ASYMPTOTIC DISTRIBUTION OF NODAL INTERSECTIONS FOR ARITHMETIC RANDOM WAVES

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Abstract. We study the nodal intersections number of random Gaussian toral Laplace eigenfunctions ("arithmetic random waves") against a fixed smooth reference curve. The expected intersection number is proportional to the the square root of the eigenvalue times the length of curve, independent of its geometry. The asymptotic behaviour of the variance was addressed by Rudnick-Wigman; they found a precise asymptotic law for "generic" curves with nowhere vanishing curvature, depending on both its geometry and the angular distribution of lattice points lying on circles corresponding to the Laplace eigenvalue. They also discovered that there exist peculiar "static" curves, with variance of smaller order of magnitude, though did not prescribe what the true asymptotic behaviour is in this case.

In this paper we study the finer aspects of the limit distribution of the nodal intersections number. For "generic" curves we prove the Central Limit Theorem (at least, for "most" of the energies). For the aforementioned static curves we establish a non-Gaussian limit theorem for the distribution of nodal intersections, and on the way find the true asymptotic behaviour of their fluctuations, under the well-separatedness assumption on the corresponding lattice points, satisfied by most of the eigenvalues.

Keywords and Phrases: Arithmetic random waves, nodal intersections, limit theorems, lattice points.

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1. Introduction and main results

1.1. Toral nodal intersections. Let $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$ be the two-dimensional standard torus and Δ the Laplacian on $\mathbb{T}$. It is well-known that the eigenvalues of $-\Delta$ ("energy levels") are all the number of the form $E_n = 4\pi^2 n$ where n is an integer expressible as a sum of two squares

$$n \in S := \{a^2 + b^2 : \ a, b \in \mathbb{Z}\}.$$

Given a number $n \in S$ we denote

$$\Lambda_n = \{\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^2 : \ \|\lambda\|^2 := \lambda_1^2 + \lambda_2^2 = n\}$$

to be the collection of lattice points lying on the radius-$\sqrt{n}$ centred circle in $\mathbb{R}^2$; the eigenspace of $-\Delta$ corresponding to E_n then admits the orthonormal basis

$$\left\{e_\lambda(x) := e^{i2\pi\langle\lambda,x\rangle}\right\}_{\lambda\in\Lambda_n},$$

$x = (x_1, x_2) \in \mathbb{T}$. Equivalently, we may express every (complex-valued) function T_n satisfying

$$\Delta T_n + E_n T_n = 0$$

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as a linear combination

$$T_n(x) := \frac{1}{\sqrt{\mathcal{N}_n}} \sum_{\lambda \in \Lambda_n} a_\lambda \mathrm{e}_\lambda(x) \tag{1.1}$$

(the meaning of the normalizing constant on the r.h.s. of (1.1) will clear in §1.2); T_n is real-valued if and only if for every $\lambda \in \Lambda_n$ we have

$$a_{-\lambda} = \overline{a_\lambda}. \tag{1.2}$$

From now on, we assume T_n in (1.1) to be real-valued. The *nodal line* of T_n is the zero set $T_n^{-1}(0)$; under some generic assumptions $T_n^{-1}(0)$ is a *smooth curve.* Given a fixed reference curve $\mathcal{C} \subset \mathbb{T}$ one is interested in the number $\mathcal{Z}_\mathcal{C}(T_n)$ of *nodal intersection*, i.e. the number of intersections of $T_n^{-1}(0)$ with $\mathcal{C}$; one expects that for n sufficiently big $\mathcal{Z}_\mathcal{C}(T_n)$ is *finite*, and it is believed that their number should be commensurable with $\sqrt{n}$. That it is indeed so was verified by Bourgain and Rudnick [BR12] for $\mathcal{C}$ with nowhere vanishing curvature; they showed that in this case

$$n^{1/2-o(1)} \ll \mathcal{Z}_\mathcal{C}(T_n) \ll \sqrt{n}, \tag{1.3}$$

and go rid of the $o(1)$ in the exponent of the lower bound for "most" n [BR15].

1.2. **Nodal intersections for arithmetic random waves.** We endow the linear space (1.1) with a Gaussian probability measure by taking the coefficients a_λ in (1.1) random variables. Namely, we assume that the a_λ are standard complex-Gaussian i.i.d. save to (1.2), all defined on the same probability space; equivalently, T_n is a real-valued centered Gaussian field on $\mathbb{T}$ with covariance function

$$r_n(x,y) := \mathbb{E}[T_n(x) \cdot T_n(y)] = \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \cos(2\pi \langle \lambda, x-y \rangle). \tag{1.4}$$

The random fields T_n are the "arithmetic random waves" [ORW, KKW13, MPRW16]; by (1.4) above, T_n are unit variance and stationary, and by the standard abuse of notation we may denote

$$r_n(x-y) := r_n(x,y).$$

Given a smooth (finite) length-L curve $\mathcal{C} \subset \mathbb{T}^2$ we define $\mathcal{Z}_n = \mathcal{Z}_\mathcal{C}(T_n)$ to be the number of nodal intersections of T_n against $\mathcal{C}$; it is a (a.s. finite [RW14]) random variable whose distribution is the primary focus of this paper. Rudnick and Wigman [RW14] have computed its expected number to be

$$\mathbb{E}[\mathcal{Z}_n] = \frac{\sqrt{E_n}}{\pi\sqrt{2}} L \tag{1.5}$$

independent of the geometry of $\mathcal{C}$, and also studied the asymptotic behaviour of the variance of $\mathcal{Z}_n$ for large values of n; in order to be able to exhibit their results we require some number theoretic preliminaries. First, denote $\mathcal{N}_n = |\Lambda_n|$ to be the number of lattice points lying on the radius-$\sqrt{n}$ circle. While on one hand, along $n \in S$ the number $\mathcal{N}_n$ grows [Lan08] *on average* as $c_{RL} \cdot \sqrt{\log n}$ with $c_{RL} > 0$ the Ramanujan-Landau constant, on the other hand $\mathcal{N}_n$ is subject to large and erratic fluctuations; for example for the (thin) sequence of primes $p \equiv 1 \mod 4$ the corresponding $\mathcal{N}_p = 8$ does not grow at all. From this point on we will assume that $\mathcal{N}_n \to \infty$ (also holding for density-1 sequence $\{n\} \subset S$); it is also easy to derive the bound

$$\mathcal{N}_n = O(n^{o(1)}). \tag{1.6}$$

We will also need to consider the angular distribution of Λ_n; to this end we define the probability measures

$$\mu_n := \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \delta_{\lambda/\sqrt{n}}, \tag{1.7}$$

$n \in S$, on the unit circle $\mathcal{S}^1 \subset \mathbb{R}^2$. It is well-known that for a density-1 subsequence $\{n_j\} \subset S$ the corresponding lattice points Λ_{n_j} are asymptotically equidistributed in the sense that the corresponding measures are weak-$*$ convergent

$$\mu_{n_j} \Rightarrow \frac{d\theta}{2\pi} \tag{1.8}$$

to the uniform measure on $\mathcal{S}^1$. To the other extreme, there exists [Cil93] a (thin) sequence $\{n_j\} \subset S$ with angles all concentrated around $\pm 1, \pm i$

$$\mu_{n_j} \Rightarrow \frac{1}{4}\left(\delta_{\pm 1} + \delta_{\pm i}\right)$$

(thinking of $\mathcal{S}^1 \subset \mathbb{C}$), and the other partial weak-$*$ limits were partially classified [KKW13, KW16].

Back to the variance of $\mathcal{Z}_n$, let $\gamma : [0, L] \to \mathbb{T}$ be the arc-length parametrization of $\mathcal{C}$. Rudnick and Wigman [RW14, Theorem 1.1] found that[1]

$$\mathrm{Var}(\mathcal{Z}_n) = (4B_{\mathcal{C}}(\Lambda_n) - L^2) \cdot \frac{n}{\mathcal{N}_n} + O\left(\frac{n}{\mathcal{N}_n^{3/2}}\right), \tag{1.9}$$

where

$$\begin{aligned} B_{\mathcal{C}}(\Lambda_n) &:= \int_0^L \int_0^L \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \left\langle \frac{\lambda}{|\lambda|}, \dot\gamma(t_1) \right\rangle^2 \cdot \left\langle \frac{\lambda}{|\lambda|}, \dot\gamma(t_2) \right\rangle^2 dt_1 dt_2 \\ &= \int_0^L \int_0^L \int_{\mathcal{S}^1} \langle \theta, \dot\gamma(t_1) \rangle^2 \cdot \langle \theta, \dot\gamma(t_2) \rangle^2 \, d\mu_n(\theta) dt_1 dt_2. \end{aligned} \tag{1.10}$$

The leading term

$$4B_{\mathcal{C}}(\Lambda_n) - L^2$$

fluctuates [RW14, §7] in the interval $[0, L^2]$ depending on both the angular distribution of the lattice points Λ_n and the geometry of $\mathcal{C}$. One may also define $B_{\mathcal{C}}(\mu)$ for any probability measure μ on $\mathcal{S}^1$ with μ in place of μ_n on the r.h.s. of (1.10).

In order for the variance (1.9) to observe an asymptotic law we need to split S into sequences $\{n_j\} \subset S$ with corresponding Λ_{n_j} admitting limit angular law. That is, $\mu_{n_j} \Rightarrow \mu$ for some μ probability measure on $\mathcal{S}^1$; in this case

$$B_{\mathcal{C}}(\Lambda_{n_j}) = B_{\mathcal{C}}(\mu_{n_j}) \to B_{\mathcal{C}}(\mu),$$

so that if $B_{\mathcal{C}}(\mu) > L^2/4$, in this case (1.9) is

$$\mathrm{Var}(\mathcal{Z}_{n_j}) \sim (4B_{\mathcal{C}}(\mu) - L^2) \frac{n_j}{\mathcal{N}_{n_j}}.$$

[1] Initially under another technical assumption, subsequently lifted in [RWY15].

Rudnick and Wigman [RW14] observed that there exist "special" curves (see Definition 1 in §1.3) for which

$$B_{\mathcal{C}}(\mu) \equiv \frac{L^2}{4},$$

so that the leading term vanishes irrespective of the limit measure (though under a very restrictive scenario it might still vanish for other curves); for these (1.9) gives no clue as of what is the true behaviour of $\mathrm{Var}(\mathcal{Z}_n)$ other than that it is of lower order of magnitude than the "typical" $n/\mathcal{N}_n$.

1.3. **Statement of main results: limiting laws for $\mathcal{Z}_n$.** Our two principal results below concern the limit laws for $\mathcal{Z}_n$. Theorem 1.1 asserts the Central Limit Theorem for $\mathcal{Z}_n$, under the "generic" scenario that the leading terms of the variance in (1.9) are bounded away from zero. Indeed, if we assume that the lattice points Λ_n are equidistributed (i.e. the generic assumption (1.8) on the energy levels), then the assumptions of Theorem 1.1 hold for generic curves [RW14, Corollary 7.2], also see the discussion in §1.4 below. Theorem 1.3 investigates the peculiar alternative situation (the slightly more restrictive aforementioned "static" curves) with a non-Gaussian limit law, and among other things determines the true (lower order) asymptotic law of the variance (1.9) in the latter case.

Conventions. The notation $\to^d$ means convergence in distribution of random variables, $=^d$ denotes equality in law between two random variables or random fields, and $\mathcal{N}(m, \sigma^2)$ is the Gaussian distribution with mean m and variance σ^2. Throughout this manuscript we will assume that $\mathcal{C} \subset \mathbb{T}$ is a given (fixed) curve of length L, and $\gamma : [0, L] \to \mathbb{T}$ an arc-length parametrization of $\mathcal{C}$.

First we formulate the Central Limit Theorem holding under "generic" assumptions.

Theorem 1.1. *Let $\mathcal{C} \subset \mathbb{T}$ be a smooth curve on the torus with nowhere zero curvature of total length L, and $\{n\} \subset S$ such that $\mathcal{N}_n \to +\infty$, and $\{4B_{\mathcal{C}}(\mu_n) - L^2\}$ is bounded away from zero. Then the limiting distribution of the nodal intersections number is Gaussian, i.e.*

$$\frac{\mathcal{Z}_n - \mathbb{E}[\mathcal{Z}_n]}{\sqrt{\mathrm{Var}(\mathcal{Z}_n)}} \xrightarrow{d} Z,$$

where $Z \sim \mathcal{N}(0, 1)$.

Next investigate the (non-generic) situation when the variance is of lower order. In order to formulate Theorem 1.3 we will have to restrict $\mathcal{C}$ to be static (Definition 1), and the sequence $\{n\} \subset S$ to be δ-separated (Definition 2).

Definition 1 (Static curves)**.** *A smooth curve $\mathcal{C} \subset \mathbb{T}$ with nowhere zero curvature is called static if for every probability measure μ on $\mathcal{S}^1$*

$$4B_{\mathcal{C}}(\mu) - L^2 = 0. \tag{1.11}$$

For example, any semi-circle or circle are static [RW14, §7.2]. In Appendix F we show that any smooth curve with nowhere vanishing curvature and invariant under some nontrivial rotation of finite order, is static. We thank D. Panov for pointing this out.

Definition 2 (δ-separated sequences)**.** *Let $\delta > 0$. A sequence $\{n\} \subset S$ of energy levels is δ-separated if*

$$\min_{\lambda \neq \lambda' \in \Lambda_n} \|\lambda - \lambda'\| \gg n^{1/4+\delta}. \tag{1.12}$$

Bourgain and Rudnick [BR11, Lemma 5] showed that "most" n satisfy the δ-separatedness property for every $0<\delta<\frac{1}{4}$. In fact they have a strong quantitative estimate on the number of the exceptions; a precise estimate on these was established more recently [GW16].

Now we introduce some more notation.

Notation 1.2. (1) *For $t\in[0,L]$ set*

$$f(t):=\dot\gamma_1(t)^2-\frac{1}{L}\int_0^L\dot\gamma_1(u)^2\,du=-\dot\gamma_2(t)^2+\frac{1}{L}\int_0^L\dot\gamma_2(u)^2\,du,$$
$$g(t):=\dot\gamma_1(t)\dot\gamma_2(t)-\frac{1}{L}\int_0^L\dot\gamma_1(u)\dot\gamma_2(u)\,du. \tag{1.13}$$

(2) *For a probability measure μ on $\mathcal{S}^1$ define*

$$A_{\mathcal{C}}(\mu):=\int_{\mathcal{S}^1}\int_{\mathcal{S}^1}\left(\int_0^L\langle\theta,\dot\gamma(t)\rangle^2\cdot\langle\theta',\dot\gamma(t)\rangle^2\,dt\right)^2 d\mu(\theta)d\mu(\theta'). \tag{1.14}$$

(3) *Also define the random variable*

$$\mathcal{M}(\mu):=\frac{1}{\sqrt{16A_{\mathcal{C}}(\mu)-L^2}}\left(a_1(\mu)(Z_1^2-1)+a_2(\mu)(Z_2^2-1)+a_3(\mu)Z_1Z_2\right), \tag{1.15}$$

with Z_1,Z_2 i.i.d. standard Gaussian random variables and

$$a_1(\mu):=2(1+\widehat{\mu}(4))\int_0^L f(t)^2\,dt,\qquad a_2(\mu):=2(1-\widehat{\mu}(4))\int_0^L f(t)^2\,dt,$$
$$a_3(\mu):=4\sqrt{1-\widehat{\mu}(4)^2}\int_0^L f(t)g(t)\,dt, \tag{1.16}$$

where

$$\widehat{\mu}(4):=\int_{\mathcal{S}^1}z^{-4}\,d\mu(z)$$

is the 4th Fourier coefficient of μ.

We are now in a position to formulate our second principal theorem.

Theorem 1.3. *Let $\mathcal{C}\subset\mathbb{T}$ be a static curve of length L, and $\{n\}\subset S$ a δ-separated sequence of energies such that $\mathcal{N}_n\to+\infty$.*

(1) *The variance of $\mathcal{Z}_n$ is asymptotic to*

$$\mathrm{Var}(\mathcal{Z}_n)=\frac{n}{4\mathcal{N}_n^2}\left(16A_{\mathcal{C}}(\mu_n)-L^2\right)\cdot(1+o(1)), \tag{1.17}$$

with the leading term $16A_{\mathcal{C}}(\mu_n)-L^2$ bounded away from zero.

(2) *There exists a coupling of the random variables $\mathcal{Z}_n$ and $\mathcal{M}(\mu_n)$ (defined in (1.15)), such that*

$$\mathbb{E}\left[\left|\frac{\mathcal{Z}_n-\mathbb{E}[\mathcal{Z}_n]}{\sqrt{\mathrm{Var}(\mathcal{Z}_n)}}-\mathcal{M}(\mu_n)\right|\right]\to 0,$$

and

$$\frac{\mathcal{Z}_n-\mathbb{E}[\mathcal{Z}_n]}{\sqrt{\mathrm{Var}(\mathcal{Z}_n)}}-\mathcal{M}(\mu_n)\to 0,\quad a.s.$$

In the particular case of $\mathcal{C}$ a full circle, and $\mu_n \Rightarrow d\theta/2\pi$, Theorem 1.3 yields the following via a routine computation:

Example 1.4. *Let $\mathcal{C} \subset \mathbb{T}$ be a full circle of total length L, let $\{n\} \subset S$ be a δ-separated sequence such that $\mathcal{N}_n \to +\infty$ and $\mu_n \Rightarrow \frac{d\theta}{2\pi}$, then for the variance of the nodal intersections number we have*

$$\operatorname{Var}(\mathcal{Z}_n) \sim \frac{L^2}{32} \cdot \frac{n}{\mathcal{N}_n^2},$$

and the limiting distribution is

$$\frac{\mathcal{Z}_n - \mathbb{E}[\mathcal{Z}_n]}{\sqrt{\operatorname{Var}(\mathcal{Z}_n)}} \xrightarrow{d} 1 - \frac{Z_1^2 + Z_2^2}{2},$$

where Z_1, Z_2 are i.i.d. standard Gaussian random variables.

1.4. **Discussion.** Given a length-L toral curve $\mathcal{C}$ and its arc-length parametrization

$$\gamma : [0, L] \to \mathbb{T}$$

we may [RW14, §7.3] associate a complex number $I(\gamma) \in \mathbb{C}$ in the following way. For every $t \in [0, L]$ let $\phi(t) \in [0, 2\pi]$ be the argument of $\dot{\gamma}(t)$, where $\dot{\gamma}(t)$ is viewed as a (unit modulus) complex number, i.e.

$$\dot{\gamma}(t) =: e^{i\phi(t)};$$

we then set

$$I(\gamma) := \int_0^L e^{2i\phi(t)} dt.$$

It was shown [RW14, Corollary 7.2] that $\mathcal{C}$ is static if and only if $I(\gamma) = 0$.

Conversely, if both the imaginary and real parts

$$\Im(I(\gamma)), \Re(I(\gamma)) \neq 0 \tag{1.18}$$

of $I(\gamma)$ do not vanish, then [RW14, Corollary 7.2] show that $4B_{\mathcal{C}}(\mu) - L^2$ is bounded away from zero for *all* probability measures μ on $\mathcal{S}^1$ invariant w.r.t. rotation by $\frac{\pi}{2}$ and complex conjugation. Hence in this case the assumptions of Theorem 1.1 are satisfied for the full sequence $n \in S$ of energy levels. The condition (1.18) is a generic condition on $\mathcal{C}$ understood, for example, in the sense of *prevalence*, see[2] e.g. [OY, §6] (see also Example 3.6) and references therein; hence the scenario described by Theorem 1.1 is "generic", also including almost all energy levels for all *not static* curves.

The two only remaining cases not covered by theorems 1.1 and 1.3 are then:

(1) We have $\Re(I(\gamma)) = 0$, $\Im(I(\gamma)) \neq 0$ and the lattice points corresponding to the subsequence $\{n\} \subseteq S$ converge to

$$\mu_n \Rightarrow \frac{1}{4}(\delta_{\pm 1} + \delta_{\pm i})$$

the Cilleruelo measure.

[2] We wish to thank Michael Benedicks for pointing out [OY] to us.

(2) We have $\Re(I(\gamma)) \neq 0$, $\Im(I(\gamma)) = 0$ and the lattice points corresponding to the subsequence $\{n\} \subseteq S$ converge to

$$\mu_n \Rightarrow \frac{1}{4}\delta_{\pi/4} \star (\delta_{\pm 1} + \delta_{\pm i})$$

the tilted Cilleruelo measure, i.e. Cilleruelo measure rotated by $\frac{\pi}{4}$.

In order to analyse either of these scenarios one needs to understand which of the terms

$$\frac{n}{\mathcal{N}_n} \cdot (4B_{\mathcal{C}} - L^2)$$

(leading term of the 2nd chaotic projection, see (4.5)) or $\frac{n}{\mathcal{N}_n^2}$ (the order of magnitude of the next term) is dominant by order of magnitude, knowing that in this situation $4B_{\mathcal{C}} - L^2$ vanishes asymptotically, at least under the δ-separatedness assumption. Equivalently, whether $4B_{\mathcal{C}} - L^2$ vanishes more rapidly than $\frac{1}{\mathcal{N}_n}$; this would most certainly involve the rate of convergence of μ_n, and it is plausible that one can construct sequences $\{n\}$ observing both kinds of behaviour.

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2. Outline of the paper

2.1. **On the proof of the main results.** The proofs of theorems 1.1 and 1.3 are based on the *chaotic expansion* for the nodal intersections number (see §2.2). We first consider a unit speed parametrization of the curve $\gamma : [0, L] \to \mathcal{C}$, and set

$$f_n : [0, L] \to \mathbb{R}; \qquad t \mapsto T_n(\gamma(t)). \tag{2.1}$$

The map f_n in (2.1) defines a (non-stationary) centered Gaussian process on $[0, L]$ with covariance function

$$r_n(t_1, t_2) := \mathrm{Cov}(f_n(t_1), f_n(t_2)) = \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \cos(2\pi\langle \lambda, \gamma(t_1) - \gamma(t_2)\rangle), \quad t_1, t_2 \in [0, L], \tag{2.2}$$

see (1.4). The number $\mathcal{Z}_n$ of nodal intersections of T_n against $\mathcal{C}$ equals to the number of zero crossings of f_n in the interval $[0, L]$, and we can formally write

$$\mathcal{Z}_n = \int_0^L \delta_0(f_n(t))|f_n'(t)|\, dt, \tag{2.3}$$

where δ_0 is the Dirac delta function.

The random variable $\mathcal{Z}_n$ in (2.3) admits the Wiener-Îto chaotic expansion (see §2.2) of the form

$$\mathcal{Z}_n = \sum_{q=0} \mathcal{Z}_n[2q], \tag{2.4}$$

where the above series converges in the space $L^2(\mathbb{P})$ of random variables with finite variance. In particular, the random variables $\mathcal{Z}_n[2q]$, $\mathcal{Z}_n[2q']$ are orthogonal (uncorrelated) for $q \neq q'$, and $\mathcal{Z}_n[0] = \mathbb{E}[\mathcal{Z}_n]$.

Evaluating the second chaotic projection $\mathcal{Z}_n[2]$ yields that, under the assumptions in Theorem 1.1, the variance of the total number $\mathcal{Z}_n$ of nodal intersections is asymptotic to the variance of $\mathcal{Z}_n[2]$, both being asymptotic to (1.9). The latter and the orthogonality of the Wiener chaoses imply that the distribution of $\mathcal{Z}_n[2]$ dominates the series on the r.h.s. of (2.4), and a Central Limit Theorem result for $\mathcal{Z}_n[2]$ allows to infer the statement of Theorem 1.1.

Assume now that the curve $\mathcal{C}$ is static (Definition 1). The leading term in (1.9) vanishes, and we are left only with an upper bound for the variance of $\mathcal{Z}_n$. To obtain its precise asymptotics we need to inspect the proof of the *approximate* Kac-Rice formula [RW14, Proposition 1.3], and obtain one more term in the expansion (see §5). The main difficulty is how to control "off-diagonal" terms coming from the fourth moment of r, and specifically the quantity

$$\frac{1}{\mathcal{N}_n^2} \sum_{\substack{\lambda_1,\lambda_2,\lambda_3,\lambda_4\in\Lambda_n \\ \lambda_1+\lambda_2+\lambda_3+\lambda_4\neq 0}} \frac{1}{\|\lambda_1+\lambda_2+\lambda_3+\lambda_4\|}. \tag{2.5}$$

We will use some properties of the δ-separated sequences of energy levels (Definition 2) to show that the l.h.s. of (2.5) is $o(1)$ (see Lemma 5.2), and then prove (1.17).

It turns out that the leading term in the chaotic expansion (2.4) is no longer the projection onto the second chaos, but the projection $\mathcal{Z}_n[4]$ onto the fourth chaos. A precise analysis of the latter allows to get its asymptotic (non-Gaussian) distribution in (1.15), thus concluding the proof of Theorem 1.3 by a standard application of [Dud02, Theorem 11.7.1].

2.2. **Chaos expansion.** In this section we compute the chaotic expansion (2.4) for the nodal intersections number $\mathcal{Z}_n$. The reader can refer to [NP12] for a complete discussion on Wiener-Îto chaos expansions.

Recall the definition (2.1) of the random process f_n and the formal expression (2.3). Note that for every $t \in [0, L]$

$$f_n'(t) = \langle \nabla T_n(\gamma(t)), \dot{\gamma}(t)\rangle,$$

where ∇T_n denotes the gradient of T_n and $\dot{\gamma}$ the first derivative of γ. We have [RW14, Lemma 1.1] that

$$\mathrm{Var}(f_n'(t)) = 2\pi^2 n =: \alpha. \tag{2.6}$$

We can then rewrite (2.3) as

$$\mathcal{Z}_n = \sqrt{2\pi^2 n} \int_0^L \delta_0(f_n(t)) |\widetilde{f}_n'(t)|\, dt, \tag{2.7}$$

where

$$\widetilde{f}_n'(t) := \frac{f_n'(t)}{\sqrt{2\pi^2 n}}, \quad t \in [0, L].$$

As f_n is a unit variance process, for every $t \in [0, L]$, $f_n(t)$ and $f_n'(t)$ are independent (see e.g. [RW14, Lemma 2.2]), and so are $f_n(t)$ and $\widetilde{f}_n'(t)$. Hence [KL97, Lemma 2] we have the chaotic expansion (2.4) for (2.7)

$$\mathcal{Z}_n = \sum_{q=0}^{+\infty} \mathcal{Z}_n[2q] = \sqrt{2\pi^2 n} \sum_{q=0}^{+\infty} \sum_{\ell=0}^{q} b_{2q-2\ell} a_{2\ell} \int_0^L H_{2q-2\ell}(f_n(t)) H_{2\ell}(\widetilde{f}_n'(t))\, dt, \tag{2.8}$$

where $\{H_k, k = 0, 1, \dots\}$ denotes the Hermite polynomials [Sze75, §5.5],

$$b_{2q-2\ell} := \frac{1}{(2q-2\ell)!\sqrt{2\pi}} H_{2q-2\ell}(0) \tag{2.9}$$

are the coefficients of the (formal) chaotic expansion of the Dirac mass δ_0, and

$$a_{2\ell} := \sqrt{\frac{2}{\pi}} \frac{(-1)^{\ell+1}}{2^\ell \ell! (2\ell - 1)}, \quad \ell \geq 0 \tag{2.10}$$

are the chaotic coefficients of the absolute value $|\cdot|$. In particular from (2.8) we have for $q \geq 0$

$$\mathcal{Z}_n[2q] = \sqrt{2\pi^2 n} \sum_{\ell=0}^{q} b_{2q-2\ell} a_{2\ell} \int_0^L H_{2q-2\ell}(f_n(t)) H_{2\ell}(\widetilde{f}_n'(t))\, dt. \tag{2.11}$$

2.3. **Plan of the paper.** In §3 we will state a few key propositions instrumental in proving theorems 1.1 and 1.3, in particular, concerning $\mathcal{Z}_n[2]$ in (2.11) for "generic" curves (Theorem 1.1), and $\mathcal{Z}_n[4]$, and also the approximate Kac-Rice formula for static curves (Theorem 1.3). In §4 we will then investigate the second chaotic component for "generic" curves, whereas in §6 the fourth one in the case of static curves. The proof of the approximate Kac-Rice formula for the variance of the number of nodal intersections against static curves will be given in §5.

In the Appendix we will collect some technical results (concerning chaotic components, approximate Kac-Rice formula and bounds for certain summations over lattice points such as (2.5)) and, in particular, in §F a family of static curves will be constructed.

3. Proofs of the main results

The zeroth term in the chaos expansion (2.8) of $\mathcal{Z}_n$ is given by the following lemma (cf. (1.5)).

Lemma 3.1. *For every $n \in S$,*

$$\mathcal{Z}_n[0] = \frac{\sqrt{E_n}}{\pi\sqrt{2}} L. \tag{3.1}$$

Proof. From (2.9), (2.10) and (2.11) we have, for $q = 0$,

$$\mathcal{Z}_n[0] = \sqrt{2\pi^2 n}\, b_0\, a_0\, L = \sqrt{2\pi^2 n} \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}}\, L = \sqrt{2n} L = \frac{\sqrt{E_n}}{\pi\sqrt{2}} L.$$

∎

3.1. **Proof of Theorem 1.1.** To prove Theorem 1.1, we first need to study the asymptotic variance of the second chaotic component $\mathcal{Z}_n[2]$.

Proposition 3.2. *Let $\mathcal{C} \subset \mathbb{T}$ be a smooth curve on the torus with nowhere zero curvature, of total length L, and $\{n\} \subset S$ such that $\mathcal{N}_n \to +\infty$, and $\{4B_{\mathcal{C}}(\mu_n) - L^2\}$ is bounded away from zero. Then*

$$\mathrm{Var}(\mathcal{Z}_n) \sim \mathrm{Var}(\mathcal{Z}_n[2]),$$

i.e. the variance of $\mathcal{Z}_n[2]$ is asymptotic to the variance (1.9) of the nodal intersections $\mathcal{Z}_n$.

In light of Proposition 3.2 to be proven in §4, we are to study the asymptotic distribution of the second chaotic component.

Proposition 3.3. *Let $\mathcal{C} \subset \mathbb{T}$ be a smooth curve on the torus with nowhere zero curvature, of total length L, and $\{n\} \subset S$ such that $\mathcal{N}_n \to +\infty$, and $\{4B_{\mathcal{C}}(\mu_n) - L^2\}$ is bounded away from zero. Then*

$$\frac{\mathcal{Z}_n[2]}{\sqrt{\mathrm{Var}(\mathcal{Z}_n[2])}} \xrightarrow{d} Z,$$

where $Z \sim \mathcal{N}(0,1)$.

Proposition 3.3 will be proven in §4. We are in a position to prove our first main result.

Proof of Theorem 1.1 assuming propositions 3.2 and 3.3. By (2.8) we write

$$\frac{\mathcal{Z}_n - \mathbb{E}[\mathcal{Z}_n]}{\sqrt{\mathrm{Var}(\mathcal{Z}_n)}} = \sum_{q=1}^{+\infty} \frac{\mathcal{Z}_n[2q]}{\sqrt{\mathrm{Var}(\mathcal{Z}_n)}}.$$

Thanks to Proposition 3.2 and the orthogonality of Wiener chaoses we have, as $\mathcal{N}_n \to \infty$,

$$\frac{\mathcal{Z}_n - \mathbb{E}[\mathcal{Z}_n]}{\sqrt{\mathrm{Var}(\mathcal{Z}_n)}} = \frac{\mathcal{Z}_n[2]}{\sqrt{\mathrm{Var}(\mathcal{Z}_n[2])}} + o_{\mathbb{P}}(1),$$

where $o_{\mathbb{P}}(1)$ denotes a sequence of random variables converging to zero in probability. In particular, the distribution of the normalized total number of nodal intersections is asymptotic to the distribution of $\frac{\mathcal{Z}_n[2]}{\sqrt{\mathrm{Var}(\mathcal{Z}_n[2])}}$. The latter and Proposition 3.3 imply the statement of Theorem 1.1.

□

3.2. **Proof of Theorem 1.3.** Our first proposition asserts the variance part (1.17) of Theorem 1.3, to be proven in §5.

Proposition 3.4. *Let $\mathcal{C} \subset \mathbb{T}$ be a static curve of length L, and $\{n\} \subset S$ be a δ-separated sequence such that $\mathcal{N}_n \to +\infty$. Then*

$$\mathrm{Var}(\mathcal{Z}_n) = \frac{n}{4\mathcal{N}_n^2} \left(16A_{\mathcal{C}}(\mu_n) - L^2\right)(1 + o(1)), \tag{3.2}$$

where $A_{\mathcal{C}}(\mu_n)$ is given in (1.14) with $\mu = \mu_n$. Moreover, the leading term $16A_{\mathcal{C}}(\mu_n) - L^2$ in (3.2) is bounded away from zero.

Next we assert that, under the assumptions of Theorem 1.3, the fourth term in the chaotic series (2.8) dominates.

Proposition 3.5. *Let $\mathcal{C} \subset \mathbb{T}$ be a static curve of length L, and $\{n\} \subset S$ a δ-separated sequence such that $\mathcal{N}_n \to +\infty$. Then*

$$\operatorname{Var}(\mathcal{Z}_n[2]) = o\left(\frac{n}{\mathcal{N}_n^2}\right), \tag{3.3}$$

and

$$\operatorname{Var}(\mathcal{Z}_n[4]) \sim \frac{n}{4\mathcal{N}_n^2}\left(16A_{\mathcal{C}}(\mu_n) - L^2\right). \tag{3.4}$$

The above implies that it suffices to study the asymptotic distribution of the fourth chaotic projection.

Proposition 3.6. *Let $\mathcal{C} \subset \mathbb{T}$ be a static curve of length L, and $\{n\} \subset S$ a δ-separated sequence such that $\mathcal{N}_n \to +\infty$, and $\mu_n \Rightarrow \mu$. Then*

$$\frac{\mathcal{Z}_n[4]}{\sqrt{\operatorname{Var}(\mathcal{Z}_n[4])}} \xrightarrow{d} \mathcal{M}(\mu), \tag{3.5}$$

where $\mathcal{M}(\mu)$ is given by (1.15).

Proof of Theorem 1.3 assuming propositions 3.4-3.6. By (2.8) we write

$$\frac{\mathcal{Z}_n - \mathbb{E}[\mathcal{Z}_n]}{\sqrt{\operatorname{Var}(\mathcal{Z}_n)}} = \sum_{q=1}^{+\infty} \frac{\mathcal{Z}_n[2q]}{\sqrt{\operatorname{Var}(\mathcal{Z}_n)}}.$$

Thanks to Proposition 3.4, Proposition 3.5 and the orthogonality of Wiener chaoses we have, as $\mathcal{N}_n \to \infty$,

$$\frac{\mathcal{Z}_n - \mathbb{E}[\mathcal{Z}_n]}{\sqrt{\operatorname{Var}(\mathcal{Z}_n)}} = \frac{\mathcal{Z}_n[4]}{\sqrt{\operatorname{Var}(\mathcal{Z}_n[4])}} + o_{\mathbb{P}}(1),$$

where $o_{\mathbb{P}}(1)$ denotes convergence to zero in probability. In particular, the distribution of the normalized total number of nodal intersections is asymptotic to the one of

$$\frac{\mathcal{Z}_n[4]}{\sqrt{\operatorname{Var}(\mathcal{Z}_n[4])}}.$$

Therefore (3.5) also holds with $\mathcal{Z}_n$ in place of $\mathcal{Z}_n[4]$. In particular, this implies

$$d\left(\frac{\mathcal{Z}_n - \mathbb{E}[\mathcal{Z}_n]}{\sqrt{\operatorname{Var}(\mathcal{Z}_n)}}, \mathcal{M}(\mu_n)\right) \to 0,$$

where d is any metric which metrizes convergence in distribution of random variables, or the Kolmogorov distance (see e.g. [NP12, §C]), since $\mathcal{M}(\mu)$ in (1.15) has absolutely continuous distribution for arbitrary probability measure μ. Theorem 1.3 is then a direct consequence of Theorem 1.1, of [Dud02, Theorem 11.7.1] and of the fact that the sequence $\left\{(\mathcal{Z}_n - \mathbb{E}[\mathcal{Z}_n]) / \sqrt{\operatorname{Var}(\mathcal{Z}_n)}\right\}$ is bounded in $L^2(\mathbb{P})$.

□

4. Proofs of Proposition 3.2 and Proposition 3.3

In this section we investigate the asymptotic behaviour of the second chaotic component $\mathcal{Z}_n[2]$ of $\mathcal{Z}_n$. Our starting point is the decomposition of $\mathcal{Z}_n[2]$ into "diagonal" and "off-diagonal" terms: we define the diagonal term

$$\mathcal{Z}_n^a[2] := \frac{\sqrt{2\pi^2 n}}{2\pi}\frac{1}{\mathcal{N}_n}\, 2\sum_{\lambda\in\Lambda_n^+}(|a_\lambda|^2-1)\left(2\int_0^L\left\langle\frac{\lambda}{|\lambda|},\dot\gamma(t)\right\rangle^2 dt - L\right), \tag{4.1}$$

where if $\sqrt{n}$ is not an integer $\Lambda_n^+ := \{\lambda\in\Lambda_n : \lambda_2>0\}$, otherwise

$$\Lambda_n^+ := \{\lambda\in\Lambda_n : \lambda_2>0\}\cup\{(\sqrt{n},0)\}.$$

The off-diagonal term is

$$\mathcal{Z}_n^b[2] := \frac{\sqrt{2\pi^2 n}}{2\pi}\frac{1}{\mathcal{N}_n}\sum_{\lambda\neq\lambda'} a_\lambda\overline{a_{\lambda'}}\int_0^L\left(2\left\langle\frac{\lambda}{|\lambda|},\dot\gamma(t)\right\rangle\left\langle\frac{\lambda'}{|\lambda|},\dot\gamma(t)\right\rangle-1\right)\mathrm{e}^{i2\pi\langle\lambda-\lambda',\gamma(t)\rangle}\,dt. \tag{4.2}$$

Lemma 4.1. *For every $n\in S$ we have*

$$\mathcal{Z}_n[2] = \mathcal{Z}_n^a[2]+\mathcal{Z}_n^b[2]. \tag{4.3}$$

The proof of Lemma 4.1 will be given in Appendix A.

Proof of Proposition 3.2 assuming Lemma 4.1. Lemma 4.1 yields

$$\mathrm{Var}(\mathcal{Z}_n[2]) = \mathrm{Var}(\mathcal{Z}_n^a[2])+\mathrm{Var}(\mathcal{Z}_n^b[2])+2\mathrm{Cov}(\mathcal{Z}_n^a[2],\mathcal{Z}_n^b[2]). \tag{4.4}$$

Let us first study $\mathrm{Var}(\mathcal{Z}_n^a[2])$. By the definition (4.1) of $\mathcal{Z}_n^a[2]$ we may compute its variance to be

$$\begin{aligned}
\mathrm{Var}(\mathcal{Z}_n^a[2]) &= 2\frac{n}{\mathcal{N}_n^2}\sum_{\lambda\in\Lambda_n^+}\left(2\int_0^L\left\langle\frac{\lambda}{|\lambda|},\dot\gamma(t)\right\rangle^2 dt - L\right)^2\\
&= \frac{n}{\mathcal{N}_n^2}\sum_{\lambda\in\Lambda_n}\left(2\int_0^L\left\langle\frac{\lambda}{|\lambda|},\dot\gamma(t)\right\rangle^2 dt - L\right)^2\\
&= \frac{n}{\mathcal{N}_n}\left(4\int_0^L\int_0^L\frac{1}{\mathcal{N}_n}\sum_{\lambda\in\Lambda_n}\left\langle\frac{\lambda}{|\lambda|},\dot\gamma(t_1)\right\rangle^2\left\langle\frac{\lambda}{|\lambda|},\dot\gamma(t_2)\right\rangle^2 dt_1dt_2 - L^2\right)\\
&= \frac{n}{\mathcal{N}_n}(4B_{\mathcal{C}}(\mu_n)-L^2).
\end{aligned} \tag{4.5}$$

Next we evaluate the variance of $\mathcal{Z}_n^b[2]$: by (4.2) it is given by

$$\begin{aligned}
\mathrm{Var}(\mathcal{Z}_n^b[2]) &= \frac{n}{2\mathcal{N}_n^2}\sum_{\lambda\neq\lambda',\lambda''\neq\lambda'''}\mathbb{E}[a_\lambda\overline{a_{\lambda'}}a_{\lambda''}\overline{a_{\lambda'''}}]\times\\
&\times\int_0^L\left(2\left\langle\frac{\lambda}{|\lambda|},\dot\gamma(t)\right\rangle\left\langle\frac{\lambda'}{|\lambda|},\dot\gamma(t)\right\rangle-1\right)\mathrm{e}^{i2\pi\langle\lambda-\lambda',\gamma(t)\rangle}\,dt\times\\
&\times\int_0^L\left(2\left\langle\frac{\lambda''}{|\lambda''|},\dot\gamma(t)\right\rangle\left\langle\frac{\lambda'''}{|\lambda'''|},\dot\gamma(t)\right\rangle-1\right)\mathrm{e}^{i2\pi\langle\lambda''-\lambda''',\gamma(t)\rangle}\,dt\\
&\le C\frac{n}{2\mathcal{N}_n^2}\sum_{\lambda\neq\lambda'}\left|\left(2\left\langle\frac{\lambda}{|\lambda|},\dot\gamma(t)\right\rangle\left\langle\frac{\lambda'}{|\lambda|},\dot\gamma(t)\right\rangle-1\right)\int_0^L\mathrm{e}^{i2\pi\langle\lambda-\lambda',\gamma(t)\rangle}\,dt\right|^2,
\end{aligned} \tag{4.6}$$

for some $C > 0$.

Using the non-vanishing curvature assumption on $\mathcal{C}$, Van der Corput's Lemma (see e.g. [RW14, Lemma 5.2]) implies that for $\lambda \neq \lambda'$ the inner oscillatory integral on the r.h.s. of (4.6) may be bounded as

$$\int_0^L \mathrm{e}^{i2\pi\langle \lambda-\lambda', \gamma(t)\rangle}\, dt \ll \frac{1}{|\lambda-\lambda'|^{1/2}};$$

this together with (4.6) (and Cauchy-Schwartz) yield

$$\mathrm{Var}(\mathcal{Z}_n^b[2]) \ll \frac{n}{\mathcal{N}_n^2} \sum_{\lambda\neq\lambda'} \frac{1}{|\lambda-\lambda'|}. \tag{4.7}$$

Using the bound

$$\sum_{\lambda\neq\lambda'} \frac{1}{|\lambda-\lambda'|} \ll_\epsilon \mathcal{N}_n^\epsilon$$

for every $\epsilon > 0$ from [RW14, Proposition 5.3] to bound the r.h.s. of (4.7), and comparing the result with (4.5) we have

$$\mathrm{Var}(\mathcal{Z}_n^b[2]) = o(\mathrm{Var}(\mathcal{Z}_n^a[2])). \tag{4.8}$$

Now substituting (4.5), (4.8) into (4.4), and using Cauchy-Schwartz to bound the covariance term in (4.4), finally yield

$$\mathrm{Var}(\mathcal{Z}_n[2]) \sim \mathrm{Var}(\mathcal{Z}_n[2]^a) = \frac{n}{\mathcal{N}_n}(4B_{\mathcal{C}}(\mu_n) - L^2).$$

This taking into account (1.9), concludes the proof of Proposition 3.2.

□

Proof of Proposition 3.3 assuming Lemma 4.1. Thanks to (4.8), it suffices to investigate the asymptotic distribution of $\mathcal{Z}_n^a[2]$ as in (4.1). We have

$$\frac{\mathcal{Z}_n^a[2]}{\sqrt{\mathrm{Var}(\mathcal{Z}_n^a[2])}} = \frac{1}{\sqrt{\mathcal{N}_n/2}} \sum_{\lambda\in\Lambda_n^+} (|a_\lambda|^2 - 1) \frac{2\int_0^L \left\langle \frac{\lambda}{|\lambda|}, \dot\gamma(t)\right\rangle^2 dt - L}{\sqrt{4B_{\mathcal{C}}(\mu_n) - L^2}}. \tag{4.9}$$

Now we can apply Lindeberg's criterion (see e.g. [NP12, Theorem 11.1.2]): since, as $\mathcal{N}_n \to +\infty$, we have

$$\max_{\lambda\in\Lambda_n^+} \frac{1}{\sqrt{\mathcal{N}_n/2}} \left| \frac{2\int_0^L \left\langle \frac{\lambda}{|\lambda|}, \dot\gamma(t)\right\rangle^2 dt - L}{\sqrt{4B_{\mathcal{C}}(\mu_n) - L^2}} \right| \to 0,$$

then

$$\frac{\mathcal{Z}_n^a[2]}{\sqrt{\mathrm{Var}(\mathcal{Z}_n^a[2])}} \xrightarrow{d} Z,$$

where $Z \sim \mathcal{N}(0,1)$.

□

5. Proof of Proposition 3.4

The proof of Proposition 3.4 is inspired by the proof of Theorem 1.2 in [RW14], we refer the reader to [RW14, §1.3] and [RWY15, §1.5] for a complete discussion. We will need to inspect the proof of the *approximate* Kac-Rice formula [RW14, Proposition 1.3], which gives the asymptotic variance of the nodal intersections number in terms of an explicit integral that involves the covariance function $r = r_n$ in (2.2) and a couple of its derivatives

$$r_1 := \frac{\partial}{\partial t_1} r, \quad r_2 := \frac{\partial}{\partial t_2} r, \quad r_{12} := \frac{\partial^2}{\partial t_1 \partial t_2} r,$$

and then study higher order terms.

5.1. Auxiliary lemmas.

Lemma 5.1 (Approximate Kac-Rice, cf. [RW14], Proposition 1.3)**.** *Let $\mathcal{C} \subset \mathbb{T}$ be a static curve of total length L, and $\{n\} \subset S$ a δ-separated sequence such that $\mathcal{N}_n \to +\infty$. Then the intersection number variance is asymptotic to*

$$\begin{aligned} \mathrm{Var}(\mathcal{Z}_n) = n \int_0^L \int_0^L \Big(& \frac{3}{4} r^4 + \frac{1}{12}(r_{12}/\alpha)^4 - \frac{(r_2/\sqrt{\alpha})^4}{4} - \frac{(r_1/\sqrt{\alpha})^4}{4} \\ & + 2(r_{12}/\alpha) r (r_1/\sqrt{\alpha})(r_2/\sqrt{\alpha}) + \frac{(r_1/\sqrt{\alpha})^2 (r_2/\sqrt{\alpha})^2}{2} - \frac{3}{2} r^2 (r_2/\sqrt{\alpha})^2 \\ & - \frac{3}{2} r^2 (r_1/\sqrt{\alpha})^2 + \frac{1}{2}(r_{12}/\alpha)^2 r^2 + \frac{1}{2}(r_2/\sqrt{\alpha})^2 (r_{12}/\alpha)^2 \\ & + \frac{1}{2}(r_1/\sqrt{\alpha})^2 (r_{12}/\alpha)^2 \Big) \, dt_1 dt_2 + o\left(\frac{n}{\mathcal{N}_n^2}\right). \end{aligned} \tag{5.1}$$

To find the asymptotics of the moments of r and its derivatives appearing in the r.h.s. of (5.1), in particular to bound the contribution of "off-diagonal" terms, we will need the following whose proof is given in Appendix D.

Lemma 5.2. *Assume that $\{n\} \subset S$ is a δ-separated sequence, then*

$$\min_{\substack{\lambda_1,\lambda_2,\lambda_3,\lambda_4 \in \Lambda_n \\ \lambda_1+\lambda_2+\lambda_3+\lambda_4 \neq 0}} \|\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4\| \gg n^{2\delta},$$

where the constant involved in the "$\gg$" notation is absolute. In particular,

$$\frac{1}{\mathcal{N}_n^2} \sum_{\substack{\lambda_1,\lambda_2,\lambda_3,\lambda_4 \in \Lambda_n \\ \lambda_1+\lambda_2+\lambda_3+\lambda_4 \neq 0}} \frac{1}{\|\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4\|} = o(1). \tag{5.2}$$

Let us also introduce some more notation:

$$F_{\mathcal{C}}(\mu_n) := \int_0^L \int_0^L \frac{1}{\mathcal{N}_n^2} \sum_{\lambda,\lambda'} \left\langle \frac{\lambda}{|\lambda|}, \dot\gamma(t_1) \right\rangle \left\langle \frac{\lambda}{|\lambda|}, \dot\gamma(t_2) \right\rangle \left\langle \frac{\lambda'}{|\lambda'|}, \dot\gamma(t_1) \right\rangle \left\langle \frac{\lambda'}{|\lambda'|}, \dot\gamma(t_2) \right\rangle dt_1 dt_2. \tag{5.3}$$

We can now state the following.

Lemma 5.3. *If $\mathcal{C} \subset \mathbb{T}$ is a smooth curve with nowhere vanishing curvature, then for δ-separated sequences $\{n\}$ such that $\mathcal{N}_n \to +\infty$, we have*

$$
\begin{aligned}
&1) \int_{\mathcal{C}}\int_{\mathcal{C}} r(t_1,t_2)^4 \, dt_1 dt_2 = 3L^2 \frac{1}{\mathcal{N}_n^2} + o\left(\mathcal{N}_n^{-2}\right), \\
&2) \int_{\mathcal{C}}\int_{\mathcal{C}} \left(\frac{1}{\sqrt{\alpha}} r_1(t_1,t_2)\right)^4 \, dt_1 dt_2 = 3L^2 \frac{1}{\mathcal{N}_n^2} + o(\mathcal{N}_n^{-2}), \\
&3) \int_{\mathcal{C}}\int_{\mathcal{C}} \left(\frac{1}{\alpha} r_{12}(t_1,t_2)\right)^4 \, dt_1 dt_2 = 2^4 \cdot 3 A_{\mathcal{C}}(\mu_n) \frac{1}{\mathcal{N}_n^2} + o(\mathcal{N}_n^{-2}), \\
&4) \int_{\mathcal{C}}\int_{\mathcal{C}} (r_{12}/\alpha) r (r_1/\sqrt{\alpha})(r_2/\sqrt{\alpha}) \, dt_1 dt_2 = -4F_{\mathcal{C}}(\mu_n) \frac{1}{\mathcal{N}_n^2} + o(\mathcal{N}_n^{-2}), \\
&5) \int_{\mathcal{C}}\int_{\mathcal{C}} (r_1/\sqrt{\alpha})^2 (r_2/\sqrt{\alpha})^2 \, dt_1 dt_2 = L^2 \frac{1}{\mathcal{N}_n^2} + 4 \cdot 2 \frac{1}{\mathcal{N}_n^2} F_{\mathcal{C}}(\mu_n) + o(\mathcal{N}_n^{-2}), \\
&6) \int_{\mathcal{C}}\int_{\mathcal{C}} r^2 (r_2/\sqrt{\alpha})^2 \, dt_1 dt_2 = \frac{1}{\mathcal{N}_n^2} L^2 + o(\mathcal{N}_n^{-2}), \\
&7) \int_{\mathcal{C}}\int_{\mathcal{C}} r^2 (r_{12}/\alpha)^2 \, dt_1 dt_2 = 4 \frac{1}{\mathcal{N}_n^2} B_{\mathcal{C}}(\mu_n) + 8 \frac{1}{\mathcal{N}_n^2} F_{\mathcal{C}}(\mu_n) + o(\mathcal{N}_n^{-2}), \\
&8) \int_{\mathcal{C}}\int_{\mathcal{C}} (r_1/\sqrt{\alpha})^2 (r_{12}/\alpha)^2 \, dt_1 dt_2 = 4 \frac{1}{\mathcal{N}_n^2} B_{\mathcal{C}}(\mu_n) + o(\mathcal{N}_n^{-2}),
\end{aligned} \tag{5.4}
$$

where $B_{\mathcal{C}}(\mu_n)$, $A_{\mathcal{C}}(\mu_n)$ are given in (1.10) and (1.14) with $\mu = \mu_n$, respectively, and $F_{\mathcal{C}}(\mu_n)$ in (5.3).

5.2. **Proof of Proposition 3.4.**

Proof. Upon substituting (5.4) into (5.1), we obtain

$$
\operatorname{Var}(\mathcal{Z}_n) = \frac{n}{4\mathcal{N}_n^2} \left(16A_{\mathcal{C}}(\mu_n) + 24B_{\mathcal{C}}(\mu_n) - 7L^2\right) + o\left(\frac{n}{\mathcal{N}_n^2}\right) \tag{5.5}
$$

after some straightforward manipulations. Since $\mathcal{C}$ is static, $4B_{\mathcal{C}}(\mu_n) = L^2$ so that (5.5) is

$$
\operatorname{Var}(\mathcal{Z}_n) = \frac{n}{4\mathcal{N}_n^2} \left(16A_{\mathcal{C}}(\mu_n) - L^2\right) + o\left(\frac{n}{\mathcal{N}_n^2}\right),
$$

which is (3.2).

Now we prove that the leading term $16A_{\mathcal{C}}(\mu_n) - L^2$ is bounded away from zero. From (1.14) with $\mu = \mu_n$, write

$$
A_{\mathcal{C}}(\mu_n) = \int_{\mathcal{S}^1}\int_{\mathcal{S}^1} B(\theta_1,\theta_2)^2 d\mu_n(\theta_1) d\mu_n(\theta_2),
$$

where

$$
B(\theta_1,\theta_2) := \int_0^L \langle \theta_1, \dot{\gamma}(t)\rangle^2 \langle \theta_2, \dot{\gamma}(t)\rangle^2 \, dt.
$$

For every $\theta_1, \theta_2 \in \mathcal{S}^1$, we have

$$
B(\theta_1,\theta_2) + B(\theta_1^\perp,\theta_2) + B(\theta_1,\theta_2^\perp) + B(\theta_1^\perp,\theta_2^\perp) = L. \tag{5.6}
$$

Maximizing and minimizing the function $B(\theta_1,\theta_2)^2+B(\theta_1^\perp,\theta_2)^2+B(\theta_1,\theta_2^\perp)^2+B(\theta_1^\perp,\theta_2^\perp)^2$ under the constraint (5.6) we get

$$\frac{L^2}{4}\le B(\theta_1,\theta_2)^2+B(\theta_1^\perp,\theta_2)^2+B(\theta_1,\theta_2^\perp)^2+B(\theta_1^\perp,\theta_2^\perp)^2\le L^2,\quad \forall\theta_1,\theta_2\in\mathcal{S}^1. \tag{5.7}$$

This also gives a necessary and sufficient criterion for attaining the minimum:

$$B(\theta_1,\theta_2)^2+B(\theta_1^\perp,\theta_2)^2+B(\theta_1,\theta_2^\perp)^2+B(\theta_1^\perp,\theta_2^\perp)^2=\frac{L^2}{4},$$

if and only if

$$B(\theta_1,\theta_2)=B(\theta_1^\perp,\theta_2)=B(\theta_1,\theta_2^\perp)=B(\theta_1^\perp,\theta_2^\perp)=\frac{L}{4}.$$

Now we claim that

$$B(\theta_1,\theta_2)=\frac{L}{4},\quad \forall\theta_1,\theta_2\in\mathcal{S}^1, \tag{5.8}$$

if and only if

$$B(\theta,\theta)=\int_0^L\langle\theta,\dot\gamma(t)\rangle^4\,dt=\frac{L}{4},\quad \forall\theta\in\mathcal{S}^1. \tag{5.9}$$

Let us prove that (5.9) implies (5.8). If (5.9) holds, then we have by Cauchy-Schwartz inequality

$$B(\theta_1,\theta_2)\le\sqrt{B(\theta_1,\theta_1)\cdot B(\theta_2,\theta_2)}=\frac{L}{4},$$

that together with (5.6) allows to infer that (5.9) implies (5.8).

We prove now that (5.9) holds if and only if $\mathcal{C}$ is a straight line segment. For a unit speed parametrization $\gamma:[0,L]\to\mathcal{C}$ we have $\dot\gamma(t)=\mathrm{e}^{i\varphi(t)}$, where $\varphi(t)$ is the angle of the derivative $\dot\gamma(t)$ w.r.t. the coordinate axis. Recalling standard trigonometric identities, we write

$$\begin{aligned}\frac{L}{4}=B(\theta,\theta)&=\int_0^L\cos^4(\varphi(t)-\theta)\,dt\\ &=\frac{3}{8}L+\frac{1}{2}\int_0^L\cos(2(\varphi(t)-\theta))\,dt+\frac{1}{8}\int_0^L\cos(4(\varphi(t)-\theta))\,dt.\end{aligned} \tag{5.10}$$

From [RW14, Corollary 7.2] for static curves $\mathcal{C}$ we have

$$\int_0^L\cos(2(\varphi(t)-\theta))\,dt=0, \tag{5.11}$$

and substituting (5.11) into (5.10) we obtain

$$\int_0^L\cos(4(\varphi(t)-\theta))\,dt=-L. \tag{5.12}$$

Equality (5.12) holds true if and only if for every $t\in[0,L]$

$$\cos(4(\varphi(t)-\theta))=-1,$$

that is, $\mathcal{C}$ is a straight line segment.

Finally, let us prove that $16A_{\mathcal{C}}(\mu) - L^2 = 0$ if and only if $\mathcal{C}$ is a straight line segment. We can use the invariance property of the probability measure μ to write from (1.14)

$$A_{\mathcal{C}}(\mu) = \int_{\mathcal{S}^1/i}\int_{\mathcal{S}^1/i} 4\left(B(\theta_1,\theta_2)^2 + B(\theta_1^\perp,\theta_2)^2 + B(\theta_1,\theta_2^\perp)^2 + B(\theta_1^\perp,\theta_2^\perp)^2\right) d\mu(\theta_1)d\mu(\theta_2),$$

where $\mathcal{S}^1/i$ is a quarter of the circle of measure $\mu(\mathcal{S}^1/i) = 1/4$. By (5.7) we have

$$\frac{L^2}{16} \le A_{\mathcal{C}}(\mu) \le \frac{L^2}{4},$$

which implies

$$0 \le 16A_{\mathcal{C}}(\mu) - L^2 \le 3L^2.$$

Moreover,

$$16A_{\mathcal{C}}(\mu) - L^2 = 0$$

if and only if $B(\theta_1,\theta_2)^2 + B(\theta_1^\perp,\theta_2)^2 + B(\theta_1,\theta_2^\perp)^2 + B(\theta_1^\perp,\theta_2^\perp)^2 = \frac{L^2}{4}$, that was shown to be equivalent to (5.9) to hold, which in turn is equivalent to $\mathcal{C}$ being a straight line segment. □

The rest of this section is dedicated to proving Lemma 5.1.

5.3. **Proof of Lemma 5.1.** Our proof follows along the same path of thought as [RW14, Proposition 1.3], except that we add one more term in the expansion of the 2-point correlation function (5.18), and need to control the error terms using the more difficult to evaluate higher moments. Thereupon we are going to bring the essence of the proof, highlighting the differences, sometimes omitting some details identical to both cases.

5.3.1. *Preliminaries.* The main idea is to divide the square $[0, L]^2$ into small sub-squares and apply the usual Kac-Rice formula to "most" of them, bounding the contribution of the remaining terms (for an extensive discussion see [RW14, §1.3] and [RWY15, §1.5]).

Let us divide the interval $[0, L]$ into small sub-intervals of length roughly $1/\sqrt{E_n}$. To be more precise, let $c_0 > 0$ (chosen as in Lemma B.1) and set $k := \lfloor L\sqrt{E_n}/c_0\rfloor + 1$ and $\delta_0 := L/k$; consider the sub-intervals, for $i = 1, \dots, k$, defined as

$$I_i := [(i-1)\delta_0, i\delta_0].$$

Let us now denote by $\mathcal{Z}_i$ the number of zeros of f_n in I_i, so that

$$\mathcal{Z}_n = \sum_i \mathcal{Z}_i,$$

and therefore

$$\mathrm{Var}(\mathcal{Z}_n) = \sum_{i,j} \mathrm{Cov}(\mathcal{Z}_i, \mathcal{Z}_j). \tag{5.13}$$

Let us define, for $i, j = 1, \dots, k$ the squares

$$S_{i,j} := I_i \times I_j = [(i-1)\delta_0, i\delta_0] \times [(j-1)\delta_0, j\delta_0], \tag{5.14}$$

so that

$$[0, L]^2 = \bigcup_{i,j} S_{i,j}.$$

Definition 3. (cf. [RW14, Definition 4.5] and [RWY15, Definition 2.7]) *We say that* $S_{i,j} \subset [0,L]^2$ *in (5.14) is* singular *if there exists* $(t_1,t_2) \in S_{i,j}$ *such that*

$$r(t_1,t_2) > 1/2.$$

Since $r/\sqrt{E_n}$ is a Lipschitz function with absolute constant, if $S_{i,j}$ is singular, then

$$r(t_1,t_2) > 1/4$$

for every $(t_1,t_2) \in S_{i,j}$. Upon invoking (5.13) we may write

$$\mathrm{Var}(\mathcal{Z}_n) = \sum_{i,j: S_{i,j} \text{ non-sing.}} \mathrm{Cov}(\mathcal{Z}_i, \mathcal{Z}_j) + \sum_{i,j: S_{i,j} \text{ sing.}} \mathrm{Cov}(\mathcal{Z}_i, \mathcal{Z}_j). \tag{5.15}$$

We apply the Kac-Rice formula on the non-singular squares and bound the contribution corresponding to the singular part as follows.

Lemma 5.4. *For a* δ*-separated sequence* $\{n\} \subset S$ *such that* $\mathcal{N}_n \to +\infty$*, we have*

$$\left| \sum_{i,j: S_{i,j} \text{ sing.}} \mathrm{Cov}(\mathcal{Z}_i, \mathcal{Z}_j) \right| = o\left(\frac{n}{\mathcal{N}_n^2} \right). \tag{5.16}$$

Lemma 5.4 will be proved in §B.1. Let us now deal with the non-singular part. If $S_{i,j}$ is non-singular (Definition 3), then $r(t_1,t_2) \neq \pm 1$ for every $(t_1,t_2) \in S_{i,j}$ (note that necessarily $i \neq j$). We can apply Kac-Rice formula ([AW09], [RW14, Proposition 3.2]) to write

$$\mathrm{Cov}(\mathcal{Z}_i, \mathcal{Z}_j) = \int_{I_i} \int_{I_j} \left(K_2(t_1,t_2) - K_1(t_1) K_1(t_2) \right) \, dt_1 dt_2, \tag{5.17}$$

where ([RW14, Lemma 2.1])

$$K_1(t) := \phi_t(0) \cdot \mathbb{E}[f_n'(t) | f_n(t) = 0] = \sqrt{2}\sqrt{n},$$

ϕ_t being the density of the Gaussian random variable $f_n(t)$.

The function $K_2(t_1,t_2)$ is the 2-point correlation function of zeros of the process f_n (see [RW14, §3.2]), defined as follows: for $t_1 \neq t_2$

$$K_2(t_1,t_2) = \phi_{t_1,t_2}(0,0) \cdot \mathbb{E}\left[|f_n'(t_1)| \cdot |f_n'(t_2)| | f_n(t_1) = f_n(t_2) = 0 \right],$$

ϕ_{t_1,t_2} being the probability density of the Gaussian vector $(f_n(t_1), f_n(t_2))$. The function K_2 admits a continuation to a smooth function on the whole of S (see [RW14]), though its values at the diagonal are of no significance for our purposes.

5.3.2. *Proof of Lemma 5.1.* First we need to Taylor expand the 2-point correlation function, which will be proven in §B.2.

Lemma 5.5. *(cf.* [RW14, Proposition 3.2]*) For every $\varepsilon > 0$, the two-point correlation function K_2 satisfies, uniformly for $|r| < 1-\varepsilon$,*

(5.18)

$$\begin{aligned}K_2(t_1,t_2) = \frac{\alpha}{\pi^2}\Big(1 &+ \frac{1}{2}(r_{12}/\alpha)^2 + \frac{1}{2}r^2 - \frac{(r_2/\sqrt{\alpha})^2}{2} - \frac{(r_1/\sqrt{\alpha})^2}{2} + \frac{3}{8}r^4 + \frac{1}{24}(r_{12}/\alpha)^4 \\ &- \frac{(r_2/\sqrt{\alpha})^4}{8} - \frac{(r_1/\sqrt{\alpha})^4}{8} + (r_{12}/\alpha)r(r_1/\sqrt{\alpha})(r_2/\sqrt{\alpha}) \\ &+ \frac{(r_1/\sqrt{\alpha})^2(r_2/\sqrt{\alpha})^2}{4} - \frac{3}{4}r^2(r_2/\sqrt{\alpha})^2 - \frac{3}{4}r^2(r_1/\sqrt{\alpha})^2 + \frac{1}{4}(r_{12}/\alpha)^2r^2 \\ &+ \frac{1}{4}(r_2/\sqrt{\alpha})^2(r_{12}/\alpha)^2 + \frac{1}{4}(r_1/\sqrt{\alpha})^2(r_{12}/\alpha)^2\Big) \\ &+ \alpha O\left(r^6 + (r_1/\sqrt{\alpha})^6 + (r_2/\sqrt{\alpha})^6 + (r_{12}/\alpha)^6\right),\end{aligned}$$

where the constants involved in the "O"-notation depend only on ε.

We can now prove Lemma 5.1.

Proof of Lemma 5.1. Substituting (5.17) and (5.16) into (5.15), valid for non-singular sets, we may write

$$\mathrm{Var}(\mathcal{Z}_n) = \int_{[0,L]^2\setminus B} \left(K_2(t_1,t_2) - K_1(t_1)K_1(t_2)\right)\, dt_1dt_2 + o\left(n\mathcal{N}_n^{-2}\right),$$

where B denotes the union of all singular sets $S_{i,j}$ (see (B.2)). Outside of B, we can use Lemma 5.5 together with Lemma D.2 to bound the error term. Lemma C.1 under the assumption $4B_{\mathcal{C}}(\mu) = L^2$ gives

$$\int_0^L\int_0^L \left((r_{12}/\alpha)^2 + r^2 - (r_2/\sqrt{\alpha})^2 - (r_1/\sqrt{\alpha})^2\right) dt_1dt_2 = o(\mathcal{N}_n^{-2}),$$

and the uniform boundedness of the integrand on the r.h.s. of (5.1) allows to conclude the proof.

□

6. Proof of propositions 3.5 and 3.6

The following lemma is (3.3) of Proposition 3.5.

Lemma 6.1. *Let $\mathcal{C} \subset \mathbb{T}$ be a static smooth curve on the torus with nowhere zero curvature, of total length L. Let $\{n\} \subset S$ be a δ-separated sequence such that $\mathcal{N}_n \to +\infty$, then*

$$\mathrm{Var}(\mathcal{Z}_n[2]) = o\left(\frac{n}{\mathcal{N}_n^2}\right).$$

Proof. First, since $\mathcal{C}$ is static, $\mathcal{Z}_n^a[2] \equiv 0$ by (4.5). Hence we have

$$\mathcal{Z}_n[2] = \mathcal{Z}_n^b[2],$$

by Lemma 4.1. Concerning $\mathcal{Z}_n^b$ we invoke (4.7) to bound

$$\mathrm{Var}\left(\mathcal{Z}_n^b[2]\right) \ll \frac{n}{\mathcal{N}_n^2}\sum_{\lambda\neq\lambda'}\frac{1}{|\lambda-\lambda'|} \ll \frac{n}{\mathcal{N}_n^2}\cdot\frac{\mathcal{N}_n^2}{n^\delta} = o\left(\frac{n}{\mathcal{N}_n^2}\right),$$

since our sequence of energy levels is assumed to be δ-separated (1.12), and (1.6).

□

In what follows we study the limiting distribution of $\mathcal{Z}_n[4]$. We have the following explicit formula for the fourth chaotic component $\mathcal{Z}_n[4]$ upon substituting $q = 2$ in (2.11):

$$\begin{aligned}\mathcal{Z}_n[4] = \sqrt{2\pi^2 n}\Big(b_4 a_0 \int_0^L H_4(f_n(t))\,dt \\ + b_2 a_2 \int_0^L H_2(f_n(t)) H_2(\widetilde{f}_n'(t))\,dt + b_0 a_4 \int_0^L H_4(\widetilde{f}_n'(t))\,dt\Big).\end{aligned} \tag{6.1}$$

6.1. **Preliminaries.** Let us define the random variables ($n \in S$)

$$W_1(n) := \frac{1}{\sqrt{\mathcal{N}_n/2}} \sum_{\lambda \in \Lambda_n^+} (|a_\lambda|^2 - 1), \tag{6.2}$$

and the random processes

$$W_2^t(n) := \frac{1}{\sqrt{\mathcal{N}_n/2}} \sum_{\lambda \in \Lambda_n^+} (|a_\lambda|^2 - 1)\, 2 \left\langle \frac{\lambda}{|\lambda|}, \dot\gamma(t) \right\rangle^2. \tag{6.3}$$

on $t \in [0, L]$. In Lemma 6.3 below we will find an explicit formula for $\mathcal{Z}_n[4]$ in terms of the $W_1(n)$ and $W_2(n)$ and their by-products. To this end we first express each of the three terms in (6.1) in terms of $W_1(n)$ and $W_2(n)$ (proven in Appendix E).

Lemma 6.2. *For a δ-separated sequence $\{n\} \subset S$ such that $\mathcal{N}_n \to +\infty$, we have*

$$\begin{aligned}\int_0^L H_4(f_n(t))\,dt &= X_n^a + X_n^b, \\ \int_0^L H_4(f_n'(t))\,dt &= Y_n^a + Y_n^b, \\ \int_0^L H_2(f_n(t)) H_2(f_n'(t))\,dt &= Z_n^a + Z_n^b,\end{aligned}$$

where for $n \in S$

$$\begin{aligned}X_n^a &:= \frac{6L}{\mathcal{N}_n}(W_1(n)^2 - 1), \\ Y_n^a &:= \frac{6L}{\mathcal{N}_n}\left(\int_0^L W_2^t(n)^2\,dt - 4\frac{1}{\mathcal{N}_n}\sum_\lambda \int_0^L \left\langle \frac{\lambda}{|\lambda|}, \dot\gamma(t)\right\rangle^4 dt\right), \\ Z_n^a &:= \frac{2}{\mathcal{N}_n}\left(W_1(n)\int_0^L W_2^t(n)\,dt - L\right),\end{aligned} \tag{6.4}$$

and as $\mathcal{N}_n \to +\infty$

$$\mathrm{Var}(X_n^b) = o\left(\frac{1}{\mathcal{N}_n^2}\right), \mathrm{Var}(Y_n^b) = o\left(\frac{1}{\mathcal{N}_n^2}\right), \mathrm{Var}(Z_n^b) = o\left(\frac{1}{\mathcal{N}_n^2}\right). \tag{6.5}$$

We then have the following result.

Lemma 6.3. *For a δ-separated sequence $\{n\} \subset S$ such that $\mathcal{N}_n \to +\infty$, we have*

$$\mathcal{Z}_n[4] = \mathcal{Z}_n^a[4] + \mathcal{Z}_n^b[4], \tag{6.6}$$

with

$$\mathcal{Z}_n^a[4] = \frac{\sqrt{2n}}{24}\Big(3X_n^a - Y_n^a - 6Z_n^a\Big), \tag{6.7}$$

where X_n^a, Y_n^a *and* Z_n^a *are as in (6.4),*

$$\mathrm{Var}(\mathcal{Z}_n^a[4]) \sim \frac{n}{4\mathcal{N}_n^2}\Big(16A_{\mathcal{C}}(\mu_n) + 24B_{\mathcal{C}}(\mu_n) - 7L^2\Big), \tag{6.8}$$

and

$$\mathrm{Var}(\mathcal{Z}_n^b[4]) = o\left(\frac{n}{\mathcal{N}_n^2}\right). \tag{6.9}$$

Proof of Lemma 6.3 assuming Lemma 6.2. We have the explicit values

$$\begin{aligned} a_0 &= \sqrt{\frac{2}{\pi}}, \quad a_2 = \sqrt{\frac{2}{\pi}}\frac{1}{2}, \quad a_4 = \sqrt{\frac{2}{\pi}}\frac{-1}{2\cdot 2^2\cdot 3}, \\ b_0 &= \frac{1}{\sqrt{2\pi}}, \quad b_2 = \frac{-1}{2\sqrt{2\pi}}, \quad b_4 = \frac{3}{4!\sqrt{2\pi}} \end{aligned} \tag{6.10}$$

by (2.9) and (2.10). Substituting (6.10) into (6.1), we have

$$\begin{aligned} \mathcal{Z}_n[4] &= \frac{\sqrt{2\pi^2 n}}{\pi}\Big(\frac{3}{4!}\int_0^L H_4(f_n(t))\,dt - \frac{1}{4}\int_0^L H_2(f_n(t))H_2(\widetilde{f}_n'(t))\,dt - \frac{1}{4!}\int_0^L H_4(\widetilde{f}_n'(t))\,dt\Big) \\ &= \frac{\sqrt{2n}}{24}\Big(3X_n^a - Y_n^a - 6Z_n^a + 3X_n^b - Y_n^b - 6Z_n^b\Big), \end{aligned}$$

by Lemma 6.2; the latter is (6.6) with

$$\mathcal{Z}_n^b[4] := \frac{\sqrt{2n}}{24}\left(3X_n^b - Y_n^b - 6Z_n^b\right),$$

and (6.9) follows from (6.5).

Let us now prove (6.8). First, observe that we can write

$$\begin{aligned} 3X_n^a - Y_n^a - 6Z_n^a = \frac{6}{\mathcal{N}_n}\Big[&\frac{1}{\mathcal{N}_n/2}\sum_{\lambda,\lambda'\in\Lambda_n^+}(|a_\lambda|^2-1)(|a_{\lambda'}|^2-1)\times \\ &\times\int_0^L\Big(3 - 4\langle\frac{\lambda}{|\lambda|},\dot\gamma(t)\rangle^2 - 4\langle\frac{\lambda}{|\lambda|},\dot\gamma(t)\rangle^2\langle\frac{\lambda'}{|\lambda'|},\dot\gamma(t)\rangle^2\Big)\,dt \\ &- L + 4\frac{1}{\mathcal{N}_n}\sum_{\lambda\in\Lambda_n}\int_0^L\langle\lambda,\dot\gamma(t)\rangle^4\,dt\Big]. \end{aligned} \tag{6.11}$$

Equality (6.11) and some straightforward computations yield

$$\mathrm{Var}(3X_n^a - Y_n^a - 6Z_n^a) = \frac{36}{\mathcal{N}_n^2}(A_n + B_n + 24C_n), \tag{6.12}$$

where

$$
\begin{aligned}
A_n &= \frac{1}{\mathcal{N}_n^2/4} \sum_{\lambda,\lambda' \in \Lambda_n^+} \left| \int_0^L \left(3 - 4\langle \frac{\lambda}{|\lambda|}, \dot\gamma(t)\rangle^2 - 4\langle \frac{\lambda}{|\lambda|}, \dot\gamma(t)\rangle^2 \langle \frac{\lambda'}{|\lambda'|}, \dot\gamma(t)\rangle^2 \right) dt \right|^2, \\
B_n &= \frac{1}{\mathcal{N}_n^2/4} \sum_{\lambda,\lambda' \in \Lambda_n^+} \int_0^L \left(3 - 4\langle \frac{\lambda}{|\lambda|}, \dot\gamma(t)\rangle^2 - 4\langle \frac{\lambda}{|\lambda|}, \dot\gamma(t)\rangle^2 \langle \frac{\lambda'}{|\lambda'|}, \dot\gamma(t)\rangle^2 \right) dt \\
&\quad \times \int_0^L \left(3 - 4\langle \frac{\lambda'}{|\lambda'|}, \dot\gamma(s)\rangle^2 - 4\langle \frac{\lambda'}{|\lambda'|}, \dot\gamma(s)\rangle^2 \langle \frac{\lambda}{|\lambda|}, \dot\gamma(s)\rangle^2 \right) ds, \\
C_n &= \frac{1}{\mathcal{N}_n^2/4} \sum_{\lambda \in \Lambda_n^+} \left| \int_0^L \left(3 - 4\langle \frac{\lambda}{|\lambda|}, \dot\gamma(t)\rangle^2 - 4\langle \frac{\lambda}{|\lambda|}, \dot\gamma(t)\rangle^4 \right) dt \right|^2.
\end{aligned}
\tag{6.13}
$$

Now, squaring out the respective terms on the l.h.s. of (6.13), with some straightforward computations we obtain

$$
A_n = -9L^2 + 32B_{\mathcal{C}}(\mu_n) + 16A_{\mathcal{C}}(\mu_n), \tag{6.14}
$$

$$
B_n = -5L^2 + 16B_{\mathcal{C}}(\mu_n) + 16A_{\mathcal{C}}(\mu_n), \tag{6.15}
$$

and that

$$
C_n = O\left(\frac{1}{\mathcal{N}_n}\right). \tag{6.16}
$$

Substituting (6.14), (6.15) and (6.16) into (6.12) we obtain

$$
\mathrm{Var}(3X_n^a - Y_n^a - 6Z_n^a) = \frac{36}{\mathcal{N}_n^2}(-14L^2 + 48B_{\mathcal{C}}(\mu_n) + 32A_{\mathcal{C}}(\mu_n) + o(1)), \tag{6.17}
$$

which, in turn, yields (6.8), bearing in mind (6.7) with (6.17).

□

Proof of Proposition 3.5. First, (3.3) is the statement Lemma 6.1, and the leading term $16A_{\mathcal{C}}(\mu_n) - L^2$ was shown to be bounded away from zero as part of Proposition 3.4.

We now turn to proving (3.4). Since the curve is assumed to be static, we have $4B_{\mathcal{C}}(\mu_n) = L^2$. From (6.8) we have

$$
\mathrm{Var}(\mathcal{Z}_n[4]) = \frac{n}{4\mathcal{N}_n^2}\left(16A_{\mathcal{C}}(\mu_n) - L^2\right) + o\left(\frac{n}{\mathcal{N}_n^2}\right),
$$

which is (3.4).

□

6.2. **Proof of Proposition 3.6.**

6.2.1. *Auxiliary results.* Lemma 6.3 implies that for δ-separated sequences $\{n\}$ such that $\mathcal{N}_n \to +\infty$,

$$
\frac{\mathcal{Z}_n[4]}{\sqrt{\mathrm{Var}(\mathcal{Z}_n[4])}} = \frac{\mathcal{Z}_n^a[4]}{\sqrt{\mathrm{Var}(\mathcal{Z}_n^a[4])}} + o_{\mathbb{P}}(1), \tag{6.18}
$$

where $\mathcal{Z}_n^a[4]$ is defined in (6.7) and $o_{\mathbb{P}}(1)$ denotes a sequence of random variables converging to 0 in probability. We may then infer results on the limit distribution of $\mathcal{Z}_n[4]$ from the corresponding results on $\mathcal{Z}_n^a[4]$ for static curves.

Lemma 6.4. *If $4B_{\mathcal{C}}(\mu_n) - L^2 = 0$, then for $W_1(n)$ and $W_2(n)$ defined in (6.2) and (6.3) respectively we have the identity*

$$W_1(n) = \frac{1}{L}\int_0^L W_2^t(n)\,dt. \tag{6.19}$$

Proof. The covariance matrix $\Sigma(n)$ of the Gaussian random vector $\left(W_1(n), \int_0^L W_2^t(n)\,dt\right)$

$$\Sigma(n) = \begin{pmatrix} 1 & L \\ L & 4B_{\mathcal{C}}(\mu_n) \end{pmatrix}$$

satisfies

$$\det \Sigma(n) = 4B_{\mathcal{C}}(\mu_n) - L^2 = 0.$$

Therefore $W_1(n)$ is a multiple of $\int_0^L W_2^t(n)\,dt$, which one may evaluate as given by (6.19). □

Lemma 6.5. *For a static curve $\mathcal{C} \subset \mathbb{T}$ we have*

$$\begin{aligned} \mathcal{Z}_n^a[4] = \frac{\sqrt{2n}}{4\mathcal{N}_n}\Bigg(&-\int_0^L \left(W_2^t(n) - \frac{1}{L}\int_0^L W_2^u(n)\,du\right)^2 dt \\ &+ 4\frac{1}{\mathcal{N}_n}\sum_{\lambda\in\Lambda_n}\int_0^L \left\langle \frac{\lambda}{|\lambda|}, \dot\gamma(t)\right\rangle^4 dt - L\Bigg). \end{aligned} \tag{6.20}$$

Lemma 6.5 follows directly from (6.7) and Lemma 6.4 and is omitted here. In order to study the asymptotic distribution of the r.h.s. of (6.20), we need to study the asymptotic behavior of the sequence of stochastic processes $\{W_2(n)\}_n$, when $\mu_n \Rightarrow \mu$. The natural candidate to be the limiting process is the centred Gaussian process $W_2(\mu) = \{W_2^t(\mu)\}_t$ on $[0, L]$ uniquely defined by the covariance function

$$k_\mu(s,t) := \mathbb{E}[W_2^s(\mu)\cdot W_2^t(\mu)] = 4\int_{\mathcal{S}^1} \langle\theta, \dot\gamma(t)\rangle^2 \langle\theta, \dot\gamma(s)\rangle^2\, d\mu(\theta),$$

$s, t \in [0, L]$. The kernel k_μ above is positive-definite, hence the existence of such a $W_2(\mu)$ is guaranteed by the virtue of Kolmogorov's Theorem.

Proposition 6.6. *Let $\mathcal{C} \subset \mathbb{T}$ be a static curve of length L, and $\{n\} \subset S$ a δ-separated sequence such that $\mathcal{N}_n \to +\infty$, and $\mu_n \Rightarrow \mu$. Then*

$$\frac{\mathcal{Z}_n^a[4]}{\sqrt{\mathrm{Var}(\mathcal{Z}_n^a[4])}} \xrightarrow{d} \mathcal{I}(\mu), \tag{6.21}$$

where

$$\mathcal{I}(\mu) := \frac{-\int_0^L \left(W_2^t(\mu) - \frac{1}{L}\int_0^L W_2^u(\mu)\,du\right)^2 dt + \left(4\int_{\mathcal{S}^1}\int_0^L \langle\theta, \dot\gamma(t)\rangle^4\, dt d\mu(\theta) - L\right)}{\sqrt{16A_{\mathcal{C}}(\mu) - L^2}}, \tag{6.22}$$

and $A_{\mathcal{C}}(\mu)$ is as in (1.14).

Proof. Thanks to Lemma 6.5 and (6.8) for static curves, it suffices to prove that the stochastic processes $W_2(n)$ weakly converge to $W_2(\mu)$. It is immediate that the finite dimensional distributions of $W_2(n)$ convergence to those of $W_2(\mu)$, so that a standard application of Prokhorov's Theorem (see e.g. [Dud02]) allows to conclude the proof. □

6.2.2. *Proof of Proposition 3.6.* Before giving a proof for Proposition 3.6, we need to introduce some more notation. Let us think of a probability measure μ on $\mathcal{S}^1$ as a probability measure on $[0, 2\pi]$. There exists a centered Gaussian process $\widetilde{B} = \widetilde{B}(\mu)$ indexed by $[0, 2\pi]$ such that

$$\mathrm{Cov}\left(\int_0^{2\pi} 1_A(a)\, d\widetilde{B}_a, \int_0^{2\pi} 1_B(a)\, d\widetilde{B}_a\right) = \mu(A \cap B), \tag{6.23}$$

for any $A, B \in \mathcal{B}([0, 2\pi])$ – the Borel σ-field on the interval $[0, 2\pi]$, 1_E denoting the indicator function of the set $E \in \mathcal{B}([0, 2\pi])$.

Let us also introduce the following three centred, jointly Gaussian, random variables

$$N_1 := \int_0^{2\pi} (\cos a)^2\, d\widetilde{B}_a, \quad N_2 := \int_0^{2\pi} (\sin a)^2\, d\widetilde{B}_a, \quad N_3 := \int_0^{2\pi} \cos a \cdot \sin a\, d\widetilde{B}_a, \tag{6.24}$$

defined as stochastic integrals on $[0, 2\pi]$ with respect to $\widetilde{B}$.

Proof of Proposition 3.6. First, let us denote by $G(d\mu)$ a Gaussian measure on $\mathcal{S}^1$ with control μ (see e.g. [NP12]), i.e. a centered Gaussian family

$$G = \{G(A) : A \in \mathcal{B}(\mathcal{S}^1)\}$$

such that

$$\mathbb{E}[G(A) \cdot G(B)] = \mu(A \cap B).$$

We have the following equality in law of stochastic processes

$$W_2^{\cdot} = 2 \int_{\mathcal{S}^1} G(\mu(d\theta)) \langle \theta, \dot\gamma(\cdot)\rangle^2, \tag{6.25}$$

where the r.h.s. of (6.25) denotes the Wiener-Itô integral on the unit circle with respect to the Gaussian measure $G(d\mu)$.

From (6.25) we deduce that

$$W_2^t - \frac{1}{L}\int_0^L W_2^u\, du = 2 \int_{\mathcal{S}^1} G(\mu(d\theta)) \left(\langle \theta, \dot\gamma(t)\rangle^2 - \frac{1}{L}\int_0^L \langle \theta, \dot\gamma(u)\rangle^2\, du\right), \tag{6.26}$$

again equality in law. We parameterize the unit circle as

$$[0, 2\pi] \ni a \mapsto (\cos a, \sin a) \in \mathcal{S}^1.$$

Recalling (6.23) and (6.24), we have by (6.26)

(6.27)
$$\begin{aligned}
&W_2^t - \frac{1}{L}\int_0^L W_2^u\,du \\
&\overset{d}{=} 2\int_0^{2\pi} d\widetilde{B}_a \left((\cos a\cdot\dot\gamma_1(t)+\sin a\cdot\dot\gamma_2(t))^2 - \frac{1}{L}\int_0^L (\cos a\cdot\dot\gamma_1(u)+\sin a\cdot\dot\gamma_2(u))^2\,du \right) \\
&= 2\int_0^{2\pi} d\widetilde{B}_a(\cos a)^2\left(\dot\gamma_1(t)^2 - \frac{1}{L}\int_0^L \dot\gamma_1(u)^2\,du\right) \\
&+ 2\int_0^{2\pi} d\widetilde{B}_a(\sin a)^2\left(\dot\gamma_2(t)^2 - \frac{1}{L}\int_0^L \dot\gamma_2(u)^2\,du\right) \\
&+ 4\int_0^{2\pi} d\widetilde{B}_a \cos a\cdot\sin a\left(\dot\gamma_1(t)\dot\gamma_2(t) - \frac{1}{L}\int_0^L \dot\gamma_1(u)\dot\gamma_2(u)\,du\right) \\
&= 2N_1 f(t) - 2N_2 f(t) + 4N_3 g(t),
\end{aligned}$$

where f and g are as in (1.13). The covariance matrix of $N := (N_1, N_2, N_3)$ is

$$\Sigma_N := \begin{pmatrix} \frac{3+\widehat{\mu}(4)}{8} & \frac{1-\widehat{\mu}(4)}{8} & 0 \\ \frac{1-\widehat{\mu}(4)}{8} & \frac{3+\widehat{\mu}(4)}{8} & 0 \\ 0 & 0 & \frac{1-\widehat{\mu}(4)}{8} \end{pmatrix}.$$

Note that N_3 is independent of N_1 and N_2.

The eigenvalues of Σ_N are $\frac{1+\widehat{\mu}(4)}{4}$, $\frac{1}{2}$ and $\frac{1-\widehat{\mu}(4)}{8}$, it is then immediate that

$$\begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} \overset{d}{=} \begin{pmatrix} \frac{1}{2}Z_1 + \frac{1}{\sqrt{2}}\sqrt{\frac{1+\widehat{\mu}(4)}{4}}Z_2 \\ \frac{1}{2}Z_1 - \frac{1}{\sqrt{2}}\sqrt{\frac{1+\widehat{\mu}(4)}{4}}Z_2 \\ \sqrt{\frac{1-\widehat{\mu}(4)}{8}}Z_3 \end{pmatrix}, \tag{6.28}$$

where Z_1, Z_2, Z_3 are i.i.d. standard Gaussian random variables. Substituting (6.28) into (6.27), thanks to Proposition 6.6, we can conclude the proof.

□

Appendix A. Computations for the 2nd chaotic component

Proof of Lemma 4.1. From (2.11) with $q = 1$ we have

$$\mathscr{Z}_n[2] = \sqrt{2\pi^2 n}\left(b_2 a_0 \int_0^L H_2(f_n(t))\,dt + b_0 a_2 \int_0^L H_2(\widetilde{f}_n'(t))\,dt\right). \tag{A.1}$$

We evaluate the first summand in the r.h.s. of (A.1) to be

$$\begin{aligned}
\int_0^L H_2(f_n(t))\,dt &= \int_0^L (f_n(t)^2 - 1)\,dt = \int_0^L (T_n(\gamma(t))^2 - 1)\,dt \\
&= \frac{1}{\mathcal{N}_n}\sum_{\lambda,\lambda'\in\Lambda_n} a_\lambda \overline{a_{\lambda'}} \int_0^L \mathrm{e}^{i2\pi\langle\lambda-\lambda',\gamma(t)\rangle}\,dt - L.
\end{aligned} \tag{A.2}$$

Now we are left to simplify the second summand in the right-hand side of (A.1). We get

(A.3)

$$\begin{aligned}\int_0^L H_2(\widetilde{f}_n'(t))\,dt &= \int_0^L \left(\left(\widetilde{f}_n'(t)\right)^2 - 1\right)\,dt = \frac{1}{2\pi^2 n}\int_0^L \langle \nabla T_n(\gamma(t)), \dot\gamma(t)\rangle^2\,dt - L\\ &= \frac{1}{2\pi^2 n}\int_0^L \frac{4\pi^2}{\mathcal{N}_n}\sum_{\lambda,\lambda'\in\Lambda_n} a_\lambda\overline{a_{\lambda'}}\langle\lambda,\dot\gamma(t)\rangle\langle\lambda',\dot\gamma(t)\rangle \mathrm{e}^{i2\pi\langle\lambda-\lambda',\gamma(t)\rangle}\,dt - L\\ &= 2\frac{1}{\mathcal{N}_n}\sum_{\lambda,\lambda'\in\Lambda_n} a_\lambda\overline{a_{\lambda'}}\int_0^L \left\langle\frac{\lambda}{|\lambda|},\dot\gamma(t)\right\rangle\left\langle\frac{\lambda'}{|\lambda|},\dot\gamma(t)\right\rangle \mathrm{e}^{i2\pi\langle\lambda-\lambda',\gamma(t)\rangle}\,dt - L.\end{aligned}$$

Using (A.2) and (A.3) in (A.1) we obtain, taking into account that

$$b_2a_0 = -1/(2\pi) = -b_0a_2,$$

(A.4)

$$\begin{aligned}\mathscr{Z}_n[2] &= \sqrt{2\pi^2 n}\left(b_2a_0\int_0^L H_2(f_n(t))\,dt + b_0a_2\int_0^L H_2(\widetilde{f}_n'(t))\,dt\right)\\ &= \frac{\sqrt{2\pi^2 n}}{2\pi}\left(-\frac{1}{\mathcal{N}_n}\sum_{\lambda,\lambda'\in\Lambda_n} a_\lambda\overline{a_{\lambda'}}\int_0^L \mathrm{e}^{i2\pi\langle\lambda-\lambda',\gamma(t)\rangle}\,dt + L\right)\\ &+ \frac{\sqrt{2\pi^2 n}}{2\pi}\left(2\frac{1}{\mathcal{N}_n}\sum_{\lambda,\lambda'\in\Lambda_n} a_\lambda\overline{a_{\lambda'}}\int_0^L \left\langle\frac{\lambda}{|\lambda|},\dot\gamma(t)\right\rangle\left\langle\frac{\lambda'}{|\lambda|},\dot\gamma(t)\right\rangle \mathrm{e}^{i2\pi\langle\lambda-\lambda',\gamma(t)\rangle}\,dt - L\right)\\ &= \frac{\sqrt{2\pi^2 n}}{2\pi}\left(-L\frac{1}{\mathcal{N}_n}\sum_{\lambda\in\Lambda_n}|a_\lambda|^2 + L + 2\frac{1}{\mathcal{N}_n}\sum_{\lambda\in\Lambda_n}|a_\lambda|^2\int_0^L\left\langle\frac{\lambda}{|\lambda|},\dot\gamma(t)\right\rangle^2 dt - L\right)\\ &+ \frac{\sqrt{2\pi^2 n}}{2\pi}\frac{1}{\mathcal{N}_n}\sum_{\lambda\neq\lambda'} a_\lambda\overline{a_{\lambda'}}\int_0^L\left(2\left\langle\frac{\lambda}{|\lambda|},\dot\gamma(t)\right\rangle\left\langle\frac{\lambda'}{|\lambda|},\dot\gamma(t)\right\rangle - 1\right)\mathrm{e}^{i2\pi\langle\lambda-\lambda',\gamma(t)\rangle}\,dt.\end{aligned}$$

Now, thanks to [RW08, Lemma 2,3], we can write (A.4) as

$$\begin{aligned}\mathscr{Z}_n[2] &= \frac{\sqrt{2\pi^2 n}}{2\pi}\frac{1}{\mathcal{N}_n}\sum_{\lambda\in\Lambda_n}(|a_\lambda|^2 - 1)\left(2\int_0^L\left\langle\frac{\lambda}{|\lambda|},\dot\gamma(t)\right\rangle^2 dt - L\right)\\ &+ \frac{\sqrt{2\pi^2 n}}{2\pi}\frac{1}{\mathcal{N}_n}\sum_{\lambda\neq\lambda'} a_\lambda\overline{a_{\lambda'}}\int_0^L\left(2\left\langle\frac{\lambda}{|\lambda|},\dot\gamma(t)\right\rangle\left\langle\frac{\lambda'}{|\lambda|},\dot\gamma(t)\right\rangle - 1\right)\mathrm{e}^{i2\pi\langle\lambda-\lambda',\gamma(t)\rangle}\,dt,\end{aligned}$$

which equals (4.3).

□

Appendix B. Auxiliary results for the approximate Kac-Rice formula

B.1. **Contribution of singular squares.** In this section we prove Lemma 5.4, following [RW14, §4]. We first need the following result, whose proof is similar to the proof of Lemma 2.4 in [RWY15] and hence omitted.

Lemma B.1. *There exists a constant $c_0 > 0$ sufficiently small such that for every $t_1, t_2 \in [0, L]$ with*

$$0 < |t_1 - t_2| < c_0/\sqrt{E_n},$$

we have

$$r(t_1, t_2) \neq \pm 1.$$

Proposition 4.4 in [RW14] also asserts that for $t_1 \in [0, L]$ and $0 < |t_2 - t_1| < c_0/\sqrt{n}$ one has the uniform estimate

$$K_2(t_1, t_2) = O(n). \tag{B.1}$$

Lemma B.1 and (B.1) allows to prove the following as in the proof of [RW14, Proposition 4.1].

Lemma B.2. *We have*

$$\mathrm{Var}(\mathcal{Z}_i) = O(1),$$

uniformly for $i \leq k$, where the constants involved in the "O"-notation depend only on c_0.

The following follows upon applying Cauchy-Schwartz with Lemma B.2.

Corollary B.3. *We have*

$$\mathrm{Cov}(\mathcal{Z}_i, \mathcal{Z}_j) = O(1),$$

uniformly for $i, j \leq k$, where the constants involved in the "O"-notation depend only on c_0.

Let us now denote by B the union of all singular cubes

$$B = \bigcup_{S_{i,j} \text{ singular}} S_{i,j}. \tag{B.2}$$

The proof of the following is similar to the proof of Lemma 4.7 in [RW14].

Lemma B.4. *The total area of the singular set is, for a δ-separated sequence $\{n\} \subset S$ such that $\mathcal{N}_n \to +\infty$,*

$$meas(B) = o\left(\mathcal{N}_n^{-2}\right).$$

Proof. We apply Chebyshev-Markov inequality to the measure of B to get

$$\mathrm{meas}(B) \ll \int_0^L r(t_1, t_2)^6 \, dt_1 dt_2.$$

Hence bounding the measure of the singular set B is reduced to bounding the 6th moment and its derivatives. An application of Lemma C.2 below then concludes the proof of Lemma B.4. □

We are now in a position to prove Lemma 5.4.

Proof of Lemma 5.4. Since the number of singular cubes is $O\left(E_n \mathrm{meas}(B)\right)$, Corollary B.3 bounds the contribution of singular cubes in (5.15) as

$$\left| \sum_{i,j: S_{i,j} \text{ sing.}} \mathrm{Cov}(\mathcal{Z}_i, \mathcal{Z}_j) \right| = O\left(E_n \cdot \mathrm{meas}(B)\right); \tag{B.3}$$

The statement of Lemma 5.4 then follows upon an application of Lemma B.4. □

B.2. **Taylor expansion for the two-point correlation function.**

Proof of Lemma 5.5. Recall that $\alpha = 2\pi^2 n$ as in (2.6). Lemma 3.2 in [RW14] asserts that

$$K_2 = \frac{1}{\pi^2(1-r^2)^{3/2}} \cdot \mu \cdot (\sqrt{1-\rho^2} + \rho \arcsin \rho), \tag{B.4}$$

where

$$\mu = \sqrt{\alpha(1-r^2) - r_1^2} \cdot \sqrt{\alpha(1-r^2) - r_2^2},$$

$$\rho = \frac{r_{12}(1-r^2) + r r_1 r_2}{\sqrt{\alpha(1-r^2) - r_1^2} \cdot \sqrt{\alpha(1-r^2) - r_2^2}}.$$

We now set

$$G(\rho) := \frac{2}{\pi}\left(\sqrt{1-\rho^2} + \rho \arcsin \rho\right),$$

then, as $\rho \to 0$,

$$G(\rho) = \frac{2}{\pi}\left(1 + \frac{\rho^2}{2} + \frac{\rho^4}{24} + O(\rho^6)\right). \tag{B.5}$$

Let us now expand μ around 0.

$$\begin{aligned} \mu = \alpha\Big(1 - r^2 - \frac{(r_2/\sqrt{\alpha})^2}{2} - \frac{(r_1/\sqrt{\alpha})^2}{2} - \frac{(r_2/\sqrt{\alpha})^4}{8} \\ - \frac{(r_1/\sqrt{\alpha})^4}{8} + \frac{(r_1/\sqrt{\alpha})^2 (r_2/\sqrt{\alpha})^2}{4} + O((r^2 + (r_2/\sqrt{\alpha})^2)^3)\Big). \end{aligned} \tag{B.6}$$

Moreover,

$$\frac{1}{(1-r^2)^{3/2}} = 1 + \frac{3}{2}r^2 + \frac{15}{8}r^4 + O(r^6). \tag{B.7}$$

Let us now Taylor expand ρ.

$$\begin{aligned} \rho =& (r_{12}/\alpha) + (r_{12}/\alpha)\frac{(r_2/\sqrt{\alpha})^2}{2} + (r_{12}/\alpha)\frac{(r_1/\sqrt{\alpha})^2}{2} + r(r_1/\sqrt{\alpha})(r_2/\sqrt{\alpha}) \\ &+ O(r_{12}(r^2 + (r_2/\sqrt{\alpha})^2)^2). \end{aligned} \tag{B.8}$$

Substituting (B.8) into (B.5) we obtain

(B.9)

$$\begin{aligned} G(\rho) = \frac{2}{\pi}\Big(1 + \frac{1}{2}\left((r_{12}/\alpha)^2 + (r_{12}/\alpha)^2 (r_2/\sqrt{\alpha})^2) + (r_{12}/\alpha)^2 (r_1/\sqrt{\alpha})^2)\right) \\ + 2(r_{12}/\alpha) r (r_1/\sqrt{\alpha})(r_2/\sqrt{\alpha}) + \frac{1}{24}(r_{12}/\alpha)^4 + O(r_{12}^2(r^2 + (r_2/\sqrt{\alpha})^2)^2)\Big). \end{aligned}$$

Finally, using (B.7), (B.6) and (B.9) in (B.4) we get

$$
\begin{aligned}
K = \frac{\alpha}{\pi^2}\Big(1 &+ \frac{1}{2}(r_{12}/\alpha)^2 + \frac{1}{2}r^2 - \frac{(r_2/\sqrt{\alpha})^2}{2} - \frac{(r_1/\sqrt{\alpha})^2}{2} \\
&+ \frac{3}{8}r^4 + \frac{1}{24}(r_{12}/\alpha)^4 - \frac{(r_2/\sqrt{\alpha})^4}{8} - \frac{(r_1/\sqrt{\alpha})^4}{8} \\
&+ (r_{12}/\alpha)r(r_1/\sqrt{\alpha})(r_2/\sqrt{\alpha}) + \frac{(r_1/\sqrt{\alpha})^2(r_2/\sqrt{\alpha})^2}{4} \\
&- \frac{3}{2}r^2\frac{(r_2/\sqrt{\alpha})^2}{2} - \frac{3}{2}r^2\frac{(r_1/\sqrt{\alpha})^2}{2} + \frac{1}{2}(r_{12}/\alpha)^2\frac{1}{2}r^2 \\
&+ \frac{1}{4}(r_2/\sqrt{\alpha})^2(r_{12}/\alpha)^2 + \frac{1}{4}(r_1/\sqrt{\alpha})^2(r_{12}/\alpha)^2 \\
&+ O(r^6 + (r_1/\sqrt{\alpha})^6 + (r_2/\sqrt{\alpha})^6 + (r_{12}/\alpha)^6)\Big),
\end{aligned}
$$

which is (5.18).

□

Appendix C. Moments of r and its derivatives

Lemma C.1. *If $\mathcal{C} \subset \mathbb{T}$ is a smooth curve with nowhere vanishing curvature, then for a δ-separated sequence $\{n\}$ such that $\mathcal{N}_n \to +\infty$, we have*

$$
\begin{aligned}
&1) \int_0^L \int_0^L r(t_1,t_2)^2\, dt_1 dt_2 = \frac{L^2}{\mathcal{N}_n} + o\left(\frac{1}{\mathcal{N}_n^2}\right), \\
&2) \int_0^L \int_0^L \left|\frac{1}{\sqrt{4\pi^2 n}} r_1(t_1,t_2)\right|^2 dt_1 dt_2 = \frac{L^2}{2\mathcal{N}_n} + o\left(\frac{1}{\mathcal{N}_n^2}\right), \\
&3) \int_0^L \int_0^L \left|\frac{1}{4\pi^2 n} r_{12}(t_1,t_2)\right|^2 dt_1 dt_2 = \frac{B_{\mathcal{C}}(\mu_n)}{\mathcal{N}_n} + o\left(\frac{1}{\mathcal{N}_n^2}\right),
\end{aligned}
$$

where $B_{\mathcal{C}}(\mu_n)$ is given in (1.10).

Proof. Let us start with 1). Squaring out, we have (on separating the diagonal $\lambda = \lambda'$)

$$
\int_0^L \int_0^L r(t_1,t_2)^2\, dt_1 dt_2 = \frac{L^2}{\mathcal{N}_n} + \frac{1}{\mathcal{N}_n^2} \sum_{\lambda \neq \lambda'} \left| \int_0^L \mathrm{e}^{i2\pi\langle \lambda-\lambda', \gamma(t)\rangle}\, dt \right|^2. \tag{C.1}
$$

Lemma 5.2 in [RW14] yields

$$
\int_0^L \mathrm{e}^{i2\pi\langle \lambda-\lambda', \gamma(t)\rangle}\, dt \ll \frac{1}{|\lambda-\lambda'|^{1/2}},
$$

therefore the contribution of the off-diagonal pairs in (C.1) is

$$
\frac{1}{\mathcal{N}_n^2} \sum_{\lambda \neq \lambda'} \left| \int_0^L \mathrm{e}^{i2\pi\langle \lambda-\lambda', \gamma(t)\rangle}\, dt \right|^2 \ll \frac{1}{\mathcal{N}_n^2} \sum_{\lambda \neq \lambda'} \frac{1}{|\lambda-\lambda'|}.
$$

Condition (1.12) then allows to conclude part 1). The remaining terms can be dealt in a similar way to the proof of [RW14, Proposition 5.1], taking into account (1.12) to control the contribution of the “off-diagonal terms”.

□

Proof of Lemma 5.3. Let us prove 1). We can write

$$\begin{aligned}\int_{\mathcal{C}}\int_{\mathcal{C}} r(t_1,t_2)^4\,dt_1dt_2 &= \int_{\mathcal{C}}\int_{\mathcal{C}}\left(\frac{1}{\mathcal{N}_n}\sum_{\lambda}\mathrm{e}^{i2\pi\langle\lambda,\gamma(t_1)-\gamma(t_2)\rangle}\right)^4 dt_1dt_2\\ &= \int_{\mathcal{C}}\int_{\mathcal{C}}\frac{1}{\mathcal{N}_n^4}\sum_{\lambda_1,\dots,\lambda_4}\mathrm{e}^{i2\pi\langle\lambda_1-\lambda_2+\lambda_3-\lambda_4,\gamma(t_1)-\gamma(t_2)\rangle}\,dt_1dt_2\end{aligned}\tag{C.2}$$

Now let us split the summation on the r.h.s. of (C.2) into two sums: one over quadruples $(\lambda_1,\dots,\lambda_4)$ such that $\lambda_1+\dots+\lambda_4=0$ and the other one over quadruples $(\lambda_1,\dots,\lambda_4)$ such that $\lambda_1+\dots+\lambda_4\neq 0$:

$$\begin{aligned}\int_{\mathcal{C}}\int_{\mathcal{C}} r(t_1,t_2)^4\,dt_1dt_2 &= L^2\frac{|S_4(n)|}{\mathcal{N}_n^4}+\int_{\mathcal{C}}\int_{\mathcal{C}}\frac{1}{\mathcal{N}_n^4}\sum_{\lambda_1-\lambda_2+\lambda_3-\lambda_4\neq 0}\mathrm{e}^{i2\pi\langle\lambda_1-\lambda_2+\lambda_3-\lambda_4,\gamma(t_1)-\gamma(t_2)\rangle}\,dt_1dt_2\\ &= L^2\frac{|S_4(n)|}{\mathcal{N}_n^4}+\frac{1}{\mathcal{N}_n^4}\sum_{\lambda_1-\lambda_2+\lambda_3-\lambda_4\neq 0}\left|\int_{\mathcal{C}}\mathrm{e}^{i2\pi\langle\lambda_1-\lambda_2+\lambda_3-\lambda_4,\gamma(t_1)\rangle}\,dt_1\right|^2,\end{aligned}\tag{C.3}$$

where

$$S_4(n) := \{(\lambda_1,\dots,\lambda_4)\in\Lambda_n^4 : \lambda_1+\dots+\lambda_4=0\}.$$

Recall that [KKW13]

$$|S_4(n)| = 3\mathcal{N}_n(\mathcal{N}_n-1),\tag{C.4}$$

and moreover [RW14, Lemma 5.2]

$$\frac{1}{\mathcal{N}_n^4}\sum_{\lambda_1-\lambda_2+\lambda_3-\lambda_4\neq 0}\left|\int_0^L\mathrm{e}^{i2\pi\langle\lambda_1-\lambda_2+\lambda_3-\lambda_4,\gamma(t_1)\rangle}\,dt_1\right|^2 \ll \frac{1}{\mathcal{N}_n^4}\sum_{\lambda_1-\lambda_2+\lambda_3-\lambda_4\neq 0}\frac{1}{|\lambda_1+\dots+\lambda_4|}.\tag{C.5}$$

Substituting (C.4) and (C.5) into (C.3) we get

$$\int_{\mathcal{C}}\int_{\mathcal{C}} r(t_1,t_2)^4\,dt_1dt_2 = 3L^2\frac{1}{\mathcal{N}_n^2}+O\left(\frac{1}{\mathcal{N}_n^4}\sum_{\lambda_1-\lambda_2+\lambda_3-\lambda_4\neq 0}\frac{1}{|\lambda_1+\dots+\lambda_4|}\right).\tag{C.6}$$

Lemma 5.2, in particular (2.5), then allows to estimate the error on the r.h.s. of (C.6), thus concluding the proof of 1).

Let us now deal with 4).

(C.7)

$$
\begin{aligned}
&\int_{\mathcal{C}}\int_{\mathcal{C}}(r_{12}/\alpha)r(r_1/\sqrt{\alpha})(r_2/\sqrt{\alpha})\,dt_1dt_2\\
&= 4\cdot\int_{\mathcal{C}}\int_{\mathcal{C}}\frac{1}{\mathcal{N}_n^4}\sum_{\lambda_1,\dots,\lambda_4}\left\langle\frac{\lambda_1}{|\lambda_1|},\dot\gamma(t_1)\right\rangle\left\langle\frac{\lambda_1}{|\lambda_1|},\dot\gamma(t_2)\right\rangle\left\langle\frac{\lambda_3}{|\lambda_3|},\dot\gamma(t_1)\right\rangle\left\langle\frac{\lambda_4}{|\lambda_4|},\dot\gamma(t_2)\right\rangle\times\\
&\times \mathrm{e}^{i2\pi\langle\lambda_1+\lambda_2+\lambda_3+\lambda_4,\gamma(t_1)-\gamma(t_2)\rangle}\,dt_1dt_2\\
&= -4\cdot\int_{\mathcal{C}}\int_{\mathcal{C}}\frac{1}{\mathcal{N}_n^4}\sum_{\lambda,\lambda'}\left\langle\frac{\lambda}{|\lambda|},\dot\gamma(t_1)\right\rangle\left\langle\frac{\lambda}{|\lambda|},\dot\gamma(t_2)\right\rangle\left\langle\frac{\lambda'}{|\lambda'|},\dot\gamma(t_1)\right\rangle\left\langle\frac{\lambda'}{|\lambda'|},\dot\gamma(t_2)\right\rangle\,dt_1dt_2\\
&- 8\cdot\int_{\mathcal{C}}\int_{\mathcal{C}}\frac{1}{\mathcal{N}_n^4}\underbrace{\sum_{\lambda,\lambda'}\left\langle\frac{\lambda}{|\lambda|},\dot\gamma(t_1)\right\rangle^2\left\langle\frac{\lambda}{|\lambda|},\dot\gamma(t_2)\right\rangle\left\langle\frac{\lambda'}{|\lambda'|},\dot\gamma(t_2)\right\rangle}_{=0}\,dt_1dt_2\\
&+ 4\cdot\int_{\mathcal{C}}\int_{\mathcal{C}}\frac{1}{\mathcal{N}_n^4}\sum_{\lambda_1+\dots+\lambda_4\neq0}\left\langle\frac{\lambda_1}{|\lambda_1|},\dot\gamma(t_1)\right\rangle\left\langle\frac{\lambda_1}{|\lambda_1|},\dot\gamma(t_2)\right\rangle\left\langle\frac{\lambda_3}{|\lambda_3|},\dot\gamma(t_1)\right\rangle\left\langle\frac{\lambda_4}{|\lambda_4|},\dot\gamma(t_2)\right\rangle\times\\
&\times \mathrm{e}^{i2\pi\langle\lambda_1+\lambda_2+\lambda_3+\lambda_4,\gamma(t_1)-\gamma(t_2)\rangle}\,dt_1dt_2\\
&= -4F_{\mathcal{C}}(\mu_n)\frac{1}{\mathcal{N}_n^2}+o(\mathcal{N}_n^{-2}),
\end{aligned}
$$

where in the last step we used (1.13), and Cauchy-Schwartz inequality and again (5.2) to bound the contribution of "off-diagonal" terms. Let us now study 5).

(C.8)

$$
\begin{aligned}
&\int_{\mathcal{C}}\int_{\mathcal{C}}(r_1/\sqrt{\alpha})^2(r_2/\sqrt{\alpha})^2\,dt_1dt_2\\
&= 4\cdot\int_{\mathcal{C}}\int_{\mathcal{C}}\frac{1}{\mathcal{N}_n^4}\sum_{\lambda_1,\dots,\lambda_4}\left\langle\frac{\lambda_1}{|\lambda_1|},\dot\gamma(t_1)\right\rangle\left\langle\frac{\lambda_2}{|\lambda_2|},\dot\gamma(t_1)\right\rangle\left\langle\frac{\lambda_3}{|\lambda_3|},\dot\gamma(t_2)\right\rangle\left\langle\frac{\lambda_4}{|\lambda_4|},\dot\gamma(t_2)\right\rangle\times\\
&\times \mathrm{e}^{i2\pi\langle\lambda_1-\lambda_2+\lambda_3-\lambda_4,\gamma(t_1)-\gamma(t_2)\rangle}\,dt_1dt_2\\
&= 4\cdot\int_{\mathcal{C}}\int_{\mathcal{C}}\frac{1}{\mathcal{N}_n^4}\sum_{\lambda,\lambda'}\left\langle\frac{\lambda}{|\lambda|},\dot\gamma(t_1)\right\rangle^2\left\langle\frac{\lambda'}{|\lambda'|},\dot\gamma(t_2)\right\rangle^2\,dt_1dt_2\\
&+ 4\cdot2\int_{\mathcal{C}}\int_{\mathcal{C}}\frac{1}{\mathcal{N}_n^4}\sum_{\lambda,\lambda'}\left\langle\frac{\lambda}{|\lambda|},\dot\gamma(t_1)\right\rangle\left\langle\frac{\lambda'}{|\lambda'|},\dot\gamma(t_1)\right\rangle\left\langle\frac{\lambda}{|\lambda|},\dot\gamma(t_2)\right\rangle\left\langle\frac{\lambda'}{|\lambda'|},\dot\gamma(t_2)\right\rangle\,dt_1dt_2\\
&+ 4\cdot\int_{\mathcal{C}}\int_{\mathcal{C}}\frac{1}{\mathcal{N}_n^4}\sum_{\lambda_1+\dots+\lambda_4\neq0}\left\langle\frac{\lambda_1}{|\lambda_1|},\dot\gamma(t_1)\right\rangle\left\langle\frac{\lambda_2}{|\lambda_2|},\dot\gamma(t_1)\right\rangle\left\langle\frac{\lambda_3}{|\lambda_3|},\dot\gamma(t_2)\right\rangle\left\langle\frac{\lambda_4}{|\lambda_4|},\dot\gamma(t_2)\right\rangle\times\\
&\times \mathrm{e}^{i2\pi\langle\lambda_1-\lambda_2+\lambda_3-\lambda_4,\gamma(t_1)-\gamma(t_2)\rangle}\,dt_1dt_2\\
&= L^2\frac{1}{\mathcal{N}_n^2}+4\cdot2\frac{1}{\mathcal{N}_n^2}F_n+o(\mathcal{N}_n^{-2}),
\end{aligned}
$$

where for the last step we used the well-known equality $\frac{1}{\mathcal{N}_n}\sum_\lambda \left\langle \frac{\lambda}{|\lambda|}, v\right\rangle^2 = \frac{1}{2}$ which holds for every unit vector v, and still Cauchy-Schwartz inequality and then (5.2) to bound the contribution of "off-diagonal" terms.

The proof of 2) is analogous to that of 1), whereas the proofs of 3), 6)-8) are analogous to that of 5) above, and hence omitted. □

Lemma C.2. *If $\mathcal{C} \subset \mathbb{T}$ is a smooth curve with nowhere vanishing curvature, then for δ-separated sequences $\{n\}$ such that $\mathcal{N}_n \to +\infty$, we have*

$$\begin{aligned} &1) \int_{\mathcal{C}}\int_{\mathcal{C}} r(t_1,t_2)^6\, dt_1 dt_2 = o(\mathcal{N}_n^{-2}),\\ &2) \int_{\mathcal{C}}\int_{\mathcal{C}} (r_1/\sqrt{\alpha})^6\, dt_1 dt_2 = o(\mathcal{N}_n^{-2}),\\ &3) \int_{\mathcal{C}}\int_{\mathcal{C}} (r_{12}/\alpha)^6\, dt_1 dt_2 = o(\mathcal{N}_n^{-2}). \end{aligned} \tag{C.9}$$

Proof. Let us prove 1).

$$\begin{aligned} \int_{\mathcal{C}}\int_{\mathcal{C}} r(t_1,t_2)^6\, dt_1 dt_2 &= \int_{\mathcal{C}}\int_{\mathcal{C}} \left(\frac{1}{\mathcal{N}_n}\sum_{\lambda\in\Lambda_n} \mathrm{e}_\lambda(\gamma(t_1)-\gamma(t_2))\right)^6 dt_1 dt_2\\ &= \int_{\mathcal{C}}\int_{\mathcal{C}} \frac{1}{\mathcal{N}_n^6}\sum_{\lambda_1,\dots,\lambda_6\in\Lambda_n} \mathrm{e}_{\lambda_1+\dots+\lambda_6}(\gamma(t_1)-\gamma(t_2))\, dt_1 dt_2\\ &= \frac{1}{\mathcal{N}_n^6}\sum_{\lambda_1,\dots,\lambda_6\in\Lambda_n} \left|\int_{\mathcal{C}} \mathrm{e}_{\lambda_1+\dots+\lambda_6}(\gamma(t))\, dt\right|^2\\ &= L^2\frac{|S_6(n)|}{\mathcal{N}_n^6} + \frac{1}{\mathcal{N}_n^6}\sum_{\lambda_1+\dots+\lambda_6\neq 0} \left|\int_{\mathcal{C}} \mathrm{e}_{\lambda_1+\dots+\lambda_6}(\gamma(t))\, dt\right|^2\\ &\ll \frac{|S_6(n)|}{\mathcal{N}_n^6} + \frac{1}{\mathcal{N}_n^6}\sum_{\lambda_1+\dots+\lambda_6\neq 0} \frac{1}{|\lambda_1+\dots+\lambda_6|}, \end{aligned} \tag{C.10}$$

where

$$S_6(n) := \{(\lambda_1,\dots,\lambda_6)\in\Lambda_n^6 : \lambda_1+\dots+\lambda_6 = 0\},$$

and for the last step we used Lemma 5.2 in [RW14]. Recall now that [BB15]

$$|S_6(n)| = O\left(\mathcal{N}_n^{7/2}\right).$$

Using the latter together with Lemma D.2 (just below) in (C.10), we can conclude the proof of 1). The proofs of the remaining cases 2), 3) are similar to that of 1) and hence omitted.

□

Appendix D. Contribution of "off-diagonal" terms

D.1. **Proof of Lemma 5.2.** Let us start with the following simple lemma.

Lemma D.1. *Let A, B, C, D be four points on $\mathbb{S}^1$ such that the segment AC intersects the segment BD, and O be the centre of the circle. Then the angle between AC and BD equals*

$$\frac{\angle AOB + \angle COD}{2}.$$

Proof. Let E be the intersection point of AC with BD and α the corresponding angle. Then, as α is the external angle in the triangle AED we have that

$$\alpha = \angle ADE + \angle EAD = \frac{\angle AOB + \angle COD}{2}, \tag{D.1}$$

since for any chord XY on the circle, the angle $\angle XOY$ is twice the angle $\angle XZY$ subtended by another point Z on the circle.

□

We are now in a position to prove Lemma 5.2.

Proof of Lemma 5.2. First suppose that the segment $\lambda_1\lambda_3$ intersects the segment $\lambda_2\lambda_4$. For $u = \lambda_3 - \lambda_1$ and $w = \lambda_4 - \lambda_2$, $v := u - w$, and α the angle between u and w we have

$$\begin{aligned}\|v\|^2 &= \|u\|^2 + \|w\|^2 - 2\|u\| \cdot \|w\| \cos\alpha = (\|u\| - \|w\|)^2 + 2\|u\| \cdot \|w\|(1 - \cos\alpha) \\ &\geq 2\|u\|\|w\|(1 - \cos\alpha) \gg 2\|u\| \cdot \|w\| \cdot \alpha^2,\end{aligned}$$

provided that $\alpha \in [0, \pi]$ (say). Since $\|u\| \cdot \|w\| \gg n^{1/2+2\delta}$ by assumption (1.12), and

$$\alpha \gg n^{-1/4+\delta}$$

by Lemma D.1, we have

$$\|v\| \gg n^{1+2\delta} \cdot n^{-1/2+2\delta} = n^{4\delta},$$

which, in case $\lambda_1\lambda_3$ intersects $\lambda_2\lambda_4$, is stronger than claimed.

Otherwise, let us assume that $\lambda_1\lambda_3$ does not intersect $\lambda_2\lambda_4$. That means that both λ_1, λ_3 are lying on the same arc $\overset{\frown}{\lambda_2\lambda_4}$ (one of two choices) and either the arcs $\overset{\frown}{\lambda_2\lambda_1}$, $\overset{\frown}{\lambda_1\lambda_3}$, $\overset{\frown}{\lambda_3\lambda_4}$ are pairwise disjoint or the arcs $\overset{\frown}{\lambda_4\lambda_1}$, $\overset{\frown}{\lambda_1\lambda_3}$ and $\overset{\frown}{\lambda_3\lambda_2}$ are pairwise disjoint; we assume w.l.o.g that the former holds. Since

$$\lambda_3 - \lambda_1 = (-\lambda_1) - (-\lambda_3),$$

and upon replacing (λ_1, λ_3) by $(-\lambda_3, -\lambda_1)$ if necessary, we may assume that λ_1, λ_3 are lying in the smaller of the arcs $\overset{\frown}{\lambda_2\lambda_4}$, i.e. the angle $\lambda_2 O \lambda_4 < \pi$.

Denote as before $u = \lambda_3 - \lambda_1$ and $w = \lambda_4 - \lambda_2$, $v := u - w$. Here we claim that $|\|u\| - \|w\||$ is not too small, and then, by the triangle inequality, so is

$$\|v\| \geq |\|u\| - \|w\||.$$

Let α be the angle $\alpha = \angle\lambda_2 O \lambda_4$, and the angle $\beta = \lambda_1 O \lambda_3 < \alpha$. We have $\|u\| = 2\sqrt{n}\sin(\alpha/2)$, $\|w\| = 2\sqrt{n}\sin(\beta/2)$, so that

$$\|u\| - \|w\| = 2\sqrt{n}\sin\left(\frac{\alpha - \beta}{4}\right)\cos\left(\frac{\alpha + \beta}{4}\right).$$

However, by the assumption (1.12) we have that both $\alpha - \beta \gg n^{-1/4+\delta}$ and $2\pi - (\alpha + \beta) \gg n^{-1/4+\delta}$ as at least one of α or β falls short of π by at least $\gg n^{-1/4+\delta}$. Hence

$$\|u\| - \|w\| \gg \sqrt{n} \cdot n^{-1/4+\delta} \cdot n^{-1/4+\delta} = n^{2\delta}$$

which yields the statement of Lemma 5.2 in this case.

□

Lemma D.2. *For δ-separated sequences $\{n\} \subset S$ we have the bound*

$$\frac{1}{\mathcal{N}_n^4} \sum_{\lambda_1+\dots+\lambda_6\neq 0} \frac{1}{\|\lambda_1+\dots+\lambda_6\|} = o_{\mathcal{N}_n\to\infty}(1)\,. \tag{D.2}$$

Proof. Let $\epsilon > 0$ be positive number, and set $A := \frac{1}{2}\mathcal{N}_n^{2+\epsilon}$ (say). We distinguish between two cases:
i) $\|\lambda_1+\dots+\lambda_6\| \geq A$, ii) $\|\lambda_1+\dots+\lambda_6\| < A$. The contribution of all summands with i) holding is

$$\ll \frac{1}{A} \ll \frac{1}{\mathcal{N}_n^{\epsilon}},$$

hence we are only to bound the contribution of summands satisfying ii).

Assume here that ii) indeed holds. We claim that for fixed $\lambda_1,\dots,\lambda_4$ such that $\lambda_1 + \ldots + \lambda_4 \neq 0$, there exist $O(1)$-choices for λ_5, λ_6. Indeed, if we assume that both $\|\lambda_1 + \dots + \lambda_4 + \lambda_5 + \lambda_6\| < A$ and $\|\lambda_1 + \dots + \lambda_4 + \lambda_5' + \lambda_6'\| < A$, then, by the triangle inequality, we obtain

$$\|\lambda_5 + \lambda_6 - \lambda_5' - \lambda_6'\| < 2A. \tag{D.3}$$

By Lemma 5.2 and in light of (1.6), (D.3) is valid only if $\lambda_5 + \lambda_6 - \lambda_5' - \lambda_6' = 0$. This in turn implies that either $\lambda_5 = -\lambda_6$ or $\lambda_5 = \lambda_5'$ or $\lambda_5 = \lambda_6'$). The possibility $\lambda_5 = -\lambda_6$ is not valid, since then $\lambda_1 + \dots + \lambda_6 = \lambda_1 + \dots + \lambda_4$, and therefore given $\lambda_1, \dots \lambda_4$ there could be at most two choices for λ_5, λ_6 such that ii) holds.

Let $B > 0$ be a large (but fixed) parameter, and assume that ii) holds. We distinguish between further two cases: a) $\|\lambda_1 + \dots + \lambda_6\| \geq B$, or b) $\|\lambda_1 + \dots + \lambda_6\| < B$. By the above, the contribution of summands satisfying a) to the l.h.s. of (D.2) is $O\left(\frac{1}{B}\right)$.

To treat terms that satisfy b) we recall that [KKW13, Theorem 2.2] shows that the number of 6-tuples $(\lambda_1, \dots, \lambda_6) \in \Lambda_n^6$ satisfying $\lambda_1 + \ldots + \lambda_6 = 0$ is $o(\mathcal{N}_n^4)$. A slight modification[3] of the proof of [KKW13, Theorem 2.2] yields that the same holds for an arbitrary *fixed* $v \in \mathbb{Z}^2$ in place of 0, i.e. the number of tuples such 6-tuples satisfying

$$\lambda_1 + \ldots + \lambda_6 = v$$

is $o(\mathcal{N}_n^4)$. Therefore the total contribution to (D.2) of summands satisfying b) is $o_B(1)$. Consolidating the various bounds we encountered, we have that

$$\frac{1}{\mathcal{N}_n^4} \sum_{\lambda_1+\dots+\lambda_6\neq 0} \frac{1}{\|\lambda_1+\dots+\lambda_6\|} \ll \frac{1}{\mathcal{N}_n^{\epsilon}} + \frac{1}{B} + o_B(1),$$

which certainly implies (D.2). □

[3]In Eq. 59, consider $y_1 + y_2 \in A + v$ and in Eq. 60, consider $y_1 + y_2 \in A + v$ so that we have $\langle 1_A \star 1_A, 1_{A+v}\rangle$

Appendix E. Auxiliary computations for the fourth chaos

Proof of Lemma 6.2. Let us start with 1). Recall the expression for the fourth Hermite polynomial: $H_4(t) = t^4 - 6t^2 + 3$.

$$
\begin{aligned}
1) \int_0^L H_4(f_n(t))\,dt &= \int_0^L \left(f_n(t)^4 - 6f_n(t)^2 + 3\right)\,dt \\
&= \frac{1}{\mathcal{N}_n^2} \sum_{\lambda,\lambda',\lambda'',\lambda'''} a_\lambda \overline{a_{\lambda'}} a_{\lambda''} \overline{a_{\lambda'''}} \int_0^L \mathrm{e}^{i2\pi\langle \lambda-\lambda'+\lambda''-\lambda''',\gamma(t)\rangle}\,dt \\
&\quad - 6\frac{1}{\mathcal{N}_n} \sum_{\lambda,\lambda'} a_\lambda \overline{a_{\lambda'}} \int_0^L \mathrm{e}^{i2\pi\langle \lambda-\lambda',\gamma(t)\rangle}\,dt + 3L.
\end{aligned} \tag{E.1}
$$

Let us divide the first term on the r.h.s. of (E.1) into two summation, one over the quadruples in $S_4(n)$ and the other one over the quadruples not belonging to $S_4(n)$.

$$
\begin{aligned}
\int_0^L H_4(f_n(t))\,dt &= 3L\frac{1}{\mathcal{N}_n^2} \sum_{\lambda,\lambda'} |a_\lambda|^2 |a_{\lambda'}|^2 - 3L\frac{1}{\mathcal{N}_n^2} \sum_{\lambda} |a_\lambda|^4 \\
&\quad + \frac{1}{\mathcal{N}_n^2} \sum_{\lambda,\lambda',\lambda'',\lambda''' \notin S_n(4)} a_\lambda \overline{a_{\lambda'}} a_{\lambda''} \overline{a_{\lambda'''}} \int_0^L \mathrm{e}^{i2\pi\langle \lambda-\lambda'+\lambda''-\lambda''',\gamma(t)\rangle}\,dt \\
&\quad - 6\frac{1}{\mathcal{N}_n} \sum_{\lambda,\lambda'} a_\lambda \overline{a_{\lambda'}} \int_0^L \mathrm{e}^{i2\pi\langle \lambda-\lambda',\gamma(t)\rangle}\,dt + 3L.
\end{aligned} \tag{E.2}
$$

Now let us divide the fourth term on the r.h.s. of (E.2) into two sums, one for the diagonal terms and the other one for the off-diagonal terms. That is,

$$
\begin{aligned}
\int_0^L H_4(f_n(t))\,dt &= 3L\frac{1}{\mathcal{N}_n^2} \sum_{\lambda,\lambda'} |a_\lambda|^2 |a_{\lambda'}|^2 - 6L\frac{1}{\mathcal{N}_n} \sum_{\lambda} |a_\lambda|^2 + 3L \\
&\quad - 6\frac{1}{\mathcal{N}_n} \sum_{\lambda\neq\lambda'} a_\lambda \overline{a_{\lambda'}} \int_0^L \mathrm{e}^{i2\pi\langle \lambda-\lambda',\gamma(t)\rangle}\,dt - 3L\frac{1}{\mathcal{N}_n^2} \sum_{\lambda} |a_\lambda|^4 \\
&\quad + \frac{1}{\mathcal{N}_n^2} \sum_{\lambda,\lambda',\lambda'',\lambda''' \notin S_n(4)} a_\lambda \overline{a_{\lambda'}} a_{\lambda''} \overline{a_{\lambda'''}} \int_0^L \mathrm{e}^{i2\pi\langle \lambda-\lambda'+\lambda''-\lambda''',\gamma(t)\rangle}\,dt.
\end{aligned} \tag{E.3}
$$

Note that the first three terms on the r.h.s. of (E.3) can be rewritten as

$$
\begin{aligned}
\int_0^L H_4(f_n(t))\,dt &= 3\cdot 2\cdot L\frac{1}{\mathcal{N}_n}\left(\frac{1}{\sqrt{\mathcal{N}_n/2}}\sum_{\lambda\in\Lambda_n^+}(|a_\lambda|^2-1)\right)^2\\
&\quad - 3L\frac{1}{\mathcal{N}_n^2}\sum_\lambda |a_\lambda|^4 - 6\frac{1}{\mathcal{N}_n}\sum_{\lambda\neq\lambda'} a_\lambda\overline{a_{\lambda'}}\int_0^L \mathrm{e}^{i2\pi\langle\lambda-\lambda',\gamma(t)\rangle}\,dt\\
&\quad + \frac{1}{\mathcal{N}_n^2}\sum_{\lambda,\lambda',\lambda'',\lambda'''\notin S_n(4)} a_\lambda\overline{a_{\lambda'}}a_{\lambda''}\overline{a_{\lambda'''}}\int_0^L \mathrm{e}^{i2\pi\langle\lambda-\lambda'+\lambda''-\lambda''',\gamma(t)\rangle}\,dt\\
&= 6\cdot L\frac{1}{\mathcal{N}_n}W_1(n)^2 - 3L\frac{1}{\mathcal{N}_n^2}\sum_\lambda |a_\lambda|^4 - 6\frac{1}{\mathcal{N}_n}\sum_{\lambda\neq\lambda'} a_\lambda\overline{a_{\lambda'}}\int_0^L \mathrm{e}^{i2\pi\langle\lambda-\lambda',\gamma(t)\rangle}\,dt\\
&\quad + \frac{1}{\mathcal{N}_n^2}\sum_{\lambda,\lambda',\lambda'',\lambda'''\notin S_n(4)} a_\lambda\overline{a_{\lambda'}}a_{\lambda''}\overline{a_{\lambda'''}}\int_0^L \mathrm{e}^{i2\pi\langle\lambda-\lambda'+\lambda''-\lambda''',\gamma(t)\rangle}\,dt\\
&= X_n^a + X_n^b,
\end{aligned}
$$

where $W_1(n)$ is defined as in (6.2), X_n^a is given in (6.4) and

$$
\begin{aligned}
X_n^b :=& -3L\frac{1}{\mathcal{N}_n^2}\sum_\lambda(|a_\lambda|^4-2) - 6\frac{1}{\mathcal{N}_n}\sum_{\lambda\neq\lambda'} a_\lambda\overline{a_{\lambda'}}\int_0^L \mathrm{e}^{i2\pi\langle\lambda-\lambda',\gamma(t)\rangle}\,dt\\
&+ \frac{1}{\mathcal{N}_n^2}\sum_{\lambda,\lambda',\lambda'',\lambda'''\notin S_n(4)} a_\lambda\overline{a_{\lambda'}}a_{\lambda''}\overline{a_{\lambda'''}}\int_0^L \mathrm{e}^{i2\pi\langle\lambda-\lambda'+\lambda''-\lambda''',\gamma(t)\rangle}\,dt.
\end{aligned}
\tag{E.4}
$$

Let us prove that, as $\mathcal{N}_n\to+\infty$,

$$
\mathrm{Var}(X_n^b) = o\left(\frac{1}{\mathcal{N}_n^2}\right). \tag{E.5}
$$

For the first term on the r.h.s. of (E.4) we have

$$
\mathrm{Var}\left(\frac{1}{\mathcal{N}_n^2}\sum_\lambda(|a_\lambda|^4-2)\right) = O(\mathcal{N}_n^{-3}) \tag{E.6}
$$

whereas for the second one [RW14, (5.18)] and (1.12) give

$$
\mathrm{Var}\left(\frac{1}{\mathcal{N}_n}\sum_{\lambda\neq\lambda'} a_\lambda\overline{a_{\lambda'}}\int_0^L \mathrm{e}^{i2\pi\langle\lambda-\lambda',\gamma(t)\rangle}\right) = o\left(\frac{1}{\mathcal{N}_n^2}\right). \tag{E.7}
$$

For the last term

$$
\mathrm{Var}\left(\frac{1}{\mathcal{N}_n^2}\sum_{\lambda,\lambda',\lambda'',\lambda'''\notin S_n(4)} a_\lambda\overline{a_{\lambda'}}a_{\lambda''}\overline{a_{\lambda'''}}\int_0^L \mathrm{e}^{i2\pi\langle\lambda-\lambda'+\lambda''-\lambda''',\gamma(t)\rangle}\right) = o\left(\frac{1}{\mathcal{N}_n^2}\right). \tag{E.8}
$$

follows from [RW14, (5.18)] and Lemma 5.2. This concludes the proof of 1).

Let us now study the second summand in (6.1). The argument given below is similar to the one above concerning the first summand in (6.1); accordingly we will omit some technical details.

(E.9)

$$\begin{aligned}\int_0^L H_4(f_n'(t))\,dt &= \int_0^L \left(f_n'(t)^4 - 6f_n'(t)^2 + 3\right)\,dt\\ &= \frac{1}{(2\pi^2 n)^2}\int_0^L \frac{(2\pi)^4}{\mathcal{N}_n^2}\sum_{\lambda,\lambda',\lambda'',\lambda'''} a_\lambda\overline{a_{\lambda'}}a_{\lambda''}\overline{a_{\lambda'''}}\times\\ &\times \langle\lambda,\dot\gamma(t)\rangle\langle\lambda',\dot\gamma(t)\rangle\langle\lambda'',\dot\gamma(t)\rangle\langle\lambda''',\dot\gamma(t)\rangle \mathrm{e}^{i2\pi\langle\lambda-\lambda'+\lambda''-\lambda''',\gamma(t)\rangle}\,dt\\ &- 6\frac{1}{2\pi^2 n}\int_0^L \frac{(2\pi)^2}{\mathcal{N}_n}\sum_{\lambda,\lambda'} a_\lambda\overline{a_{\lambda'}}\langle\lambda,\dot\gamma(t)\rangle\langle\lambda',\dot\gamma(t)\rangle \mathrm{e}^{i2\pi\langle\lambda-\lambda',\gamma(t)\rangle}\,dt + 3L.\end{aligned}$$

Dealing with (E.9) as in (E.3) we obtain

$$\begin{aligned}\int_0^L H_4(f_n'(t))\,dt &= 3\int_0^L \left(2\frac{1}{\mathcal{N}_n}\sum_\lambda (|a_\lambda|^2-1)\left\langle\frac{\lambda}{|\lambda|},\dot\gamma(t)\right\rangle^2\right)^2 dt\\ &- 3\frac{1}{(2\pi^2 n)^2}\frac{(2\pi)^4}{\mathcal{N}_n^2}\sum_\lambda |a_\lambda|^4\int_0^L \langle\lambda,\dot\gamma(t)\rangle^4\,dt\\ &+ \frac{1}{(2\pi^2 n)^2}\int_0^L \frac{(2\pi)^4}{\mathcal{N}_n^2}\sum_{\lambda,\lambda',\lambda'',\lambda'''\notin S_n(4)} a_\lambda\overline{a_{\lambda'}}a_{\lambda''}\overline{a_{\lambda'''}}\times\\ &\times \langle\lambda,\dot\gamma(t)\rangle\langle\lambda',\dot\gamma(t)\rangle\langle\lambda'',\dot\gamma(t)\rangle\langle\lambda''',\dot\gamma(t)\rangle \mathrm{e}^{i2\pi\langle\lambda-\lambda'+\lambda''-\lambda''',\gamma(t)\rangle}\,dt\\ &- 6\frac{1}{2\pi^2 n}\int_0^L \frac{(2\pi)^2}{\mathcal{N}_n}\sum_{\lambda\neq\lambda'} a_\lambda\overline{a_{\lambda'}}\langle\lambda,\dot\gamma(t)\rangle\langle\lambda',\dot\gamma(t)\rangle \mathrm{e}^{i2\pi\langle\lambda-\lambda',\gamma(t)\rangle}\,dt\\ &= Y_n^a + Y_n^b,\end{aligned}$$

where Y_n^a is defined as in (6.4) and

$$\begin{aligned}Y_n^b :=& - 3\frac{1}{(2\pi^2 n)^2}\frac{(2\pi)^4}{\mathcal{N}_n^2}\sum_\lambda (|a_\lambda|^4 - 2)\int_0^L \langle\lambda,\dot\gamma(t)\rangle^4\,dt\\ &+ \frac{1}{(2\pi^2 n)^2}\int_0^L \frac{(2\pi)^4}{\mathcal{N}_n^2}\sum_{\lambda,\lambda',\lambda'',\lambda'''\notin S_n(4)} a_\lambda\overline{a_{\lambda'}}a_{\lambda''}\overline{a_{\lambda'''}}\times\\ &\times \langle\lambda,\dot\gamma(t)\rangle\langle\lambda',\dot\gamma(t)\rangle\langle\lambda'',\dot\gamma(t)\rangle\langle\lambda''',\dot\gamma(t)\rangle \mathrm{e}^{i2\pi\langle\lambda-\lambda'+\lambda''-\lambda''',\gamma(t)\rangle}\,dt\\ &- 6\frac{1}{2\pi^2 n}\int_0^L \frac{(2\pi)^2}{\mathcal{N}_n}\sum_{\lambda\neq\lambda'} a_\lambda\overline{a_{\lambda'}}\langle\lambda,\dot\gamma(t)\rangle\langle\lambda',\dot\gamma(t)\rangle \mathrm{e}^{i2\pi\langle\lambda-\lambda',\gamma(t)\rangle}\,dt.\end{aligned}$$

We are now left with the third summand in (6.1).

(E.10)

$$\begin{aligned}\int_0^L H_2(f_n(t))H_2(f_n'(t))\,dt &= \int_0^L (f_n(t)^2-1)(f_n'(t)^2-1)\,dt\\
&= \frac{(2\pi)^2}{2\pi^2 n\mathcal{N}_n^2}\int_0^L \sum_{\lambda,\lambda',\lambda'',\lambda'''} a_\lambda\overline{a_{\lambda'}}a_{\lambda''}\overline{a_{\lambda'''}}\times\\
&\times\langle\lambda'',\dot\gamma(t)\rangle\langle\lambda''',\dot\gamma(t)\rangle \mathrm{e}^{i2\pi\langle\lambda-\lambda'+\lambda''-\lambda''',\gamma(t)\rangle}\,dt\\
&-\frac{1}{\mathcal{N}_n}\sum_{\lambda,\lambda'} a_\lambda\overline{a_{\lambda'}}\int_0^L \mathrm{e}^{i2\pi\langle\lambda-\lambda',\gamma(t)\rangle}\,dt\\
&-\frac{1}{2\pi^2 n}\int_0^L \frac{(2\pi)^2}{\mathcal{N}_n}\sum_{\lambda,\lambda'} a_\lambda\overline{a_{\lambda'}}\langle\lambda,\dot\gamma(t)\rangle\langle\lambda',\dot\gamma(t)\rangle \mathrm{e}^{i2\pi\langle\lambda-\lambda',\gamma(t)\rangle}\,dt + L\end{aligned}$$

Finally, we can rewrite (E.10) as

$$\begin{aligned}&\int_0^L H_2(f_n(t))H_2(f_n'(t))\,dt\\
&= \left(\frac{1}{\mathcal{N}_n}\sum_{\lambda}(|a_\lambda|^2-1)\right)\left(\frac{1}{\mathcal{N}_n}\sum_{\lambda'}(|a_\lambda'|^2-1)\int_0^L 2\left\langle\frac{\lambda'}{|\lambda'|},\dot\gamma(t)\right\rangle^2\,dt\right)\\
&-\frac{(2\pi)^2}{2\pi^2 n\mathcal{N}_n^2}\sum_\lambda |a_\lambda|^4\int_0^L\langle\lambda,\dot\gamma(t)\rangle^2\,dt\\
&+\frac{(2\pi)^2}{2\pi^2 n\mathcal{N}_n^2}\sum_{\lambda,\lambda',\lambda'',\lambda'''\notin S_n(4)} a_\lambda\overline{a_{\lambda'}}a_{\lambda''}\overline{a_{\lambda'''}}\int_0^L\langle\lambda'',\dot\gamma(t)\rangle\langle\lambda''',\dot\gamma(t)\rangle\times\\
&\times \mathrm{e}^{i2\pi\langle\lambda-\lambda'+\lambda''-\lambda''',\gamma(t)\rangle}\,dt - \frac{1}{\mathcal{N}_n}\sum_{\lambda\ne\lambda'} a_\lambda\overline{a_{\lambda'}}\int_0^L \mathrm{e}^{i2\pi\langle\lambda-\lambda',\gamma(t)\rangle}\,dt\\
&-\frac{1}{2\pi^2 n}\int_0^L\frac{(2\pi)^2}{\mathcal{N}_n}\sum_{\lambda\ne\lambda'} a_\lambda\overline{a_{\lambda'}}\langle\lambda,\dot\gamma(t)\rangle\langle\lambda',\dot\gamma(t)\rangle \mathrm{e}^{i2\pi\langle\lambda-\lambda',\gamma(t)\rangle}\,dt\\
&= Z_n^a + Z_n^b,\end{aligned}$$

where Z_n^a is defined as in (6.4) and

$$\begin{aligned}Z_n^b :=& -\frac{(2\pi)^2}{2\pi^2 n\mathcal{N}_n^2}\sum_\lambda(|a_\lambda|^4-2)\int_0^L\langle\lambda,\dot\gamma(t)\rangle^2\,dt\\
&+\frac{(2\pi)^2}{2\pi^2 n\mathcal{N}_n^2}\sum_{\lambda,\lambda',\lambda'',\lambda'''\notin S_n(4)} a_\lambda\overline{a_{\lambda'}}a_{\lambda''}\overline{a_{\lambda'''}}\int_0^L\langle\lambda'',\dot\gamma(t)\rangle\langle\lambda''',\dot\gamma(t)\rangle \mathrm{e}^{i2\pi\langle\lambda-\lambda'+\lambda''-\lambda''',\gamma(t)\rangle}\,dt\\
&-\frac{1}{\mathcal{N}_n}\sum_{\lambda\ne\lambda'} a_\lambda\overline{a_{\lambda'}}\int_0^L \mathrm{e}^{i2\pi\langle\lambda-\lambda',\gamma(t)\rangle}\,dt\\
&-\frac{1}{2\pi^2 n}\int_0^L\frac{(2\pi)^2}{\mathcal{N}_n}\sum_{\lambda\ne\lambda'} a_\lambda\overline{a_{\lambda'}}\langle\lambda,\dot\gamma(t)\rangle\langle\lambda',\dot\gamma(t)\rangle \mathrm{e}^{i2\pi\langle\lambda-\lambda',\gamma(t)\rangle}\,dt.\end{aligned}$$

□

Appendix F. A family of static curves

Proposition F.1. *Let $\mathcal{C} \subset \mathbb{T}$ be a smooth closed curve with nowhere vanishing curvature, invariant w.r.t. rotation by $2\pi/k$ for some $k \geq 3$. Then $\mathcal{C}$ is static.*

Proof. To show that $\mathcal{C}$ is static it is sufficient [RW14, Corollary 7.2] to see that

$$B_{\mathcal{C}}\left(\frac{d\theta}{2\pi}\right) = \frac{L^2}{4}, \tag{F.1}$$

i.e. check the condition (1.11) with $\mu = \frac{d\theta}{2\pi}$ only. By inverting the order of integration in (1.10) we have that (F.1) is equivalent to

$$\int_0^{2\pi} \left(\int_0^L \langle \theta, \dot{\gamma}(t) \rangle^2 dt \right)^2 \frac{d\theta}{2\pi} = \frac{L^2}{4}. \tag{F.2}$$

We claim that for every $\theta \in [0, 2\pi]$ the integrand above is

$$\int_0^L \langle \theta, \dot{\gamma}(t) \rangle^2 dt = \frac{L}{2}, \tag{F.3}$$

which is certainly sufficient for (F.2).

To prove it we denote $\dot{\gamma}(t) =: e^{i\varphi(t)}$ so that the integrand on the l.h.s. of (F.3) is

$$\langle \theta, \dot{\gamma}(t) \rangle^2 = \cos(\theta - \varphi(t))^2,$$

and by the assumed invariance of $\mathcal{C}$ w.r.t. rotations by $2\pi/k$ we have that

$$\int_0^L \cos(\theta - \varphi(t))^2 dt = \int_0^{L/k} \left(\sum_{j=0}^{k-1} \cos\left(\theta - \varphi(t) + j \cdot \frac{2\pi}{k}\right)^2 \right) dt. \tag{F.4}$$

Now, since, by assumption, $k \geq 3$, we have that

$$\sum_{j=0}^{k-1} \cos\left(\theta - \varphi(t) + j \cdot \frac{2\pi}{k}\right)^2 = \frac{k}{2} + \sum_{j=0}^{k-1} \cos\left(2\left(\theta - \varphi(t) + \frac{2\pi}{k}\right)\right) = \frac{k}{2},$$

which we substitute into (F.4) to obtain

$$\int_0^L \cos(\theta - \varphi(t))^2 dt = \frac{k}{2} \cdot \frac{L}{k} = \frac{L}{2},$$

that is, we obtain (F.3), which, as it was mentioned above, is sufficient to yield the statement of Proposition F.1.

□

References

[AW09] J.-M. Azaïs and M. Wschebor. *Level sets and extrema of random processes and fields*, John Wiley & Sons Inc., Hoboken, NJ, 2009.

[BB15] E. Bombieri and J. Bourgain. A problem on the sum of two squares. *International Math. Res. Notices*, 11, 3343–3407 (2015).

[BR11] J. Bourgain and Z. Rudnick. *On the geometry of the nodal lines of eigenfunctions on the two-dimensional torus*, Ann. Henri Poincaré 12, no. 6, 1027–1053 (2011).

[BR12] J. Bourgain and Z. Rudnick. *Restriction of toral eigenfunctions to hypersurfaces and nodal sets*, Geometric and Functional Analysis 22, no. 4, 878–937 (2012).

[BR15] J. Bourgain and Z. Rudnick. *Nodal intersections and L^p restriction theorems on the torus*, Israel Journal of Mathematics, 207, 479–505 (2015).

[Cil93] J. Cilleruelo. *The distribution of the lattice points on circles*, Journal of Number Theory 43, no. 2, 198–202 (1993).

[Dud02] R.-M. Dudley. Real analysis and probability. *Cambridge University Press.*

[ET15] L. El-Hajj and J.-A. Toth. *Intersection bounds for nodal sets of planar Neumann eigenfunctions with interior analytic curves.* J. Differential Geom. 100, no. 1, 1–53 (2015).

[GW16] A. Granville and I. Wigman. *Planck-scale mass equidistribution of toral Laplace eigenfunctions.* Preprint, arXiv: 1612.07819 (2016).

[KKW13] M. Krishnapur, P. Kurlberg and I. Wigman. *Nodal length fluctuations for arithmetic random waves*, Annals of Mathematics (2) 177, no. 2, 699–737 (2013).

[KL97] M. Kratz and J.-R. León. *Hermite polynomial expansion for non-smooth functionals of stationary Gaussian processes: crossings and extremes.* Stochastic Process. Appl. 66, no. 2, 237–252 (1997).

[KW16] P. Kurlberg and I. Wigman. *On probability measures arising from lattice points on circles*, Mathematische Annalen, in press, arXiv: 1501.01995 (2015).

[Lan08] E. Landau. *Uber die Einteilung der positiven Zahlen nach vier Klassen nach der Mindestzahl der zu ihrer addition Zusammensetzung erforderlichen Quadrate*, Archiv der Mathematik und Physics III (1908).

[MPRW16] D. Marinucci, G. Peccati, M. Rossi and I. Wigman. *Non-universality of nodal length distribution for arithmetic random waves.* Geometric and Functional Analysis, 26, no. 3, 926–960 (2016).

[NP12] I. Nourdin and G. Peccati. Normal approximations with Malliavin calculus. From Steins method to universality. *Cambridge University Press* (2012).

[ORW] F. Oravecz, Z. Rudnick, I. Wigman. *The Leray measure of nodal sets for random eigenfunctions on the torus*, Ann. Inst. Fourier (Grenoble) 58 (2008), 299–335.

[OY] W. Ott, J. A. Yorke. *Prevalence*, Bull. Amer. Math. Soc. 42 (2005), 263–290.

[RW08] Z. Rudnick and I. Wigman. *On the volume of nodal sets for eigenfunctions of the Laplacian on the torus*, Annales de lInstitut Henri Poincaré, 9, no. 1, 109–130 (2008).

[RW14] Z. Rudnick and I. Wigman. *Nodal intersections for random eigenfunctions on the torus*, Amer. J. of Math., in press, arXiv:1402.3621 (2014).

[RWY15] Z. Rudnick, I. Wigman and N. Yesha. *Nodal intersections for random waves on the* 3-*dimensional torus*, Annales de lInstitut Fourier, in press, arXiv: 1501.07410 (2015).

[Sze75] G. Szego . Orthogonal polynomials. *American Mathematical Society*, Colloquium Publications, Vol. XXIII (1975).

[TZ09] J. Toth and S. Zelditch. *Counting nodal lines which touch the boundary of an analytic domain.* J. Differential Geom. 81, no. 3, 649686 (2009).

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