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ASYMPTOTIC DISTRIBUTION OF EIGENVALUES OF SCHRÖDINGER OPERATORS WITH NONCLASSICAL POTENTIALS

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1. Introduction. Let Δ be the Laplacian in the Euclidean space $\boldsymbol{R}^n$, that is, $\Delta=\sum_{i=1}^{n}\partial^2/\partial z_i^2$. Let $V(z)$ be a nonnegative function defined on $\boldsymbol{R}^n$. Suppose that the set $\{z\in\boldsymbol{R}^n;\ V(z)=0\}$ is an unbounded subset of $\boldsymbol{R}^n$. Our aim is to give an estimate for the asymptotic distribution of eigenvalues of the Schrödinger operator $-\Delta+V(z)$. Several results on this problem are known (cf. for example, Robert [8], Simon [10] and Solomyak [12]).

In this paper we restrict our attention to the potential of the form

$$V(x, y)=C\prod_{i=1}^{p} f_i(|x|)^{\alpha_i}\cdot\prod_{j=1}^{q} g_j(|y|)^{\beta_j}\cdot|x|^{\gamma}|y|^{\delta}, \tag{1.1}$$

where $x=(x_1, \cdots, x_{m_1})\in\boldsymbol{R}^{m_1}$, $y=(y_1, \cdots, y_{m_2})\in\boldsymbol{R}^{m_2}$, $|x|=(\sum_{i=1}^{m_1}x_i^2)^{1/2}$, $|y|=(\sum_{j=1}^{m_2}y_j^2)^{1/2}$ and $m_1+m_2=n$ with some conditions on f_i, g_j, α_i, β_j, γ and δ.

Our main result is given in Section 3. Special cases of our estimates are closely related to some results studied by Robert, Simon and others.

The case $V(x, y)=C\prod_{i=1}^{p}f_i(|x|)^{\alpha_i}\cdot\prod_{j=1}^{q}g_j(|y|)^{\beta_j}$ is a classical one and the asymptotic distribution of eigenvalues is given by the well-known formula (cf. Rozenbljum [9]).

The case $V(x, y)=(1+|x|^2)^{\alpha}|y|^{2\beta}$ is studied by Robert [8] by means of pseudo-differential operator calculus with operator symbols. Our method is quite different from his. The results will be given as corollaries when $\alpha m_2\geq\beta m_1$ in Section 3.

The case $V(x, y)=|x|^{\alpha}|y|^{\beta}$ is studied by Simon [10] when $m_1=m_2=1$. The case $m_1m_2\geq 2$ is included in the results of Solomyak [12]. Our method gives another proof of their results when $\alpha m_2=\beta m_1$. The result is given in Corollary 3.1.

In order to prove the main theorem we shall use classical Dirichlet-Neumann bracketing method formulated by Edmunds and Evans [2]. We shall also apply a simple modification of Theorem 2 of Fefferman [3; p. 144], where he gives several estimates for the eigenvalues of Schrödinger operators with polynomial potentials. We shall apply Fefferman's theorem to operators with A_∞-weight potentials and use it in the proof of Lemmas 3.2 and 3.2′.

In Section 2 we shall show some properties of A_∞-weights. These properties will be used in Sections 3 and 4. In Section 3 we shall state our main theorem and give the proof assuming several lemmas. In Sections 4 and 5 we shall prove these lemmas in Section 3.

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2. A_∞-weight potentials. Let Ω be an open set in $\boldsymbol{R}^n$. By $L^2(\Omega)$ we shall denote the Lebesgue space of all square integrable functions in Ω. By $H^1(\Omega)$ we shall denote the Sobolev space

$$H^1(\Omega)=\left\{u\in L^2(\Omega);\ \frac{\partial u}{\partial x_i}\in L^2(\Omega),\ i=1,\cdots,n\right\}$$

where $\partial/\partial x_i$ denote distributional derivatives. We put

$$|\nabla u(z)|^2=\sum_{i=1}^{n}\left|\frac{\partial u}{\partial x_i}(z)\right|^2$$

for $u\in H^1(\Omega)$ and $z\in\Omega$. By $C_0^\infty(\Omega)$ we shall denote the space of all infinitely differentiable functions with compact support in Ω. For a set S in $\boldsymbol{R}^n$, $|S|$ denotes the Lebesgue measure of S. By cubes in $\boldsymbol{R}^n$ we shall mean closed cubes whose sides are parallel to the coordinate axes.

Let us recall the definition of A_∞-weights.

DEFINITION. A nonnegative locally integrable function $w(z)$ on $\boldsymbol{R}^n$ is called an A_∞-weight on $\boldsymbol{R}^n$ if there exist positive constants C and δ such that

$$\frac{\int_S w(z)dz}{\int_Q w(z)dz}\leq C\left(\frac{|S|}{|Q|}\right)^\delta \tag{2.1}$$

for all cubes Q in $\boldsymbol{R}^n$ and for all measurable subsets S of Q. We call the pair (C,δ) of constants A_∞-constants of w. We denote the space of all A_∞-weights on $\boldsymbol{R}^n$ by $A_\infty(\boldsymbol{R}^n)$ or A_∞.

We now mention some properties of A_∞-weights which are useful in proving that our potential V belongs to A_∞. For the proof we refer to [4; Chap. IV].

LEMMA 2.1. *Let $w(z)\geq 0$ be locally integrable on $\boldsymbol{R}^n$. Then the following conditions are equivalent*:

(1) $w\in A_\infty$.

(2) *There exist $0<C_1, C_2<1$ such that*

$$\left|\left\{z\in Q;\ w(z)\le C_1\frac{1}{|Q|}\int_Q w(y)dy\right\}\right|\le C_2|Q|$$

for every cube Q.

(3) *There exists $C>0$ such that*

$$\frac{1}{|Q|}\int_Q w(z)dz\le C\exp\left(\frac{1}{|Q|}\int_Q \log w(z)dz\right)$$

for every cube Q.

(4) *There exist $C>0$ and $\varepsilon>0$ such that*

$$\left(\frac{1}{|Q|}\int_Q w(z)^{1+\varepsilon}dz\right)^{1/(1+\varepsilon)}\le\frac{C}{|Q|}\int_Q w(z)dz$$

for every cube Q.

REMARK 2.1. By Hölder's and Jensen's inequalities, $w\in A_\infty$ is equivalent to saying that

$$\frac{1}{|Q|}\int_Q w(z)dz\sim\left(\frac{1}{|Q|}\int_Q w(z)^{1+\varepsilon}dz\right)^{1/(1+\varepsilon)}\sim\exp\left(\frac{1}{|Q|}\int_Q \log w(z)dz\right)$$

for every Q, where the bounds are independent of Q.

LEMMA 2.2. *Let u and v be A_∞-weights. Then we have the following*:

(1) *If α, $\beta>0$, then $\alpha u+\beta v\in A_\infty$.*

(2) *If $0<\alpha<1$, then $u^\alpha\in A_\infty$.*

(3) *If u^2, $v^2\in A_\infty$, then $uv\in A_\infty$.*

Lemma 2.2 is a direct consequence of Lemma 2.1 but we give a proof for convenience.

PROOF. (1) follows from the Hardy-Littlewood maximal theorem with weights, but follows directly from the definition of A_∞-weights. Let (C', δ') and (C'', δ'') be A_∞-constants of u and v, respectively. Then $C'|S|^{\delta'}\int_Q udz\ge|Q|^{\delta'}\int_S udz$ for every subset S of a cube Q and a similar inequality holds for v with constants (C'', δ''). Thus, adding both sides, we get (2.1) for $\alpha u+\beta v$ with constants $C=\max(C', C'')$ and $\delta=\min(\delta', \delta'')$.

(2) Assume $0<\alpha<1$. Fix a cube Q. By Hölder's inequality

$$\frac{1}{|Q|}\int_Q u(z)^\alpha dz\le\left(\frac{1}{|Q|}\int_Q u(z)dz\right)^\alpha,$$

which, by Lemma 2.1 (3), does not exceed

$$\left(C\exp\left(\frac{1}{|Q|}\int_Q \log u(z)dz\right)\right)^\alpha = C^\alpha \exp\left(\frac{1}{|Q|}\int_Q \log u(z)^\alpha dz\right).$$

Thus $u^\alpha \in A_\infty$.

(3) By Schwartz's inequality

$$\frac{1}{|Q|}\int_Q u(z)v(z)dz \le \left(\frac{1}{|Q|}\int_Q u(z)^2 dz\right)^{1/2}\left(\frac{1}{|Q|}\int_Q v(z)^2 dz\right)^{1/2}.$$

Applying Lemma 2.1 (3) to each term on the right hand side, we get

$$\begin{aligned}\frac{1}{|Q|}\int_Q u(z)v(z)dz &\le C\exp\left(\frac{1}{2|Q|}\int_Q (\log u(z)^2 + \log v(z)^2)dz\right)\\ &= C\exp\left(\frac{1}{|Q|}\int_Q \log u(z)v(z)ds\right),\end{aligned}$$

which proves (3). q.e.d.

LEMMA 2.3. *Let $P_{ij}(z)$ be polynomials on $\boldsymbol{R}^n$ of degrees d_{ij}, where $i=1, \cdots, q$ and $j=1, \cdots, r$. Let α_{ij}, β_{ij} and γ_{ij} be positive numbers. Let*

$$f_i(z) = \sum_{j=1}^{r} \alpha_{ij} |P_{ij}(z)|^{\beta_{ij}}, \qquad i=1, \cdots, q,$$

and

$$w(z) = \prod_{i=1}^{q} f_i(z)^{\gamma_i}.$$

Then $w(z)$ is an A_∞-weight on $\boldsymbol{R}^n$ and the A_∞-constants depend only on n, d_{ij}, β_{ij}, γ_{ij} and q.

PROOF. First we observe the following: if $P(z)$ is a polynomial on $\boldsymbol{R}^n$ *and* $\alpha>0$, then $|P(z)|^\alpha \in A_\infty$. Indeed, we have

$$\frac{1}{|Q|}\int_Q |P(z)|dz \le \max_{z\in Q} |P(z)| \le \frac{C}{|Q|}\int_Q |P(z)|dz \tag{2.2}$$

for every cube Q, where C is a constant depending only on n and the degree of P (cf. [3; p. 146]). Thus Lemma 2.1 (4) holds for every $\varepsilon>0$. Thus $|P(z)|^\alpha \in A_\infty$ for $\alpha=1, 2, \cdots$. By Lemma 2.2 (2), this holds for every $\alpha>0$.

Next we observe the following: if $P_j(z)$, $j=1, \cdots, h$ are polynomials on $\boldsymbol{R}^n$, then $\prod_{j=1}^{h} |P_j(z)|^{\alpha_j} \in A_\infty$ for every $\alpha_j>0$. Since $|P_1(z)|^{2\alpha_1} \in A_\infty$ and $|P_2(z)|^{2\alpha_2} \in A_\infty$, we have $|P_1(z)|^{\alpha_1} |P_2(z)|^{\alpha_2} \in A_\infty$ by Lemma 2.2 (3). The case $h>2$ is shown similarly.

Therefore, by Lemma 2.2 (1), $f_i(z)^\gamma \in A_\infty$ for $\gamma=1, 2, \cdots$. By Lemma 2.2 (2),

$f_i(z)^\gamma \in A_\infty$ for every $\gamma > 0$. Applying the preceding argument, we can show $w(z) \in A_\infty$. q.e.d.

COROLLARY TO LEMMA 2.3. *Let $w(z)$ be the function given in Lemma 2.3. Then there exists a positive constant C depending only on n, d_{ij}, β_{ij}, γ_i and q such that*

$$\frac{1}{|Q|}\int_Q w(z)dz \le \max_{z\in Q} w(z) \le C\frac{1}{|Q|}\int_Q w(z)dz$$

for all cubes Q in $\boldsymbol{R}^n$.

PROOF. It suffices to show the second inequality. By the definition of w we have

$$\max_{z\in Q} w(z) \le \prod_{i=1}^{q}\left(\sum_{j=1}^{r} \alpha_{ij}\left(\max_{z\in Q}|P_{ij}(z)|\right)^{\beta_{ij}}\right)^{\gamma_i}.$$

Since $|P_{ij}|$ are A_∞-weights, by (2.2) and Lemma 2.1 (3), the last term does not exceed

$$\prod_{i=1}^{q}\left(\sum_{j=1}^{r}\alpha_{ij}\left(C_{1ij}\frac{1}{|Q|}\int_Q |P_{ij}(z)|dz\right)^{\beta_{ij}}\right)^{\gamma_i}$$

$$\le \prod_{i=1}^{q}\left(\sum_{j=1}^{r}\alpha_{ij}\left(C_{1ij}C_{2ij}\exp\left(\frac{1}{|Q|}\int_Q \log|P_{ij}(z)|dz\right)\right)^{\beta_{ij}}\right)^{\gamma_i}$$

$$\le C_3\prod_{i=1}^{q}\left(\sum_{j=1}^{r}\alpha_{ij}\left(\exp\left(\frac{1}{|Q|}\int_Q \log|P_{ij}(z)|dz\right)\right)^{\beta_{ij}}\right)^{\gamma_i},$$

where C_{1ij} depend only on n and d_{ij}, while $C_3 = \prod_{i=1}^{q}(\max_j(C_{1ij}C_{2ij})^{\beta_{ij}})^{\gamma_i}$. By Jensen's inequality the last term does not exceed

$$C_3\prod_{i=1}^{q}\left(\frac{1}{|Q|}\int_Q \sum_{j=1}^{r}\alpha_{ij}|P_{ij}(z)|^{\beta_{ij}}dz\right)^{\gamma_i} = C_3\prod_{i=1}^{q}\left(\frac{1}{|Q|}\int_Q f_i(z)dz\right)^{\gamma_i}.$$

Note that f_i are A_∞-weights. Applying Lemma 2.1 (3) again to the last term and arguing similarly as above, we get an estimate

$$\max_{z\in Q} w(z) \le C_4\frac{1}{|Q|}\int_Q \prod_{i=1}^{q} f_i(z)^{\gamma_i}dz \le C_4\frac{1}{|Q|}\int_Q w(z)dz,$$

where C_4 depends only on n, d_{ij}, β_{ij}, γ_i and q. q.e.d.

The following Lemmas 2.4 and 2.6 are modifications of Theorems 2 and 3 in Fefferman [3; p. 144], respectively.

LEMMA 2.4. *Let $U(z)$ be an A_∞-weight on $\boldsymbol{R}^n$. Put*

$$\lambda_1 = \inf_{\substack{a>0\\ \xi\in\boldsymbol{R}^n}}\left(a^{-2} + a^{-n}\int_{|z-\xi|<a/2} U(z)dz\right).$$

Suppose that $\lambda_1 > 0$. Then

$$C\lambda_1 \int_Q |v(z)|^2 dz \leq \int_Q (|\nabla v(z)|^2 + U(z)|v(z)|^2) dz$$

for all cubes Q *in* $\boldsymbol{R}^n$ *with side length* $2(\lambda_1)^{-1/2}$ *and for all* $v \in H^1(\mathring{Q})$, *where* C *is a positive constant depending only on* n *and the* A_∞*-constants for* $U(z)$, *and* $\mathring{Q}$ *denotes the interior of* Q.

To prove Lemma 2.4 we use the following lemma.

LEMMA 2.5 (Morimoto [6]). *Let* Q *be a cube in* $\boldsymbol{R}^n$ *and let* $U(z)$ *be a nonnegative measurable function on* Q. *Suppose that there exist positive constants* C_1 *and* C_2 *such that*

$$C_1 |Q| \leq |\{z \in Q;\, C_2 l(Q)^{-2} \leq U(z)\}| , \tag{2.3}$$

where $l(Q)$ *denotes the side length of* Q. *Then we have*

$$Cl(Q)^{-2} \int_Q |v(z)|^2 dz \leq \int_Q (|\nabla v(z)|^2 + U(z)|v(z)|^2) dz$$

for all $v \in H^1(\mathring{Q})$, *where* C *is a positive constant depending only on* n, C_1 *and* C_2.

PROOF OF LEMMA 2.4. Let Q be a cube in $\boldsymbol{R}^n$ *with* $l(Q) = 2(\lambda_1)^{-1/2}$ and center z^0. Put $a = 2\lambda_1^{-1/2}$ and $\xi = z^0$. Then, by the definition of λ_1, we get

$$\lambda_1 \leq \frac{1}{4}\lambda_1 + \frac{1}{|Q|}\int_Q U dz .$$

Therefore

$$\frac{3}{4}\lambda_1 \leq \frac{1}{|Q|}\int_Q U dz .$$

Thus

$$3l(Q)^{-2} \leq \frac{1}{|Q|}\int_Q U dz . \tag{2.4}$$

Since U is an A_∞-weight in $\boldsymbol{R}^n$, we have, by Lemma 2.1 (2),

$$C_1 |Q| \leq \left|\left\{z \in Q;\, C_2 \frac{1}{|Q|}\int_Q U dz \leq U(z)\right\}\right|$$

where C_1 and C_2 are positive constants depending only on n and the A_∞-constants of U. Combining this with (2.4), we have

$$C_1 |Q| \leq |\{z \in Q;\, 3C_2 l(Q)^{-2} \leq U(z)\}| .$$

Therefore U and Q satisfy the inequality (2.3). Thus Lemma 2.4 follows from Lemma 2.5. q.e.d.

Now, in order to consider the distribution of the eigenvalues of Schrödinger operators with A_∞-weight potentials, we introduce some notation.

Let U be an A_∞-weight. Suppose that an operator $-\Delta+U$ which is defined on $C_0^\infty(\boldsymbol{R}^n)$ is essentially selfadjoint in $L^2(\boldsymbol{R}^n)$ and L is a selfadjoint realization of $-\Delta+U$. Assume that L has only discrete spectrum. Let λ be a positive number and let $N(\lambda, U)$ be the number of eigenvalues of L less than λ. Let $\mathscr{F}_\lambda$ be a tesselation of $\boldsymbol{R}^n$ by cubes whose side length is $\lambda^{-1/2}$ and whose vertices are points in $\lambda^{-1/2}\boldsymbol{Z}^n$ where $\boldsymbol{Z}$ is the set of integers. Let $N_1(\lambda, U)$ be the number of cubes Q in $\mathscr{F}_\lambda$ such that

$$\frac{1}{|Q|}\int_Q U(z)dz<\lambda .$$

LEMMA 2.6. *Assume that U satisfies the above conditions. Then we have*

$$N_1(C_1\lambda, U)\leq N(\lambda, U)\leq N_1(C_2\lambda, U)$$

for every positive number λ, where C_1 is a constant depending only on n, while C_2 is a constant depending only on n and the A_∞-constants of U.

We omit the proof of Lemma 2.6. The reader may follow the arguments of the proof of Theorem 3 in [3; p. 148] if he applies Lemma 2.5 in place of Main Lemma in [3; p. 146].

REMARK 2.2. Lemma 2.4 shows that Theorem 2 in [3; p. 144] is also valid for A_∞-weight potentials. This follows easily from the proof of Theorem 2 in [3].

REMARK 2.3. Let $U(z)$ be an A_∞-weight on $\boldsymbol{R}^n$. Suppose that $-\Delta+U$ defined on $C_0^\infty(\boldsymbol{R}^n)$ is essentially selfadjoint in $L^2(\boldsymbol{R}^n)$ and L is a selfadjoint realization of $-\Delta+U$. If $N_1(\lambda, U)<\infty$ for all $\lambda>0$, then L has only discrete spectrum. This fact is verified in a manner similar to the proof for Remark 4 in Simon [11; p. 215].

REMARK 2.4. Let $w(z)$ be the function given in Lemma 2.3. Let $N_2(\lambda, w)$ be the number of cubes in $\mathscr{F}_\lambda$ such that

$$\max_{z\in Q} w(z)<\lambda .$$

for $\lambda>0$. Then

$$N_2(\lambda, w)\leq N_1(\lambda, w)\leq N_2(C\lambda, w)$$

for every positive number λ, where C is a constant independent of λ. The first inequality is obvious and the second inequality follows from Corollary to Lemma 2.3.

3. Main theorem. Let p and q be positive integers. Let

$$f_i(t)=\sum_{k=0}^{d_i} a_{ik}t^k\,, \qquad (i=1,\cdots,p)\,,$$

$$g_j(t)=\sum_{s=0}^{h_j} b_{js}t^s\,, \qquad (j=1,\cdots,q)\,,$$

where d_i and h_j are nonnegative integers. We assume that a_{ik}, $b_{js}\geq 0$, $(0<k<d_i, 0<s<h_j; 1\leq i\leq p, 1\leq j\leq q)$ a_{i0}, $b_{j0}>0$ and $a_{id_i}=b_{jh_j}=1$. We put

$$V(z)=V(x,y)=C\prod_{i=1}^{p} f_i(|x|)^{\alpha_i}\cdot\prod_{j=1}^{q} g_j(|y|)^{\beta_j}\cdot|x|^{\gamma}|y|^{\delta}\,, \tag{3.1}$$

where $z=(x,y)\in \boldsymbol{R}^{m_1}\times\boldsymbol{R}^{m_2}=\boldsymbol{R}^n$, $m_1>0$, $m_2>0$, and α_i, β_j, γ and δ are nonnegative numbers and $C>0$ is a constant. To avoid trivial cases, we assume $\sum_{i=1}^{p}\alpha_i d_i+\gamma\neq 0$ and $\sum_{j=1}^{q}\beta_j h_j+\delta\neq 0$.

By Lemma 2.3 V is an A_∞-weight on $\boldsymbol{R}^n$. The operator $-\Delta+V$ defined on $C_0^\infty(\boldsymbol{R}^n)$ is essentially selfadjoint in $L^2(\boldsymbol{R}^n)$, since $V\geq 0$ and $V\in L^2_{\mathrm{loc}}(\boldsymbol{R}^n)$ (cf. Kato [5]), where $L^2_{\mathrm{loc}}(\boldsymbol{R}^n)$ denotes the set of all functions square integrable on every compact subset of $\boldsymbol{R}^n$. Let L be a selfadjoint realization of $-\Delta+V$. Then L has only discrete spectrum. Indeed, we can show easily that $N_2(\lambda, V)<\infty$ for all $\lambda>0$ where $N_2(\lambda, V)$ is the quantity defined in Remark 2.4. Thus, by Remarks 2.3 and 2.4, the assertion follows.

Now we give an asymptotic formula for $N(\lambda, V)$ which, by definition, is the number of eigenvalues of L less than λ and denoted simply by $N(\lambda)$. Our main result is the following:

THEOREM. *Let V be the potential given by* (3.1). *Suppose that $\gamma m_2\leq(\sum_{j=1}^{q}\beta_j h_j+\delta)m_1$ and $\delta m_1\leq(\sum_{i=1}^{p}\alpha_i d_i+\gamma)m_2$. Set $\mu_1=2^{-1}(2+\delta)(\sum_{i=1}^{p}\alpha_i d_i+\gamma)^{-1}$ and $\mu_2=2^{-1}(2+\gamma)(\sum_{j=1}^{q}\beta_j h_j+\delta)^{-1}$. Then*

$$N(\lambda)\sim\frac{\omega_n}{(2\pi)^n}\int_A(\lambda-V)^{n/2}\,dxdy \qquad as \quad \lambda\to\infty\,,$$

where ω_n is the volume of the unit ball in $\boldsymbol{R}^n$ and the set A is defined as follows:

(1) *If $\gamma\neq 0$ and $\delta\neq 0$, then*

$$A=\{(x,y)\in\boldsymbol{R}^{m_1}\times\boldsymbol{R}^{m_2};\ V(x,y)\leq\lambda,\ |x|\leq C_1\lambda^{\mu_1},\ |y|\leq C_2\lambda^{\mu_2}\}$$

where C_1 is a positive constant depending only on m_2, C, d_i, α_i, b_{j0}, β_j, γ and δ, while C_2 is a positive constant depending only on m_1, C, h_j, β_j, a_{i0}, α_i, γ and δ.

(2) *If $\gamma=0$ and $\delta\neq 0$, then*

$$A=\{(x,y)\in\boldsymbol{R}^{m_1}\times\boldsymbol{R}^{m_2};\ V(x,y)\ \leq\lambda,\ |x|\leq C_3\lambda^{\mu_1}\}$$

where C_3 is a positive constant depending only on m_2, C, d_i, α_i, b_{j0}, β_j and δ.

(3) *If $\gamma\neq 0$ and $\delta=0$, then*

$$A=\{(x, y)\in \boldsymbol{R}^{m_1}\times \boldsymbol{R}^{m_2};\ V(x, y)\leq\lambda,\ |y|\leq C_4\lambda^{\mu_2}\}$$

where C_4 is a positive constant depending only on m_1, C, h_j, β_j, a_{i0}, α_i and γ.

(4) *If $\gamma=0$ and $\delta=0$, then*

$$A=\{(x, y)\in \boldsymbol{R}^{m_1}\times \boldsymbol{R}^{m_2};\ V(x, y)\leq\lambda\}\ .$$

COROLLARY 3.1. *Let α, $\beta>0$ and $\alpha m_2=\beta m_1$. Let*

$$V(x, y)=|x|^{\alpha}|y|^{\beta}$$

for $(x, y)\in \boldsymbol{R}^{m_1}\times \boldsymbol{R}^{m_2}$. Then

$$N(\lambda)\sim a\lambda^{\theta}\log\lambda \qquad as \quad \lambda\to\infty\ ,$$

where $\theta=n/2+m_1/\alpha$ and

$$a=\frac{(2+\alpha+\beta)\Gamma(m_1/\alpha)}{2^{n-1}\alpha\beta\Gamma(m_1/2)\Gamma(m_2/2)\Gamma(n/2+m_1/\alpha+1)}\ .$$

COROLLARY 3.2 ([8; Theorem 3.2 (i)]). *Let α, $\beta>0$ and $\alpha m_2>\beta m_1$. Let*

$$V(x, y)=(1+|x|^2)^{\alpha}|y|^{2\beta}$$

for $(x, y)\in \boldsymbol{R}^{m_1}\times \boldsymbol{R}^{m_2}=\boldsymbol{R}^n$. Then

$$N(\lambda)\sim a\lambda^{\theta} \qquad as \quad \lambda\to\infty\ ,$$

where $\theta=n/2+m_2/(2\beta)$ and

$$a=\frac{\Gamma(m_2/(2\beta))}{2^n\pi^{m_1/2}\beta\Gamma(m_2/2)\Gamma(\theta+1)}\int_{\boldsymbol{R}^{m_1}}(1+|x|^2)^{-\alpha m_2/(2\beta)}dx\ .$$

COROLLARY 3.3 ([8; Theorem 3.2 (ii)]). *Let α, $\beta>0$ and $\alpha m_2=\beta m_1$.* Let

$$V(x, y)=(1+|x|^2)^{\alpha}|y|^{2\beta}$$

for $(x, y)\in \boldsymbol{R}^{m_1}\times \boldsymbol{R}^{m_2}=\boldsymbol{R}^n$. Then

$$N(\lambda)\sim a\lambda^{\theta}\log\lambda \qquad as \quad \lambda\to\infty\ ,$$

where $\theta=n/2+m_2/(2\beta)$ and

$$a=\frac{(1+\beta)\Gamma(m_2/(2\beta))}{2^n\alpha\beta\Gamma(m_1/2)\Gamma(m_2/2)\Gamma(\theta+1)}\ .$$

REMARK. Our constants in the corollaries are different from those in Robert [8]. A careful calculation will lead to our constants.

Let Ω be an open set in $\boldsymbol{R}^n$ and V be the function given by (3.1). Define

$$t[u, v] = \int_{\Omega} (\nabla u \cdot \overline{\nabla v} + V u \bar{v}) dx dy$$

and

$$\|u\|_{t,\Omega}^2 = t[u, u] + \int_{\Omega} |u|^2 dx dy$$

for appropriate functions u and v. By $D_{\mathscr{D},\Omega}$ and $D_{\mathscr{N},\Omega}$ we denote the completions with respect to the norm $\| \ \|_{t,\Omega}$ of $C_0^\infty(\Omega)$ and the restriction of $C_0^\infty(\boldsymbol{R}^n)$ to Ω, respectively. Let $t_{\mathscr{D}}$ and $t_{\mathscr{N}}$ be sesquilinear extensions of t to $D_{\mathscr{D},\Omega}$ and $D_{\mathscr{N},\Omega}$, respectively. Then $t_{\mathscr{D}}$ and $t_{\mathscr{N}}$ are closed and semibounded forms. Let $T_{\mathscr{D},\Omega}$ and $T_{\mathscr{N},\Omega}$ be associated selfadjoint operators with respect to $t_{\mathscr{D}}$ and $t_{\mathscr{N}}$, respectively (cf. [2; p. 139]). Let $\Delta_{\mathscr{D},\Omega}$ and $\Delta_{\mathscr{N},\Omega}$ be the Dirichlet and Neumann Laplacian on Ω, respectively. If there is no confusion, we drop the notation Ω, for example, and we denote $T_{\mathscr{D}}$ instead of $T_{\mathscr{D},\Omega}$.

Let T be a selfadjoint operator in $L^2(\Omega)$. For $\lambda > 0$ let

$$N(\lambda, T, \Omega) = \operatorname{rank} \int_{-\infty}^{\lambda} dE_\mu(T),$$

where $E_\mu(T)$ is the resolution of the identity corresponding to T.

In this notation we prove the theorem.

PROOF OF THEOREM. First we prove (1). Let λ be a large positive number. Let $\mathscr{F}'_\lambda$ be a tesselation of $\boldsymbol{R}^n$ by cubes Q whose side length is $\lambda^{-1/2}(\log \lambda)^{1/n}$ and whose vertices are points in $\lambda^{-1/2}(\log \lambda)^{1/n}\boldsymbol{Z}^n$.

Let $B^i = \{(x, y) \in \boldsymbol{R}^{m_1} \times \boldsymbol{R}^{m_2}; x_i = 0\}$ $(i = 1, \cdots, m_1)$, $B_j = \{(x, y) \in \boldsymbol{R}^{m_1} \times \boldsymbol{R}^{m_2}; y_j = 0\}$ $(j = 1, \cdots, m_2)$ and $B = (\bigcup_{i=1}^{m_1} B^i) \cup (\bigcup_{j=1}^{m_2} B_j)$. Let $\mathscr{I}_1$ be all cubes Q in $\mathscr{F}'_\lambda$ such that $\min_{z \in Q} V(z) \leq \lambda$ and $Q \cap B = \emptyset$. Let $\mathscr{I}_2$ be all cubes in $\mathscr{F}'_\lambda$ such that $\max_{z \in Q} V(z) \leq \lambda$. Let K_1 and K_2 be positive constants which will be determined later. Let $\mathscr{I}_3$ be all cubes Q in $\mathscr{F}'_\lambda \setminus \mathscr{I}_1$ such that $\min_{z \in Q} V(z) \leq \lambda$ and

$$Q \subset \{(x, y) \in \boldsymbol{R}^{m_1} \times \boldsymbol{R}^{m_2}; |x| \leq K_1 \lambda^{\mu_1}, |y| \leq K_2 \lambda^{\mu_2}\},$$

where μ_1 and μ_2 are constants defined in the theorem.

Let K_3 and K_4 be positive constants which will be determined later and put

$$F_1 = \left\{(x, y) \in \boldsymbol{R}^{m_1} \times \boldsymbol{R}^{m_2}; (x, y) \notin \bigcup_{Q \in \mathscr{I}_1 \cup \mathscr{I}_3} Q, |x_i| < K_3 \lambda^{-1/2}, i = 1, \cdots, m_1\right\},$$

$$F_2 = \left\{(x, y) \in \boldsymbol{R}^{m_1} \times \boldsymbol{R}^{m_2}; (x, y) \notin \bigcup_{Q \in \mathscr{I}_1 \cup \mathscr{I}_3} Q, |y_j| < K_4 \lambda^{-1/2}, j = 1, \cdots, m_2\right\},$$

and

$$F_3=\boldsymbol{R}^n \setminus \text{the closure of } \left(\left(\bigcup_{Q \in \mathscr{I}_1 \cup \mathscr{I}_3} Q \right) \cup F_1 \cup F_2 \right).$$

Note that if λ is sufficiently large, then $F_1 \cap F_2 = \varnothing$.

Now we estimate $N(\lambda)$. Remark that $N(\lambda)=N(\lambda, L, \boldsymbol{R}^n)$ by definition. We have $L=T_{\mathscr{D},\boldsymbol{R}^n}=T_{\mathscr{N},\boldsymbol{R}^n}$ since $-\Delta+V$ defined on $C_0^\infty(\boldsymbol{R}^n)$ is essentially selfadjoint in $L^2(\boldsymbol{R}^n)$. Therefore

$$N(\lambda)=N(\lambda, T_{\mathscr{N}}, \boldsymbol{R}^n)=N(\lambda, T_{\mathscr{D}}, \boldsymbol{R}^n). \tag{3.2}$$

Let Ω_1, Ω_2, Ω_3 and Ω_4 be open sets in $\boldsymbol{R}^n$ and let Ω be the interior of the closure of $\Omega_1 \cup \Omega_2$. Suppose that $\Omega_1 \cap \Omega_2 = \varnothing$, $|\Omega \setminus (\Omega_1 \cup \Omega_2)|=0$ and $\Omega_3 \subset \Omega_4$. By an argument similar to that in Edmunds and Evans [2; p. 143], we get

$$N(\lambda, T_{\mathscr{N}}, \Omega) \le N(\lambda, T_{\mathscr{N}}, \Omega_1)+N(\lambda, T_{\mathscr{N}}, \Omega_2),$$

$$N(\lambda, T_{\mathscr{D}}, \Omega) \ge N(\lambda, T_{\mathscr{D}}, \Omega_1)+N(\lambda, T_{\mathscr{D}}, \Omega_2),$$

and

$$N(\lambda, T_{\mathscr{D}}, \Omega_3) \le N(\lambda, T_{\mathscr{D}}, \Omega_4).$$

Therefore, by (3.2),

$$\sum_{Q \in \mathscr{I}_2} N(\lambda, T_{\mathscr{D}}, \mathring{Q}) \le N(\lambda) \le \sum_{Q \in \mathscr{I}_1} N(\lambda, T_{\mathscr{N}}, \mathring{Q})+\sum_{Q \in \mathscr{I}_3} N(\lambda, T_{\mathscr{N}}, \mathring{Q})+\sum_{i=1}^{3} N(\lambda, T_{\mathscr{N}}, F_i). \tag{3.3}$$

We have the following three estimates for $N(\cdot, \cdot, \cdot)$.

Lemma 3.1. $N(\lambda, T_{\mathscr{N}}, F_3)=0$.

Lemma 3.2. $N(\lambda, T_{\mathscr{N}}, F_1)=N(\lambda, T_{\mathscr{N}}, F_2)=0$.

Lemma 3.3.

$$\sum_{Q \in \mathscr{I}_2} N(\lambda, T_{\mathscr{D}}, \mathring{Q}) \sim \sum_{Q \in \mathscr{I}_1} N(\lambda, T_{\mathscr{N}}, \mathring{Q})+\sum_{Q \in \mathscr{I}_3} N(\lambda, T_{\mathscr{N}}, \mathring{Q})$$

$$\sim \frac{\omega_n}{(2\pi)^n} \int_A (\lambda-V)^{n/2}\, dxdy \qquad \text{as} \quad \lambda \to \infty$$

where

$$A=\{(x, y) \in \boldsymbol{R}^{m_1} \times \boldsymbol{R}^{m_2};\ V(x, y) \le \lambda,\ |x| \le K_1 \lambda^{\mu_1},\ |y| \le K_2 \lambda^{\mu_2}\}.$$

We shall postpone the proof of these three lemmas to the following sections. By (3.3), Lemmas 3.1, 3.2 and 3.3

$$N(\lambda) \sim \frac{\omega_n}{(2\pi)^n} \int_A (\lambda-V)^{n/2}\, dxdy \qquad \text{as} \quad \lambda \to \infty.$$

Thus the proof of (1) is complete.

We prove (2). Let $\mathscr{I}_1$ and $\mathscr{I}_2$ be the subsets of $\mathscr{F}'_\lambda$ defined in the proof of (1). Let K'_1 be a positive constant which will be determined later. Let $\mathscr{I}'_3$ be the set of all cubes Q in $\mathscr{F}'_\lambda \setminus \mathscr{I}_1$ such that $\min_{z\in Q} V(z)\leq\lambda$ and

$$Q\subset\{(x, y)\in \boldsymbol{R}^{m_1}\times\boldsymbol{R}^{m_2};\, |x|\leq K'_1\lambda^{\mu_1}\},$$

where μ_1 is a constant defined in the theorem. Let K'_4 be a positive constant which will be determined later and put

$$F'_2=\{(x, y)\in\boldsymbol{R}^{m_1}\times\boldsymbol{R}^{m_2};\,(x, y)\notin\bigcup_{Q\in\mathscr{I}_1\cup\mathscr{I}'_3} Q,\, |y_j|<K'_4\lambda^{-1/2}, j=1, \cdots, m_2\},$$

and

$$F'_3=\boldsymbol{R}^n\setminus\text{the closure of}\left(\left(\bigcup_{Q\in\mathscr{I}_1\cup\mathscr{I}'_3} Q\right)\cup F'_2\right).$$

An argument similar to that in the proof of (1) shows that

$$\begin{aligned}(3.4)\qquad \sum_{Q\in\mathscr{I}_2} N(\lambda, T_{\mathscr{D}}, \mathring{Q})&\leq N(\lambda)\\ &\leq\sum_{Q\in\mathscr{I}_1} N(\lambda, T_{\mathscr{N}}, \mathring{Q})+\sum_{Q\in\mathscr{I}'_3} N(\lambda, T_{\mathscr{N}}, \mathring{Q})+\sum_{i=2}^{3} N(\lambda, T_{\mathscr{N}}, F'_i).\end{aligned}$$

As before we have:

LEMMA 3.1′.

$$N(\lambda, T_{\mathscr{N}}, F'_3)=0.$$

LEMMA 3.2′.

$$N(\lambda, T_{\mathscr{N}}, F'_2)=0.$$

LEMMA 3.3′.

$$\begin{aligned}\sum_{Q\in\mathscr{I}_2} N(\lambda, T_{\mathscr{D}}, \mathring{Q})&\sim\sum_{Q\in\mathscr{I}_1} N(\lambda, T_{\mathscr{N}}, \mathring{Q})+\sum_{Q\in\mathscr{I}'_3} N(\lambda, T_{\mathscr{N}}, \mathring{Q})\\ &\sim\frac{\omega_n}{(2\pi)^n}\int_{A'}(\lambda-V)^{n/2}\,dxdy\qquad as\quad \lambda\to\infty\end{aligned}$$

where

$$A'=\{(x, y)\in\boldsymbol{R}^{m_1}\times\boldsymbol{R}^{m_2};\, V(x, y)\leq\lambda,\, |x|\leq K'_1\lambda^{\mu_1}\}.$$

We shall postpone the proof of these three lemmas again to the following sections. By (3.4), Lemmas 3.1′, 3.2′ and 3.3′

$$N(\lambda)\sim\frac{\omega_n}{(2\pi)^n}\int_{A'}(\lambda-V)^{n/2}\,dxdy \qquad \text{as} \quad \lambda\to\infty\,.$$

Thus the proof of (2) is complete.

We get the proof for (3) if we interchange x and y in the definition of $V(x, y)$.

We now prove (4). Let $\mathscr{I}_1$ and $\mathscr{I}_2$ be the subsets of $\mathscr{F}'_\lambda$ defined in the proof of (1). Let $\mathscr{I}''_3$ be the set of all cubes Q in $\mathscr{F}'_\lambda\diagdown\mathscr{I}_1$ such that $\min_{z\in Q} V(z)\le\lambda$. Let

$$F''_3=\boldsymbol{R}^n\diagdown\text{the closure of } \bigcup_{Q\in\mathscr{I}_1\cup\mathscr{I}''_3} Q\,.$$

An argument similar to that in the proof of (1) shows that

$$\sum_{Q\in\mathscr{I}_2} N(\lambda, T_{\mathscr{D}}, \mathring{Q})\le N(\lambda)\le \sum_{Q\in\mathscr{I}_1} N(\lambda, T_{\mathscr{N}}, \mathring{Q})+\sum_{Q\in\mathscr{I}''_3} N(\lambda, T_{\mathscr{N}}, \mathring{Q})\,. \tag{3.5}$$

LEMMA 3.3″.

$$\sum_{Q\in\mathscr{I}_2} N(\lambda, T_{\mathscr{D}}, \mathring{Q})\sim \sum_{Q\in\mathscr{I}_1} N(\lambda, T_{\mathscr{N}}, \mathring{Q})+\sum_{Q\in\mathscr{I}''_3} N(\lambda, T_{\mathscr{N}}, \mathring{Q})$$

$$\sim\frac{\omega_n}{(2\pi)^n}\int_{A''}(\lambda-V)^{n/2}\,dxdy \qquad \textit{as} \quad \lambda\to\infty$$

where

$$A''=\{(x, y)\in\boldsymbol{R}^{m_1}\times\boldsymbol{R}^{m_2};\ V(x, y)\le\lambda\}\,.$$

This lemma is proved in Section 5. By (3.5) and Lemma 3.3″

$$N(\lambda)\sim\frac{\omega_n}{(2\pi)^n}\int_{A''}(\lambda-V)^{n/2}\,dxdy \qquad \text{as} \quad \lambda\to\infty\,.$$

Thus the proof of (4) is complete. q.e.d.

4. Proof of Lemmas 3.1, 3.2, 3.1′ and 3.2′.

PROOF OF LEMMA 3.1. First we assume that

$$V(x, y)>\lambda \qquad \text{for all} \quad (x, y)\in F_3\,. \tag{4.1}$$

Then we obviously have

$$\int_{F_3}(|\nabla u|^2+V|u|^2)dxdy>\lambda\int_{F_3}|u|^2dxdy\,,$$

for all $u\in H^1(F_3)$, $u\neq0$. This proves Lemma 3.1.

Now we prove (4.1). Suppose contrarily that there exists a point (x_0, y_0) in F_3 such that

(4.2) $$V(x_0, y_0) \leq \lambda\,.$$

Then there exists a cube Q in $\mathscr{F}'_\lambda$ such that $(x_0, y_0) \in Q$ and $\min_{z \in Q} V(z) \leq \lambda$. By the definition of F_3 this cube Q does not belong to $\mathscr{I}_1 \cup \mathscr{I}_3$. Therefore, there exists a point $(x_1, y_1) \in Q$ such that $|x_1| > K_1 \lambda^{\mu_1}$ or $|y_1| > K_2 \lambda^{\mu_2}$, where K_1, K_2, μ_1 and μ_2 are constants given in the definition of $\mathscr{I}_3$.

Suppose $|x_1| > K_1 \lambda^{\mu_1}$. Since the side length of Q is $\lambda^{-1/2}(\log \lambda)^{1/n}$,

$$\inf\{|x|; (x, y) \in Q\} \geq |x_1| - m_1^{1/2} \lambda^{-1/2} (\log \lambda)^{1/n}\,.$$

Observe that the right hand side is not less than

$$K_1 \lambda^{\mu_1} - m_1^{1/2} \lambda^{-1/2} (\log \lambda)^{1/n} > (K_1/2) \lambda^{\mu_1}$$

if λ is sufficiently large. Therefore

(4.3) $$\inf\{|x|; (x, y) \in Q\} \geq (K_1/2) \lambda^{\mu_1}\,.$$

Thus, by (4.2) and the assumptions on f_i and g_j,

$$\lambda \geq V(x_0, y_0) = C \prod_{i=1}^{p} f_i(|x_0|)^{\alpha_i} \cdot \prod_{j=1}^{q} g_j(|y_0|)^{\beta_j} \cdot |x_0|^{\gamma} |y_0|^{\delta} \geq C \prod_{j=1}^{q} b_{j0}^{\beta_j} \cdot |x_0|^{\Sigma \alpha_i d_i + \gamma} |y_0|^{\delta}\,.$$

By (4.3) the last term is not less than

$$C \prod_{j=1}^{q} b_{j0}^{\beta_j} \cdot ((K_1/2) \lambda^{\mu_1})^{\Sigma \alpha_i d_i + \gamma} |y_0|^{\delta} = C_1 K_1^{\Sigma \alpha_i d_i + \gamma} \lambda^{1 + \delta/2} |y_0|^{\delta}\,,$$

where $C_1 = C \prod_{j=1}^{q} b_{j0}^{\beta_j} \cdot 2^{-(\Sigma \alpha_i d_i + \gamma)}$. Therefore

$$|y_0| \leq C_2 K_1^{-(\Sigma \alpha_i d_i + \gamma)/\delta} \lambda^{-1/2}$$

where $C_2 = C_1^{-1/\delta}$.

If we choose K_1 and K_4 so that

(4.4) $$C_2 K_1^{-(\Sigma \alpha_i d_i + \gamma)/\delta} < K_4\,,$$

then

$$|y_0| < K_4 \lambda^{-1/2}\,.$$

Thus, for all components y_{0j} $(j = 1, \cdots, m_2)$ of y_0 we have $|y_{0j}| < K_4 \lambda^{-1/2}$. Hence $(x_0, y_0) \in F_2$. This contradicts $(x_0, y_0) \in F_3$.

If $|y_1| > K_2 \lambda^{\mu_2}$, then a similar argument shows that

$$|x_{0i}| < K_3 \lambda^{-1/2}\,, \qquad x_0 = (x_{01}, \cdots, x_{0m_1})\,,$$

under the condition

(4.5) $$C_3 K_2^{-(\Sigma \beta_j h_j + \delta)/\gamma} < K_3\,,$$

where $C_3=(C\prod_{i=1}^{p} a_{i0}^{\alpha_i}\cdot 2^{-(\Sigma\beta_j h_j+\delta)})^{-1/\gamma}$. Therefore $(x_0, y_0)\in F_1$ and this contradicts $(x_0, y_0)\in F_3$. Thus (4.1) holds under the conditions (4.4) and (4.5). We shall give exact values of K_1, K_2, K_3 and K_4 satisfying (4.4) and (4.5) later. q.e.d.

PROOF OF LEMMA 3.2. We prove $N(\lambda, T_{\mathcal{N}}, F_2)=0$. First we show

$$\inf\{|x|; (x, y)\in F_2\}>(K_1/2)\lambda^{\mu_1}. \tag{4.6}$$

Let $(x, y)\in F_2$. Choose Q in $\mathscr{F}'_\lambda$ so that $(x, y)\in Q$. Since

$$|y_j|<K_4\lambda^{-1/2}, \qquad y=(y_1, \cdots, y_{m_2})$$

by the definition of F_2 and since the side length of Q is $\lambda^{-1/2}(\log\lambda)^{1/n}$, we have $(x, 0)\in Q$ if λ is sufficiently large. Therefore $0=\min_{z\in Q} V(z)\leq\lambda$. Since $Q\notin\mathscr{I}_1\cup\mathscr{I}_3$, there exists a point $(x_0, y_0)\in Q$ such that

$$|x_0|>K_1\lambda^{\mu_1} \tag{4.7}$$

or

$$|y_0|>K_2\lambda^{\mu_2}. \tag{4.8}$$

(4.8) is impossible if λ is sufficiently large. Therefore (4.7) holds and

$$|x|\geq|x_0|-m_1^{1/2}\lambda^{-1/2}(\log\lambda)^{1/n}>(K_1/2)\lambda^{\mu_1}$$

if λ is sufficiently large. Thus we have (4.6).

Applying arguments similar to those in the proof of Lemma 3.1, we have, by (4.6),

$$\begin{aligned} V(x, y)&=C\prod_{i=1}^{p} f_i(|x|)^{\alpha_i}\cdot\prod_{j=1}^{q} g_j(|y|)^{\beta_j}\cdot|x|^{\gamma}|y|^{\delta}\geq C\prod_{j=1}^{q} b_{j0}^{\beta_j}\cdot|x|^{\Sigma\alpha_i d_i+\gamma}\cdot|y|^{\delta} \\ &\geq C\prod_{j=1}^{q} b_{j0}^{\beta_j}\cdot((K_1/2)\lambda^{\mu_1})^{\Sigma\alpha_i d_i+\gamma}\cdot|y|^{\delta}=C_4K_1^{\Sigma\alpha_i d_i+\gamma}\lambda^{1+\delta/2}|y|^{\delta}, \end{aligned}$$

for all $(x, y)\in F_2$, where $C_4=C\prod_{j=1}^{q} b_{j0}^{\beta_j}\cdot 2^{-(\Sigma\alpha_i d_i+\gamma)}$. Therefore

$$\begin{aligned} \int_{F_2}(|\nabla u|^2+V|u|^2)dxdy&\geq\int_{F_2}(|\nabla u|^2+C_4K_1^{\Sigma\alpha_i d_i+\gamma}\lambda^{1+\delta/2}|y|^{\delta}|u|^2)dxdy \\ &\geq\int_{F_{2x}}\left(\int_G(|\nabla_y u|^2+C_4K_1^{\Sigma\alpha_i d_i+\gamma}\lambda^{1+\delta/2}|y|^{\delta}|u|^2)dy\right)dx \end{aligned} \tag{4.9}$$

for all $u\in H^1(F_2)$ where $|\nabla_y u|^2=\sum_{j=1}^{m_2}|\partial u/\partial y_j|^2$, $F_{2x}=\{x\in\boldsymbol{R}^{m_1}; (x, y)\in F_2\}$ and $G=\{y\in\boldsymbol{R}^{m_2}; |y_j|<K_4\lambda^{-1/2}, j=1, \cdots, m_2\}$.

Remark that the function $C_4K_1^{\Sigma\alpha_i d_i+\gamma}\lambda^{1+\delta/2}|y|^{\delta}$ is an A_∞-weight on $\boldsymbol{R}^{m_2}$ by Lemma 2.3. Set

$$\lambda_1 = \inf_{\substack{a>0 \\ \xi \in \boldsymbol{R}^{m_2}}} \left(a^{-2} + a^{-m_2} \int_{|x-\xi|<a/2} C_4 K_1^{\sum \alpha_i d_i + \gamma} \lambda^{1+\delta/2} |y|^\delta dy \right).$$

Then, by elementary calculus,

$$\lambda_1 = C_5 K_1^{1/\mu_1} \lambda\,, \tag{4.10}$$

where

$$C_5 = 2^{-2/(2+\delta)}(2+\delta)\delta^{-\delta/(2+\delta)}\{2^{-(\delta+m_2)} C_4 m_2 (\delta+m_2)^{-1} \omega_{m_2}\}^{2/(2+\delta)}$$

and ω_{m_2} is the volume of the unit ball in $\boldsymbol{R}^{m_2}$.

By Lemma 2.4

$$\int_{G'} (|\nabla_y v|^2 + C_4 K_1^{\sum \alpha_i d_i + \gamma} \lambda^{1+\delta/2} |y|^\delta |v|^2) dy \geq C_6 \lambda_1 \int_{G'} |v|^2 dy \tag{4.11}$$

for all $v \in H^1(G')$, where C_6 is a constant depending only on m_2 and δ, while $G' = \{y \in \boldsymbol{R}^{m_2}; |y_j| < \lambda_1^{-1/2}, j=1, \cdots, m_2\}$.

Choosing K_1 and K_4 so that

$$C_5 K_1^{1/\mu_1} = K_4^{-2}\,, \tag{4.12}$$

we get $G=G'$ by (4.10). Therefore we have

$$\int_{F_2} (|\nabla u|^2 + V|u|^2) dx dy \geq C_5 C_6 K_1^{1/\mu_1} \lambda \int_{F_2} |u|^2 dx dy \tag{4.13}$$

for all $u \in H^1(F_2)$. Choose K_1 so that

$$C_5 C_6 K_1^{1/\mu_1} > 1\,. \tag{4.14}$$

Then we have

$$\int_{F_2} (|\nabla u|^2 + V|u|^2) dx dy > \lambda \int_{F_2} |u|^2 dx dy$$

for all $u \in H^1(F_2)$, $u \neq 0$. Hence $N(\lambda, T_{\mathcal{N}}, F_2) = 0$.

Similar arguments show that $N(\lambda, T_{\mathcal{N}}, F_1) = 0$ if we choose K_2 and K_3 so that

$$C_7 K_2^{1/\mu_2} = K_3^{-2} \tag{4.15}$$

and

$$C_7 C_8 K_2^{1/\mu_2} > 1\,, \tag{4.16}$$

where C_7 is a positive constant depending only on m_1, γ, C, β_j, h_j, α_i and a_{i0}, while C_8 is the constant given in Lemma 2.4 for the function $|x|^\gamma$.

Now we choose K_1, K_2, K_3 and K_4 so that they satisfy (4.4), (4.5), (4.12), (4.14), (4.15) and (4.16). We may put

$$K_1 = \max\{(C_2 C_5^{1/2})^{\delta\mu_1}, (C_5 C_6)^{-\mu_1}\} + 1\,, \tag{4.17}$$

$$K_2 = \max\{(C_3 C_7^{1/2})^{\gamma\mu_2}, (C_7 C_8)^{-\mu_2}\} + 1\,, \tag{4.18}$$

and define K_3 and K_4 so that they satisfy (4.12) and (4.15), respectively. Then all conditions in the proofs of Lemmas 3.1 and 3.2 are satisfied. q.e.d.

Proof of Lemma 3.1′. If we set $\gamma = 0$ and replace F_2, F_3, $\mathscr{I}_3$, K_1, K_4 in the proof of Lemma 3.1 by F'_2, F'_3, $\mathscr{I}'_3$, K'_1, K'_4, respectively, then we get the proof of Lemma 3.1′. The different point is that the argument on the inequality $|y_1| > K_2 \lambda^{\mu_2}$ does not occur. The condition on K'_1 and K'_4 is

$$C_9 K_1'^{-(\sum \alpha_i d_i)/\delta} < K'_4\,, \tag{4.4$'$}$$

where C_9 is a positive constant corresponding to C_2. We shall give exact values of K'_1 and K'_4 later. q.e.d.

Proof of Lemma 3.2′. If we set $\gamma = 0$ and replace F_2, $\mathscr{I}_3$, K_1, K_4 in the proof of $N(\lambda, T_{\mathscr{N}}, F_2) = 0$ in Lemma 3.2 by F'_2, $\mathscr{I}'_3$, K'_1, K'_4, respectively, then we get the proof of Lemma 3.2′. The different point is that the inequality (4.8) does not occur. The conditions on K'_1 and K'_4 are

$$C_{10} K_1'^{1/\mu_1} = K_4'^{-2} \tag{4.12$'$}$$

and

$$C_{10} C_{11} K_1'^{1/\mu_1} > 1\,, \tag{4.14$'$}$$

where C_{10} and C_{11} are positive constants corresponding to C_5 and C_6. If we put

$$K'_1 = \max\{(C_9 C_{10}^{1/2})^{\delta\mu_1}, (C_{10} C_{11})^{-\mu_1}\} + 1\,, \tag{4.17$'$}$$

then all conditions (4.4)′, (4.12)′ and (4.14)′ are satisfied. q.e.d.

5. Proof of Lemmas 3.3, 3.3′ and 3.3″. First we prove Lemma 3.3. Let l be the side length of cubes in $\mathscr{F}'_\lambda$, that is, $l = \lambda^{-1/2}(\log \lambda)^{1/n}$. In order to prove Lemma 3.3, we show the following three inequalities:

$$\text{(1)}\quad \sum_{Q \in \mathscr{I}_1} N(\lambda, T_{\mathscr{N}}, \mathring{Q}) \le \frac{\omega_n}{(2\pi)^n} \int_A (\lambda - V)^{n/2} dx dy + O(M_1 (\log \lambda)^{1-1/n})$$

as $\lambda \to \infty$, where $M_1 = \sharp \mathscr{I}_1$.

$$\text{(2)}\qquad \sum_{Q \in \mathscr{I}_3} N(\lambda, T_{\mathscr{N}}, \mathring{Q}) \le O(M_3 \log \lambda)$$

as $\lambda \to \infty$, where $M_3 = \sharp \mathscr{I}_3$.

(3)
$$\sum_{Q\in\mathscr{I}_2} N(\lambda, T_{\mathscr{D}}, \mathring{Q}) \geq \frac{\omega_n}{(2\pi)^n}\int_A (\lambda - V)^{n/2}dxdy - m_1\lambda^{n/2}|S^1| - m_2\lambda^{n/2}|S_1| - O(M_2(\log\lambda)^{1-1/n})$$

as $\lambda\to\infty$, where $M_2 = \sharp\mathscr{I}_2$, $S^1 = \{(x, y)\in A; |x_1| < l\}$ and $S_1 = \{(x, y)\in A; |y_1| < l\}$.

PROOF OF (1). Let Q be a cube in $\mathscr{I}_1$. Since

$$\int_Q (|\nabla u|^2 + V|u|^2)dxdy \geq \int_Q (|\nabla u|^2 + \min_Q V\cdot|u|^2)dxdy$$

for all $u\in H^1(\mathring{Q})$,

$$N(\lambda, T_{\mathscr{N}}, \mathring{Q}) \leq N(\lambda - \min_Q V, -\Delta_{\mathscr{N}}, \mathring{Q})$$

by the min-max principle in Reed-Simon [7; p. 78]. Following Edmunds and Evans [2; p. 143], we get

$$N(\lambda - \min_Q V, -\Delta_{\mathscr{N}}, \mathring{Q}) \leq \frac{\omega_n}{(2\pi)^n}|Q|\left(\lambda - \min_Q V\right)^{n/2} + C_1\{1 + (|Q|\lambda^{n/2})^{1-1/n}\},$$

where C_1 is a positive constant depending only on m_1 and m_2. Therefore

(5.1)
$$\sum_{Q\in\mathscr{I}_1} N(\lambda, T_{\mathscr{N}}, \mathring{Q}) \leq \frac{\omega_n}{(2\pi)^n}\sum_{Q\in\mathscr{I}_1}|Q|\left(\lambda - \min_Q V\right)^{n/2} + C_1\{M_1 + M_1(\log\lambda)^{1-1/n}\},$$

since the side length of Q is $l = \lambda^{-1/2}(\log\lambda)^{1/n}$.

Let $\xi_1, \cdots, \xi_n$ be positive integers. Let Q be a cube in $\mathscr{I}_1$ with center $(l(\xi_1 + 1/2), \cdots, l(\xi_n + 1/2))$ and let Q' be a cube in $\mathscr{F}'_\lambda$ with center $(l(\xi_1 - 1/2), \cdots, l(\xi_n - 1/2))$. Then

$$V(x, y) = C\prod_{i=1}^{p} f_i(|x|)^{\alpha_i}\cdot\prod_{j=1}^{q} g_j(|y|)^{\beta_j}\cdot|x|^{\gamma}|y|^{\delta} \leq \min_Q V \leq \lambda$$

for all $(x, y)\in Q'$. Therefore $Q'\in\mathscr{I}_2$ and

$$|Q|\left(\lambda - \min_Q V\right)^{n/2} \leq \int_{Q'}(\lambda - V)^{n/2}dxdy.$$

Note that $Q\to Q'$ is a one-to-one correspondence from cubes in $\mathscr{I}_1$ with centers in the first orthant to cubes in $\mathscr{I}_2$ with centers in the first orthant. Then we get, by the symmetry property of V,

(5.2)
$$\sum_{Q\in\mathscr{I}_1}|Q|\left(\lambda - \min_Q V\right)^{n/2} \leq \int_I (\lambda - V)^{n/2}dxdy,$$

where $I=\bigcup_{Q\in\mathscr{I}_2} Q$. Note that

$$I\subset A . \tag{5.3}$$

Indeed, by the definition of $\mathscr{I}_2$,

$$I\subset\{(x, y)\in \boldsymbol{R}^{m_1}\times \boldsymbol{R}^{m_2};\ V(x, y)\leq\lambda\} .$$

Furthermore, following the argument in the proof of Lemma 3.1, we get $\mathscr{I}_2\subset\mathscr{I}_1\cup\mathscr{I}_3$. Thus we get (5.3). Hence

$$\sum_{Q\in\mathscr{I}_1}|Q|\left(\lambda-\min_Q V\right)^{n/2}\leq\int_A(\lambda-V)^{n/2}dxdy .$$

Applying this to (5.1), we get

$$\sum_{Q\in\mathscr{I}_1}N(\lambda, T_{\mathscr{N}}, \mathring{Q})\leq\frac{\omega_n}{(2\pi)^n}\int_A(\lambda-V)^{n/2}dxdy+O(M_1(\log\lambda)^{1-1/n}) ,$$

where the bound of the error term is independent of λ. q.e.d.

Proof of (2). Applying the argument in the proof of (1), we get

$$\begin{aligned}\sum_{Q\in\mathscr{I}_3}N(\lambda, T_{\mathscr{N}}, \mathring{Q})\leq\sum_{Q\in\mathscr{I}_3}N(\lambda, -\Delta_{\mathscr{N}}, \mathring{Q})&\leq\frac{\omega_n}{(2\pi)^n}\sum_{Q\in\mathscr{I}_3}|Q|\lambda^{n/2}+C_1\{M_3+M_3(\log\lambda)^{1-1/n}\}\\&=O(M_3\log\lambda) .\end{aligned}$$

q.e.d.

Proof of (3). Let Q be a cube in $\mathscr{I}_2$. Since

$$\int_Q(|\nabla u|^2+V|u|^2)dxdy\leq\int_Q\left(|\nabla u|^2+\max_Q V\cdot|u|^2\right)dxdy$$

for all $u\in H^1(\mathring{Q})$,

$$N(\lambda, T_{\mathscr{D}}, \mathring{Q})\geq N\left(\lambda-\max_Q V, -\Delta_{\mathscr{D}}, \mathring{Q}\right)$$

by the min-max principle. Following Edmunds and Evans [2; p. 143] as before, we get

$$N\left(\lambda-\max_Q V, -\Delta_{\mathscr{D}}, \mathring{Q}\right)\geq\frac{\omega_n}{(2\pi)^n}|Q|\left(\lambda-\max_Q V\right)^{n/2}-C_2\{1+(|Q|\lambda^{n/2})^{1-1/n}\} ,$$

where C_2 is a positive constant depending only on m_1 and m_2. Therefore

$$\sum_{Q\in\mathscr{I}_2}N(\lambda, T_{\mathscr{D}}, \mathring{Q})\geq\frac{\omega_n}{(2\pi)^n}\sum_{Q\in\mathscr{I}_2}|Q|\left(\lambda-\max_Q V\right)^{n/2}-C_2\{M_2+M_2(\log\lambda)^{1-1/n}\} . \tag{5.4}$$

Applying an argument similar to that in the proof of (1), we get

$$\sum_{Q\in\mathscr{I}_2} |Q|\left(\lambda-\max_Q V\right)^{n/2} \geq \int_J (\lambda-V)^{n/2}dxdy\,, \tag{5.5}$$

where $J=\{(x, y)\in \boldsymbol{R}^{m_1}\times \boldsymbol{R}^{m_2}; (x, y)\in \bigcup_{Q\in\mathscr{I}_1} Q,\ V(x, y)\leq\lambda\}$. Recall the definition of $\mathscr{I}_1$ and apply the argument in the proof of Lemma 3.1. Then we get

$$\left(\bigcup_{Q\in\mathscr{I}_1} Q\right)\cap\{(x, y)\in \boldsymbol{R}^{m_1}\times \boldsymbol{R}^{m_2}; |x|>K_1\lambda^{\mu_1} \text{ or } |y|>K_2\lambda^{\mu_2}\}=\varnothing\,.$$

Therefore, by the definition of A,

$$\begin{aligned} J&=\{(x, y)\in A; |x_i|\geq l,\ i=1, \cdots, m_1, |y_j|\geq l,\ j=1, \cdots, m_2\} \\ &=A\diagdown\left(\bigcup_{i=1}^{m_1}\{(x, y)\in A; |x_i|<l\}\cup\bigcup_{j=1}^{m_2}\{(x, y)\in A; |y_j|<l\}\right) \\ &=A\diagdown\left(\bigcup_{i=1}^{m_1} S^i\cup\bigcup_{j=1}^{m_2} S_j\right), \quad \text{say}\,. \end{aligned}$$

Thus by (5.5)

$$\begin{aligned} &\sum_{Q\in\mathscr{I}_2} |Q|\left(\lambda-\max_Q V\right)^{n/2} \geq \int_A (\lambda-V)^{n/2}dxdy-\sum_{i=1}^{m_1}\int_{S^i}(\lambda-V)^{n/2}dxdy- \\ &\sum_{j=1}^{m_2}\int_{S_j}(\lambda-V)^{n/2}dxdy\geq\int_A(\lambda-V)^{n/2}dxdy-\lambda^{n/2}\sum_{i=1}^{m_1}|S^i|-\lambda^{n/2}\sum_{j=1}^{m_2}|S_j| \\ &\geq\int_A(\lambda-V)^{n/2}dxdy-\lambda^{n/2}m_1|S^1|-\lambda^{n/2}m_2|S_1|\,, \end{aligned}$$

where we used the symmetry property of V. Therefore, by (5.4),

$$\begin{aligned} \sum_{Q\in\mathscr{I}_2} N(\lambda, T_{\mathscr{D}}, \mathring{Q})\geq&\frac{\omega_n}{(2\pi)^n}\int_A(\lambda-V)^{n/2}dxdy \\ &-m_1\lambda^{n/2}|S^1|-m_2\lambda^{n/2}|S_1|-O(M_2(\log\lambda)^{1-1/n})\,. \end{aligned}$$

q.e.d.

Therefore, by (1), (2) and (3), Lemma 3.3 follows from the following three lemmas.

LEMMA 5.1.

$$M_1(\log\lambda)^{1-1/n}=M_2(\log\lambda)^{1-1/n}=o\left(\int_A(\lambda-V)^{n/2}dxdy\right) \quad \textit{as} \quad \lambda\to\infty\,.$$

LEMMA 5.2.

$$\lambda^{n/2}|S^1|=o\left(\int_A(\lambda-V)^{n/2}dxdy\right) \quad \textit{as} \quad \lambda\to\infty\,,$$

and

$$\lambda^{n/2}|S_1|=o\left(\int_A(\lambda-V)^{n/2}dxdy\right)\quad as\quad \lambda\to\infty\,.$$

LEMMA 5.3.

$$M_3(\log\lambda)=o\left(\int_A(\lambda-V)^{n/2}dxdy\right)\quad as\quad \lambda\to\infty\,.$$

To prove Lemmas 5.1, 5.2 and 5.3, we use the following lemma, where $f(\lambda)\approx g(\lambda)$ means that $f(\lambda)=O(g(\lambda))$ and $g(\lambda)=O(f(\lambda))$ as $\lambda\to\infty$.

LEMMA 5.4. *Let V be the function defined by* (3.1). *Set* $\nu_1=n/2+m_1(\sum_{i=1}^p\alpha_id_i+\gamma)^{-1}$, $\nu_2=n/2+m_2(\sum_{j=1}^q\beta_jh_j+\delta)^{-1}$, $\nu_3=m_1/2+2^{-1}(2+\delta)m_1(\sum_{i=1}^p\alpha_id_i+\gamma)^{-1}$, *and* $\nu_4=m_2/2+2^{-1}(2+\gamma)m_2(\sum_{j=1}^q\beta_jh_j+\delta)^{-1}$.

(1) *If* $\gamma m_2<(\sum_{i=1}^q\beta_jh_j+\delta)m_1$, $\delta m_1<(\sum_{i=1}^p\alpha_id_i+\gamma)m_2$ *and* $\nu_1\neq\nu_2$, *then*

$$\int_A(\lambda-V)^{n/2}dxdy\approx\lambda^{\nu_1}+\lambda^{\nu_2}\,.$$

(2) *If* $\delta m_1>(\sum_{i=1}^p\alpha_id_i+\gamma)m_2$, *then*

$$\int_A(\lambda-V)^{n/2}dxdy\approx\lambda^{\nu_3}\,.$$

(3) *If* $\gamma m_2>(\sum_{j=1}^q\beta_jh_j+\delta)m_1$, *then*

$$\int_A(\lambda-V)^{n/2}dxdy\approx\lambda^{\nu_4}\,.$$

(4) *In the other cases,*

$$\int_A(\lambda-V)^{n/2}dxdy\approx(\lambda^{\nu_1}+\lambda^{\nu_2})\log\lambda\,.$$

These estimates are given by elementary calculus, so we omit the proof of Lemma 5.4.

As a consequence of Lemma 5.4, we get

$$\int_A(\lambda-V)^{n/2}dxdy=O((\lambda^{\nu_1}+\lambda^{\nu_2})\log\lambda+\lambda^{\nu_3}+\lambda^{\nu_4})\,. \tag{5.6}$$

Remark that an easy calculation shows that the order of $\int_A(\lambda-V)^{n/2}dxdy$ is the same as that of $\lambda^{n/2}|A|$.

PROOF OF LEMMA 5.1. Since the argument before (5.2) shows that $M_1=M_2$, it suffices to estimate $M_2(\log\lambda)^{1-1/n}$.

Since the side length of $Q \in \mathscr{I}_2$ is $l=\lambda^{-1/2}(\log \lambda)^{1/n}$,

$$M_2(\log \lambda)^{1-1/n}=l^{-n}(\log \lambda)^{1-1/n}\left|\bigcup_{Q\in\mathscr{I}_2} Q\right|=(\log \lambda)^{-1/n}\lambda^{n/2}|I|\,.$$

By (5.3) the term on the right hand side does not exceed $(\log \lambda)^{-1/n}\lambda^{n/2}|A|$. Since $\lambda^{n/2}|A|=O(\int_A(\lambda-V)^{n/2}dxdy)$, the assertion of Lemma 5.1 is valid. q.e.d.

PROOF OF LEMMA 5.2. First we prove

$$\lambda^{n/2}|S^1|=o\left(\int_A(\lambda-V)^{n/2}dxdy\right). \tag{5.7}$$

If $m_1>1$, then

$$\lambda^{n/2}|S^1|\le 2\lambda^{n/2}l|S'|=2(\log \lambda)^{1/n}\lambda^{(n-1)/2}|S'|\,,$$

where S' is the set of all points $(x', y)\in \boldsymbol{R}^{m_1-1}\times \boldsymbol{R}^{m_2}$ such that

$$C\prod_{i=1}^{p} f_i(|x'|)^{\alpha_i}\cdot\prod_{j=1}^{q} g_j(|y|)^{\beta_j}\cdot|x'|^{\gamma}|y|^{\delta}\le\lambda\,,$$

$$|x'|\le K_1\lambda^{\mu_1} \quad\text{and}\quad |y|\le K_2\lambda^{\mu_2}\,,$$

where K_1, K_2, μ_1 and μ_2 are constants given in the definition of $\mathscr{I}_3$. By an argument similar to that in the note after Lemma 5.4, we can show that the order of $\lambda^{(n-1)/2}|S'|$ is the same as that of $\int_{S'}(\lambda-V'(x', y))^{(n-1)/2}dx'dy$, where $V'(x', y)=V(0, x', y)$. If we replace m_1 by m_1-1 in Lemma 5.4, we get the order of $\int_{S'}(\lambda-V')^{(n-1)/2}dx'dy$. Thus, replacing m_1 by m_1-1 in (5.6), we get

$$(\log \lambda)^{1/n}\int_{S'}(\lambda-V)^{(n-1)/2}dx'dy=O((\lambda^{\nu_1'}+\lambda^{\nu_2'})(\log \lambda)^{1+1/n}+(\lambda^{\nu_3'}+\lambda^{\nu_4'})(\log \lambda)^{1/n})\,, \tag{5.8}$$

where $\nu_1'=(n-1)/2+(m_1-1)(\sum_{i=1}^{p}\alpha_i d_i+\gamma)^{-1}$, $\nu_2'=(n-1)/2+m_2(\sum_{j=1}^{q}\beta_j h_j+\delta)^{-1}$, $\nu_3'=(m_1-1)/2+2^{-1}(2+\delta)(m_1-1)(\sum_{i=1}^{p}\alpha_i d_i+\gamma)^{-1}$, and $\nu_4'=m_2/2+2^{-1}(2+\gamma)m_2(\sum_{j=1}^{q}\beta_j h_j+\delta)^{-1}$. If we compare the order of $\int_A(\lambda-V)^{n/2}dxdy$ in Lemma 5.4 with the one on the right hand side of (5.8), then we get

$$\lambda^{n/2}|S^1|=o\left(\int_A(\lambda-V)^{n/2}dxdy\right)\qquad\text{as}\quad \lambda\to\infty\,.$$

If $m_1=1$, then, by the definition of S^1,

$$\lambda^{(1+m_2)/2}|S^1|\le C\lambda^{(1+m_2)/2}l\lambda^{m_2\mu_2}=C\lambda^{m_2/2+m_2\mu_2}(\log \lambda)^{1/n}\,,$$

where C is a constant independent of λ. Therefore, by Lemma 5.4, we can show

$$\lambda^{n/2}|S^1|=o\left(\int_A(\lambda-V)^{n/2}dxdy\right).$$

Thus we get (5.7).

Similarly, we can prove

$$\lambda^{n/2}|S_1|=o\left(\int_A(\lambda-V)^{n/2}dxdy\right).$$

q.e.d.

PROOF OF LEMMA 5.3. Let B^i and B_j be the subsets of $\boldsymbol{R}^n$ and $\mathscr{I}_3$ be the set of cubes as defined in the proof of the Theorem. Let $\{i_1,\cdots,i_s\}$ and $\{j_1,\cdots,j_t\}$ be subsets of $\{1,\cdots,m_1\}$ and $\{1,\cdots,m_2\}$, respectively. For $\{i_1,\cdots,i_s\}$ and $\{j_1,\cdots,j_t\}$, denote

$$\begin{aligned}\mathscr{Q}^{i_1,\cdots,i_s}=\{Q\in\mathscr{I}_3;\ &Q\cap B^i\neq\emptyset,\ i=i_1,\cdots,i_s,\ Q\cap B^i=\emptyset,\ i\neq i_1,\cdots,i_s,\\ &Q\cap B_j=\emptyset, j=1,\cdots,m_2\},\\ \mathscr{Q}_{j_1,\cdots,j_t}=\{Q\in\mathscr{I}_3;\ &Q\cap B_j\neq\emptyset, j=j_1,\cdots,j_t,\ Q\cap B_j=\emptyset, j\neq j_1,\cdots,j_t,\\ &Q\cap B^i=\emptyset,\ i=1,\cdots,m_1\}\end{aligned}$$

and

$$\begin{aligned}\mathscr{Q}^{i_1,\cdots,i_s}_{j_1,\cdots,j_t}=\{Q\in\mathscr{I}_3;\ &Q\cap B^i\neq\emptyset,\ i=i_1,\cdots,i_s,\ Q\cap B^i=\emptyset,\ i\neq i_1,\cdots,i_s,\\ &Q\cap B_j\neq\emptyset, j=j_1,\cdots,j_t,\ Q\cap B_j=\emptyset, j\neq j_1,\cdots,j_t\}.\end{aligned}$$

Then we get a disjoint decomposition of $\mathscr{I}_3$:

$$\mathscr{I}_3=\left(\bigcup_{i_1<\cdots<i_s}\mathscr{Q}^{i_1,\cdots,i_s}\right)\cup\left(\bigcup_{j_1<\cdots<j_t}\mathscr{Q}_{j_1,\cdots,j_t}\right)\cup\left(\bigcup_{\substack{i_1<\cdots<i_s\\ j_1<\cdots<j_t}}\mathscr{Q}^{i_1,\cdots,i_s}_{j_1,\cdots,j_t}\right).\tag{5.9}$$

Now we show that

$$(\#\mathscr{Q}^{i_1,\cdots,i_s})\log\lambda=o\left(\int_A(\lambda-V)^{n/2}dxdy\right)\tag{5.10}$$

for any $i_1<\cdots<i_s$ in $\{1,\cdots,m_1\}$.

Fix $i_1<\cdots<i_s$ and simply denote $\mathscr{Q}$ instead of $\mathscr{Q}^{i_1,\cdots,i_s}$.

First suppose $s<m_1$. Let $\mathscr{Q}'$ be the set of Q in $\mathscr{Q}$ which are contained in the first orthant. Let R be the set of all points $(x,y)\in\boldsymbol{R}^{m_1}\times\boldsymbol{R}^{m_2}$ such that

$$\begin{aligned}&0\le x_i\le l, && i=i_1,\cdots,i_s,\\ &l\le x_i, && i\neq i_1,\cdots,i_s,\\ &l\le y_j, && j=1,\cdots,m_2,\\ &|x^*|\le K_1\lambda^{\mu_1}, && |y|\le K_2\lambda^{\mu_2}\end{aligned}$$

and

$$C\prod_{i=1}^{p} f_i(|x^*-le_1|)^{\alpha_i}\cdot\prod_{j=1}^{q} g_j(|y-le_2|)^{\beta_j}\cdot|x^*-le_1|^{\gamma}|y-le_2|^{\delta}\leq\lambda\,,$$

where $x^*=(x_{\tau_1},\cdots,x_{\tau_{m_1-s}})$, $\tau_1<\cdots<\tau_{m_1-s}$, $\{\tau_1,\cdots,\tau_{m_1-s}\}=\{1,\cdots,m_1\}\setminus\{i_1,\cdots,i_s\}$, $e_1=(1,\cdots,1)\in R^{m_1-s}$, $e_2=(1,\cdots,1)\in R^{m_2}$ and K_1, K_2, μ_1, μ_2 are constants given in the definition of $\mathscr{I}_3$. Then, by the definitions of $\mathscr{I}_3$ and $\mathscr{Q}'$,

$$\bigcup_{Q\in\mathscr{Q}'} Q\subset R\,.$$

Therefore

$$\#\mathscr{Q}'=l^{-n}\left|\bigcup_{Q\in\mathscr{Q}'} Q\right|\leq l^{-n}|R|\leq l^{-n+s}|R'|\,,$$

where R' is the set of all points (x^*, y) in $\boldsymbol{R}^{m_1-s}\times\boldsymbol{R}^{m_2}$ such that

$$0\leq x_i^*\,,\quad i=1,\cdots,m_1-s\,,\quad x^*=(x_1^*,\cdots,x_{m_1-s}^*)\,,$$
$$0\leq y_j\,,\quad j=1,\cdots,m_2\,,\quad y=(y_1,\cdots,y_{m_2})\,,$$
$$|x^*|\leq K_1\lambda^{\mu_1}\,,\qquad |y|\leq K_2\lambda^{\mu_2}$$

and

$$C\prod_{i=1}^{p} f_i(|x^*|)^{\alpha_i}\prod_{j=1}^{q} g_j(|y|)^{\beta_j}|x^*|^{\gamma}|y|^{\delta}\leq\lambda\,.$$

Therefore, since $l=\lambda^{-1/2}(\log\lambda)^{1/n}$,

(5.11) $$\#\mathscr{Q}'\leq l^{-n+s}|R'|=(\log\lambda)^{s/n-1}\lambda^{(n-s)/2}|R'|\,.$$

By an argument similar to that in the proof of Lemma 5.2, we get

$$\lambda^{(n-s)/2}|R'|=O((\lambda^{\eta_1}+\lambda^{\eta_2})\log\lambda+\lambda^{\eta_3}+\lambda^{\eta_4})\,,$$

where $\eta_1=(n-s)/2+(m_1-s)(\sum_{i=1}^{p}\alpha_i d_i+\gamma)^{-1}$, $\eta_2=(n-s)/2+m_2(\sum_{j=1}^{q}\beta_j h_j+\delta)^{-1}$, $\eta_3=(m_1-s)/2+2^{-1}(2+\delta)(m_1-s)(\sum_{i=1}^{p}\alpha_i d_i+\gamma)^{-1}$ and $\eta_4=m_2/2+2^{-1}(2+\gamma)m_2\times(\sum_{j=1}^{q}\beta_j h_j+\delta)^{-1}$. Therefore, by (5.11), we get

(5.12) $$(\#\mathscr{Q})\log\lambda=2^n(\#\mathscr{Q}')\log\lambda=O((\lambda^{\eta_1}+\lambda^{\eta_2})(\log\lambda)^{1+s/n}+(\lambda^{\eta_3}+\lambda^{\eta_4})(\log\lambda)^{s/n})\,.$$

If we compare the orders in Lemma 5.4 with the one in (5.12), then we get

$$(\#\mathscr{Q})\log\lambda=o\left(\int_A(\lambda-V)^{n/2}dxdy\right).$$

Suppose $s=m_1$. Then, by the definition of $\mathscr{I}_3$ and $\mathscr{Q}$, we get

$$\bigcup_{Q\in\mathscr{Q}} Q\subset\{(x, y)\in\boldsymbol{R}^{m_1}\times\boldsymbol{R}^{m_2};\,|x_i|\leq l,\,i=1,\cdots,m_1,\,|y|\leq K_2\lambda^{\mu_2}\}\,.$$

Therefore, by Lemma 5.4,

$$(\#\mathscr{Q})\log\lambda=l^{-n}|\mathscr{Q}|\log\lambda\leq Cl^{-n+m_1}\lambda^{m_2\mu_2}\log\lambda$$
$$=C\lambda^{-m_2/2+m_2\mu_2}(\log\lambda)^{m_1/n}=o\left(\int_A(\lambda-V)^{n/2}dxdy\right),$$

where C is a constnt independent of λ. Therefore (5.10) holds.

Similarly, we can show that

$$(\#\mathscr{Q}_{j_1,\dots,j_t})\log\lambda=o\left(\int_A(\lambda-V)^{n/2}dxdy\right)$$

and

$$(\#\mathscr{Q}^{i_1,\dots,i_s}_{j_1,\dots,j_t})\log\lambda=o\left(\int_A(\lambda-V)^{n/2}dxdy\right).$$

Therefore, Lemma 5.3 follows from (5.9). q.e.d.

Thus we proved Lemma 3.3. If we set $\gamma=0$ and replace A, $\mathscr{I}_3$, K_1 in the proof of Lemma 3.3 by A', $\mathscr{I}'_3$, K'_1, respectively, then we get the proof of Lemma 3.3′ after simple modification. If we set $\gamma=\delta=0$ and replace A, $\mathscr{I}_3$ in the proof of Lemma 3.3 by A'', $\mathscr{I}''_3$, respectively, then we get the proof of Lemma 3.3″. The differences caused by these modifications are inessential.

REMARK 5.1. The above method does not give an asymptotic estimate for $N(\lambda)$ when $\gamma m_2>(\sum_{j=1}^q\beta_jh_j+\delta)m_1$ or $\delta m_1>(\sum_{i=1}^p\alpha_id_i+\gamma)m_2$. Indeed, we cannot get good estimates for error terms in that case.

REMARK 5.2. We also have the asymptotic formula for the potential

$$V(x,y)=|x|^\alpha|y|^\beta|y-1|^\gamma$$

where $(x,y)\in\boldsymbol{R}\times\boldsymbol{R}$, α, β, $\gamma>0$, $\beta\leq\alpha$, $\gamma\leq\alpha$ and $\alpha\leq\beta+\gamma$. Let $\mu_1=\max\{(2+\beta)(2\alpha)^{-1},(2+\gamma)(2\alpha)^{-1}\}$ and $\mu_2=(2+\alpha)2^{-1}(\beta+\gamma)^{-1}$. Then

$$N(\lambda)\sim\frac{\omega_n}{(2\pi)^n}\int_A(\lambda-V)dxdy\qquad\text{as}\quad\lambda\to\infty,$$

where

$$A=\{(x,y)\in\boldsymbol{R}\times\boldsymbol{R};\ V(x,y)\leq\lambda,\ |x|\leq C_1\lambda^{\mu_1},\ |y|\leq C_2\lambda^{\mu_2}\}$$

and C_1, C_2 are positive constants depending only on α, β and γ. The proof of this result is a modification of the proof of the Theorem.

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