l} b_{j h}(t) T_{h l}(t)-T_{j l}(t)\left[b_{l l}(t)+\sum_{h \neq l} b_{l h}(t) T_{h l}(t)\right], \quad j \neq l
$$

for $t \in I_{0}$.
From Theorem $\mathrm{S}^{\prime}$, we know that the asymptotic solution of (2.31) depends on $T_{j k}, \quad(j, k=1,2, \cdots, n)$, but to find $T_{j k}, \quad(j, k=1,2, \cdots, n)$, we need to solve a nonlinear system of integral equations which is at present very difficult. An
important fact is that to find the asymptotic solution of (2.31) we need only to find suitable approximations of $T_{j k},(j, k=1,2, \cdots, n)$, and these approximations can be easily obtained by finite iterations. This is the idea behind the proof of Theorem 2.1.1.

Proof of Theorem 2.1.1. Under Assumption 2.1.2, Assumption 2.3.1 is satisfied. Since $p(t) \in L^{m}\left(I_{0}\right)$, by Hölder's inequality, we have

$$
\sup _{x \geq t}(1+x-t)^{-1} \int_{t}^{x} p(\tau) d \tau<\sup _{x \geq t}\left[\int_{t}^{x} p^{m}(\tau) d \tau\right]^{\frac{1}{m}} \rightarrow 0 \text {, as } t \rightarrow+\infty
$$

that is Assumption 2.3.2 is satisfied. So by Theorem $\mathrm{S}^{\prime}$, it follows that there exists a linear transformation:

$$
\begin{equation*}
y_{j}=z_{j}+\sum_{k \neq j} T_{j k}(t) z_{k}, \quad(j=1,2, \cdots, n) \tag{2.35}
\end{equation*}
$$

with scalars $T_{j k}(t)$, such that

1. for every pair of indices j . and $\mathrm{k},(j \neq k)$, the derivative $T_{j k}^{\prime}(t)$ exists, and both $T_{j k}(t)$ and $T_{j k}^{\prime}(t)$ are continuous on the interval $I_{0} ;$
2. $\lim _{t \rightarrow+\infty} T_{j k}(t)=0, \quad(j \neq k) ;$
3. the transformation (2.35) reduces the system

$$
\begin{equation*}
\frac{d y}{d t}=[\Lambda(t)+P(t)] y \tag{2.36}
\end{equation*}
$$

to

$$
\begin{equation*}
\frac{d z_{j}}{d t}=\left[\lambda_{j}(t)+p_{j j}(t)+\sum_{h \neq j} p_{j h}(t) T_{h j}(t)\right] z_{j}, \quad(j=1,2, \cdots, n) \tag{2.37}
\end{equation*}
$$

where $p_{i j}(t),(i, j=1,2, \cdots, n)$, denote the entries of $n \times n$ matrix $P(t)$,

$$
\begin{equation*}
T_{j l}(t)=\int_{\tau_{j l}}^{t} \exp \left\{\int_{s}^{t}\left(\lambda_{j}(\tau)-\lambda_{l}(\tau)\right) d \tau\right\} U_{j l}(s, T(s)) d s, j \neq l \tag{2.38}
\end{equation*}
$$

and

$$
\begin{equation*}
U_{j l}(t, T)=p_{j l}(t)+\sum_{h \neq l} p_{j h}(t) T_{h l}(t)-T_{j l}(t)\left[p_{l l}(t)+\sum_{h \neq l} p_{l h}(t) T_{h l}(t)\right] . \tag{2.39}
\end{equation*}
$$

Let

$$
\left\{\begin{array}{l}
T_{j l, k+1}(t)=\int_{\tau_{j l}}^{t} \exp \left\{\int_{s}^{t}\left(\lambda_{j}(\tau)-\lambda_{l}(\tau)\right) d \tau\right\} U_{j l}\left(s, T_{k}(s)\right) d s  \tag{2.40}\\
T_{j l, 0}(t)=0 \\
U_{j l l}\left(t, T_{k}(t)\right)=p_{j l}(t)+\sum_{h \neq l} p_{j h}(t) T_{h l, k}(t) \\
\\
\quad-T_{j l, k}(t)\left[p_{l l}(t)+\sum_{h \neq l} p_{l h}(t) T_{h l, k}(t)\right]
\end{array}\right.
$$

for $j \neq l$ and $k=0,1,2, \cdots, r-1$, where r is the smallest integer such that $r \geq m-2$, and $m$ is defined in Assumption 2.1.3. From (3.39) and (3.40), we have:

$$
T_{j l}(t)-T_{j l, k+1}(t)=\int_{\tau_{j l}}^{t} \exp \left\{\int_{s}^{t}\left(\lambda_{j}(\tau)-\lambda_{l}(\tau)\right) d \tau\right\}\left[U_{j l}(s, T(s))-U_{j l}\left(s, T_{k}(s)\right)\right] d s
$$

where $j \neq l$ and

$$
\begin{align*}
& U_{j l}(t, T(t))-U_{j l}\left(t, T_{k}(t)\right)= \sum_{l \neq h} p_{j h}(t)\left(T_{h l}(t)-T_{h l, k}(t)\right) \\
&-\left(T_{j l}(t)-T_{j l, k}(t)\right) p_{l l}+\left(T_{j l, k}(t)-T_{j l}(t)\right) \sum_{h \neq l} p_{l h}(t) T_{h l, k}(t) \\
&+T_{j l}(t) \sum_{h \neq l} p_{l h}(t)\left(T_{h l, k}(t)-T_{h l}(t)\right) \tag{2.41}
\end{align*}
$$

In order to complete the proof, we establish:
Lemma 2.3.1. Under Assumptions 2.1.2 and 2.1.3, we have:

1. $T_{j l}(t)$ is in $L^{m}\left(I_{0}\right)$, for $j, l=1,2, \cdots, n, \quad j \neq l$ and for $m$ defined in Assumption 2.1.3.
2. If

$$
\begin{equation*}
\left|T_{j l}(t)-T_{j l, k}(t)\right| \in L^{s}\left(I_{0}\right) \tag{2.42}
\end{equation*}
$$

for all $j, l=1,2, \cdots, n$, then

$$
\begin{equation*}
\left|T_{j l}(t)-T_{j l, k+1}(t)\right| \in L^{\frac{\alpha m}{a+m}}\left(I_{0}\right) \tag{2.43}
\end{equation*}
$$

$$
\text { for all } j, l=1,2, \cdots, n
$$

Proof of Lemma 2.3.1. We first prove conclusion 1. Since $\lim _{t \rightarrow+\infty} T_{h l}(t)=0$ and $T_{h l}(t)$ are continuous on $I_{0}$, for all $h, l=1,2, \cdots, n$, and $\|P(t)\| \leq p(t)$ with $p(t) \in L^{m}\left(I_{0}\right)$, from (2.39), we have

$$
\left|U_{j l}(t, T)\right| \in L^{m}\left(I_{0}\right), \quad j, l=1,2, \cdots, n, \quad j \neq l
$$

In case $j>l$, we have

$$
\begin{gather*}
\left|T_{j l}(t)\right|=\left|\int_{+\infty}^{t} \exp \left\{\int_{s}^{t}\left[\lambda_{j}(\tau)-\lambda_{l}(\tau)\right] d \tau\right\} U_{j l}(s, T(s)) d s\right| \\
\leq \int_{t}^{+\infty} e^{\delta(t-s)}\left|U_{j l}(s, T(s))\right| d s=\int_{0}^{+\infty} e^{-\delta x}\left|U_{j l}(x+t, T(x+t))\right| d x \tag{2.44}
\end{gather*}
$$

Let

$$
f(x)= \begin{cases}e^{-\delta x}, & \text { if } x>0 \\ 0, & \text { if } x \leq 0\end{cases}
$$

In case $j<l$, we have

$$
\begin{gather*}
\left|T_{j l}(t)\right|=\left|\int_{t_{0}}^{t} \exp \left\{\int_{s}^{t}\left[\lambda_{j}(\tau)-\lambda_{l}(\tau)\right] d \tau\right\} U_{j l}(s, T(s)) d s\right| \\
\leq \int_{t_{0}}^{t} e^{-\delta(t-s)}\left|U_{j l}(s, T(s))\right| d s=\int_{t_{0}-t}^{0} e^{\delta x}\left|U_{j l}(x+t, T(x+t))\right| d x \tag{2.45}
\end{gather*}
$$

Let

$$
f(x)= \begin{cases}e^{\delta x}, & \text { if } x<0 \\ 0, & \text { if } x \geq 0\end{cases}
$$

Set $U_{j l}(t, T(t))=0$, for $t \notin I_{0},(j, l=1,2, \cdots, n)$, then in both cases $j>l$ and $j<l$, we have:

$$
\begin{equation*}
\left|T_{j l}(t)\right| \leq \int_{-\infty}^{+\infty} f(x)\left|U_{j l}(x+t, T(x+t))\right| d x \tag{2.46}
\end{equation*}
$$

By these extensions of $U_{j l}(t)$ and $f(t)$, in both cases the function $f(x) \in L^{1}(-\infty, \infty)$, and $U_{j l}(x+t, T(x+t)) \in L^{m}(-\infty, \infty)$, for all $j, l=1,2, \cdots, n$, and for each fixed $x \in(-\infty, \infty)$. Thus, by the following Minkowski inequality([20], Theorem 203):

$$
\left\{\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} f(x, y) d y\right]^{p} d x\right\}^{1 / p} \leq \int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{+\infty}[f(x, y)]^{p} d x\right\}^{1 / p} d y
$$

for $p>1$, and $f(x, y) \geq 0$, we have

$$
\begin{gather*}
\left(\int_{-\infty}^{+\infty}\left|T_{j l}(t)\right|^{m} d t\right)^{1 / m} \leq \int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{+\infty}\left|U_{j l}(x+t, T(x+t))\right|^{m} d t\right\}^{1 / m} f(x) d x \\
\leq\|f\|_{1}\left\|U_{j l}\right\|_{m} \tag{2.47}
\end{gather*}
$$

Hence

$$
\left|T_{j l}(t)\right| \in L^{m}\left(I_{0}\right), \text { for all } j, l=1,2, \cdots, n, j \neq l
$$

To prove conclusion 2, first notice that by mathematical induction and Hölder's inequality we have:

$$
T_{j l, k}(t) \rightarrow 0, \text { as } t \rightarrow \infty, \text { for all } j, l=1,2, \cdots, n, \quad k=1,2, \cdots
$$

Since

$$
T_{j l, k}(t) \rightarrow 0, \quad T_{j l}(t) \rightarrow 0, \text { as } t \rightarrow \infty, \text { and }\left|p_{j l}(t)\right| \in L^{m}\left(I_{0}\right)
$$

for all $j, l=1,2, \cdots, n, \quad k=1,2, \cdots$, from (2.41), if $\left|T_{j l}(t)-T_{j l, k}(t)\right| \in L^{s}\left(I_{0}\right)$, then by Hölder's inequality, we have

$$
\begin{equation*}
\left|U_{j l}(t, T(t))-U_{j l}\left(t, T_{k}(t)\right)\right| \in L^{\frac{s m}{s+m}}\left(I_{0}\right) \tag{2.48}
\end{equation*}
$$

As in the proof for conclusion 1 , from (2.48) we thus have conclusion 2.
The proof of Theorem 2.1.1 will be completed if we can show that

$$
\begin{equation*}
d_{j}=\sum_{h \neq j} p_{j h}(t)\left(T_{h j}(t)-T_{h j, r}(t)\right) \in L^{1}\left(I_{0}\right) \tag{2.49}
\end{equation*}
$$

for all $j=1,2, \cdots, n$, where r is the smallest integer such that $r \geq m-2$. Since $T_{j l, 0}(t)=0$, for all $j, l=1,2, \cdots, n,(2.49)$ follows from Lemma 2.3.1 immediately.

Remark. Notice that for $1<m \leq 2$, we have $r=0$ and

$$
\sum_{h \neq j} p_{j h}(t) T_{h j, r}(t)=0
$$

for all $j, l=1,2, \cdots, n$, so Theorem 2.1.1 includes Theorem HW as a special case, in fact, in this case, we have

$$
\sum_{h \neq j} p_{j h}(t) T_{h j}(t) \in L^{1}\left(I_{0}\right),
$$

for all $j, l=1,2, \cdots, n$.

### 2.4 The Proof of Lemma 2.1.1

The proof is to be given in four steps.
STEP 1: By differentiating both sides of (2.8) and by (2.1), we obtain

$$
\begin{equation*}
\frac{d Q}{d t} z+\left[I_{n}+Q\right] \frac{d z}{d t}=[\Lambda(t)+P(t)+R(t)]\left[I_{n}+Q\right] z \tag{2.50}
\end{equation*}
$$

Hence if $Q^{\prime}(t)$ exists, then by (2.9), $Q$ should satisfy the linear differential equation:

$$
\begin{equation*}
\frac{d Q}{d t}=[\Lambda(t)+P(t)+R(t)]\left[I_{n}+Q\right]-\left[I_{n}+Q\right][\Lambda(t)+P(t)] \tag{2.51}
\end{equation*}
$$

or, equivalently,

$$
\begin{equation*}
\frac{d Q}{d t}=[\Lambda(t)+P(t)] Q-Q[\Lambda(t)+P(t)]+[R(t)]\left[I_{n}+Q\right] \tag{2.52}
\end{equation*}
$$

A general solution $Q(t)$ of (2.52) can be written in the form

$$
\begin{equation*}
Q(t)=\Phi(t) C \Psi(t)^{-1}+\int^{t} \Phi(t) \Phi(s)^{-1} R(t) \Psi(s) \Psi(t)^{-1} d s \tag{2.53}
\end{equation*}
$$

where C is an arbitrary constant matrix, $\Phi(t)$ is an $n \times n$ fundamental matrix of

$$
\begin{equation*}
\frac{d \Phi}{d t}=[\Lambda(t)+P(t)+R(t)] \Phi \tag{2.54}
\end{equation*}
$$

and $\Psi(t)$ is an $n \times n$ fundamental matrix of

$$
\begin{equation*}
\frac{d \Psi}{d t}=[\Lambda(t)+P(t)] \Psi \tag{2.55}
\end{equation*}
$$

Thus $Q(t)$ exists on $I_{0}$ satisfying the first condition ol' Lemma 2.1.1. We shall prove the existence of the solution of (2.52) satisfying condition $\lim _{t \rightarrow+\infty} Q(t)=0$, in an interval $I=\left\{t: t_{1} \leq t<\infty\right\}$ for a large $t_{1}$, then, by (2.53), $Q(t)$ exists on $I_{0}$ satisfying the required conditions.

STEP 2: We shall construct $Q(t)$ by means of equation (2.52) and the condition $\lim _{t \rightarrow+\infty} Q(t)=0$. To do this, let $\Phi(t, s)$ be the unique solution of the initial value problem:

$$
\begin{equation*}
\frac{d Y}{d t}=[\Lambda(t)+P(t)] Y, \quad Y(s)=I_{n}+G(s) \tag{2.56}
\end{equation*}
$$

where $G(t)$ is the transformation matrix given in the assumption of the Lemma. Then, (2.52) is equivalent to the following linear integral equation:

$$
\begin{equation*}
Q(t)=\int^{t} \Phi(t, s)\left[I_{n}+G(s)\right]^{-1} R(s)\left[I_{n}+Q(s)\right]\left[I_{n}+G(s)\right] \Phi(t, s)^{-1} d s \tag{2.57}
\end{equation*}
$$

where by assumption,

$$
\begin{equation*}
\Phi(t, s)=\left[I_{n}+G(t)\right] \exp \left\{\int_{s}^{t} \Lambda(\tau) d \tau\right\} \tag{2.58}
\end{equation*}
$$

Let

$$
\begin{equation*}
\left[I_{n}+G(s)\right]^{-1}\left[I_{n}+Q(s)\right]\left[I_{n}+G(s)\right]=I_{n}+H(s) \tag{2.59}
\end{equation*}
$$

and let

$$
\begin{equation*}
\tilde{R}(s)=\left[I_{n}+G(s)\right]^{-1} R(s)\left[I_{n}+G(s)\right] \tag{2.60}
\end{equation*}
$$

then we can rewrite (2.57) as

$$
\begin{equation*}
H(t)=\int^{t} \exp \left\{\int_{s}^{t} \Lambda(\tau) d \tau\right\} \tilde{R}(s)\left[I_{n}+H(s)\right] \exp \left\{-\int_{s}^{t} \Lambda(\tau) d \tau\right\} d s \tag{2.61}
\end{equation*}
$$

STEP 3: Let

$$
H(t)=\left(h_{i j}(t)\right)_{n \times n} \text { and } \tilde{R}(t)=\left(\tilde{r}_{i j}(t)\right)_{n \times n},
$$

we can write the integral equation (2.61) in the following form:

$$
\left\{\begin{array}{l}
h_{j j}(t)=\int_{\tau_{j j}}^{t}\left[\tilde{r}_{j j}(s)+\sum_{h=1, h \neq j}^{n} \tilde{r}_{j h}(s) h_{h j}(s)\right] d s  \tag{2.62}\\
h_{j k}(t)=\int_{\tau_{j k}}^{t} \exp \left\{\int_{s}^{t} \lambda_{j k}(\tau) d \tau\right\}\left[\tilde{r}_{j k}(s)+\sum_{h=1, h \neq j}^{n} \tilde{r}_{j h}(s) h_{h j}(s)\right] d s, j \neq k
\end{array}\right.
$$

where $j, k=1,2, \cdots, n$, and the initial points $\tau_{j k}$ are chosen as the follows:

$$
\tau_{j k}= \begin{cases}+\infty & \text { if } j=k  \tag{2.63}\\ t_{2} & \text { if } j<k \\ +\infty & \text { if } j>k\end{cases}
$$

where $t_{2}\left(\geq t_{1}\right)$ is a suitable large number. As in the proof of Theorem $L$, by using successive approximations in a usual manner (see [31], [28]), it can be shown that

$$
\begin{equation*}
H(t) \rightarrow 0, \text { as } t \rightarrow+\infty \tag{2.64}
\end{equation*}
$$

STEP 4: From (2.59), we have

$$
\begin{equation*}
I_{n}+Q(t)=\left[I_{n}+G(t)\right]\left[I_{n}+H(t)\right]\left[I_{n}+G(t)\right]^{-1} \tag{2.65}
\end{equation*}
$$

Since

$$
G(t), H(t) \rightarrow 0, \text { as } t \rightarrow+\infty
$$

it follows that

$$
I_{n}+Q(t) \rightarrow I_{n}, \text { as } t \rightarrow+\infty
$$

So

$$
Q(t) \rightarrow 0, \text { as } t \rightarrow+\infty .
$$

## CHAPTER III

## ASYMPTOTIC BEHAVIOR FOR SYSTEM HAVING MULTIPLE EIGENVALUES

### 3.1 Introduction

In last chapter, we studied the asymptotic behavior of the solutions for the system of differential equations (1.1) in which the real parts of the eigenvalues of $A(t)$ are separated from each other by a fixed number in the sense of Assumption 2.1.2 (see the Remark at the end of Section 2.1). In this chapter, we will consider the asymptotic behavior of the solutions for the system of differential equations (1.1) in which the real parts of some eigenvalues of $A(t)$ have same limits as t goes to $+\infty$. As mentioned in Chapter I, this problem has a close relationship with that of finding the deficiency index of linear differential operators.

Consider the linear system of differential equations

$$
\begin{equation*}
\frac{d y}{d t}=[A+V(t)+\tilde{R}(t)] y \tag{3.1}
\end{equation*}
$$

where A is an $n \times n$ constant matrix, $V(t)$ and $\tilde{R}(t)$ are $n \times n$ continuous matrices on $I_{0}=\left[t_{0}, \infty\right)$.

In Chapter I, we mentioned that by using Theorem L, Levinson proved the following theorem.
Theorem LE [31]. Assume that (i) $A$ is a constant matrix with distinct eigenvalues; (ii) $V(t)$ satisfies Assumption 1.4; (iii) $\tilde{R}^{\prime}(t)$ is continuous for $t \geq t_{0}$, $\tilde{R}(t) \in L^{1}\left(I_{0}\right)$; and (iv) the eigenvalues of the matrix $A+V(t)$ satisfy Assumption 1.2. Then there exists an $n \times n$ matrix $Q(t)$ such that

1. the derivative $Q^{\prime}(t)$ exists, and the entries of $Q(t)$ and $Q^{\prime}(t)$ are continuous in $t$ on the interval $I_{0}$,
2. $\lim _{t \rightarrow+\infty} Q(t)=0$,
3. the transformation:

$$
y=P\left[I_{n}+Q(t)\right] z
$$

changes system (3.1) into

$$
\frac{d z}{d t}=\Lambda(t) z
$$

on the interval $I_{0}$, where $P$ is a constant matrix such that $P^{-1} A P$ is a diagonal matrix and $\Lambda(t)$ is a diagonal matrix with diagonal elements the eigenvalues of $A+V(t)$.

If the eigenvalues of $A$ are not all distinct, the analysis of the asymptotic behavior of the solution for above system involves much more delicate techniques. Here are some important results due to Devinatz and Kaplan [9] and Chiba and Kimura [3]. In order to introduce their theorems we need some assumptions. Let

$$
K=\sum_{i=1}^{s} \sum_{j=1}^{e_{i}} \bigoplus K_{i j}, \text { with } K_{i j}=\left[\begin{array}{ccccc}
\lambda_{i} & 1 & 0 & \cdots & 0 \\
0 & \lambda_{i} & 1 & \cdots & 0 \\
& & \cdots & & \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & \lambda_{i}
\end{array}\right]
$$

be Jordan's canonical form of A, where the $\lambda_{i}^{\prime} s$ are the distinct eigenvalues of $A$, $K_{i j}$ are $n_{i j}$ by $n_{i j}$ matrices, and $\Sigma \oplus$ denotes the direct sum of matrices.
Assumption 3.1.1. The minimal polynomial of A is of degree n , that is $e_{i}=1$, for all $i=1,2, \cdots, s$.

Remark: Under Assumption 3.1.1, the minimal polynomial of $A$ is of the form

$$
P(\lambda)=\prod_{i=1}^{s}\left(\lambda-\lambda_{i}\right)^{n_{i 1}}, \text { with } \lambda_{i} \neq \lambda_{k} \text { if } i \neq k, \text { and } \sum_{i=1}^{s} n_{i 1}=n .
$$

Let

$$
r+1=\max \left\{n_{i 1}, \quad i=1,2, \cdots, s\right\} .
$$

Assumption 3.1.2. $V(t)$ is absolutely continuous on each compact subinterval in $\left(t_{0}, \infty\right), t^{r}\left|V^{\prime}(t)\right| \in L^{1}\left(I_{0}\right)$, and $V(t) \rightarrow 0$, as $t \rightarrow+\infty$.

Let $\left\{\lambda_{i j}(t)\right\}$ denote the eigenvalues of $A+V(t)$, where $1 \leq i \leq s$, and $1 \leq j \leq n_{i 1}$.

Assumption 3.1.3. The eigenvalues $\left\{\lambda_{i j}(t)\right\}$ are absolutely continuous and satisfy:
(1) $\lambda_{i j}(t) \rightarrow \lambda_{i}$, as $t \rightarrow+\infty$, for $1 \leq j \leq n_{i 1}$;
(2) $\int_{t_{0}}^{\infty} t^{r}\left|\lambda_{i j}^{\prime}(t)\right| d t<+\infty, \quad 1 \leq i \leq s, 1 \leq j \leq n_{i 1}$.

Assumption 3.1.4. The matrix $\tilde{R}(t)$ satisfies

$$
t^{r} \tilde{R}(t) \in L^{1}\left(I_{0}\right)
$$

For any positive integer $q$, let

$$
\mu_{k q}=\lambda_{k 1}(t)+\frac{q-1}{t} .
$$

Devinatz and Kaplan [9] proved the following:
Theorem DK [9]. Suppose Assumptions 3.1.1, 3.1.2, 3.1.3 and 3.1.4 are all satisfied by system (3.1). Suppose, furthermore, that for each fixed $(k, q), 1 \leq k \leq$ $s, 1 \leq q \leq n_{k 1}$, all of pairs $(j, p), 1 \leq j \leq s, 1 \leq p \leq n_{j 1}$, fall into one of two classes $I_{k q}$ and $J_{k q}$, where
(1) $(j, p) \in I_{k q}$, if

$$
\int_{t_{0}}^{t} \mathcal{R}\left[\mu_{k q}(\tau)-\mu_{j p}(\tau)\right] d \tau \rightarrow \infty, \text { as } t \rightarrow \infty
$$

and

$$
\int_{s}^{t} \mathcal{R}\left[\mu_{k q}(\tau)-\mu_{j p}(\tau)\right] d \tau>-L, \text { for } t \geq s>0
$$

(2) $(j, p) \in J_{k q}$, if

$$
\int_{s}^{t} \mathcal{R}\left[\mu_{k q}(\tau)-\mu_{j p}(\tau)\right] d \tau<L, \text { for } t \geq s>0
$$

where $L$ is a positive constant.
Then there exists a positive constant $t_{1},\left(t_{1} \geq t_{0}\right)$, eigenvectors $q_{i}$ corresponding to $\lambda_{i}, 1 \leq i \leq s$, and $n$ linearly independent functions $y_{i j}(t), 1 \leq i \leq s, 1 \leq j \leq n_{i 1}$, which are solutions of (3.1) such that

$$
y_{i j}(t) t^{-j+1} \exp \left\{-\int_{t_{1}}^{t} \lambda_{i 1}(\tau) d \tau\right\} \rightarrow q_{i}, \text { as } t \rightarrow \infty
$$

If not all $e_{i}^{\prime} s$ are equal to 1 , we have the following theorem obtained by Chiba and Kimura [3]. To state it, we need:

Assumption 3.1.5. The Jordan's canonical form $J(t)$ of $A+V(t)$ is written as follows:

$$
J(t)=\sum_{i=1}^{s} \sum_{j=1}^{e_{i}} \bigoplus J_{i j}(t), \quad J_{i j}(t)=\left[\begin{array}{ccccc}
\lambda_{i}(t) & 1 & 0 & \cdots & 0 \\
0 & \lambda_{i}(t) & 1 & \cdots & 0 \\
& & \cdots & & \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & \lambda_{i}(t)
\end{array}\right]
$$

for $t \geq t_{0}$, where $J_{i j}(t)$ must also be $n_{i j}$ by $n_{i j}$ matrices.
Let

$$
n_{i}=\max _{k}\left\{n_{i k}\right\}, \quad n_{0}=\max _{i}\left\{n_{i}\right\} .
$$

Chiba and Kimura [3] proved the following:
Theorem CK [3]. Let $(h, l)$ be a pair of indices such that $1 \leq h \leq s$ and $1 \leq l \leq n_{h k}$, for some $k$. Suppose that:
(A) Assumption 3.1.5 is satisfied.
(B) There exist $2 s+1$ real-valued functions $\underline{\alpha}_{1}(t), \cdots, \underline{\alpha}_{s}(t) ; \bar{\alpha}_{1}(t), \cdots, \bar{\alpha}_{s}(t)$ and $\beta(t)$ continuous on $\left[t_{0}, \infty\right)$ and a partitioning $\left\{I_{1}, I_{2}\right\}$ of the set $\{1,2, \cdots, n\}$ with the following properties:
(i) for $t_{0} \leq t<\infty$, we have

$$
\underline{\alpha}_{i}(t) \leq \int_{t_{0}}^{t} \operatorname{Re}\left(\lambda_{h}(\tau)-\lambda_{i}(\tau)\right) d \tau \leq \bar{\alpha}_{i}(t), \quad \bar{\alpha}_{i}(t)-\underline{\alpha}_{i}(t) \leq \beta(t), \quad i=1, \cdots, s
$$

(ii) If $i \in I_{1}$, then $\underline{\alpha}_{i}(t)-\left(n_{i}-1\right) \log (t)$ is an increasing function on $\left[t_{0}, \infty\right)$ and tends to $\infty$ as $t \rightarrow \infty$. If $i \in I_{2}$, then $\underline{\alpha}_{i}(t)$ is an increasing or decreasing function and $\bar{\alpha}_{i}(t) \leq \beta(t)$.
(C)

$$
\int_{t_{0}}^{\infty} \tau^{n_{0}-1+l-1} e^{\beta(\tau)}\left\|V^{\prime}(\tau)\right\| d \tau<\infty
$$

(D)

$$
\int_{t_{0}}^{\infty} \tau^{n_{0}-1+l-1} e^{\beta(\tau)}\|\tilde{R}(\tau)\| d \tau<\infty
$$

where $\|\cdot\|$ denotes any matrix norm.
Then for any $j$ such that $n_{h j} \geq l$, there exists a solution $y(t)$ of (3.1) with the asymptotic expression

$$
y=\left(q_{h j 1}+o(1)\right) \frac{t^{l-1}}{(l-1)!} \exp \left\{\int_{t_{0}}^{t} \lambda_{h}(\tau) d \tau\right\}
$$

where $q_{h j 1}$ are the eigenvectors of $A$ associated with $\lambda_{h}$.
From Assumption 3.1.2 and condition (C) in Theorem CK, it is easy to see that Theorems DK and CK cannot be applied to the system (3.1) with $V^{\prime}(t) \notin$ $L^{1}\left(I_{0}\right)$. The range of the applications of Theorems DK and CK are also greatly limited by Assumptions 3.1.1 and 3.1.5 as well. Assumption 3.1.1 requires that the eigenvalues of $A$ are all distinct when $A$ is diagonalizable, and Assumption 3.1.5 requires that the diagonal block $J_{i j}(t)$ of the Jordan's canonical form $J(t)$ of
$A+V(t)$ has the same order as the diagonal block $K_{i j}$ of the Jordan's canonical form $J$ of $A$. Theorem HX in [28] generalized above results. In Theorem HX, the condition (C) of Theorem CK, Assumptions 3.1.1, Assumptions 3.1.2 and 3.1.5 are removed. In this chapter we will generalize Theorem HX.

### 3.2 The Main Theorem

Consider the following system of equations:

$$
\begin{equation*}
\frac{d y}{d t}=[\Lambda(t)+R(t)] y \tag{3.2}
\end{equation*}
$$

where $\Lambda(t)=\operatorname{diag}\left\{\lambda_{1}(t), \lambda_{2}(t), \cdots, \lambda_{n}(t)\right\}$ is a diagonal matrix, and the entries of matrix $R(t)$ are small in certain sense to be seen below.

Note when $A$ is diagonalizable, system (3.1) can be changed into (3.2). For the sake of convenience, we assume, without loss of generality, that the diagonal elements of $R(t)$ are all zero.

As in Chapter I, let

$$
\begin{equation*}
R(t)=\left\{r_{j k}(t)\right\}_{j, k=1}^{n} \tag{3.3}
\end{equation*}
$$

and

$$
\begin{equation*}
r_{1}(t)=\max _{j>k}\left|r_{j k}(t)\right|, \quad r_{2}(t)=\max _{j<k}\left|r_{j k}(t)\right| . \tag{3.4}
\end{equation*}
$$

In order to obtain the asymptotic formula for the solution of the systems with $D_{j k}(t)=0\left(t^{-1}\right), t \geq t_{0}$, where $D_{j k}(t)$ are defined in Assumption 1.2, we assume the following:

Assumption 3.2.1. There exist two positive constants $\beta$, $(\beta>1)$ and K such that for each pair of indices j and $\mathrm{k},(j, k=1,2, \cdots, n ; k \neq j)$, either

$$
\begin{equation*}
\exp \left\{\int_{t}^{s} D_{j k}(\tau) d \tau\right\} \leq K\left(\frac{t}{s}\right)^{\beta}, \text { for all } s \geq t \tag{3.5}
\end{equation*}
$$

or

$$
\begin{equation*}
\exp \left\{\int_{t}^{s} D_{j k}(\tau) d \tau\right\} \leq K\left(\frac{s}{t}\right)^{\beta}, \text { for all } s \leq t \tag{3.6}
\end{equation*}
$$

Assumption 3.2.2. The matrix $R(t)$ is continuous and satisfies:

$$
\begin{equation*}
|R(t)|=o\left(t^{-1}\right), \text { as } \quad t \rightarrow+\infty, \tag{3.7}
\end{equation*}
$$

and either

$$
\begin{equation*}
\int_{t_{0}}^{+\infty} t^{-1} \max _{s \geq t^{1 / 2^{n}}}\left\{s r_{1}(s)\right\} d t<+\infty \tag{3.8}
\end{equation*}
$$

or

$$
\begin{equation*}
\int_{t_{0}}^{+\infty} t^{-1} \max _{s \geq t^{1 / 2^{n}}}\left\{s r_{2}(s)\right\} d t<+\infty \tag{3.9}
\end{equation*}
$$

Theorem 3.2.1. Under Assumption 3.2.1 and Assumption 3.2.2, there exists an $n \times n$ matrix $Q(t)$ such that

1. the derivative $Q^{\prime}(t)$ exists, and the entries of $Q(t)$ and $Q^{\prime}(t)$ are continuous in $t$ on the interval $I_{0}$,
2. $\lim _{t \rightarrow+\infty} Q(t)=0$,
3. the transformation:

$$
\begin{equation*}
y=\left[I_{n}+Q(t)\right] z \tag{3.10}
\end{equation*}
$$

changes systems (3.2) into

$$
\begin{equation*}
\frac{d z}{d t}=\Lambda(t) z \tag{3.11}
\end{equation*}
$$

on the interval $I_{0}$, where $I_{n}$ is the $n \times n$ identity matrix.

This generalizes Theorem HX in the sense that $D_{j k}(t)$ can be of order $O\left(t^{-1}\right)$ for some or all $j \neq k$. The proof of Theorem 3.2.1 will be given in Section 3.3, and two examples will be given in Section 3.4 to illustrate the application of the Theorem. Here, applying "shearing" transformations (see W. Wasow [36]) we will have two immediate results from Theorem HX and Theorem 3.2.1.

Consider the following $2 \times 2$ system:

$$
\frac{d y}{d t}=\left[\begin{array}{cc}
\lambda_{1}(t) & r_{12}(t)  \tag{3.12}\\
& \\
r_{21}(t) & \lambda_{2}(t)
\end{array}\right] y
$$

we have:
Proposition 3.2.1. If system (3.12) satisfies one of the following two conditions:
(a) $\lambda_{1}(t)$ and $\lambda_{2}(t)$ satisfy Assumption 1.7 for some $\alpha,(0 \leq \alpha<1)$, and $r_{12}(t)=$ $0\left(t^{-p}\right), \quad r_{21}(t)=0\left(t^{-q}\right)$, with $p+q>1+\alpha$; or
(b) $\lambda_{1}(t)$ and $\lambda_{2}(t)$ satisfy Assumption 3.2.1 and $r_{12}(t)=0\left(t^{-1}(\ln t)^{-p}\right), \quad r_{21}(t)=$ $0\left(t^{-1}(\ln t)^{-q}\right)$, with $p+q>1$,
then there exist a $2 \times 2$ matrix $Q(t)$ and a $2 \times 2$ diagonal matrix $S(t)$ such that
(1) the derivative $Q^{\prime}(t)$ exists, and the entries of $Q(t)$ and $Q^{\prime}(t)$ are continuous in $t$ on the interval $I_{0}$,
(2) $\lim _{t \rightarrow+\infty} Q(t)=0$,
(3) the transformation:

$$
y=S(t)\left[I_{2}+Q(t)\right] z
$$

changes systems (3.12) into

$$
\begin{equation*}
\frac{d z}{d t}=\left[\Lambda(t)-S^{-1}(t) S^{\prime}(t)\right] z \tag{3.13}
\end{equation*}
$$

on the interval $I_{0}$, where $S(t)=\operatorname{diag}\left\{1, t^{-\delta}\right\}$ in case $(a)$; and $S(t)=$ $\operatorname{diag}\left\{1,(\ln t)^{-\gamma}\right\}$ in case (b), with some real numbers $\delta$, and $\gamma$.

Proof of Proposition 3.2.1: (The proof is given under condition (a) by using Theorem HX. Similar proof can be obtained under condition (b) by using Theorem 3.2.1.) Let $c=p+q-\alpha-1, \delta=1+\frac{c}{2}-p$, and let $S(t)=\operatorname{diag}\left\{1, t^{-\delta}\right\}$, then system (3.12) is transformed into

$$
\frac{d \tilde{y}}{d t}=\left[\begin{array}{cc}
\lambda_{1}(t) & \tilde{r}_{12}(t)  \tag{3.14}\\
& \lambda_{2}(t)+\frac{\delta}{t}
\end{array}\right] \tilde{y}
$$

by $y=S(t) \tilde{y}$, where

$$
\tilde{r}_{12}(t)=0\left(t^{-\left(1+\frac{c}{2}\right)}\right),
$$

and

$$
\tilde{r}_{21}(t)=0\left(t^{-p-q+1+\frac{c}{2}}\right)=o\left(t^{-\alpha}\right)
$$

By Theorem HX, there exists an $2 \times 2$ matrix $Q(t)$ satisfing condition (1) and (2) in Proposition 3.3.1 such that the transformation $\tilde{y}=\left[I_{2}+Q(t)\right] z$ changes (3.14) into (3.13).

In general, since for

$$
S(t)=\operatorname{diag}\left\{1, t^{-\delta_{1}}, t^{-\delta_{2}}, \cdots, t^{-\delta_{n-1}}\right\}
$$

where $\delta_{1}, \delta_{2}, \cdots, \delta_{n-1}$ are all non-zero real numbers, if we perform the transformation

$$
y(t)=S(t) \cdot z(t)
$$

then system (3.2) can be changed into

$$
\begin{equation*}
\frac{d z}{d t}=\left[\left(S^{-1}(t) \Lambda(t) S(t)-S^{-1}(t) S^{\prime}(t)\right)+S^{-1}(t) R(t) S^{\prime}(t)\right] z \tag{3.15}
\end{equation*}
$$

where

$$
\begin{gathered}
S^{-1}(t) \Lambda(t) S(t)-S^{-1}(t) S^{\prime}(t) \\
=\left[\begin{array}{cccc}
\lambda_{1}(t) & 0 & \cdots & 0 \\
0 & \lambda_{2}(t)+\delta_{1} t^{-1} & \cdots & 0 \\
& & \cdots & \\
0 & 0 & \cdots & \lambda_{n}(t)+\delta_{n-1} t^{-1}
\end{array}\right]
\end{gathered}
$$

and

$$
\begin{gather*}
\cdot S^{-1}(t) R(t) S(t) \\
=\left[\begin{array}{cccc}
0 & r_{12}(t) t^{-\delta_{1}} & \cdots & r_{1 n}(t) t^{\delta_{n-1}} \\
r_{21}(t) t^{\delta_{1}} & 0 & \cdots & r_{2 n}(t) t^{\delta_{1}-\delta_{n-1}} \\
& & \cdots & \\
& & \cdots & 0
\end{array}\right] . \tag{3.16}
\end{gather*}
$$

Thus we have the following proposition:
Proposition 3.2.2. If there exist $n-1$ non-zero real numbers $\delta_{1}, \delta_{2}, \cdots, \delta_{n-1}$ such that system (3.15) satisfies either the Assumptions 1.7 and 1.8 or Assumptions 3.2.1 and 3.2.2, then there exists an $n \times n$ matrix $Q(t)$ such that

1. the derivative $Q^{\prime}(t)$ exists, and the entries of $Q(t)$ and $Q^{\prime}(t)$ are continuous in $t$ on the interval $I_{0}$,
2. $\lim _{t \rightarrow+\infty} Q(t)=0$,

## 3. the transformation:

$$
z=\left[I_{n}+Q(t)\right] w
$$

changes systems (3.15) into

$$
\begin{equation*}
\frac{d w}{d t}=\left[\Lambda(t)-S^{-1}(t) S^{\prime}(t)\right] w \tag{3.17}
\end{equation*}
$$

on the interval $I_{0}$, where $S^{-1}(t) S^{\prime}(t)=-\operatorname{diag}\left\{0, \delta_{1} t^{-1}, \delta_{2} t^{-1}, \cdots, \delta_{n-1} t^{-1}\right\}$.

### 3.3 Proof of Theorem 3.2.1

We will prove Theorem 3.2.1 under the condition (3.9) in Assumption 3.2.2. The proof is to be given in seven steps. A similar proof is valid also under condition (3.8). Here we use the successive approximation method similar to Gauss-Seidel iterations(e.g. see G.H. Golub and C.F. VanLoan [19] and R.S. Varga [34]) developed in Hsieh and Xie [28].

STEP 1: By differentiating both sides of (3.10) and by (3.2), we obtain

$$
\begin{equation*}
\frac{d Q}{d t} z+\left[I_{n}+Q\right] \frac{d z}{d t}=[\Lambda(t)+R(t)]\left[I_{n}+Q\right] z \tag{3.18}
\end{equation*}
$$

Hence, by (3.11), $Q$ should satisfy the linear differential equation:

$$
\begin{equation*}
\frac{d Q}{d t}=[\Lambda(t)+R(t)]\left[I_{n}+Q\right]-\left[I_{n}+Q\right] \Lambda(t) \tag{3.19}
\end{equation*}
$$

or, equivalently,

$$
\begin{equation*}
\frac{d Q}{d t}=\Lambda(t) Q-Q \Lambda(t)+R(t)\left[I_{n}+Q\right] \tag{3.20}
\end{equation*}
$$

A general solution $Q(t)$ of (3.20) can be written in the form

$$
\begin{equation*}
Q(t)=\Phi(t) C \Psi(t)^{-1}+\int^{t} \Phi(t) \Phi(s)^{-1} R(s) \Psi(s) \Psi(t)^{-1} d s \tag{3.21}
\end{equation*}
$$

where C is an arbitrary constant matrix, $\Phi(t)$ is an $n \times n$ fundamental matrix of

$$
\begin{equation*}
\frac{d \Phi}{d t}=[\Lambda(t)+R(t)] \Phi \tag{3.22}
\end{equation*}
$$

and $\Psi(t)$ is an $n \times n$ fundamental matrix of

$$
\begin{equation*}
\frac{d \Psi}{d t}=\Lambda(t) \Psi \tag{3.23}
\end{equation*}
$$

Thus $Q(t)$ exists on $I_{0}$ satisfying condition (1) of Theorem 3.2.1. We shall prove the existence of the solution of (3.20) satisfying condition $\lim _{t \rightarrow+\infty} Q(t)=0$, in an interval $I=\left\{t: t_{2} \leq t<\infty\right\}$ for a large $t_{2}$, then, by (3.21), $Q(t)$ exists on $I_{0}$ satisfying the required conditions.

STEP 2: We shall construct $Q(t)$ by means of equation (3.19) and the condition $\lim _{t \rightarrow+\infty} Q(t)=0$. To do this, let $\Phi(t, s)$ be the unique solution of the initial value problem:

$$
\begin{equation*}
\frac{d Y}{d t}=\Lambda(t) Y, \quad Y(s)=I_{n} \tag{3.24}
\end{equation*}
$$

Also, let

$$
\begin{equation*}
Q(t)=\left(q_{j k}(t)\right)_{j, k=1}^{n} \tag{3.25}
\end{equation*}
$$

and

$$
\begin{equation*}
\lambda_{j k}(t)=\lambda_{j}(t)-\lambda_{k}(t), \quad j, k=1,2, \cdots, n,(j \neq k) . \tag{3.26}
\end{equation*}
$$

Then, (3.19) is equivalent to the following linear integral equation:

$$
\begin{equation*}
Q(t)=\int^{t} \Phi(t, s) R(s)\left[I_{n}+Q(s)\right] \Phi(t, s)^{-1} d s \tag{3.27}
\end{equation*}
$$

where

$$
\begin{equation*}
\Phi(t, s)=\exp \left\{\int_{s}^{t} \Lambda(\tau) d \tau\right\} \tag{3.28}
\end{equation*}
$$

with $\Lambda(t)=\operatorname{diag}\left\{\lambda_{1}(t), \lambda_{2}(t), \cdots, \lambda_{n}(t)\right\}$.
STEP 3: Let

$$
\begin{equation*}
\tilde{r}_{1}(t)=\max _{s \geq t}\left[s r_{1}(s)\right] \quad \text { and } \quad \tilde{r}_{2}(t)=\max _{s \geq t}\left[s r_{2}(s)\right] . \tag{3.29}
\end{equation*}
$$

Then by (3.7), $\tilde{r}_{1}(t)$ and $\tilde{r}_{2}(t)$ are monotonic decreasing and tending to zero as $t \rightarrow+\infty$. Moreover, by (3.9), we have

$$
\begin{equation*}
\int_{t_{0}}^{\infty} t^{-1} \tilde{r}_{2}(t) d t<+\infty \tag{3.30}
\end{equation*}
$$

By the notation (3.25) and (3.26), we can write the integral equation (3.27) in the following form:

$$
\left\{\begin{array}{l}
q_{j j}(t)=\int_{\tau_{j j}}^{t}\left[\sum_{h=1, h \neq j}^{n} r_{j h}(s) q_{h j}(s)\right] d s  \tag{3.31}\\
q_{j k}(t)=\int_{\tau_{j k}}^{t} \exp \left\{\int_{s}^{t} \lambda_{j k}(\tau) d \tau\right\}\left[r_{j k}(s)+\sum_{h=1, h \neq j}^{n} r_{j h}(s) q_{h k}(s)\right] d s, \quad(j \neq k)
\end{array}\right.
$$

where $j, k=1,2, \cdots, n$, and the initial points $\tau_{j k}$ are chosen as follows:

$$
\tau_{j k}= \begin{cases}+\infty & \text { if } j=k  \tag{3.32}\\ t_{2} & \text { for the case }(3.5), \quad(j \neq k) \\ +\infty & \text { for the case }(3.6), \quad(j \neq k)\end{cases}
$$

where $t_{2}$ is a suitable large number.
STEP 4: In order to utilize the condition (3.9), namely (3.30), define the "rowwise" successive approximations as follows:

$$
\left\{\begin{array}{l}
q_{0, j k}(t) \equiv 0,  \tag{3.33}\\
q_{p, j j}=\int_{\tau_{j j}}^{t}\left[\sum_{h=1}^{j-1} r_{j h}(s) q_{p, h j}(s)+\sum_{h=j+1}^{n} r_{j h}(s) q_{p-1, h j}(s)\right] d s \\
q_{p, j k}(t)=\int_{\tau_{j k}}^{t} \exp \left\{\int_{s}^{t} \lambda_{j k}(\tau) d \tau\right\}\left[r_{j k}(s)+\sum_{h=1}^{j-1} r_{j h}(s) q_{p, h k}(s)\right. \\
\\
\left.\quad+\sum_{h=j+1}^{n} r_{j h}(s) q_{p-1, h k}(s)\right] d s, \quad(j \neq k)
\end{array}\right.
$$

where $j, k=1,2, \cdots, n, p=1,2, \cdots$. Note that $q_{p, 1 k}(t)$ depends only on $q_{p-1, j k}(t)$, and $q_{p, j k}(t)$ dependents only on $q_{p-1, h k}(t)$ for $h>j$ and $q_{p, h k}(t)$ for $h<j$. Therefore, $q_{p, j k}(t)$ is obtained in the increasing order of $p$ and, for each $p$, in the increasing order of $j$, namely from the first row down, and thus the $q_{p, j k}(t)$ are defined successively.

STEP 5: We will see that $q_{p, j k}(t)$ defined by (3.33) is uniformly bounded for all p on the interval $\left[t_{2},+\infty\right)$ for large enough $t_{2}$; namely,

$$
\begin{equation*}
\left|q_{p, j k}(t)\right| \leq G, \quad t \in\left[t_{2},+\infty\right) \tag{3.34}
\end{equation*}
$$

for $j, k=1,2, \cdots, n ; p=1,2, \cdots$. In order to prove this we provide the following lemma, valid for each fixed $p,(p=1,2, \cdots)$.
Lemma 3.3.1. Suppose that there exist two positive constants $G$ and $t_{2},\left(t_{2}>1\right.$, large enough), such that
(a) $\left|q_{p-1, j k}(t)\right| \leq G$, for $t \in\left[t_{2},+\infty\right), j, k=1,2, \cdots, n$;
(b) $\quad \int_{t_{2}}^{+\infty} s^{-1} \tilde{r}_{2}\left(s^{1 / 2^{n}}\right) d s<\frac{1}{2 n} ;$
(c) $\quad \tilde{r}_{1}\left(\sqrt{t_{2}}\right)+\tilde{r}_{2}\left(t_{2}^{1 / 2^{n}}\right) \leq \frac{G}{2^{n+1}(1+n G) K_{1}}, \quad\left(K_{1}=2^{\beta} K\right)$.

Then we have

$$
\begin{align*}
& \left|q_{p, j k}(t)\right| \leq \\
& \left\{\begin{array}{l}
2 n G\left[\tilde{r}_{1}\left(t_{2}\right)+\tilde{r}_{2}\left(t_{2}\right)\right] t^{-1 / 2^{j}}+n G \int_{t}^{+\infty} s^{-1} \tilde{r}_{2}\left(s^{1 / 2^{j}}\right) d s, \quad(j=k), \\
(1+n G) K_{1}\left\{\left[\tilde{r}_{1}\left(t_{2}\right)+\tilde{r}_{2}\left(t_{2}\right)\right] t^{-1 / 2}+\tilde{r}_{1}(\sqrt{t})+\tilde{r}_{2}(\sqrt{t})\right\}, \quad(j>k), \\
2(1+n G) K_{1}\left\{\left[\tilde{r}_{1}\left(t_{2}\right)+\tilde{r}_{2}\left(t_{2}\right)\right] t^{-1 / 2^{j}}+\tilde{r}_{2}\left(t^{1 / 2^{j}}\right)\right\}, \quad(j<k),
\end{array}\right. \tag{3.35}
\end{align*}
$$

for $t \in\left[t_{2}, \infty\right)$, and $j, k=1,2, \cdots, n$.

Proof of Lemma 3.3.1: We will prove (3.35) in the increasing order of $\mathbf{j}$.
Case I: $j=1$. For $k=1$, since

$$
\begin{equation*}
r_{2}(s) \leq s^{-1} \max _{t \geq s}\left[s r_{2}(t)\right] \leq s^{-1} \max _{t \geq s}\left[t r_{2}(t)\right]=s^{-1} \tilde{r}_{2}(s) \tag{3.36}
\end{equation*}
$$

by (3.33), we have

$$
\begin{align*}
\left|q_{p, 11}(t)\right| & \leq \int_{t}^{\infty} \sum_{h=2}^{n} r_{1 h}(s) q_{p-1, h 1}(s) \mid d s  \tag{3.37}\\
& \leq(n-1) G \int_{t}^{\infty} r_{2}(s) d s \leq(n-1) G \int_{t}^{\infty} s^{-1} \tilde{r}_{2}(s) d s
\end{align*}
$$

For $k=2,3, \cdots, n$, in case of (3.6), for $t \in\left[t_{2}, \infty\right)$, by (3.33) and (3.32), we have

$$
\begin{align*}
\left|q_{p, 1 k}(t)\right| & \leq \int_{t}^{\infty} K\left(\frac{t}{s}\right)^{\beta}\left|r_{1 k}(s)+\sum_{h=2}^{n} r_{1 h}(s) q_{p-1, h k}(s)\right| d s \\
& \leq[1+(n-1) G] K \int_{t}^{\infty} K\left(\frac{t}{s}\right)^{\beta} r_{2}(s) d s  \tag{3.38}\\
& \leq[1+(n-1) G] K\left\{\max _{s \geq t}\left[s r_{2}(s)\right]\right\}=[1+(n-1) G] K \tilde{r}_{2}(t)
\end{align*}
$$

${ }_{n}$ In case of (3.5), for $t \in\left[t_{2}, \infty\right)$, by (3.33) and (3.32), we have

$$
\begin{align*}
\left|q_{p, 1 k}(t)\right| \leq & \int_{t_{2}}^{t} K\left(\frac{s}{t}\right)^{\beta}\left|r_{1 k}(s)+\sum_{h=2}^{n} r_{1 h}(s) q_{p-1, h k}(s)\right| d s \\
\leq & \int_{t_{2}}^{\sqrt{t}+t_{2}} K\left(\frac{s}{t}\right)^{\beta}\left|r_{1 k}(s)+\sum_{h=2}^{n} r_{1 h}(s) q_{p-1, h k}(s)\right| d s \\
& +\int_{\sqrt{t}}^{t} K\left(\frac{s}{t}\right)^{\beta}\left|r_{1 k}(s)+\sum_{h=2}^{n} r_{1 h}(s) q_{p-1, h k}(s)\right| d s  \tag{3.39}\\
\leq & {[1+(n-1) G]\left[\max _{s \geq t_{2}} s r_{2}(s)\right] \int_{t_{2}}^{\sqrt{t}+t_{2}} K s^{-1}\left(\frac{s}{t}\right)^{\beta} d s } \\
& +[1+(n-1) G]\left[\max _{s \geq \sqrt{t}}\left[s r_{2}(s)\right]\right] \int_{\sqrt{t}}^{t} K s^{-1}\left(\frac{s}{t}\right)^{\beta} d s .
\end{align*}
$$

Since

$$
\begin{equation*}
\int_{t_{2}}^{\sqrt{t}+t_{2}} K s^{-1}\left(\frac{s}{t}\right)^{\beta} d s=\frac{K}{\beta}\left[\left(\frac{1}{\sqrt{t}}+\frac{t_{2}}{t}\right)^{\beta}-\left(\frac{t_{2}}{t}\right)^{\beta}\right] \leq 2^{\beta-1} K \frac{1}{\sqrt{t}}, \tag{3.40}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{\sqrt{t}}^{t} K s^{-1}\left(\frac{s}{t}\right)^{\beta} d s \leq \frac{K}{\beta}\left(1-\left(\frac{\sqrt{t}}{t}\right)^{\beta}\right)<K \tag{3.41}
\end{equation*}
$$

we have

$$
\begin{equation*}
\left|q_{p, 1 k}(t)\right| \leq K_{1}[1+(n-1) G] \tilde{r}_{2}\left(t_{2}\right) \frac{1}{\sqrt{t}}+K_{1}[1+(n-1) G] \tilde{r}_{2}(\sqrt{t}) . \tag{3.42}
\end{equation*}
$$

Hence,

$$
\left|q_{p, 1 k}(t)\right| \leq \begin{cases}(n-1) G \int_{t}^{\infty} s^{-1} \tilde{r}_{2}(s) d s, & k=1  \tag{3.43}\\ (1+(n-.1) G) K_{1}\left[\tilde{r}_{2}\left(t_{2}\right) \frac{1}{\sqrt{t}}+\tilde{r}_{2}(\sqrt{t})\right], & k \neq 1\end{cases}
$$

Thus, (3.35) is true for $j=1$ and $k=1,2, \cdots, n$.
Case II: Assume that (3.35) is true for $j<m,(m>1)$, we want to show that (3.35) is true for $j=m$.

Case IIa. For $k<m$, in case of (3.6), from (3.33) and the assumption (3.35) for this case, (i.e. (3.34) is true for $j<m$ ), we have

$$
\begin{align*}
& \left.\left|q_{p, m k}(t)\right| \leq \int_{t}^{\infty} K\left(\frac{t}{s}\right)^{\beta} \right\rvert\, r_{m k}(s)+\sum_{h=1}^{m-1} r_{m h}(s) q_{p, h k}(s) \\
& +\sum_{h=m+1}^{n} r_{m h}(s) q_{p-1, h k}(s) \mid d s \\
& \quad \leq[1+(m-1) G]\left[\max _{s \geq t}\left\{s r_{1}(s)\right\}\right] \int_{t}^{\infty} K s^{-1}\left(\frac{t}{s}\right)^{\beta} d s  \tag{3.44}\\
& \quad+(n-m) G\left[\max _{s \geq t}\left\{s r_{2}(s)\right\}\right] \int_{t}^{\infty} K s^{-1}\left(\frac{t}{s}\right)^{\beta} d s \\
& \quad=K_{1}[1+(m-1) G] \tilde{r}_{1}(t)+K_{1}(n-m) G \tilde{r}_{2}(t) .
\end{align*}
$$

In case of (3.5), from (3.33) and by (3.40) and (3.41), with similar reasons as for (3.39), we have

$$
\begin{aligned}
& \left.\left|q_{p, m k}(t)\right| \leq \int_{t_{2}}^{t} K\left(\frac{t}{s}\right)^{\beta} \right\rvert\, r_{m k}(s)+\sum_{h=1}^{m-1} r_{m h}(s) q_{p, h k}(s) \\
& +\sum_{h=m+1}^{n} r_{m h}(s) q_{p-1, h k}(s) \mid d s \\
& \leq \quad\left\{[1+(m-1) G]\left[\max _{s \geq t_{2}}\left\{s r_{1}(s)\right\}\right]+(n-m) G\left[\max _{s \geq t_{2}}\left\{s r_{2}(s)\right\}\right]\right\} \\
& \quad \cdot \int_{t_{2}}^{\sqrt{t}+t_{2}} K s^{-1}\left(\frac{t}{s}\right)^{\beta} d s
\end{aligned}
$$

$$
\begin{align*}
& +\left\{[1+(m-1) G]\left[\max _{s \geq \sqrt{t}}\left\{s r_{1}(s)\right\}\right]+(n-m) G\left[\max _{s \geq t_{2}}\left\{s r_{2}(s)\right\}\right]\right\} \\
& \quad \cdot \int_{\sqrt{t}}^{t} K s^{-1}\left(\frac{t}{s}\right)^{\beta} d s \\
& \leq K_{1}\left[\{1+(m-1) G\} \tilde{r}_{1}\left(t_{2}\right)+(n-m) G \tilde{r}_{2}\left(t_{2}\right)\right] \frac{1}{\sqrt{t}}  \tag{3.45}\\
& \\
& \quad+K\left[\{1+(m-1) G\} \tilde{r}_{1}(\sqrt{t})+(n-m) G \tilde{r}_{2}(\sqrt{t})\right] \\
& \leq
\end{align*}
$$

Thus, (3.35) is true for $k<j=m$.
CaseIIb. For $k=m$, from (3.33), we have

$$
\begin{align*}
& \left|q_{p, m m}(t)\right| \leq \int_{i}^{\infty}\left\{\sum_{h=1}^{m-1}\left|r_{m h}(s) q_{p, h m}(s)\right|+\sum_{h=m+1}^{n}\left|r_{m h}(s) q_{p-1, h m}(s)\right|\right\} d s  \tag{3.46}\\
& \quad \leq\left\{\sum_{h=1}^{m-1} \int_{t}^{\infty} s^{-1}\left|q_{p, h m}(s)\right| d s\right\}\left[\max _{s \geq t}\left\{s \tilde{r}_{1}(s)\right\}\right]+(n-m) G \int_{t}^{\infty} r_{2}(s) d s
\end{align*}
$$

Since $h<m$ in the first summation, by the assumption (3.35) of this case, we have

$$
\begin{equation*}
\left|q_{p, h m}(t)\right| \leq 2(1+n G) K_{1}\left[\left[\tilde{r}_{1}\left(t_{2}\right)+\tilde{r}_{2}\left(t_{2}\right)\right] t^{-\frac{1}{2^{h}}}+\tilde{r}_{2}\left(t^{\frac{1}{2^{\hbar}}}\right)\right] . \tag{3.47}
\end{equation*}
$$

Hence,

$$
\begin{align*}
& \int_{t}^{\infty} s^{-1}\left|q_{p, h m}(s)\right| d s \\
& \quad \leq 2(1+n G) K_{1}\left[\left[\tilde{r}_{1}\left(t_{2}\right)+\tilde{r}_{2}\left(t_{2}\right)\right] \int_{t}^{\infty} s^{-1} s^{-\frac{1}{2^{\hbar}}} d s+\int_{t}^{\infty} s^{-1} \tilde{r}_{2}\left(s^{\frac{1}{2^{\hbar}}}\right) d s\right] \tag{3.48}
\end{align*}
$$

Since

$$
\begin{equation*}
\int_{t}^{\infty} s^{-1} s^{-\frac{1}{2^{h}}} d s=2^{h} t^{-\frac{1}{2^{h}}} \tag{3.49}
\end{equation*}
$$

we have

$$
\begin{align*}
& \int_{t}^{\infty} s^{-1}\left|q_{p, h m}(s)\right| d s  \tag{3.50}\\
& \quad \leq 2^{h+1}(1+n G) K_{1}\left[\left[\tilde{r}_{1}\left(t_{2}\right)+\tilde{r}_{2}\left(t_{2}\right)\right] t^{-\frac{1}{2^{h}}}+\int_{t}^{\infty} s^{-1} \tilde{r}_{2}\left(s^{\frac{1}{2^{h}}}\right) d s\right]
\end{align*}
$$

for all $h<m$. Note here that $\tilde{r}_{1}(t)$ and $\tilde{r}_{2}(t)$ are decreasing functions. Substituting (3.50) into (3.46), we obtain

$$
\left|q_{p, m m}(t)\right|
$$

$$
\begin{align*}
& \leq 2^{m+1} K_{1}(m-1)(1+n G) \tilde{r}_{1}(t)\left[\tilde{r}_{1}\left(t_{2}\right)+\tilde{r}_{2}\left(t_{2}\right)\right] t^{-\frac{1}{m}}  \tag{3.51}\\
+ & 2^{m} K_{1}(m-1)(1+n G) \tilde{r}_{1}(t) \int_{t}^{\infty} s^{-1} \tilde{r}_{2}\left(s^{\frac{1}{2 m}}\right) d s+(n-m) G \int_{t}^{\infty} r_{2}(s) d s
\end{align*}
$$

By condition (c) of the lemma, we have

$$
\begin{equation*}
2^{n} K_{1}(1+n G) \tilde{r}_{1}(t) \leq \frac{G}{2}<G \tag{3.52}
\end{equation*}
$$

for $t \in\left[t_{2}, \infty\right)$. Hence,

$$
\left|q_{p, m m}(t)\right|
$$

$$
\begin{align*}
\leq & 2(m-1) G\left[\tilde{r}_{1}\left(t_{2}\right)+\tilde{r}_{2}\left(t_{2}\right)\right] t^{-\frac{1}{2^{m}}}+(m-1) G \int_{t}^{\infty} s^{-1} \tilde{r}_{2}\left(s^{\frac{1}{2} \pi}\right) d s  \tag{3.53}\\
& +(n-m) G \int_{t}^{\infty} r_{2}(s) d s \\
\leq & 2(m-1) G\left[\tilde{r}_{1}\left(t_{2}\right)+\tilde{r}_{2}\left(t_{2}\right)\right] t^{-\frac{1}{2^{m}}}+n G \int_{t}^{\infty} s^{-1} \tilde{r}_{2}\left(s^{\frac{1}{2} m}\right) d s .
\end{align*}
$$

Thus, (3.35) holds for $j=k=m$.
CaseIIc. For $k>m$, in case of (3.6), from (3.33), we have

$$
\begin{align*}
& \left|q_{p, m k}(t)\right| \\
& \leq \\
& \left.\quad \int_{t}^{\infty} K\left(\frac{t}{s}\right)^{\beta} \right\rvert\, r_{m k}(s)+\sum_{h=1}^{m-1} r_{m h}(s) q_{p, h k}(s) \\
& \quad+\sum_{h=m+1}^{n} r_{m h}(s) q_{p-1, h k}(s) \mid d s  \tag{3.54}\\
& \leq \\
& \leq\left\{\left[\max _{s \geq t}\left\{s r_{2}(s)\right\}\right]+\sum_{h=1}^{m-1}\left[\max _{s \geq t}\left|s r_{m h}(s) q_{p, h k}(s)\right|\right]\right. \\
& \\
& \left.+(n-m) G\left[\max _{s \geq t}\left\{s r_{2}(s)\right\}\right]\right\} \int_{t}^{\infty} K s^{-1}\left(\frac{t}{s}\right)^{\beta} d s \\
& \leq
\end{align*}
$$

Since $h<m$, by the assumption (3.35) of this case, we have

$$
\begin{align*}
& \left|q_{p, h k}(t)\right| \\
& \quad \leq 2(1+n G) K_{1}\left[\left[\tilde{r}_{1}\left(t_{2}\right)+\tilde{r}_{2}\left(t_{2}\right)\right] t^{-\frac{1}{2^{\hbar}}}+\tilde{r}_{2}\left(t^{\frac{1}{2^{\hbar}}}\right)\right] \tag{3.55}
\end{align*}
$$

Substituting these into (3.54), we obtain

$$
\begin{align*}
& \left|q_{p, m k}(t)\right| \leq K\left[\tilde{r}_{2}(t)+2 K_{1}(m-1)(1+n G) \tilde{r}_{1}(t)\left(\left[\tilde{r}_{1}(t)+\tilde{r}_{2}(t)\right] t^{-\frac{1}{2 m}}\right.\right.  \tag{3.56}\\
& \left.\left.\quad+\tilde{r}_{2}\left(t^{\frac{1}{2 m}}\right)\right)+(n-m) G \tilde{r}_{2}(t)\right] .
\end{align*}
$$

By (3.52), we have

$$
\begin{align*}
& \left|q_{p, m k}(t)\right| \leq K_{1}\left[\tilde{r}_{2}(t)+(m-1) G\left[\tilde{r}_{1}(t)+\tilde{r}_{2}(t)\right] t^{-\frac{1}{2^{m}}}\right. \\
& \left.\quad \quad+(m-1) G \tilde{r}_{2}\left(t \frac{1}{2^{m}}\right)+(n-m) G \tilde{r}_{2}(t)\right]  \tag{3.57}\\
& \leq \\
& \quad K_{1}(m-1) G\left[\tilde{r}_{1}(t)+\tilde{r}_{2}(t)\right] t^{-\frac{1}{2^{m}}}+K_{1}(1+n G) \tilde{r}_{2}\left(t^{\frac{1}{2^{m}}}\right) \\
& \leq \\
& 2 K_{1}(1+n G)\left\{\left[\tilde{r}_{1}(t)+\tilde{r}_{2}(t)\right] t^{-\frac{1}{2^{m}}}+\tilde{r}_{2}\left(t^{\frac{1}{2^{m}}}\right)\right\}
\end{align*}
$$

Thus, (3.35) holds for this case.
In case of (3.5), from (3.33) and the assumption (3.35) of this case, since $h<m<k$, we have

$$
\begin{align*}
& \left.\left|q_{p, m k}(t)\right| \leq \int_{t_{2}}^{t} K\left(\frac{s}{t}\right)^{\beta} \right\rvert\, r_{m k}(s)+\sum_{h=1}^{m-1} r_{m h}(s) q_{p, h k}(s) \\
& \quad+\sum_{h=m+1}^{n} r_{m h}(s) q_{p-1, h k}(s) \mid d s \\
& \quad \leq\left\{\max _{s \geq t_{2}}\left\{s r_{2}(s)\right\}+(m-1) G\left[\max _{s \geq t_{2}}\left\{s r_{1}(s)\right\}\right]\right.  \tag{3.58}\\
& \left.\quad+(n-m) G\left[\max _{s \geq t_{2}}\left\{s r_{2}(s)\right\}\right]\right\} \int_{t_{2}}^{\sqrt{t}+t_{2}} K s^{-1}\left(\frac{s}{t}\right)^{\beta} d s \\
& \quad+\left\{\max _{s \geq \sqrt{t}}\left\{s r_{2}(s)\right\}+(m-1)\left[\max _{s \geq \sqrt{t}, 1 \leq h \leq m-1}\left|q_{p, h k}(s)\left\{s r_{1}(s)\right\}\right|\right]\right. \\
& \left.\quad+(n-m) G\left[\max _{s \geq \sqrt{t}}\left\{s r_{2}(s)\right\}\right]\right\} \int_{\sqrt{t}}^{t} K s^{-1}\left(\frac{s}{t}\right)^{\beta} d s
\end{align*}
$$

$$
\begin{aligned}
& \leq K_{1}\left\{\tilde{r}_{2}\left(t_{2}\right)+(m-1) G \tilde{r}_{1}\left(t_{2}\right)+(n-m) G \tilde{r}_{2}\left(t_{2}\right)\right\} t^{-\frac{1}{2^{\prime} m}} \\
& +K_{1}\left[\tilde{r}_{2}(\sqrt{t})+2 K_{1}(m-1)(1+n G) \tilde{r}_{1}(\sqrt{t})\left[\tilde{r}_{1}\left(t_{2}\right)+\tilde{r}_{2}\left(t_{2}\right)\right] t^{-\frac{1}{2^{\frac{1}{m}}}}\right. \\
& \left.+2 K_{1}(m-1)(1+n G) \tilde{r}_{1}(\sqrt{t}) \tilde{r}_{2}\left(t^{\frac{1}{2} m}\right)+(n-m) G \tilde{r}_{2}(\sqrt{t})\right]
\end{aligned}
$$

By (3.52) and the fact that $\tilde{r}_{2}(t)$ is monotonic decreasing, we have

$$
\begin{align*}
& \left|q_{p, m k}(t)\right| \\
& \leq \\
& \leq K_{1}\left[\tilde{r}_{2}\left(t_{2}\right)+(m-1) G \tilde{r}_{1}\left(t_{2}\right)+(n-m) G \tilde{r}_{2}\left(t_{2}\right)\right] t^{-\frac{1}{2^{m}}} \\
& \quad+K_{1}\left[\tilde{r}_{2}(\sqrt{t})+(m-1) G\left[\tilde{r}_{1}\left(t_{2}\right)+\tilde{r}_{2}\left(t_{2}\right)\right] t^{-\frac{1}{2^{m}}}\right.  \tag{3.59}\\
& \left.\quad+(m-1) G \tilde{r}_{2}\left(t^{\frac{1}{2^{m}}}\right)+(n-m) G \tilde{r}_{2}(\sqrt{t})\right] \\
& \leq \\
& \leq \\
& \quad K_{1}\left[\tilde{r}_{2}\left(t_{2}\right)+n G \tilde{r}_{1}\left(t_{2}\right)+n G \tilde{r}_{2}\left(t_{2}\right)+n G\left[\tilde{r}_{1}\left(t_{2}\right)+\tilde{r}_{2}\left(t_{2}\right)\right]\right] t^{-\frac{1}{2^{m m}}} \\
& \\
& \quad+K_{1}\left[\tilde{r}_{2}(\sqrt{t})+(m-1) G \tilde{r}_{2}\left(t^{\frac{1}{2^{m}}}\right)+(n-m) G \tilde{r}_{2}(\sqrt{t})\right] \\
& \leq
\end{align*}
$$

Thus, by (3.44), (3.45), (3.53), (3.57) and (3.59), we have for $j=m$,

$$
\begin{align*}
& \left|q_{p, m k}(t)\right| \leq \\
& \left\{\begin{array}{l}
2 n G\left[\tilde{r}_{1}\left(t_{2}\right)+\tilde{r}_{2}\left(t_{2}\right)\right] t^{-1 / 2^{m}}+n G \int_{t}^{+\infty} s^{-1} \tilde{r}_{2}\left(s^{1 / 2^{m}}\right) d s, \quad(m=k), \\
(1+n G) K_{1}\left\{\left[\tilde{r}_{1}\left(t_{2}\right)+\tilde{r}_{2}\left(t_{2}\right)\right] \frac{1}{\sqrt{t}}+\tilde{r}_{1}(\sqrt{t})+\tilde{r}_{2}(\sqrt{t})\right\}, \quad(m>k), \\
2(1+n G) k_{1}\left\{\left[\tilde{r}_{1}\left(t_{2}\right)+\tilde{r}_{2}\left(t_{2}\right)\right] t^{-1 / 2^{m}}+\tilde{r}_{2}\left(t^{1 / 2^{m}}\right)\right\}, \quad(m<k),
\end{array}\right. \tag{3.60}
\end{align*}
$$

for $t \in\left[t_{2}, \infty\right)$, and $m, k=1,2, \cdots, n$.
STEP 6: In this step, we will show that, for large enough $t_{2}$, the sequences $\left\{q_{p, j k}(t) \mid p=1,2, \cdots\right\}$ converges uniformly to functions $q_{j k}(t)$ on $\left[t_{2}, \infty\right)$ satisfying
the expressions (3.31), for $j, k=1,2, \cdots, n$. In order to do that, let

$$
\begin{equation*}
\left\|q_{p}-q_{p-1}\right\|_{t}=\max _{s>t, 1 \leq j, k \leq n}\left|q_{p, j k}(s)-q_{p-1, j k}(s)\right| . \tag{3.61}
\end{equation*}
$$

Note that, from (3.33), we have

$$
\begin{gather*}
q_{p+1, j k}(t)-q_{p, j k}(t)= \\
\left\{\begin{array}{l}
\int_{\infty}^{t}\left\{\sum_{h=1}^{j-1} r_{j h}(s)\left[q_{p+1, h j}(s)-q_{p, h j}(s)\right]\right. \\
\left.+\sum_{h=j+1}^{n} r_{j h}(s)\left[q_{p, h j}(s)-q_{p-1, h j}(s)\right]\right\} d s, \quad(j=k), \\
\int_{\tau_{j k}}^{t} \exp \left[\int_{s}^{t} \lambda_{j k}(\tau) d \tau\right]\left\{\sum_{h=1}^{j-1} r_{j h}(s)\left[q_{p+1, h j}(s)-q_{p, h j}(s)\right]\right. \\
\left.+\sum_{h=j+1}^{n} r_{j h}(s)\left[q_{p, h j}(s)-q_{p-1, h j}(s)\right]\right\} d s, \quad(j \neq k)
\end{array}\right. \tag{3.62}
\end{gather*}
$$

We will establish:
Lemma 3.3.2. If $t_{2}$ is large enough such that
(a)

$$
\int_{t}^{+\infty} s^{-1} \tilde{r}_{2}\left(s^{\frac{1}{2^{x}}}\right) d s<\frac{1}{4 n}, \text { for } t \geq t_{2}
$$

and
(b)

$$
\tilde{r}_{1}\left(\sqrt{t_{2}}\right)+\tilde{r}_{2}\left(t_{2}^{\frac{1}{2^{n}}}\right)<\frac{1}{2^{n+2} n K_{1}}
$$

then we have

$$
\begin{align*}
& \left|q_{p+1, j k}(t)-q_{p, j k}(t)\right| \leq \\
& \left\{\begin{array}{l}
\left\{2 n\left[\tilde{r}_{1}\left(t_{2}\right)+\tilde{r}_{2}\left(t_{2}\right)\right] t^{-1 / 2^{j}}+n \int_{t}^{+\infty} s^{-1} \tilde{r}_{2}\left(s^{1 / 2^{j}}\right) d s\right\}\left\|q_{p}-q_{p-1}\right\|_{t_{2}}, \quad(j=k), \\
n K_{1}\left\{\left[\tilde{r}_{1}\left(t_{2}\right)+\tilde{r}_{2}\left(t_{2}\right)\right] \frac{1}{\sqrt{t}}+\tilde{r}_{1}(\sqrt{t})+\tilde{r}_{2}(\sqrt{t})\right\}\left\|q_{p}-q_{p-1}\right\|_{t_{2}}, \quad(j>k), \\
2 n K_{1}\left\{\left[\tilde{r}_{1}\left(t_{2}\right)+\tilde{r}_{2}\left(t_{2}\right)\right] t^{-1 / 2^{j}}+\tilde{r}_{2}\left(t^{1 / 2^{j}}\right)\right\}\left\|q_{p}-q_{p-1}\right\|_{t_{2}}, \quad(j<k),
\end{array}\right. \tag{3.63}
\end{align*}
$$

for $t \geq t_{2}$ and $j, k=1,2, \cdots, n ; p=1,2, \cdots$.
Now for $t_{2}$ satisfying Lemma 3.3.2, we have

$$
\begin{equation*}
\left|q_{p+1, j k}(t)-q_{p, j k}(t)\right| \leq \frac{1}{2}\left\|q_{p}-q_{p-1}\right\|_{t_{2}} \tag{3.64}
\end{equation*}
$$

for $t \geq t_{2}$ and $j, k=1,2, \cdots, n ; p=1,2, \cdots$. Hence,

$$
\begin{equation*}
\left\|q_{p+1}-q_{p}\right\|_{t_{2}} \leq \frac{1}{2}\left\|q_{p}-q_{p-1}\right\|_{t_{2}} \tag{3.65}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|q_{p+1}-q_{p}\right\|_{t_{2}} \leq \frac{1}{2^{p}}\left\|q_{1}-q_{0}\right\|_{t_{2}}=\frac{1}{2^{p}} G \tag{3.66}
\end{equation*}
$$

for $p=1,2, \cdots$. Therefore, the sequences $\left\{q_{p, j k}(t) \mid p=1,2, \cdots\right\}$ converge uniformly to functions $q_{j k}(t)$ on $\left[t_{2}, \infty\right)$, for $j, k=1,2, \cdots, n$, and furthermore, they satisfy the expressions (3.31).

Lemma 3.3.2 can be proved in a fashion similar to that for Lemma 3.3.1 with slight modifications.

STEP 7: In order to show that

$$
\begin{equation*}
\lim _{t \rightarrow \infty} q_{j k}(t)=0, \quad j, k=1,2, \cdots, n \tag{3.67}
\end{equation*}
$$

note from Lemma 3.3.1, we have

$$
\begin{align*}
& \left|q_{j k}(t)\right|=\lim _{p \rightarrow \infty}\left|q_{p, j k}(t)\right| \leq \\
& \quad\left\{\begin{array}{l}
2 n G\left[\tilde{r}_{1}\left(t_{2}\right)+\tilde{r}_{2}\left(t_{2}\right)\right] t^{-\frac{1}{2 j}}+n G \int_{t}^{+\infty} s^{-1} \tilde{r}_{2}\left(s^{1 / 2^{j}}\right) d s, \quad(j=k), \\
(1+n G) K_{1}\left\{\left[\tilde{r}_{1}\left(t_{2}\right)+\tilde{r}_{2}\left(t_{2}\right)\right] \frac{1}{\sqrt{t}}+\tilde{r}_{1}(\sqrt{t})+\tilde{r}_{2}(\sqrt{t})\right\}, \quad(j>k), \\
2(1+n G) K_{1}\left\{\left[\tilde{r}_{1}\left(t_{2}\right)+\tilde{r}_{2}\left(t_{2}\right)\right] t^{-\frac{1}{2 j}}+\tilde{r}_{2}\left(t^{1 / 2^{j}}\right)\right\}, \quad(j<k),
\end{array}\right. \tag{3.68}
\end{align*}
$$

for $t \in\left[t_{2}, \infty\right)$, and $j, k=1,2, \cdots, n$. By (3.7), we have

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \tilde{r}_{j}(t)=0, \quad(j=1,2) \tag{3.69}
\end{equation*}
$$

Thus, by (3.9), (3.68) and (3.69), (3.67) follows. This completes the proof of Theorem 3.2.1.

### 3.4 Examples

In this section, we will use some examples to illustrate the application of Theorem 3.2.1.

Example 1. Consider

$$
\begin{equation*}
\frac{d y}{d t}=A(t) y \tag{3.70}
\end{equation*}
$$

with

$$
A(t)=\left[\begin{array}{cc}
1 & \frac{\sin (t)}{t \ln ^{2}(t)} \\
\frac{\cos \left(t^{3}\right)}{t \ln (\ln (t))} & 1+2 t^{-1}
\end{array}\right]
$$

for $t \in[2,+\infty)$.
$A(t)$ satisfies Assumption 3.2.1 and 3.2.2 with $\beta=2$, so by Theorem 3.2.1 we know that (3.70) can be changed into

$$
\frac{d z}{d t}=\left[\begin{array}{cc}
1 & 0 \\
0 & 1+2 t^{-1}
\end{array}\right] z
$$

by $y=\left[I_{2}+Q(t)\right] z$, where $Q(t) \rightarrow 0$, as $t \rightarrow+\infty$.

Theorem L can not be applied to this example, because the corresponding $R(t)$ does not satisfy $R(t) \in L^{1}\left(I_{0}\right)$; Theorem LE can not be applied to this example, because the corresponding matrix $A$ has multiple eigenvalues; Theorem WH and Theorem HL can not be applied to this example, because the corresponding $A(t)$ does not satisfy Assumption 1.5; Theorem HX can not be applied to this
example, because the corresponding $A(t)$ does not satisfy Assumption 1.7; Theorem 2.1.1 and Theorem 2.1.2 can not be applied to this example, because the corresponding $A(t)$ does not satisfy Assumption 2.1.2; Theorem 3.1.1 and Theorem 3.1.2 can not be applied to this example, because the corresponding $V(t)$ does not satisfy $V^{\prime}(t) \in L^{1}\left(I_{0}\right)$.

Example 2. Consider

$$
\begin{equation*}
\frac{d y}{d t}=A(t) y \tag{3.71}
\end{equation*}
$$

with

$$
A(t)=\left[\begin{array}{cc}
1 & 1 \\
t^{-3 / 2} & 1
\end{array}\right]
$$

for $t \in[1,+\infty)$.

System (3.71) can be changed into

$$
\frac{d \tilde{y}}{d t}=\left[\begin{array}{cc}
1+t^{-3 / 4}+\frac{3}{8} t^{-1} & -\frac{3}{8} t^{-1}  \tag{3.72}\\
-\frac{3}{8} t^{-1} & 1-t^{-3 / 4}+\frac{3}{8} t^{-1}
\end{array}\right] \tilde{y}
$$

by

$$
y=\left[\begin{array}{cc}
1 & 1 \\
t^{-3 / 4} & -t^{-3 / 4}
\end{array}\right] \tilde{y} .
$$

By Proposition 3.2.1, we know that (3.72) can be changed into

$$
\frac{d z}{d t}=\left[\begin{array}{cc}
1+t^{-3 / 4}+\frac{3}{8} t^{-1} & 0 \\
0 & 1-t^{-3 / 4}-\frac{5}{8} t^{-1}
\end{array}\right] z
$$

by $\tilde{y}=S(t)\left[I_{2}+Q(t)\right] z$, where $Q(t) \rightarrow 0$, as $t \rightarrow+\infty$, and

$$
S(t)=\left[\begin{array}{cc}
1 & 0 \\
0 & t^{-\delta}
\end{array}\right], \text { with } 0<\delta<\frac{1}{4}
$$

For similar reasons to those in Example 1, Theorem L, Theorem LE, Theorem WH, Theorem HL, Theorem HX, Theorem 2.1.1 and Theorem 2.1.2 can not be applied to this example. Theorem 3.1.1 can not be applied to this example, because the eigenvalues of $A(t)$ do not satisfy Assumption 3.1.3. Note in this case $r=1$. Theorem 3.1.2 can not be applied to this example, because the corresponding $R(t)$ does not satisfy condition (D) in Theorem 3.1.2. Note in this case $n_{0}=2$, to satisfy Assumption 3.1.5, we have to let $V(t)$ be the zero matrix and let $R(t)$ be

$$
R(t)=\left[\begin{array}{cc}
0 & 0 \\
t^{-3 / 2} . & 0
\end{array}\right]
$$

## CHAPTER IV

## THE DEFICIENCY INDEX OF CERTAIN FOURTH ORDER ORDINARY SELF-ADJOINT DIFFERENTIAL OPERATORS

### 4.1 Introduction

Consider the self-adjoint ordinary differential operator

$$
\begin{equation*}
l(y)=\sum_{k=0}^{n}(-1)^{k}\left(p_{n-k}(t) y^{(k)}\right)^{(k)} \tag{4.1}
\end{equation*}
$$

defined on some interval $(a, b)$ of the real axis, where b is either finite or $+\infty$, the coefficients $p_{k}(t), \quad(k=0,1,2, \cdots, n)$, are all real functions and $p_{0}(t)>0$.

The differential operator (4.1) is self-adjoint in the sense that for any test functions $u, v \in C_{0}^{\infty}(a, b)$,

$$
\int_{a}^{b} l(u) \bar{v} d t=\int_{a}^{b} u \overline{l(v)} d t .
$$

Hence, when restricted to the test functions, $l$ is a densely defined symmetric operator in the Hilbert-Lebesgue space $L^{2}(a, b)$, so it has a symmetric closed extension designated by $L_{0}$ and called the minimal operator associated with $l$.

## Let

$$
k=\operatorname{dim}\left\{\left(L_{0}-\bar{z} I\right) D\left(L_{0}\right)\right\}^{\perp}, \quad m=\operatorname{dim}\left\{\left(L_{0}-z I\right) D\left(L_{0}\right)\right\}^{\perp}, \operatorname{Im}\{z\}>0,
$$

where $D\left(L_{0}\right)$ is the domain of $L_{0}$. The pair $(k, m)$ is called the deficiency index of $L_{0}$. The knowledge of the deficiency index will give information about the spectra of self-adjoint extensions. Since the differential operator (4.1) has real coefficients, we have $k=m$. It is proved that, for (4.1), m can take any value between n and 2 n . Thus the deficiency index problem for (4.1) is then the problem of determining the
integer m in $[\mathrm{n}, 2 \mathrm{n}]$ for a certain class of real coefficients $p_{k}(t), \quad(k=0,1,2, \cdots, n)$. We may formulate this in the following way:

Find the dimension of the linear space of square integrable solutions to the equation

$$
\begin{equation*}
l(y)=z y, \quad \operatorname{Im}\{z\} \neq 0 \tag{4.2}
\end{equation*}
$$

In the late 1960 's, W.N. Everitt ([15], [16]) first investigated the limit point problem for the following fourth order differential operator

$$
y^{(4)}-\left(p_{1} y^{\prime}\right)^{\prime}+p_{2} y
$$

Stimulated by the work of Everitt, many other authors, including A. Devinnatz ([7], [8]), S.P. Eastham ([11], [12], [13]), R.B. Paris and A.D. Wood [32], P.W. Walker [35], and A.D. Wood [37], studied the deficiency index problem of the minimal operator associated with the following fourth order self-adjoint differential operator

$$
\begin{equation*}
l(y)=y^{(4)}(t)-a\left\{t^{\alpha} y^{\prime}(t)\right\}^{\prime}+b t^{\beta} y(t), \quad t \in(1, \infty), \tag{4.3}
\end{equation*}
$$

where a , and b are real numbers. In the $\alpha \beta$-plane, the solution of differential equation (4.2) for differential operator (4.3) has special properties on the half-ray $\beta=\alpha-2, \alpha>2$, and most methods in determining deficiency index fail to apply to this case.

In this chapter, we will use the result in Chapter III to investigate the deficiency index of the minimal operator associated with the following differential operator

$$
\begin{equation*}
l(y)=y^{(4)}(t)-a\left\{t^{\alpha} y^{\prime}(t)\right\}^{\prime}+c(t) t^{\beta} y(t), \quad t \in(1, \infty) \tag{4.4}
\end{equation*}
$$

where $c(t)=b(\ln t)^{\eta}$, and $\alpha, \beta, \eta, a, b$ are any real numbers. We are especially interested in the half-ray $\beta=\alpha-2, \alpha>2$. The purpose of this Chapter is to show that the new result in Chapter III can be used to do a delicate analysis.

The main theorems of this Chapter are the following:
Theorem 4.1.1. If $\beta<\alpha-2, \alpha>2$, or $\beta=\alpha-2, \alpha>2, \eta<-1$, then the deficiency index for the minimal operator associated with the self-adjoint operator (4.4) is (2,2) when $a>0$, and is $(3,3)$ when $a<0$.

Theorem 4.1.2. If $\alpha-2<\beta<\frac{3 \alpha-2}{2}, \alpha>2$, or $\beta=\alpha-2, \alpha>2, \quad \eta>1$, then the deficiency index for the minimal operator associated with the self-adjoint operator (4.4) is (2, 2) when $a>0, b>0$; and is (3,3) when $a<0, b<0$.

### 4.2 Asymptotic Analysis

By using quasi-derivatives [8], equation

$$
\begin{equation*}
y^{(4)}(t)-a\left\{t^{\alpha} y^{\prime}(t)\right\}^{\prime}+c(t) t^{\beta} y(t)=z y(t), \quad \operatorname{Im}\{z\} \neq 0 \tag{4.5}
\end{equation*}
$$

can be changed equivalently to

$$
\begin{equation*}
\frac{d U}{d t}=B(t) U \tag{4.6}
\end{equation*}
$$

where

$$
\begin{gathered}
B(t)=\left[\begin{array}{cccc}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & a t^{\alpha} & 0 & -1 \\
c(t) t^{\beta}-z & 0 & 0 & 0
\end{array}\right], \\
U(t)=\left(y, y^{\prime}, y^{(2)}, y^{[3]}\right)
\end{gathered}
$$

and

$$
y^{[3]}=a t^{\alpha} y^{\prime}-y^{(3)}
$$

Let $Q_{0}(t)$ be a positive measurable function such that $1 / Q_{0}(t)$ is integrable on every finite subinterval of $(0, \infty)$, but is divergent on the entire half-axis. Let us set

$$
\begin{equation*}
s(t)=\int_{0}^{t} \frac{d \tau}{Q_{0}(\tau)} \tag{4.7}
\end{equation*}
$$

and denote its inverse by $t=t(s)$. Further, if we set $V(s)=U(t(s))$, then (4.6) takes the form

$$
\begin{equation*}
\frac{d V}{d t}=\left[Q_{0}(t(s)) B(t(s))\right] V \tag{4.8}
\end{equation*}
$$

Now let $Q_{1}(t)$ and $Q_{2}(t)$ be positive functions which are absolutely continuous on every finite subinterval of $(0, \infty)$. Set

$$
\begin{equation*}
C(s)=\operatorname{diag}\left\{Q_{2}(t(s)), Q_{1}(t(s)), Q_{1}^{-1}(t(s)), Q_{2}^{-1}(t(s))\right\} \tag{4.9}
\end{equation*}
$$

and make the transformation:

$$
\begin{equation*}
V(s)=C(s) W(s) \tag{4.10}
\end{equation*}
$$

then the differential equation (4.8) becomes

$$
\begin{equation*}
\frac{d W}{d s}=A(s) W \tag{4.11}
\end{equation*}
$$

where

$$
A(s)=\left[\begin{array}{cccc}
-b_{2} & d_{1} & 0 & 0 \\
0 & -b_{1} & a_{0} & 0 \\
0 & a_{1} & b_{1} & -d_{1} \\
a_{2}-z Q_{0} Q_{2}^{2} & 0 & 0 & b_{2}
\end{array}\right],
$$

and

$$
\begin{gathered}
a_{0}=\frac{Q_{0}}{Q_{1}^{2}}, a_{1}=a Q_{0} Q_{1}^{2} t(s)^{\alpha}, a_{2}=c(t(s)) Q_{0} Q_{2}^{2} t(s)^{\beta}, \\
b_{k}=\frac{1}{Q_{k}} \cdot \frac{d Q_{k}}{d s},(k=1,2), d_{1}=\frac{Q_{0} Q_{1}}{Q_{2}} .
\end{gathered}
$$

If we choose

$$
\begin{equation*}
Q_{0}(t)=\frac{1}{\alpha_{0}+1} t^{-\alpha_{0}}, \quad\left(\alpha_{0} \geq 0\right), \quad Q_{k}(t)=t^{-\alpha_{k}}, \quad(k=1,2) \tag{4.12}
\end{equation*}
$$

and take $a_{0}(s)=1, a_{1}(s)=a, d_{1}(s)=1$, then we have

$$
\alpha_{0}=\frac{1}{2} \alpha, \alpha_{1}=\frac{1}{4} \alpha, \quad \alpha_{2}=\frac{3}{4} \alpha .
$$

Thus $A(s)$ is now of the form

$$
A(s)=\frac{1}{\alpha_{0}+1}\left[\begin{array}{cccc}
\frac{3 \alpha}{4 s} & 1 & 0 & 0  \tag{4.13}\\
0 & \frac{\alpha}{4 s} & 1 & 0 \\
0 & a & -\frac{\alpha}{4 s} & -1 \\
\frac{c(s)}{s^{\gamma}}-\frac{z}{s^{b}} & 0 & 0 & -\frac{3 \alpha}{4 s}
\end{array}\right]
$$

where

$$
\gamma(\alpha, \beta)=\frac{4 \alpha-2 \beta}{2+\alpha}>1, \text { and } \delta(\alpha)=\frac{4 \alpha}{\alpha+2}>2, \text { for } \beta<\frac{3 \alpha-2}{2}, \alpha>2
$$

Write $A(s)$ as

$$
A(s)=A+V(s)
$$

with

$$
A=\frac{1}{\alpha_{0}+1}\left[\begin{array}{rrrr}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & a & 0 & -1 \\
0 & 0 & 0 & 0
\end{array}\right]
$$

then A has a double eigenvalue at $\lambda=0$ and two simple eigenvalues at $\pm \sqrt{a} /\left(\alpha_{0}+\right.$ 1). Hence there is an invertible matrix $P$ such that

$$
P^{-1} A P=\Lambda\left[\begin{array}{cccc}
\lambda_{1} & 0 & 0 & 0 \\
0 & \lambda_{2} & 0 & 0 \\
0 & 0 & 0 & 1 /\left(\alpha_{0}+1\right) \\
0 & 0 & 0 & 0
\end{array}\right]
$$

where $\lambda_{1}=-\lambda_{2}=\sqrt{a} /\left(\alpha_{0}+1\right)$.
Denote the eigenvalues of $A(s)$ by $\lambda_{k}(s), \quad(k=1,2,3,4)$, solve

$$
\begin{equation*}
\left(\alpha_{0}+1\right) A(s) P(s)=P(s) \tilde{\Lambda}(s) \tag{4.14}
\end{equation*}
$$

where

$$
\tilde{\Lambda}(s)=\left[\begin{array}{cccc}
\tilde{\lambda}_{1}(s) & 0 & 0 & 0 \\
0 & \tilde{\lambda}_{2}(s) & 0 & 0 \\
0 & 0 & \tilde{\lambda}_{3}(s) & 1 \\
0 & 0 & 0 & \tilde{\lambda}_{4}(s)
\end{array}\right]
$$

with $\tilde{\lambda}_{k}(s)=\left(\alpha_{0}+1\right) \lambda_{k}(s), \quad k=1,2,3,4$, we have

$$
P(s)=\left[\begin{array}{cccc}
1 & 1 & 1 & 0  \tag{4.15}\\
\tilde{\lambda}_{1}(s)-\frac{3 \alpha}{4 s} & \tilde{\lambda}_{2}(s)-\frac{3 \alpha}{4 s} & \tilde{\lambda}_{3}(s)-\frac{3 \alpha}{4 s} & 1 \\
f_{1}(s) & f_{2}(s) & f_{3}(s) & -\frac{\alpha}{s} \\
g_{1}(s) & g_{2}(s) & g_{3}(s) & a+\left(\frac{\alpha}{4 s}\right)^{2}-\tilde{\lambda}_{4}^{2}(s)
\end{array}\right]
$$

where

$$
f_{k}(s)=\tilde{\lambda}_{k}^{2}(s)-\frac{\alpha}{s} \tilde{\lambda}_{k}(s)+3\left(\frac{\alpha}{4 s}\right)^{2}, \quad g_{k}(s)=\frac{s^{\delta} c(s)-z s^{\gamma}}{s^{\delta+\gamma}\left(\tilde{\lambda}_{k}(s)+\frac{3 \alpha}{4 s}\right)}, \quad k=1,2,3 .
$$

Now let us compute $\tilde{\lambda}_{k}(s), \quad k=1,2,3,4$. The characteristic equation of the matrix $\left(\alpha_{0}+1\right) A(s)$ is

$$
\begin{equation*}
\lambda^{4}-\left\{\frac{5}{8}\left(\frac{\alpha}{s}\right)^{2}+a\right\} \lambda^{2}+\left(\frac{3 \alpha}{4 s}\right)^{2}\left\{\left(\frac{\alpha}{4 s}\right)^{2}+a\right\}+\frac{c(s)}{s^{\gamma}}-\frac{z}{s^{\delta}}=0 . \tag{4.16}
\end{equation*}
$$

Case I: When $\beta<\alpha-2, \alpha>2$, or $\beta=\alpha-2, \quad \alpha>2, \quad \eta<-1$, by solving above equation, we can have

$$
\begin{equation*}
\tilde{\lambda}^{2}(s)=\frac{1}{2}\left\{\frac{5}{8}\left(\frac{\alpha}{s}\right)^{2}+a\right\}\left[1 \pm\left\{1-\frac{4}{a^{2}}\left(a\left(\frac{3 \alpha}{4 s}\right)^{2}+\frac{c(s)}{s^{\gamma}}-\frac{z}{s^{\delta}}\right)+h(s)\right\}^{\frac{1}{2}}\right] \tag{4.17}
\end{equation*}
$$

where $h(s)=O\left(\frac{1}{9^{4}}\right)$. If we choose the plus sign in (4.17) and use Taylor expansion, we have

$$
\tilde{\lambda}^{2}(s)=a+h_{1}(s)
$$

where $h_{1}(s)=O\left(\frac{1}{s^{2}}\right)$ and $h_{1}^{\prime}(s)=O\left(\frac{1}{s^{3}}\right)$. If we choose the negative sign in (4.17) and use Taylor expansion again, we have

$$
\tilde{\lambda}^{2}(s)=\left(\frac{3 \alpha}{4 s}\right)^{2}+h_{2}(s)
$$

where $h_{2}(s)=o\left(\frac{1}{s^{2}}\right)$ and $h_{2}^{\prime}(s)=o\left(\frac{1}{s^{3}}\right)$. Thus the eigenvalues of $\left(\alpha_{0}+1\right) A(s)$ are

$$
\tilde{\lambda}_{1}(s)=-\tilde{\lambda}_{2}(s)=\sqrt{a}+h_{3}(s)
$$

and

$$
\tilde{\lambda}_{3}(s)=-\tilde{\lambda}_{4}(s)=\frac{3 \alpha}{4 s}+h_{4}(s)
$$

where $h_{3}(s)=O\left(\frac{1}{s^{2}}\right), \quad h_{3}^{\prime}(s)=O\left(\frac{1}{s^{3}}\right), \quad h_{4}(s)=o\left(\frac{1}{s}\right)$ and $h_{4}^{\prime}(s)=o\left(\frac{1}{s^{2}}\right)$. Hence,

$$
P(s)=P+O\left(\frac{1}{s}\right) \text { with } P=\left[\begin{array}{cccc}
1 & 1 & 1 & 0  \tag{4.18}\\
\sqrt{a} & -\sqrt{a} & 0 & 1 \\
a & a & 0 & 0 \\
0 & 0 & 0 & a
\end{array}\right]
$$

It follows that for large $\mathrm{s}, P(s)$ is invertible, and

$$
\begin{equation*}
P^{-1}(s)=P^{-1}+O\left(\frac{1}{s}\right) \tag{4.19}
\end{equation*}
$$

with

$$
P^{-1}=\frac{1}{2 a \sqrt{a}}\left[\begin{array}{cccc}
0 & a & \sqrt{a} & -1 \\
0 & -a & \sqrt{a} & 1 \\
2 a \sqrt{a} & 0 & -2 \sqrt{a} & 0 \\
0 & 0 & 0 & 2 \sqrt{a}
\end{array}\right]
$$

Make the transformation

$$
\begin{equation*}
W(s)=P(s) X(s) \tag{4.20}
\end{equation*}
$$

then the differential equation (4.11) with $A(s)$ given by (4.13) becomes

$$
\begin{equation*}
\frac{d X}{d s}=\left\{\tilde{\Lambda}(s) /\left(\alpha_{0}+1\right)-P^{-1}(s) P^{\prime}(s)\right\} X \tag{4.21}
\end{equation*}
$$

Notice that $2 \leq \gamma<\delta$. Thus

$$
P^{\prime}(s)=\left[\begin{array}{cccc}
0 . & 0 & 0 & 0 \\
O\left(\frac{1}{s^{2}}\right) & O\left(\frac{1}{s^{2}}\right) & O\left(\frac{1}{s^{2}}\right) & 0 \\
O\left(\frac{1}{s^{2}}\right) & O\left(\frac{1}{s^{2}}\right) & O\left(\frac{1}{s^{3}}\right) & O\left(\frac{1}{s^{2}}\right) \\
O\left(\frac{|c(s)|]}{s^{1+\gamma}}\right) & O\left(\frac{|c(s)|}{s^{1}+\gamma}\right) & O\left(\frac{|c(s)|]}{s^{\gamma}}\right) & O\left(\frac{1}{s^{3}}\right)
\end{array}\right]
$$

and

If we now take

$$
Q(s)=\operatorname{diag}\left\{1,1,1, \frac{1}{\sin (\ln s)}\right\}
$$

and set

$$
\begin{equation*}
X(s)=Q(s) Z(s) \tag{4.23}
\end{equation*}
$$

then (4.21) becomes

$$
\begin{equation*}
\frac{d Z}{d s}=\left[\operatorname{diag}\left\{\sqrt{a},-\sqrt{a}, \frac{3 \alpha}{4 s},-\frac{3 \alpha}{4 s}+\frac{\ln (\ln s)+(\ln s)^{-1}}{s \ln (\ln s)}\right\}+R(s)\right] Z \tag{4.24}
\end{equation*}
$$

where

$$
R(s)=\left[\begin{array}{cccc}
O\left(\frac{1}{s^{2}}\right) & O\left(\frac{1}{s^{2}}\right) & O\left(\frac{1}{s^{2}}\right) & O\left(\frac{1}{s^{2}}\right) \\
O\left(\frac{1}{s^{2}}\right) & O\left(\frac{1}{s^{2}}\right) & O\left(\frac{1}{s^{2}}\right) & O\left(\frac{1}{s^{2}}\right) \\
O\left(\frac{1}{s^{2}}\right) & O\left(\frac{1}{s^{2}}\right) & O\left(\frac{1}{s^{3}}\right) & O\left(\frac{1}{s \ln (\ln s)}\right) \\
O\left(\frac{\ln (\ln s|c(s)|}{s^{\gamma}}\right) & O\left(\frac{\ln (\ln s) \mid c(s s| |}{s^{\gamma}}\right) & O\left(\frac{\ln (\ln s)|c(s)|}{s^{\gamma} \gamma-1}\right) & O\left(\frac{1}{s^{3}}\right)
\end{array}\right]
$$

Notice $t=s^{\frac{2}{a+2}}$, we have $c(s)=b\left(\frac{2}{\alpha+2}\right)^{\eta}(\ln s)^{\eta}$, it follows that

$$
\begin{equation*}
\text { for } \quad \gamma>2, \quad c(s) \ln (\ln s) / s^{\gamma-1} \text { is integrable, } \tag{4.25}
\end{equation*}
$$

and

$$
\begin{equation*}
\text { for } \gamma=2, \eta<-1, \frac{c(s) \ln (\ln s)}{s} \text { is integrable. } \tag{4.26}
\end{equation*}
$$

So $R(s)$ satisfies the Assumption 3.2.2 of Theorem 3.2.1. Hence if $a>0$, then from (4.25) and (4.26), by Theorem 3.2.1, there exists a $4 \times 4$ matrix $G(s)$ with $\lim _{s \rightarrow+\infty} G(s)=0$ such that the transformation:

$$
\begin{equation*}
Z(s)=\left(I_{4}+G(s)\right) T(s) \tag{4.27}
\end{equation*}
$$

changes system (4.24) into

$$
\begin{equation*}
\frac{d T}{d s}=\operatorname{diag}\left\{\sqrt{a},-\sqrt{a}, \frac{3 \alpha}{4 s},-\frac{3 \alpha}{4 s}+\frac{\ln (\ln s)+(\ln s)^{-1}}{s \ln (\ln s)}\right\} T . \tag{4.28}
\end{equation*}
$$

In case $a<0$, the system (4.24) does not satisfy Assumption 3.2.2 of Theorem 3.2.1, but from (4.25), (4.26), and the proof of Theorem 3.2.1, it is not difficult to see that there still exists a $4 \times 4$ matrix $G(s)$ with $\lim _{s \rightarrow+\infty} G(s)=0$ such that the transformation (4.27) changes system (4.24) into (4.28). Therefore when $\beta<\alpha-2, \alpha>2$, or $\beta=\alpha-2, \alpha>2, \eta<-1$, we have

$$
Z(s)=\left[\begin{array}{cccc}
(1+o(1)) Z_{1}(s) & o(1) Z_{2}(s) & o(1) Z_{3}(s) & o(1) Z_{4}(s) \\
o(1) Z_{1}(s) & (1+o(1)) Z_{2}(s) & o(1) Z_{3}(s) & o(1) Z_{4}(s) \\
o(1) Z_{1}(s) & o(1) Z_{2}(s) & (1+o(1)) Z_{3}(s) & o(1) Z_{4}(s) \\
o(1) Z_{1}(s) & o(1) Z_{2}(s) & o(1) Z_{3}(s) & (1+o(1)) Z_{4}(s)
\end{array}\right]
$$

where

$$
\begin{gathered}
Z_{1}(s)=(1+o(1)) \exp \left\{\sqrt{a}\left(s-s_{0}\right)\right\}, \quad Z_{3}(s)=(1+o(1))\left(\frac{s}{s_{0}}\right)^{\frac{3 \alpha}{4}} \\
Z_{2}(s)=(1+o(1)) \exp \left\{-\sqrt{a}\left(s-s_{0}\right)\right\}, \quad Z_{4}(s)=(1+o(1)) \frac{\ln (\ln s)}{\ln \left(\ln s_{0}\right)}\left(\frac{s_{0}}{s}\right)^{\frac{3 a}{4}-1}
\end{gathered}
$$

Since

$$
\begin{gathered}
X(s)=Q(s) Z(s), W(s)=P(s) X(s), \quad V(s)=C(s) W(s) \\
U(t)=V\left(s^{-1}(t)\right) \quad \text { and } \quad u_{1}(t)=y(t)
\end{gathered}
$$

under the giving conditions we have the following four linear independent asymptotic solutions for equation (4.5):

$$
\begin{gathered}
y_{1}(t)=(1+o(1)) t^{-\frac{3 a}{4}} \exp \left\{\sqrt{a}\left(t^{\frac{\alpha+2}{2}}-t_{0}^{\frac{\alpha+2}{2}}\right)\right\}, \\
y_{2}(t)=(1+o(1)) t^{-\frac{3 a}{4}} \exp \left\{\sqrt{a}\left(t_{0}^{\frac{\alpha+2}{2}}-t^{\frac{a+2}{2}}\right)\right\}, \\
y_{3}(t)=(1+o(1)) t^{-\frac{3 a}{4}}\left(\frac{t}{t_{0}}\right)^{\frac{3 a(a+2)}{8}},
\end{gathered}
$$

and

$$
\begin{equation*}
y_{4}(t)=o(1) t^{-\frac{3 a}{4}}\left(\frac{t_{0}}{t}\right)^{\frac{(3 a-4)(a+2)}{8}} \ln (\ln t) \tag{4.29}
\end{equation*}
$$

Case II: When $\alpha-2<\beta<\frac{3 \alpha-2}{2}, \alpha>2$, or $\beta=\alpha-2, \alpha>2, \quad \eta>0$, by solving (4.16), we can have

$$
\begin{equation*}
\tilde{\lambda}^{2}(s)=\frac{1}{2}\left\{\frac{5}{8}\left(\frac{\alpha}{s}\right)^{2}+a\right\}\left[1 \pm\left\{1-\frac{4}{a^{2}}\left(a\left(\frac{3 \alpha}{4 s}\right)^{2}+\frac{c(s)}{s^{\gamma}}-\frac{z}{s^{\delta}}\right)+h(s)\right\}^{\frac{1}{2}}\right] \tag{4.30}
\end{equation*}
$$

where $h(s)=O\left(\frac{c(s)}{s^{2}+\gamma}\right), \quad 1<\gamma \leq 2$. By Taylor expansion, if we choose the plus sign in (4.30), we have

$$
\tilde{\lambda}^{2}(s)=a+h_{1}(s)
$$

where $h_{1}(s)=O\left(\frac{c(s)}{s^{\gamma}}\right)$ and $h_{1}^{\prime}(s)=O\left(\frac{c(s)}{s^{1+\gamma}}\right)$; and if we choose the negative sign in (4.30), we have

$$
\tilde{\lambda}^{2}(s)=\left(\frac{c(s)}{a s^{\gamma}}\right)+h_{2}(s)
$$

where $h_{2}(s)=o\left(\frac{1}{s^{2}}\right)$ and $h_{2}^{\prime}(s)=o\left(\frac{1}{s^{s}}\right)$. Thus the eigenvalues of $\left(\alpha_{0}+1\right) A(s)$ are

$$
\begin{equation*}
\tilde{\lambda}_{1}(s)=-\tilde{\lambda}_{2}(s)=\sqrt{a}+h_{3}(s) \tag{4.31}
\end{equation*}
$$

and

$$
\begin{equation*}
\tilde{\lambda}_{3}(s)=-\tilde{\lambda}_{4}(s)=\sqrt{\frac{c(s)}{a s^{\gamma}}}+h_{4}(s) \tag{4.32}
\end{equation*}
$$

where $h_{3}(s)=O\left(\frac{c(s)}{s^{\gamma}}\right), \quad h_{3}^{\prime}(s)=O\left(\frac{c(s)}{s^{1}+\gamma}\right), \quad h_{4}(s)=O\left(\frac{1}{s^{2-\gamma / 2} \sqrt{c(s)}}\right), \quad$ and $h_{4}^{\prime}(s)=$ $O\left(\frac{1}{s^{3-7 / 2} \sqrt{c(s)}}\right)$. Hence,

$$
P(s)=P+O\left(\sqrt{\frac{c(s)}{s^{\gamma}}}\right) \text { with } P=\left[\begin{array}{cccc}
1 & 1 & 1 & 0  \tag{4.33}\\
\sqrt{a} & -\sqrt{a} & 0 & 1 \\
a & a & 0 & 0 \\
0 & 0 & 0 & a
\end{array}\right]
$$

it follows that for large $\mathrm{s}, P(s)$ is invertible, and

$$
\begin{equation*}
P^{-1}(s)=P^{-1}+O\left(\sqrt{\frac{c(s)}{s^{\gamma}}}\right) \tag{4.34}
\end{equation*}
$$

with

$$
P^{-1}=\frac{1}{2 a \sqrt{a}}\left[\begin{array}{cccc}
0 & a & \sqrt{a} & -1 \\
0 & -a & \sqrt{a} & 1 \\
2 a \sqrt{a} & 0 & -2 \sqrt{a} & 0 \\
0 & 0 & 0 & 2 \sqrt{a}
\end{array}\right]
$$

Make the transformation

$$
\begin{equation*}
W(s)=P(s) X(s) \tag{4.35}
\end{equation*}
$$

then the differential equation (4.11) with $A(s)$ given by (4.13) becomes

$$
\begin{equation*}
\frac{d X}{d s}=\left\{\tilde{\Lambda}(s) /\left(\alpha_{0}+1\right)-P^{-1}(s) P^{\prime}(s)\right\} X \tag{4.36}
\end{equation*}
$$

Notice $\gamma<\delta$, we have

$$
P^{\prime}(s)=\left[\begin{array}{cccc}
0 & 0 & 0 & 0 \\
O\left(\frac{1}{s^{2}}\right) & O\left(\frac{1}{s^{2}}\right) & o\left(\frac{1}{s^{3 / 2}}\right) & 0 \\
O\left(\frac{1}{s^{2}}\right) & O\left(\frac{1}{s^{2}}\right) & o\left(\frac{1}{s^{2}}\right) & O\left(\frac{1}{s^{2}}\right) \\
O\left(\frac{|c(s)|}{s^{1+\gamma}}\right) & O\left(\frac{|c(s)|}{s^{1+\gamma}}\right) & O\left(\frac{|c(s)|^{1 / 2}}{s^{1+\gamma / 2}}\right) & o\left(\frac{1}{s^{2}}\right)
\end{array}\right]
$$

Thus

$$
P^{-1}(s) P^{\prime}(s)=\left[\begin{array}{cccc}
O\left(\frac{1}{s^{2}}\right) & O\left(\frac{1}{s^{2}}\right) & o\left(\frac{1}{s^{3 / 2}}\right) & O\left(\frac{1}{s^{2}}\right)  \tag{4.37}\\
O\left(\frac{1}{s^{2}}\right) & O\left(\frac{1}{s^{2}}\right) & o\left(\frac{1}{s^{3 / 2}}\right) & O\left(\frac{1}{s^{2}}\right) \\
O\left(\frac{1}{s^{2}}\right) & O\left(\frac{1}{s^{2}}\right) & o\left(\frac{1}{s^{2}}\right) & O\left(\frac{1}{s^{2}}\right) \\
O\left(\frac{|c(s)| \mid}{s^{1+\gamma}}\right) & O\left(\frac{|c(s)| \mid}{s^{1+\gamma}}\right) & O\left(\frac{\mid c(s)]^{1 / 2}}{s^{1+\gamma}}\right) & o\left(\frac{1}{s^{2}}\right)
\end{array}\right] .
$$

Suppose that $T(s)$ is a $4 \times 4$ differentiable matrix with $T(s) \rightarrow 0$ as $s \rightarrow \infty$ such that for all sufficiently large s, $I+T(s)$ has an inverse. So we may write

$$
\{I+T(s)\}^{-1}=I-T(s)+T^{2}(s)\{I+T(s)\}^{-1}
$$

Make the transformation $X(s)=(I+T(s)) Y(s)$ so that (4.36) becomes

$$
\begin{equation*}
\frac{d Y}{d s}=\left\{\Lambda_{0}(s)+\left[\Lambda_{0}, T\right](s)-P(s)^{-1} P^{\prime}(s)+R_{1}(s)\right\} Y \tag{4.38}
\end{equation*}
$$

where $\Lambda_{0}=\tilde{\Lambda} /\left(\alpha_{0}+1\right),\left[\Lambda_{0}, T\right]=\Lambda_{0} T-T \Lambda_{0}$, the Lie product, and

$$
\begin{align*}
R_{1}(s)= & -T \Lambda_{0} T+T^{2}(I+T)^{-1} \Lambda_{0}(I+T)+T P^{-1} P^{\prime} T+  \tag{4.39}\\
& +\left[T, P^{-1} P^{\prime}\right]+T^{2}(I+T)^{-1} P^{-1} P^{\prime}(I+T)-(I+T)^{-1} T^{\prime} \tag{4.40}
\end{align*}
$$

Let $t_{j k}(s)$ be the entries of $T(s)$, and take

$$
t_{j k}(s)=0 \text { for } j=1,2,3, k=1,2,3,4, \text { and } t_{44}(s)=0
$$

Then we get

$$
\left[\Lambda_{0}, T\right]=\left[\begin{array}{cccc}
0 & 0 & 0 & 0  \tag{4.41}\\
0 & 0 & 0 & 0 \\
t_{41} /\left(\alpha_{0}+1\right) & t_{42} /\left(\alpha_{0}+1\right) & t_{43} /\left(\alpha_{0}+1\right) & 0 \\
\left(\lambda_{4}-\lambda_{1}\right) t_{41} & \left(\lambda_{4}-\lambda_{2}\right) t_{42} & \left(\lambda_{4}-\lambda_{3}\right) t_{43} & -t_{43} /\left(\alpha_{0}+1\right)
\end{array}\right]
$$

with all quantities being evaluated at s . Choose

$$
\begin{equation*}
t_{4 k}(s)=\frac{\left[P(s)^{-1} P^{\prime}(s)\right]_{4 k}}{\lambda_{4}(s)-\lambda_{k}(s)},(1 \leq k \leq 3) \tag{4.42}
\end{equation*}
$$

Recalling the order of magnitude of $P(s)^{-1} P^{\prime}(s)$, noting the values of $\lambda_{k}(s)$, and the values of $\tilde{\lambda}_{k}(s)$ given by (4.31), (4.32), and the form of $g_{k}(s)$ given after (4.15), we see that

$$
t_{4 k}(s)=O\left(|c(s)| / s^{1+\gamma}\right), \text { and } t_{4 k}^{\prime}(s)=O\left(|c(s)| / s^{2+\gamma}\right), \text { for } k=1,2
$$

For $t_{43}(s)$, we have

$$
t_{43}(s) /\left(\alpha_{0}+1\right)=\frac{g_{3}^{\prime}(s)}{a\left(\tilde{\lambda}_{4}(s)-\tilde{\lambda}_{3}(s)\right)}=\frac{\gamma}{4 s}+h(s)
$$

where $h(s)$ is integrable.

## Let us set

$$
R_{2}(s)=\left[\Lambda_{0}, T\right](s)-P(s)^{-1} P^{\prime}(s)+R_{1}(s)+\operatorname{diag}\left\{0,0,-\frac{\gamma}{4 s}, \frac{\gamma}{4 s}\right\}
$$

We see that certainly the first three elements of the last row of $R_{2}(s)$ which are in fact the first three elements of the last row of $R_{1}(s)$ are of $O\left(1 / s^{2}\right)$, and the other elements of $R_{2}(s)$ are integrable. Thus (4.38) can be written as

$$
\begin{equation*}
\frac{d Y}{d s}=\left\{\Lambda_{1}(s)+R_{2}(s)\right\} Y \tag{4.43}
\end{equation*}
$$

where

$$
\Lambda_{1}(s)=\left[\begin{array}{cccc}
\lambda_{1}(s) & 0 & 0 & 0 \\
0 & \lambda_{2}(s) & 0 & 0 \\
0 & 0 & \lambda_{3}(s)+\frac{\gamma}{4 s} & 1 /\left(\alpha_{0}+1\right) \\
0 & 0 & 0 & \lambda_{4}(s)-\frac{\gamma}{4 s}
\end{array}\right]
$$

Repeat the operation we have performed above, i.e. make a transformation of the form $\left(I+T(s)\right.$ ), but using the matrix $R_{2}(s)$ instead of $-P(s)^{-1} P^{\prime}(s)$. Because $\lambda_{4}-\lambda_{3}-\frac{\gamma}{2 s}$ is asymptotic to $d\left(\frac{|c(s)|}{s \gamma}\right)^{1 / 2}$, where $d$ is some non-zero constant and $\gamma \leq 2$, we see that
(i) if $\gamma<2$, we arrive at a differential equation

$$
\begin{equation*}
\frac{d Y}{d s}=\left\{\Lambda_{1}(s)+R(s)\right\} Y \tag{4.44}
\end{equation*}
$$

where the first three elements of the last row of $R(s)$ are $o\left(1 / s^{\xi}\right)(\xi>2)$ and the other elements of $R(s)$ are integrable;
(ii) $\gamma=2$, we arrive at a differential equation

$$
\begin{equation*}
\frac{d Y}{d s}=\left\{\Lambda_{2}(s)+R(s)\right\} Y \tag{4.45}
\end{equation*}
$$

with

$$
\Lambda_{2}(s)=\left[\begin{array}{cccc}
\lambda_{1}(s) & 0 & 0 & 0 \\
0 & \lambda_{2}(s) & 0 & 0 \\
0 & 0 & \lambda_{3}(s)+\frac{1}{2 s}+\frac{d}{s(c(s))^{1 / 2}} & 1 /\left(\alpha_{0}+1\right) \\
0 & 0 & 0 & \lambda_{4}(s)-\frac{1}{2 s}-\frac{d}{s(c(s))^{1 / 2}}
\end{array}\right]
$$

where $d$ is a positive number, the first two elements of the last row of $R(s)$ are $o\left(1 / s^{\xi}\right),(\xi>2), r_{43}(s)=O\left(1 /\left[(\ln s)^{\eta} s^{2}\right]\right)$ and the other elements of $R(s)$ are integrable.

In case (i), we take

$$
Q(s)=\operatorname{diag}\left\{1,1,1, \frac{1}{s^{\mu}}\right\}, \text { with } \xi / 2>\mu>1
$$

and set

$$
\begin{equation*}
Y(s)=Q(s) Z(s) \tag{4.46}
\end{equation*}
$$

then we have

$$
\begin{equation*}
\frac{d Z}{d s}=\left[\operatorname{diag}\left\{\sqrt{a},-\sqrt{a}, \sqrt{\frac{c(s)}{a s^{\gamma}}}+\frac{\gamma}{4 s},-\sqrt{\frac{c(s)}{a s^{\gamma}}}-\frac{\gamma}{4 s}+\frac{\mu}{s}\right\}+\tilde{R}(s)\right] Z \tag{4.47}
\end{equation*}
$$

where $\xi>2$, so $\tilde{R}(s) \in L^{1}$. Hence by Theorem L, there exists a $4 \times 4$ matrix $G(s)$ with $\lim _{s \rightarrow+\infty} G(s)=0$ such that the transformation:

$$
\begin{equation*}
Z(s)=\left(I_{4}+G(s)\right) D(s) \tag{4.48}
\end{equation*}
$$

changes system (4.47) into

$$
\begin{equation*}
\frac{d D}{d s}=\operatorname{diag}\left\{\sqrt{a},-\sqrt{a}, \sqrt{\frac{c(s)}{a s^{\gamma}}}+\frac{\gamma}{4 s},-\sqrt{\frac{c(s)}{a s^{\gamma}}}-\frac{\gamma}{4 s}+\frac{\mu}{s}\right\} D \tag{4.49}
\end{equation*}
$$

Therefore when $\alpha-2<\beta<\frac{3 \alpha-2}{2}, \alpha>2$, we have

$$
Z(s)=\left[\begin{array}{cccc}
(1+o(1)) Z_{1}(s) & o(1) Z_{2}(s) & o(1) Z_{3}(s) & o(1) Z_{4}(s) \\
o(1) Z_{1}(s) & (1+o(1)) Z_{2}(s) & o(1) Z_{3}(s) & o(1) Z_{4}(s) \\
o(1) Z_{1}(s) & o(1) Z_{2}(s) & (1+o(1)) Z_{3}(s) & o(1) Z_{4}(s) \\
o(1) Z_{1}(s) & o(1) Z_{2}(s) & o(1) Z_{3}(s) & (1+o(1)) Z_{4}(s)
\end{array}\right]
$$

where

$$
\begin{gathered}
Z_{1}(s)=(1+o(1)) \exp \left\{\sqrt{a}\left(s-s_{0}\right)\right\}, Z_{2}(s)=(1+o(1)) \exp \left\{-\sqrt{a}\left(s-s_{0}\right)\right\} \\
Z_{3}(s)=(1+o(1))\left(\frac{s}{s_{0}}\right)^{\gamma / 4} \exp \left\{\int_{s_{0}}^{s} \sqrt{\frac{c(x)}{a x^{\gamma}}} d x\right\} \\
Z_{4}(s)=(1+o(1))\left(\frac{s}{s_{0}}\right)^{\mu-\gamma / 4} \exp \left\{-\int_{s_{0}}^{s} \sqrt{\frac{c(x)}{a x^{\gamma}}} d x\right\}
\end{gathered}
$$

Since

$$
Y(s)=Q(s) Z(s), \quad X(s)=(I+T(s)) Y(s), W(s)=P(s) X(s)
$$

$$
V(s)=C(s) W(s), \quad U(t)=V\left(s^{-1}(t)\right) \text { and } u_{1}(t)=y(t)
$$

under the giving conditions, we have the following four linear independent asymptotic solutions for equation (4.5):

$$
\begin{gathered}
y_{1}(t)=(1+o(1)) t^{-\frac{3 a}{4}} \exp \left\{\sqrt{a}\left(t^{\frac{\alpha+2}{2}}-t_{0}^{\frac{\alpha+2}{2}}\right)\right\}, \\
y_{2}(t)=(1+o(1)) t^{-\frac{3 a}{4}} \exp \left\{\sqrt{a}\left(t_{0}^{\frac{\alpha+2}{2}}-t^{\frac{\alpha+2}{2}}\right)\right\} \\
y_{3}(t)=(1+o(1)) t^{-\frac{3 a}{4}}\left(\frac{t}{t_{0}}\right)^{\frac{(\alpha+2) \gamma}{8}} \exp \left\{\int_{t_{0}}^{t^{\frac{\alpha+2}{2}}} \sqrt{\frac{c\left(x^{\frac{\alpha+2}{2}}\right)}{a x^{\gamma(\alpha+2) / 2}}} d x\right\},
\end{gathered}
$$

and

$$
y_{4}(t)=o(1) t^{-\frac{3 a}{4}}\left(\frac{t}{t_{0}}\right)^{(\dot{4} \mu-\gamma)(\alpha+2) / 8} \exp \left\{-\int_{t_{0}}^{\frac{t^{\frac{\alpha+2}{2}}}{2}} \sqrt{\frac{c\left(x^{\frac{\alpha+2}{2}}\right)}{a x^{\gamma(\alpha+2) / 2}}} d x\right\}
$$

In case (ii), we take

$$
Q(s)=\operatorname{diag}\left\{1,1,1, \frac{1}{s \ln (\ln s)}\right\}
$$

and set

$$
\begin{equation*}
Y(s)=Q(s) Z(s) \tag{4.50}
\end{equation*}
$$

then from (4.45) we have

$$
\begin{equation*}
\frac{d Z}{d s}=\left[\operatorname{diag}\left\{\sqrt{a},-\sqrt{a}, \lambda_{3}^{\star}(s), \lambda_{4}^{\star}(s)\right\}+\tilde{R}(s)\right] Z, \tag{4.51}
\end{equation*}
$$

where

$$
\begin{gathered}
\lambda_{3}^{\star}(s)=\sqrt{\frac{c(s)}{a s^{\gamma}}}+\frac{1}{2 s}+\frac{d}{s(c(s))^{1 / 2}} \\
\lambda_{4}^{\star}(s)=-\sqrt{\frac{c(s)}{a s^{\gamma}}}+\frac{1}{2 s}+\frac{(\ln s)^{-1}}{s \ln (\ln s)}-\frac{d}{s(c(s))^{1 / 2}}
\end{gathered}
$$

and when $\eta>1, R(s)$ satisfies the Assumption 3.2.2 of Theorem 3.2.1. Hence if $a>0, \quad b>0$, by Theorem 3.2.1, there exists a $4 \times 4$ matrix $G(s)$ with $\lim _{s \rightarrow+\infty} G(s)=0$ such that the transformation:

$$
\begin{equation*}
Z(s)=\left(I_{4}+G(s)\right) D(s) \tag{4.52}
\end{equation*}
$$

changes system (51) into

$$
\begin{equation*}
\frac{d D}{d s}=\operatorname{diag}\left\{\sqrt{a},-\sqrt{a}, \lambda_{3}^{\star}(s), \lambda_{4}^{\star}(s)\right\} D \tag{4.53}
\end{equation*}
$$

If $a<0, b<0$, as mentioned in case I, by following the proof of Theorem 3.2.1, equation (4.53) can also be obtained. Therefore when $\beta=\alpha-2, \alpha>2$, $a b>0, \eta>1$, we have

$$
Z(s)=\left[\begin{array}{cccc}
(1+o(1)) Z_{1}(s) & o(1) Z_{2}(s) & o(1) Z_{3}(s) & o(1) Z_{4}(s) \\
o(1) Z_{1}(s) & (1+o(1)) Z_{2}(s) & o(1) Z_{3}(s) & o(1) Z_{4}(s) \\
o(1) Z_{1}(s) & o(1) Z_{2}(s) & (1+o(1)) Z_{3}(s) & o(1) Z_{4}(s) \\
o(1) Z_{1}(s) & o(1) Z_{2}(s) & o(1) Z_{3}(s) & (1+o(1)) Z_{4}(s)
\end{array}\right]
$$

where

$$
\begin{gathered}
Z_{1}(s)=(1+o(1)) \exp \left\{\sqrt{a}\left(s-s_{0}\right)\right\}, \quad Z_{2}(s)=(1+o(1)) \exp \left\{-\sqrt{a}\left(s-s_{0}\right)\right\} \\
Z_{3}(s)=(1+o(1))\left(\frac{s}{s_{0}}\right)^{1 / 2} \exp \left\{\int_{s_{0}}^{s} \frac{d}{x(c(x))^{1 / 2}}+\sqrt{\frac{c(x)}{a x^{2}}} d x\right\} \\
Z_{4}(s)=(1+o(1)) \frac{\ln (\ln s)}{\ln \left(\ln s_{0}\right)}\left(\frac{s}{s_{0}}\right)^{1 / 2} \exp \left\{-\frac{d}{x(c(x))^{1 / 2}}-\int_{s_{0}}^{s} \sqrt{\frac{c(x)}{a x^{2}}} d x\right\}
\end{gathered}
$$

Since

$$
\begin{gathered}
Y(s)=Q(s) Z(s), \quad X(s)=(I+T(s)) Y(s), W(s)=P(s) X(s) \\
V(s)=C(s) W(s), U(t)=V\left(s^{-1}(t)\right) \text { and } u_{1}(t)=y(t)
\end{gathered}
$$

under the given conditions we have the following four linear independent asymptotic solutions for equation (4.5):

$$
\begin{gathered}
y_{1}(t)=(1+o(1)) t^{-\frac{3 a}{4}} \exp \left\{\sqrt{a}\left(t^{\frac{\alpha+2}{2}}-t_{0}^{\frac{\alpha+2}{2}}\right)\right\}, \\
y_{2}(t)=(1+o(1)) t^{-\frac{3 a}{4}} \exp \left\{\sqrt{a}\left(t_{0}^{\frac{\alpha+2}{2}}-t^{\frac{\alpha+2}{2}}\right)\right\}, \\
y_{3}(t)=(1+o(1)) t^{-\frac{3 a}{4}}\left(\frac{t}{t_{0}}\right)^{\frac{\alpha+2}{4}} \exp \left\{\int _ { t _ { 0 } } ^ { \frac { \alpha + 2 + 2 } { 2 } } \left[\frac{d}{x^{\frac{\alpha+2}{2}}\left(c\left(x^{\frac{\alpha+2}{2}}\right)\right)^{1 / 2}}+\sqrt{\left.\left.\frac{c\left(x^{\frac{\alpha+2}{2}}\right.}{a x^{\alpha+2}}\right] \frac{\alpha+2}{2} x^{\frac{\alpha}{2}} d x\right\},}\right.\right.
\end{gathered}
$$

$$
\text { and } y_{4}(t)=
$$

$$
o(1) t^{t^{\frac{3 a}{4}}}\left(\frac{t}{t_{0}}\right)^{\frac{a+2}{4}} \exp \left\{\int_{t_{0} \frac{a+2}{t}}^{\frac{a+2}{\frac{\alpha}{2}}}-\left[\frac{d}{x^{\frac{a+2}{2}}\left(c\left(x^{\frac{a+2}{2}}\right)\right)^{1 / 2}}+\sqrt{\left.\left.\frac{c\left(x^{\frac{a+2}{2}}\right)}{a x^{\alpha+2}}\right] \frac{\alpha+2}{2} x^{\frac{\alpha}{2}} d x\right\} \ln (\ln t) . . . . . . . ~}\right.\right.
$$

### 4.3 Deficiency Index

Using the previous asymptotic results, we can now determine the deficiency index for the minimal operator associated with the self-adjoint operator (4.4).
(1) In case $\beta<\alpha-2, \alpha>2$, or $\beta=\alpha-2, \alpha>2, \eta<-1$ :

For $a>0$, we have $\sqrt{a}>0$, so $y_{1}(t)$ is not square integrable, but $y_{2}(t)$ is square integrable. The square integrability of $y_{3}(t)$ and $y_{4}(t)$ is independent of a. When $\alpha>2$, we have $y_{3}(t)$ is not square integrable, but $y_{4}(t)$ is square integrable. Since no non-trivial combination of $y_{1}(t)$ and $y_{3}(t)$ is square integrable, it follows that in this case the deficiency index is $(2,2)$.

For $a<0$, we have $\sqrt{a}$ purfe imaginary number, so both $y_{1}(t)$ and $y_{2}(t)$ are square integrable. So in this case the deficiency index is $(3,3)$.

This proves Theorem 4.1.1, that is if $\beta<\alpha-2, \alpha>2$, or $\beta=\alpha-2, \alpha>$ 2, $\eta<-1$, then the deficiency index for the minimal operator associated with the self-adjoint operator (4.4) is $(2,2)$ when $a>0$, and is $(3,3)$ when $a<0$.
(2) In case $\alpha-2<\beta<\frac{3 \alpha-2}{2}, \alpha>2$, or $\beta=\alpha-2, \alpha>2, a b>0, \eta>1$.

For $a>0, \quad b>0$, we have $\sqrt{a}>0$, so $y_{1}(t)$ is not square integrable, but $y_{2}(t)$ is square integrable. We can also see that $y_{3}(t)$ is not square integrable, but $y_{4}(t)$ is square integrable. Since no non-trivial combination of $y_{1}(t)$ and $y_{3}(t)$ is square integrable, it follows that in this case the deficiency index is $(2,2)$.

For $a<0, b<0$, we have $\sqrt{a}$ pure imaginary number, so both $y_{1}(t)$ and $y_{2}(t)$ are square integrable. Since in this case $y_{3}(t)$ is not square integrable, and $y_{4}(t)$ is square integrable, it follows that in this case the deficiency index is $(3,3)$.

This proves Theorem 4.1.2, that is if $\alpha-2<\beta<\frac{3 \alpha-2}{2}, \alpha>2$, or $\beta=$ $\alpha-2, \quad \alpha>2, \quad \eta>1$, then when $a>0, \quad b>0$, the deficiency index for the minimal operator associated with the self-adjoint operator $(4.4)$ is $(2,2)$; when $a<0, b<0$, the deficiency index for the minimal operator associated with the self-adjoint operator (4.4) is $(3,3)$.

Example 1. If $\beta<\alpha-2, \alpha>2$, then from Theorem 4.1.1 the deficiency index for the minimal operator associated with the self-adjoint operator $(3)$ is $(2,2)$ when $a>0$, and is ( 3,3 ) when $a<0$. This is the main result of [35].

Example 2. Let $\beta=\alpha-2, \alpha>2$, and let $b(t)=(\ln t)^{-\eta}, \eta>1$, then from Theorem 4.1.1 the deficiency index for the minimal operator associated with the self-adjoint operator (4.4) is $(2,2)$ when $a>0$, and is $(3,3)$ when $a<0$.

Example 3. Let $\beta=\alpha-2, \alpha>2$, and let $b(t)=(\ln t)^{\eta}, \eta>1$, then from Theorem 4.1.2 the deficiency index for the minimal operator associated with the self-adjoint operator (4.4) is $(2,2)$ when $a>0, \quad b>0$, and is $(3,3)$ when $a<0, b<0$.

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