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Asymptotic Dynamics of Inertial Particles with Memory

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Abstract Recent experimental and numerical observations have shown the significance of the Basset–Boussinesq memory term on the dynamics of small spherical rigid particles (or inertial particles) suspended in an ambient fluid flow. These observations suggest an algebraic decay to an asymptotic state, as opposed to the exponential convergence in the absence of the memory term. Here, we prove that the observed algebraic decay is a universal property of the Maxey–Riley equation. Specifically, the particle velocity decays algebraically in time to a limit that is $\mathcal{O}(\epsilon)$-close to the fluid velocity, where $0 < \epsilon \ll 1$ is proportional to the square of the ratio of the particle radius to the fluid characteristic length scale. These results follow from a sharp analytic upper bound that we derive for the particle velocity. For completeness, we also present a first proof of the global existence and uniqueness of mild solutions to the Maxey–Riley equation, a nonlinear system of fractional differential equations.

Keywords Inertial particles · Fractional calculus · Integro-differential equations · Maxey–Riley equation · Solid–fluid interaction

Mathematics Subject Classification 37N10

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1 Introduction

The motion of a solid body transported by an ambient Newtonian fluid flow can, in principle, be determined by solving the Navier–Stokes equations with appropriate moving boundary conditions (Galdi et al. 2008; Cartwright et al. 2010). The resulting partial differential equations are, however, too complicated for mathematical analysis. Their numerical solutions are computationally expensive and yield little insight.

For the motion of a small spherical rigid body (or inertial particle), however, one can derive a reliable model by accounting for all the forces exerted on the particle due to the solid–fluid interaction. Stokes (1851) made the first attempt to obtain such a model for the oscillatory motion of an inertial particle. Later, Basset (1888), Boussinesq (1885) and Oseen (1927) studied the settling of a solid sphere under gravity in a quiescent fluid. The resulting equation is known as the BBO equation. To study the motion of inertial particles in nonuniform unsteady flow, Tchen (1947) wrote the BBO equation in a frame of reference moving with the fluid, accounting for various inertial forces that arise in this frame.

The exact form of the forces exerted on the particle has been debated and corrected by several authors (see, e.g., Corrsin and Lumley 1956). A widely accepted form of the forces was derived by Maxey and Riley (1983) from first principles. The resulting equation, with the later correction of Auton et al. (1988) to the added mass term, is usually referred to as the Maxey–Riley (MR) equation.

To describe the MR equation, let $\mathbf{u} : \mathcal{D} \times \mathbb{R}^+ \to \mathbb{R}^n$ denote a known velocity field describing the flow of a fluid in an open spatial domain $\mathcal{D} \subseteq \mathbb{R}^n$. Here, $n = 2$ or $n = 3$ for two- and three-dimensional flows, respectively. A fluid trajectory is then the solution of the differential equation $\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x}, t)$ with some initial condition $\mathbf{x}(t_0) = \mathbf{x_0}$. An inertial particle, however, follows a different trajectory $\mathbf{y}(t) \in \mathcal{D}$. The particle velocity $\mathbf{v}(t) = \dot{\mathbf{y}}(t)$ satisfies the Maxey–Riley equation

$$
\begin{aligned}
\rho_p \dot{\mathbf{v}} =& \rho_f \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} && \text{(Force exerted by the undisturbed fluid)}\\
&+ (\rho_p - \rho_f)\mathbf{g} && \text{(Buoyancy force)}\\
&- \frac{9\nu\rho_f}{2a^2}\left(\mathbf{v} - \mathbf{u} - \frac{a^2}{6}\Delta\mathbf{u}\right) && \text{(Stokes drag)}\\
&- \frac{\rho_f}{2}\left[\dot{\mathbf{v}} - \frac{\mathrm{D}}{\mathrm{D}t}\left(\mathbf{u} + \frac{a^2}{10}\Delta\mathbf{u}\right)\right] && \text{(Added mass term)}\\
&- \frac{9\rho_f}{2a}\sqrt{\frac{\nu}{\pi}}\left[\int_{t_0}^{t} \frac{\dot{\mathbf{w}}(s)}{\sqrt{t-s}}\mathrm{d}s + \frac{\mathbf{w}(t_0)}{\sqrt{t-t_0}}\right] && \text{(Basset–Boussinesq memory term),}
\end{aligned}
\tag{1}
$$

where

$$
\mathbf{w}(t) = \mathbf{v}(t) - \mathbf{u}(\mathbf{y}(t), t) - \frac{a^2}{6}\Delta\mathbf{u}(\mathbf{y}(t), t).
$$

Here, ρ_p and ρ_f are, respectively, the particle and fluid densities; ν is the kinematic viscosity of the fluid; and a is the particle radius, and $\mathbf{g}$ is the constant gravitational acceleration vector. The initial conditions for the inertial particle are given as $\mathbf{y}(t_0) = \mathbf{y}_0$ and $\mathbf{v}(t_0) = \mathbf{v}_0$, for some $t_0 \in \mathbb{R}^+$. The material derivative $\frac{\mathrm{D}}{\mathrm{D}t} := \partial_t + \mathbf{u} \cdot \nabla$ denotes the time derivative along a fluid trajectory.

The right-hand side in (1) contains the various forces exerted on the particle. These forces have varying orders of magnitude. In particular, the Basset–Boussinesq memory term, accounting for the lagging boundary layer developed around the sphere, is routinely neglected on the grounds that it is insignificant compared with the Stokes drag and added mass (see, e.g., Maxey 1987; Balkovsky et al. 2001). Recent experimental and numerical studies, however, show that the memory term influences the dynamics of inertial particles significantly and hence cannot be generally neglected (Candelier et al. 2004; Toegel et al. 2006; Garbin et al. 2009; Daitche and Tél 2011; Guseva et al. 2013; Daitche and Tél 2014). This is the case even for heavy particles, for which the memory term becomes very small (Daitche and Tél 2011).

The numerical simulations of Daitche and Tél (2011) and Guseva et al. (2013), in particular, show the position of the particle to converge to its asymptotic limit algebraically. This is fundamentally different from the exponential convergence arising in the absence of the memory term (Rubin et al. 1995; Mograbi and Bar-Ziv 2006; Haller and Sapsis 2008). In the present paper, we prove that the observations of Daitche and Tél (2011) and Guseva et al. (2013) are a universal and generic property of the MR equation with memory, irrespective of the fluid flow carrying the particles.

The MR equation was originally derived under the assumption $\mathbf{w}(t_0) = 0$. Later, Maxey (1993) modified the original formulation to lift this unphysical restriction, obtaining Eq. (1) above. This equation can be written as a system of nonlinear fractional differential equations (Kobayashi and Coimbra 2005; Farazmand and Haller 2014) in terms of the particle position $\mathbf{y}$ and relative velocity $\mathbf{w}$ [see Eq. (7) below]. While there exist fundamental results for special classes of fractional differential equations (see, e.g., Podlubny 1998), the MR equation does not fit in any of these classes and requires separate treatment.

Even the existence and uniqueness of solutions to the MR equation is unclear. Only recently have Farazmand and Haller (2014) proved the existence, uniqueness and regularity of its *local* mild solutions. They also showed that only under the unphysical assumption $\mathbf{w}(t_0) = 0$ does the MR equation admit strong solutions. Here, we prove *global* existence and uniqueness of mild solutions to the MR equation.

We start by rewriting the MR equation in dimensionless form as a system of nonlinear fractional differential equations [see Eq. (7)] in terms of the particle position y and the function w, as defined by (1). After rescaling time, we compute the solution of the MR equation in the limit of infinitesimally small particles and then get integral equations for the MR equation for arbitrary particle sizes. We then use these integral equations to prove an analytic upper bound for the velocity $\mathbf{v}$ of a small particle of radius a. Next, we show that $\mathbf{v}$ decays algebraically to an asymptotic state that is $\mathcal{O}(\frac{a^2}{L^2})$-close to the fluid velocity $\mathbf{u}$, where L is a characteristic length scale of the fluid flow. We demonstrate these properties numerically on the double gyre flow model of Shadden et al. (2005). Finally, we construct a specific continuation method to prove global existence and uniqueness of mild solutions for the Maxey–Riley equation.

2 Preliminaries

2.1 The MR Equation in Dimensionless Variables

We rewrite the Maxey–Riley Eq. (1) in a form more appropriate for mathematical analysis. First, we rescale space, velocities and time using the characteristic length scale L, the characteristic velocity U and the characteristic time scale $T = L/U$. Using the resulting dimensionless variables $\mathbf{y} \mapsto \mathbf{y}/L$, $\mathbf{u} \mapsto \mathbf{u}/U$, $\mathbf{v} \mapsto \mathbf{v}/U$ and $t \mapsto t/T$ and rearranging various terms, we write (1) as a system of first-order integro-differential equations

$$\begin{aligned}
&\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = \mathbf{w} + \mathbf{A_u}(\mathbf{y}, t),\\
&\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}t} + \kappa\mu^{1/2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\sqrt{\pi}}\int_{t_0}^{t}\frac{\mathbf{w}(s)}{\sqrt{t-s}}\,\mathrm{d}s\right) + \mu\mathbf{w} = -\mathbf{M_u}(\mathbf{y}, t)\mathbf{w} + \mathbf{B_u}(\mathbf{y}, t),\\
&\mathbf{y}(t_0) = \mathbf{y}_0, \quad \mathbf{w}(t_0) = \mathbf{w}_0,
\end{aligned} \tag{2}$$

with

$$\mathbf{w}(t) = \mathbf{v}(t) - \mathbf{u}(\mathbf{y}(t), t) - \frac{\gamma}{6}\mu^{-1}\Delta\mathbf{u}(\mathbf{y}(t), t), \tag{3a}$$

$$\begin{aligned}
\mathbf{A_u} &= \mathbf{u} + \frac{\gamma}{6}\mu^{-1}\Delta\mathbf{u},\\
\mathbf{B_u} &= \left(\frac{3R}{2} - 1\right)\left(\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} - \mathbf{g}\right) + \left(\frac{R}{20} - \frac{1}{6}\right)\gamma\mu^{-1}\frac{\mathrm{D}}{\mathrm{D}t}\Delta\mathbf{u}\\
&\quad - \frac{\gamma}{6}\mu^{-1}\left[\nabla\mathbf{u} + \frac{\gamma}{6}\mu^{-1}\nabla\Delta\mathbf{u}\right]\Delta\mathbf{u},\\
\mathbf{M_u} &= \nabla\mathbf{u} + \frac{\gamma}{6}\mu^{-1}\nabla\Delta\mathbf{u}.
\end{aligned} \tag{3b}$$

In deriving (2), we used the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{t_0}^{t}\frac{\mathbf{w}(s)}{\sqrt{t-s}}\,\mathrm{d}s = \int_{t_0}^{t}\frac{\dot{\mathbf{w}}(s)}{\sqrt{t-s}}\mathrm{d}s + \frac{\mathbf{w}(t_0)}{\sqrt{t-t_0}},$$

obtained from carrying out the differentiation and then integrating by parts [see, e.g., (Podlubny 1998, Chapter 2)].

The dimensionless parameters in (3) are defined as

$$R = \frac{2\rho_f}{\rho_f + 2\rho_p}, \qquad \mu = \frac{R}{\mathrm{St}}, \qquad \kappa = \sqrt{\frac{9R}{2}}, \qquad \gamma = \frac{9R}{2\mathrm{Re}}, \tag{4}$$

where the Stokes (St) and the fluid Reynolds (Re) numbers are defined as

$$\mathrm{St} = \frac{2}{9}\left(\frac{a}{L}\right)^2\mathrm{Re}, \qquad \mathrm{Re} = \frac{UL}{\nu}. \tag{5}$$

Note that the vector fields $\mathbf{A_u}, \mathbf{B_u} : \mathcal{D}\times\mathbb{R}^+ \to \mathbb{R}^n$ and the tensor field $\mathbf{M_u} : \mathcal{D}\times\mathbb{R}^+ \to \mathbb{R}^{n\times n}$ are known functions of the fluid velocity field $\mathbf{u}$.

Equation (3a) defines a simple one-to-one correspondence between the particle velocity $\mathbf{v}$ and the variable $\mathbf{w}$. Once a solution $(\mathbf{y}, \mathbf{w})$ of (2) is known, the particle velocity can readily be obtained as $\mathbf{v}(t) = \mathbf{w}(t) + \mathbf{u}(\mathbf{y}(t), t) + (\gamma\mu^{-1}/6)\Delta\mathbf{u}(\mathbf{y}(t), t)$. In the absence of the Faxén correction term $(\gamma\mu^{-1}/6)\Delta\mathbf{u}$, the variable $\mathbf{w} = \mathbf{v} - \mathbf{u}$ is the relative velocity between the particle and the fluid.

The integral term in (2) is proportional to the Riemann–Liouville fractional derivative of order 1/2, which is defined as

$$\frac{\mathrm{d}^{1/2}\mathbf{w}}{\mathrm{d}t^{1/2}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\sqrt{\pi}}\int_{t_0}^{t}\frac{\mathbf{w}(s)}{\sqrt{t-s}}\,\mathrm{d}s\right), \tag{6}$$

with $t \geq t_0$ (Podlubny 1998). Using this notation, we write the initial value problem (2) in the more compact form

$$\begin{aligned}
&\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = \mathbf{w} + \mathbf{A_u}(\mathbf{y}, t),\\
&\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}t} + \kappa\mu^{1/2}\frac{\mathrm{d}^{1/2}\mathbf{w}}{\mathrm{d}t^{1/2}} + \mu\mathbf{w} = -\mathbf{M_u}(\mathbf{y}, t)\mathbf{w} + \mathbf{B_u}(\mathbf{y}, t),\\
&\mathbf{y}(t_0) = \mathbf{y}_0, \quad \mathbf{w}(t_0) = \mathbf{w}_0.
\end{aligned} \tag{7}$$

2.2 Setup and Assumptions

We use $|\cdot|$ to denote the Euclidean norm on $\mathbb{R}^m$ with $m \in \{n, 2n\}$. The induced operator norm of a square matrix acting on $\mathbb{R}^m$ is denoted by $\|\cdot\|$. We denote the supremum norm of functions by $\|\cdot\|_\infty$.

For future use, we also define the function space

$$X_K^{t,h} = \left\{f \in C\left([t, t+h]; \mathbb{R}^m\right) : \|f\|_\infty \leq K\right\}. \tag{8}$$

Since $X_K^{t,h}$ is a closed subset of $C([t, t+h]; \mathbb{R}^m)$, the metric space $(X_K^{t,h}, \|\cdot\|_\infty)$ is a complete metric space.

For the MR Eq. (2) [or its original form (1)] to make sense, the partial derivatives of the fluid velocity $\partial_x^\alpha\mathbf{u}(\mathbf{x}, t)$ and $\partial_t\partial_x^\beta\mathbf{u}(\mathbf{x}, t)$, with $|\alpha| \leq 3$ and $|\beta| \leq 2$ must exist.

The Faxén corrections (the terms involving $\Delta\mathbf{u}$) are routinely neglected in practice (Maxey 1987; Balkovsky et al. 2001). Upon neglecting the Faxén terms, the regularity assumption for the fluid velocity relaxes to the existence of the first-order partial derivative with respect to space and time, that is, $|\alpha| \leq 1$ and $\beta = 0$. In the analysis presented here, we do not neglect the Faxén terms.

For proving the global existence and uniqueness of solutions of the MR equation, we need the above partial derivatives to be uniformly bounded and Lipschitz continuous in space and time. In particular, we assume the following.

(H1) The velocity field $\mathbf{u}(\mathbf{x}, t)$ is smooth enough such that the partial derivatives $\partial_x^\alpha\mathbf{u}$ with $|\alpha| \leq 3$ and the mixed partial derivatives $\partial_t\partial_x^\beta\mathbf{u}$ with $|\beta| \leq 2$ defined over the domain $\mathcal{D} \times R^+$ are uniformly bounded.

(H2) The velocity field $\mathbf{u}(\mathbf{x}, t)$ is smooth enough such that the partial derivatives $\partial_x^\alpha \mathbf{u}$ with $|\alpha| \leq 3$ and the mixed partial derivatives $\partial_t \partial_x^\beta \mathbf{u}$ with $|\beta| \leq 2$ defined over the domain $\mathcal{D} \times R^+$ are uniformly Lipschitz continuous.

Remark Neglecting the Faxén terms, assumptions (H1) and (H2) relax, respectively, to the uniform boundedness and uniform Lipschitz continuity of the fluid velocity $\mathbf{u}$ and acceleration $\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}$.

Assumption (H1) implies the existence of constants $L_A, L_B, L_M > 0$ such that

$$\|\mathbf{A_u}\|_\infty \leq L_A, \quad \|\mathbf{B_u}\|_\infty \leq L_B, \quad \|\mathbf{M_u}\|_\infty \leq L_M. \tag{9}$$

Assumption (H2), on the other hand, implies the existence of a constant $L_c > 0$ such that

$$\begin{aligned} |\mathbf{A_u}(\mathbf{y_1}, \tau) - \mathbf{A_u}(\mathbf{y_2}, \tau)| &\leq L_c |\mathbf{y_1} - \mathbf{y_2}|, \\ |\mathbf{B_u}(\mathbf{y_1}, \tau) - \mathbf{B_u}(\mathbf{y_2}, \tau)| &\leq L_c |\mathbf{y_1} - \mathbf{y_2}|, \\ \|\mathbf{M_u}(\mathbf{y_1}, \tau) - \mathbf{M_u}(\mathbf{y_2}, \tau)\| &\leq L_c |\mathbf{y_1} - \mathbf{y_2}|, \end{aligned} \tag{10}$$

for all $\mathbf{y_1}, \mathbf{y_2} \in \mathcal{D}$ and all $\tau \in \mathbb{R}^+$. The supremum norms in (9) are taken over all $(\mathbf{y}, \tau) \in \mathcal{D} \times \mathbb{R}^+$.

Farazmand and Haller (2014) proved the following local existence and uniqueness result.

Theorem 1 (Farazmand and Haller 2014) *Assume that (H1) and (H2) hold. For any* $(\mathbf{y}_0, \mathbf{w}_0) \in \mathcal{D} \times \mathbb{R}^n$*, there exists a time increment* $\delta t > 0$ *such that, over the time interval* $[t_0, t_0 + \delta t)$*, the Maxey–Riley equation* (7) *has a unique solution* $(\mathbf{y}(t), \mathbf{w}(t))$ *satisfying* $(\mathbf{y}(t_0), \mathbf{w}(t_0)) = (\mathbf{y_0}, \mathbf{w_0})$.

2.3 The MR Equation Does Not Generate a Dynamical System

For ordinary differential equations, one may construct global solutions by continuation. In particular, given a local solution $(\mathbf{y}(t), \mathbf{w}(t))$ existing on a time interval $[t_0, t_0+\Delta_1)$, one shows that the solution does not blow up at $t = t_0 + \Delta_1$. Then initializing the ordinary differential equation from time $t = t_0 + \Delta_1$ with initial condition $(\mathbf{y}(t_0 + \Delta_1), \mathbf{w}(t_0 + \Delta_1))$, the local existence and uniqueness result is reapplied to show that the solution can be extended to an interval $[t_0, t_0 + \Delta_1 + \Delta_2)$. Repeating the above steps, the solution can be extended to a time interval $[t_0, t_0 + \Delta_1 + \Delta_2 + \Delta_3 + \cdots)$. Finally, one shows that the infinite series $\Delta_1 + \Delta_2 + \Delta_3 + \cdots$ diverges and infers global existence and uniqueness.

This continuation argument assumes that the flow map $\mathbf{F}_{t_0}^t : (\mathbf{y}_0, \mathbf{w}_0) \mapsto (\mathbf{y}(t), \mathbf{w}(t))$ has the semigroup property $\mathbf{F}_{t_0}^t = \mathbf{F}_{t_1}^t \circ \mathbf{F}_{t_0}^{t_1}$ for all $t_0 < t_1 < t$. Due to the fractional derivative, however, the flow map of the MR Eq. (7) is not a semigroup.

To see this, consider the solution $(\mathbf{y}(t), \mathbf{w}(t))$ starting from $(\mathbf{y}_0, \mathbf{w}_0)$ at time t_0. Due to the Basset history force (that is, the fractional derivative in (7)), the trajectory $(\mathbf{y}(t), \mathbf{w}(t))$ for $t > t_1$ is influenced by its entire past history. A trajectory initialized from $(\mathbf{y}(t_1), \mathbf{w}(t_1))$ is, however, ignorant of this history and therefore will follow a different path (see Fig. 1 for an illustration).

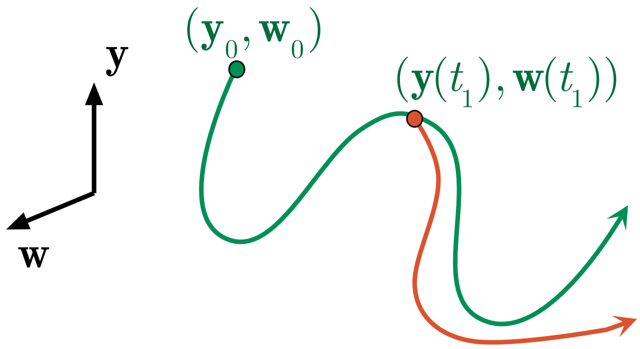

Fig. 1 A trajectory $(\mathbf{y}(t), \mathbf{w}(t))$ of the MR equation (7) initialized from $(\mathbf{y}_0, \mathbf{w}_0)$ and passing through $(\mathbf{y}(t_1), \mathbf{w}(t_1))$ at time t_1 (*green curve*). A trajectory initialized from $(\mathbf{y}(t_1), \mathbf{w}(t_1))$ at time t_1 (*red curve*) does not follow the trajectory $(\mathbf{y}(t), \mathbf{w}(t))$ (Color figure online)

As a result, the usual continuation methods for ordinary differential equations do not apply here. In Sect. 4.1, we construct a specific continuation suitable for the MR equation.

2.4 Rescaling Time

We introduce a rescaling of time that further simplifies the forthcoming analysis. Dividing the $\mathbf{w}$ component of Eq. (2) by μ and letting $\epsilon := \frac{1}{\mu}$, we get

$$\begin{aligned} &\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = \mathbf{w} + \mathbf{A_u}(\mathbf{y}, t),\\ &\epsilon\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}t} + \epsilon^{1/2}\kappa\frac{\mathrm{d}^{1/2}\mathbf{w}}{\mathrm{d}t^{1/2}} + \mathbf{w} = -\epsilon\mathbf{M_u}(\mathbf{y}, t)\mathbf{w} + \epsilon\mathbf{B_u}(\mathbf{y}, t),\\ &\mathbf{y}(t_0) = \mathbf{y}_0, \quad \mathbf{w}(t_0) = \mathbf{w}_0. \end{aligned} \tag{11}$$

Note that by (5), $\epsilon = \frac{St}{R} = \frac{2}{9R}\left(\frac{a}{L}\right)^2 \mathrm{Re}$. Since the MR equation holds for small particles ($a \ll L$), ϵ is necessarily a small and positive parameter: $0 < \epsilon \ll 1$. Thus the limit $\epsilon \to 0$ ($a \to 0$) describes the limit of infinitesimally small particles.

Rescaling time as $t = t_0 + \epsilon\tau$, we have

$$\begin{aligned} &\frac{\mathrm{d}\tilde{\mathbf{y}}}{\mathrm{d}\tau} = \epsilon\left[\tilde{\mathbf{w}} + \tilde{\mathbf{A}}_\mathbf{u}(\tilde{\mathbf{y}}, \tau)\right],\\ &\frac{\mathrm{d}\tilde{\mathbf{w}}}{\mathrm{d}\tau} + \kappa\frac{\mathrm{d}^{1/2}\tilde{\mathbf{w}}}{\mathrm{d}\tau^{1/2}} + \tilde{\mathbf{w}} = \epsilon\left[-\tilde{\mathbf{M}}_\mathbf{u}(\tilde{\mathbf{y}}, \tau)\tilde{\mathbf{w}} + \tilde{\mathbf{B}}_\mathbf{u}(\tilde{\mathbf{y}}, \tau)\right],\\ &\tilde{\mathbf{y}}(0) = \mathbf{y}_0, \quad \tilde{\mathbf{w}}(0) = \mathbf{w}_0, \end{aligned} \tag{12}$$

where

$$\begin{aligned} \tilde{\mathbf{y}}(\tau) &= \mathbf{y}(t_0 + \epsilon\tau), \ \tilde{\mathbf{w}}(\tau) = \mathbf{w}(t_0 + \epsilon\tau),\\ \tilde{\mathbf{A}}_\mathbf{u}(\tilde{\mathbf{y}}, \tau) &= \mathbf{A_u}(\mathbf{y}, t_0 + \epsilon\tau), \ \tilde{\mathbf{B}}_\mathbf{u}(\tilde{\mathbf{y}}, \tau) = \mathbf{B_u}(\mathbf{y}, t_0 + \epsilon\tau), \ \tilde{\mathbf{M}}_\mathbf{u}(\tilde{\mathbf{y}}, \tau)\\ &= \mathbf{M_u}(\mathbf{y}, t_0 + \epsilon\tau), \end{aligned}$$

and

$$\frac{\mathrm{d}^{1/2}\tilde{\mathbf{w}}}{\mathrm{d}\tau^{1/2}} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{\sqrt{\pi}}\int_0^\tau \frac{\tilde{\mathbf{w}}(s)}{\sqrt{\tau - s}}\,\mathrm{d}s\right).$$

The above rescaling of time has been previously used (Rubin et al. 1995; Mograbi and Bar-Ziv 2006; Haller and Sapsis 2008) for the asymptotic analysis of the MR equation without memory. It allows us to treat Eq. (12) as a *regular* perturbation problem with respect to ϵ, as opposed to treating Eq. (11) as a *singular* perturbation problem with respect to ϵ. To see the singular nature of the perturbation, divide the $w(t)$ equation of (11) by ϵ and take the limit $\epsilon \to 0$: The limit will be unbounded. The regularized $\epsilon \to 0$ limit of (12) is unphysical, corresponding to an inertial particle of zero radius. However, Eq. (12) is physically meaningful for any $\epsilon > 0$.

Note that a unique solution of the initial value problem (IVP) (12) over the time interval $[0, \delta)$ exists if and only if the unscaled IVP (7) has a unique solution over the time interval $[t_0, t_0 + \epsilon\delta)$. Therefore, in the following, we study the IVP (12). We will first analyze the solution of the IVP (12) in the limit $\epsilon = 0$ and then use this solution to study the IVP (12) for $\epsilon > 0$. For notational simplicity, we omit the tilde signs from all the variables.

3 Asymptotic Behavior

3.1 $\epsilon = 0$ Limit

We start with the limit $\epsilon = 0$ of Eq. (12), which as discussed in Sect. 2.4 is unphysical. In this limit, $\mathbf{y}(\tau) = \mathbf{y_0}$ remains constant for all times, and $\mathbf{w}(\tau)$ becomes

$$\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}\tau} + \kappa \frac{\mathrm{d}^{1/2}\mathbf{w}}{\mathrm{d}\tau^{1/2}} + \mathbf{w} = 0, \quad \mathbf{w}(0) = \mathbf{w}_0, \tag{13}$$

where κ is the dimensionless parameter defined by (4). Equation (13) is a linear equation tractable by Laplace transforms (Gorenflo and Mainardi 1997; Podlubny 1998). This leads to the following result.

Theorem 2 *The general solution of* (13) *is given by* $\mathbf{w}(\tau; \mathbf{w_0}) = \psi_\kappa(\tau)\mathbf{w_0}$, *where the positive, scalar function* $\psi_\kappa : [0, \infty) \to \mathbb{R}^+$ *has the following properties.*

1. ψ_κ *is given by the inverse Laplace transform*

$$\psi_\kappa(\tau) = \mathcal{L}^{-1}\left[\frac{1}{\left(\sqrt{s} + \lambda_+\right)\left(\sqrt{s} + \lambda_-\right)}\right](\tau), \tag{14}$$

where

$$\lambda_\pm = \frac{\kappa \pm \sqrt{\kappa^2 - 4}}{2}.$$

2. ψ_κ *obeys the asymptotic decay rate*

$$\psi_\kappa(\tau) \sim \frac{\kappa}{2\sqrt{\pi}}\tau^{-3/2} + \mathcal{O}\left(\tau^{-5/2}\right) \quad as \quad \tau \to \infty. \tag{15}$$

3. *There is a differentiable function* $\phi_\kappa : [0, \infty) \to \mathbb{R}^+$ *such that* $\psi_\kappa = -\phi_\kappa'$.

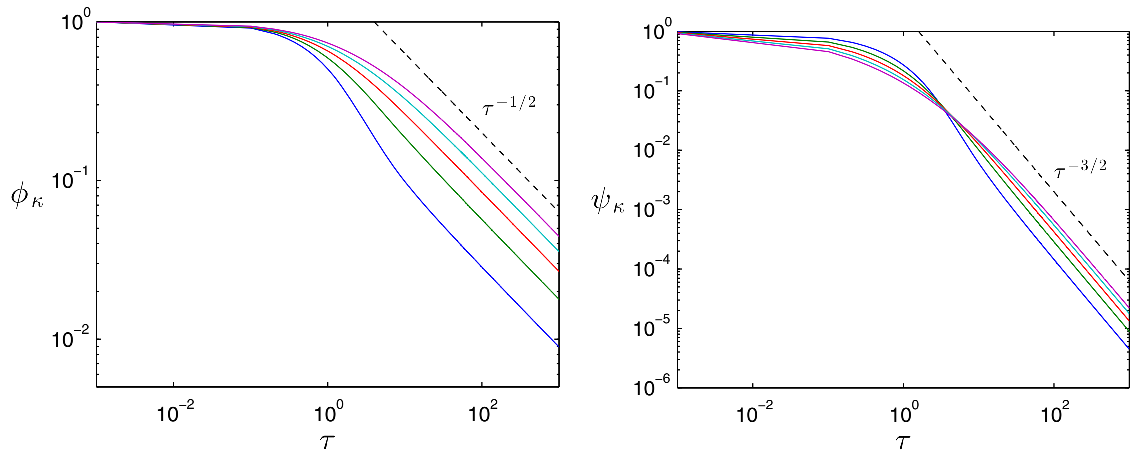

Fig. 2 Functions ϕ_κ and $\psi_\kappa = -\phi_\kappa'$. The functions are evaluated for $\kappa = 0.5$ (*blue*), $\kappa = 1$ (*green*), $\kappa = 1.5$ (*red*), $\kappa = 2$ (*cyan*) and $\kappa = 2.5$ (*magenta*) (Color figure online)

4. *The functions ψ_κ and ϕ_κ are smooth over* $\in (0, \infty)$ *and completely monotonic decreasing, i.e.,*

$$(-1)^j \psi_\kappa^{(j)}(\tau) \geq 0, \quad (-1)^j \phi_\kappa^{(j)}(\tau) \geq 0, \quad j = 0, 1, 2, \ldots, \quad \forall \tau > 0$$

5. $\psi_\kappa(0) = 1$ *and* $\phi_\kappa(0) = 1$.

Proof See "Appendix 1" for the proof of 1 and 2 and the explicit calculation of ψ_κ. For the proof of 3, 4 and 5, see the properties demonstrated for $u_\delta(t)(\psi_\kappa(\tau))$ and $u_0(t)(\phi_\kappa(\tau))$ in Gorenflo and Mainardi (1997, Section 4). □

Figure 2 shows the functions ϕ_κ and ψ_κ computed by numerically inverting their Laplace transforms. It follows from properties 2 and 3 from Theorem 2 that ϕ_κ decays asymptotically as $\tau^{-1/2}$, as confirmed by the numerics.

Since the properties of Theorem 2 hold for any $\kappa > 0$, we omit the dependence of ψ_κ and ϕ_κ on κ and write ψ and ϕ, respectively.

3.2 $\epsilon > 0$ Case

Now we analyze the general case of $\epsilon > 0$, i.e.,

$$\begin{aligned}
&\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}\tau} = \epsilon \left[\mathbf{w} + \mathbf{A_u}(\mathbf{y}, \tau)\right] \\
&\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}\tau} + \kappa \frac{\mathrm{d}^{1/2}\mathbf{w}}{\mathrm{d}\tau^{1/2}} + \mathbf{w} = \epsilon \left[-\mathbf{M_u}(\mathbf{y}, \tau)\mathbf{w} + \mathbf{B_u}(\mathbf{y}, \tau)\right], \\
&\mathbf{y}(0) = \mathbf{y}_0, \quad \mathbf{w}(0) = \mathbf{w}_0,
\end{aligned} \tag{16}$$

which is Eq. (12) with tilde signs omitted. Solutions of (16) satisfy the integral equations

$$
\begin{aligned}
\mathbf{y}(\tau) &= \mathbf{y_0} + \epsilon \int_0^\tau \mathbf{w}(s) + \mathbf{A_u}(\mathbf{y}(s), s)\,\mathrm{d}s,\\
\mathbf{w}(\tau) &= \psi(\tau)\mathbf{w_0} + \epsilon \int_0^\tau \psi(\tau - s)\left[-\mathbf{M_u}(\mathbf{y}(s), s)\mathbf{w}(s) + \mathbf{B_u}(\mathbf{y}(s), s)\right]\mathrm{d}s,
\end{aligned}
\tag{17}
$$

where $\psi(\tau)$ is given by (14) and satisfies the properties listed in Theorem 2.

This integral equation is essentially a variation-of-constants formula. The $\mathbf{y}$-equation in (17) is obtained by formal integration of the $\mathrm{d}\mathbf{y}/\mathrm{d}\tau$ equation of (16). For the $\mathbf{w}$-equation, let $\mathbf{W}(s)$ denote the Laplace transform of $\mathbf{w}(\tau)$. Taking the Laplace transform of (16) yields

$$
\mathbf{W}(s) = \frac{\mathbf{w_0}}{\left(\sqrt{s} + \lambda_+\right)\left(\sqrt{s} + \lambda_-\right)} + \frac{\mathcal{L}\left[-\mathbf{M_u}(\mathbf{y}(\tau), \tau)\mathbf{w}(\tau) + \mathbf{B_u}(\mathbf{y}(\tau), \tau)\right](s)}{\left(\sqrt{s} + \lambda_+\right)\left(\sqrt{s} + \lambda_-\right)}.
$$

Taking the inverse Laplace transform, we obtain the $\mathbf{w}$-component of Eq. (17) where $\psi(\tau)$ is given by (14).

Definition 1 A *mild* solution of the IVP (16) is a function $(\mathbf{y}, \mathbf{w}) : [0, \delta) \to \mathbb{R}^{2n}$ that solves the integral equation (17). The existence time $\delta > 0$ may be infinite.

Using the integral equation (17), we find an upper bound for $|\mathbf{w}(\tau; \mathbf{y}_0, \mathbf{w}_0)|$ and its asymptotic limit.

Theorem 3 *Assume that (H1) holds and $\epsilon < 1/L_M$. Let $(\mathbf{y}, \mathbf{w}) : [0, \delta) \to \mathbb{R}^{2n}$ be a mild solution of (16) where $[0, \delta)$ is the maximal interval of existence of such solutions.*

(i) An explicit envelope for $|\mathbf{w}(\tau; \mathbf{y}_0, \mathbf{w}_0)|$ is given by

$$
|\mathbf{w}(\tau; \mathbf{y}_0, \mathbf{w}_0)| \le |\mathbf{w}_0| \left[\sum_{j=1}^{\infty} (\epsilon L_M)^{j-1} \psi^{*j}(\tau)\right] + \epsilon L_B \left(1 - \phi(\tau)\right) + \frac{\epsilon^2 L_M L_B}{1 - \epsilon L_M},
\tag{18}
$$

where ψ^{*j} is the j-fold convolution of ψ. Moreover, the series converges uniformly and is bounded for all τ.

(ii) $|\mathbf{w}(\tau; \mathbf{y}_0, \mathbf{w}_0)|$ is bounded for all $\tau \in [0, \delta)$. Specifically,

$$
\sup_{0 \le \tau < \delta} |\mathbf{w}(\tau; \mathbf{y}_0, \mathbf{w}_0)| \le \frac{|\mathbf{w_0}| + \epsilon L_B}{1 - \epsilon L_M}.
\tag{19}
$$

(iii) If $\delta = \infty$, the asymptotic limit of $\mathbf{w}$ satisfies

$$
\limsup_{\tau \to \infty} |\mathbf{w}(\tau; \mathbf{y}_0, \mathbf{w}_0)| \le \frac{\epsilon L_B}{1 - \epsilon L_M}.
\tag{20}
$$

Proof See "Appendix 2".

In deriving the upper envelope (18) and the subsequent upper bounds (19) and (20), we have made several upper estimates. The natural question arising is how sharp these estimates are. In the following section, among other things, we show with a numerical example that these bounds are sharp by showing that they can be saturated.

3.3 Numerical Verification

We illustrate the results of Theorem 3 with an example. For the fluid flow, we use the double gyre model of Shadden et al. (2005). It is a two-dimensional velocity field with the stream function

$$\mathcal{H}(x, y, t) = A \sin(\pi f(x, t)) \sin(\pi y), \tag{21}$$

where

$$f(x, t) = \alpha \sin(\omega t)x^2 + (1 - 2\alpha \sin(\omega t))x.$$

We let $A = 0.1$, $\omega = \pi$ and $\alpha = 0.01$.

The Hamiltonian $\mathcal{H}$ defines the velocity field $\mathbf{u} = (-\partial_y \mathcal{H}, \partial_x \mathcal{H})^\top$ which we use to solve the initial value problem (7) using the numerical scheme developed by Daitche (2013). We will neglect the Faxén corrections, such that $\mathbf{A_u} = \mathbf{u}$, $\mathbf{B_u} = \left(\frac{3R}{2} - 1\right) \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}$ and $\mathbf{M_u} = \nabla \mathbf{u}$ (recall, however, that our main results also hold in the presence of the Faxén corrections).

For the parameters of the inertial particle, we let $\mathrm{St} = R/100$ resulting in $\mu = 100$ (or $\epsilon = 0.01$). This corresponds to a small inertial particle (with respect to the underlying flow) since by Eq. (5) the Stokes number is proportional to the square of the particle's radius. Three values of R are considered here: $R = 2/3$ (neutrally buoyant particle, $\rho_f = \rho_p$), $R = 1/3$ (aerosol, $\rho_f < \rho_p$) and $R = 1$ (bubble, $\rho_f > \rho_p$). In each case, we release 15 trajectories with initial conditions $\mathbf{y}_0$ uniformly distributed in the domain $[0.2, 1.8] \times [0.2, 0.8]$ (i.e., $\mathbf{y}_0 \in \{0.2, 0.6, 1.0, 1.4, 1.8\} \times \{0.2, 0.5, 0.8\}$) and identical initial relative velocities $\mathbf{w}_0 = (10, 10)^\top$. We picked large initial velocities in this example to show the algebraic decay of $|\mathbf{w}|$ more clearly.

We take the most conservative choices of the upper bounds L_B and L_M, i.e., $L_B = \|\mathbf{B_u}\|_\infty$ and $L_M = \|\mathbf{M_u}\|_\infty$. For the neutrally buoyant particle, i.e., $R = 2/3$, $\mathbf{B_u}$ vanishes identically, resulting in $L_B = 0$. The norm $\|\mathbf{M_u}\|_\infty$ is, however, independent of R, and we have $L_M \simeq 1.4237$. Theorem 3 therefore implies that for a neutrally buoyant particle, $|\mathbf{w}(t)|$ must decay to zero asymptotically, which agrees with our numerical result (see Fig. 3a). Physically, this implies that the inertial particle trajectory converges to a fluid trajectory. In the case of neutrally buoyant particles, the theoretical envelope and the numerical solutions almost coincide. This is because for $R = 2/3$, the two terms proportional to L_B vanish in the estimate (18), apparently making the upper bound close to optimal. A close-up view is shown in the inset of Fig. 3a.

Interestingly, for the neutrally buoyant particle, the evolution of the relative velocity magnitude $|\mathbf{w}|$ seems to be independent of the initial positions $\mathbf{y}_0$ as all 15 curves coincide in Fig. 3a.

For the bubble ($R = 1$) and the aerosol ($R = 1/3$), we have $L_B \simeq 0.1207$ and $L_M \simeq 1.4237$. The resulting envelope (18) and the asymptotic upper bound $\epsilon L_B/(1 - \epsilon L_M)$ are also shown (red and black dashed curves, respectively) which shows a perfect agreement with the numerical results. In plotting the envelopes, $\mathcal{O}(\epsilon^2)$-terms are neglected. The numerical solutions come very close to the analytic envelope of Theorem 3 (part (i)), indicating the tightness of the estimates.

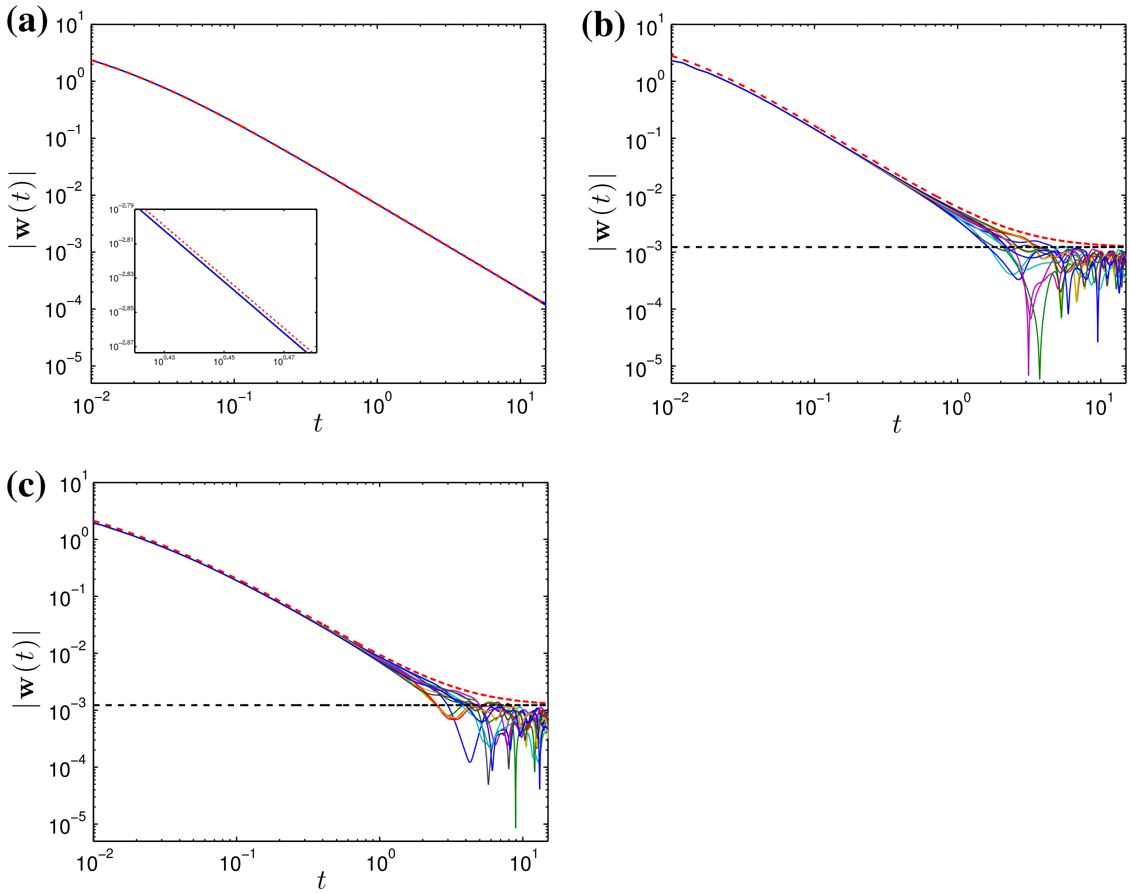

Fig. 3 Decay of the relative velocity magnitude $|\mathbf{w}(t)|$ for $R = 2/3$ (**a**), $R = 1/3$ (**b**) and $R = 1$ (**c**). The *dashed red lines* mark the analytic envelope from Theorem 3 part (i). The *dashed black lines* mark the asymptotic upper bound of $|\mathbf{w}|$, i.e., $\epsilon L_B/(1 - \epsilon L_M)$. The initial value of $|\mathbf{w}(t)|$ is $10\sqrt{2}$ in all cases. In order to focus on the asymptotics, we only plot the graphs for $t \geq 10^{-2}$ (Color figure online)

The upper envelope (18) depends on functions ϕ and ψ which in turn depend on the parameter $\kappa = \sqrt{9R/2}$. The parameter R is governed by the ratio between the particle density ρ_p and the fluid density ρ_f. As this ratio varies, the upper envelope also changes. Owing to the algebraic transient decay of ϕ and ψ (see Fig. 2), however, the envelope exhibits an algebraic decay regardless of the value of R. Figure 4 shows the behavior of the upper envelope (neglecting $\mathcal{O}(\epsilon^2)$-terms) for the double gyre parameters and various values of R. For neutrally buoyant particle ($R = 2/3$), there is a monotonic decay with the algebraic rate $t^{-3/2}$. For other values of R, the envelope decays to the asymptotic upper bound. There is still a transient algebraic decay whose rate varies, depending on the parameter R, between $t^{-1.7}$ and $t^{-1.2}$.

4 Global Existence and Uniqueness

In this section, we prove the global existence and uniqueness of mild solutions to the full Maxey–Riley equation (1) with the Faxén correction terms. In particular, we show that the equivalent reformulation (16) admits unique mild solutions for all times, that is, the integral equations (17) have a unique solution over $\mathbb{R}^+$.

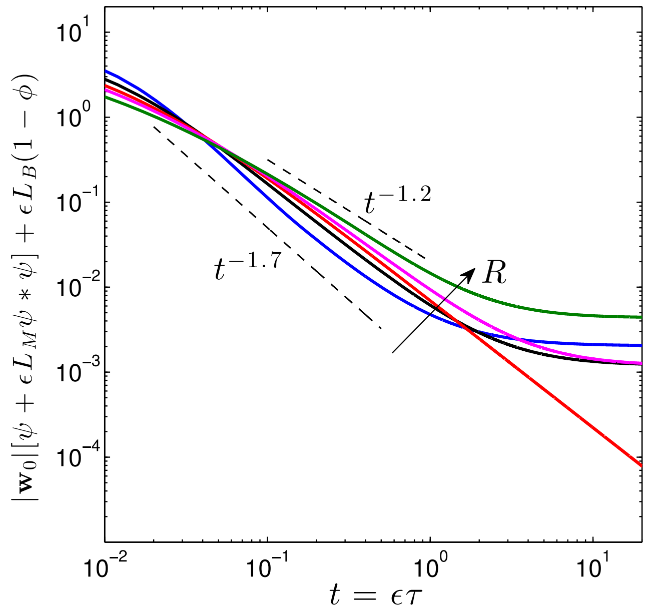

Fig. 4 Upper envelope (18), neglecting $\mathcal{O}(\epsilon^2)$-terms, for $R = 1/10$ (*blue*), $R = 1/3$ (*black*), $R = 2/3$ (*red*), $R = 1$ (*magenta*) and $R = 19/10$ (*green*) (Color figure online)

The existence of a unique local solution follows from Theorem 1. Specifically, the integral equation (17) has a unique solution over the time interval $[0, \delta t/\epsilon)$, where δt is the same time window as in Theorem 1, with the ϵ appearing due to the rescaling $t = t_0 + \epsilon\tau$ as introduced in Sect. 2.4. For notational convenience, we let $\delta = \delta t/\epsilon$.

As discussed in Sect. 2.3, the usual continuation methods used for ordinary differential equations do not apply to fractional differential equations. Therefore, we construct a specific continuation method suitable for the MR equation, which is based on the continuation method presented in the work of Kou et al. (2012) for a different class of fractional differential equations. We then show that this continuation can be repeated indefinitely to extend the solutions to the time interval $[0, \infty)$. Our approach can be summarized in the following steps.

Step 1. Showing that the local solution of the integral equation (17), defined on $[0, \delta)$, is well defined at time $\tau = \delta$.
Step 2. Defining a suitable integral operator $\mathbf{F}$ over an appropriate complete metric space whose fixed points extend the local solution of (17) from $[0, \delta)$ to $[0, \delta + h)$, for a suitable constant $h > 0$.
Step 3. Showing that the operator $\mathbf{F}$ has at least one fixed point.
Step 4. Showing that this continuation is unique.
Step 5. Showing that one can repeat steps 1 to 4 indefinitely with the same continuation window h. That is, the local solution of (17) can be continued uniquely to $\mathbb{R}^+$.

The above steps prove the following global existence and uniqueness theorem.

Theorem 4 *Assume that (H1) and (H2) hold and $\epsilon < 1/L_M$. Then the MR equation has unique, continuous, mild solutions. That is, for any $(\mathbf{y}_0, \mathbf{w}_0) \in \mathbb{R}^{2n}$, there exists a unique, continuous function $(\mathbf{y}, \mathbf{w}) : [0, \infty) \to \mathbb{R}^{2n}$ satisfying (17) and $(\mathbf{y}(0), \mathbf{w}(0)) = (\mathbf{y}_0, \mathbf{w}_0)$.*

4.1 Continuation of the Local Solution

Let us denote the local solution of the MR equation, whose existence and uniqueness is guaranteed by Theorem 1, by $\mathbf{z}_{\text{loc}} = (\mathbf{y}_{\text{loc}}, \mathbf{w}_{\text{loc}})$. We first begin by showing that this local solution defined on $[0, \delta)$ is well defined at $\tau = \delta$.

Lemma 1 *The local solution* $\mathbf{z}_{\text{loc}} : [0, \delta) \to \mathbb{R}^{2n}$ *of the MR equation is well defined at* $\tau = \delta$ *and the limit* $\lim_{\tau\to\delta^-} \mathbf{z}_{\text{loc}}(\tau)$ *is given by*

$$\mathbf{z}_{\text{loc}}(\delta) = \begin{pmatrix} \mathbf{y_0} + \epsilon \int_0^\delta \mathbf{w}_{\text{loc}}(s) + \mathbf{A_u}(\mathbf{y}_{\text{loc}}(s), s)\, ds \\ \psi(\delta)\mathbf{w_0} + \epsilon \int_0^\delta \psi(\delta - s)\left[-\mathbf{M_u}(\mathbf{y}_{\text{loc}}(s), s)\mathbf{w}_{\text{loc}}(s) + \mathbf{B_u}(\mathbf{y}_{\text{loc}}(s), s)\right] ds \end{pmatrix}. \tag{22}$$

Proof See "Appendix 3". □

Let $(\mathbf{y}_{\text{loc}}, \mathbf{w}_{\text{loc}}) : [0, \delta) \to \mathbb{R}^{2n}$ be the local solution of (17) whose existence and uniqueness is guaranteed by Theorem 1. Define

$$\mathbf{y}(\tau) = \mathbb{1}_{[0,\delta)}(\tau)\mathbf{y}_{\text{loc}}(\tau) + \mathbb{1}_{[\delta,\delta+h)}(\tau)\boldsymbol{\xi}(\tau), \tag{23a}$$

$$\mathbf{w}(\tau) = \mathbb{1}_{[0,\delta)}(\tau)\mathbf{w}_{\text{loc}}(\tau) + \mathbb{1}_{[\delta,\delta+h)}(\tau)\boldsymbol{\eta}(\tau), \tag{23b}$$

where $\mathbb{1}_A : \mathbb{R} \to \{0, 1\}$ is the indicator function of the set $A \subset \mathbb{R}$. Note that for $\tau \in [0, \delta)$, $(\mathbf{y}, \mathbf{w})$ coincides with the local solution $(\mathbf{y}_{\text{loc}}, \mathbf{w}_{\text{loc}})$. Assuming $(\mathbf{y}, \mathbf{w})$ is a continuation of this local solution to $[0, \delta + h)$, upon substitution in (17), we have

$$\begin{aligned} \boldsymbol{\eta}(\tau) &= \mathbf{y_0} + \epsilon \int_0^\delta \mathbf{w}_{\text{loc}}(s) + \mathbf{A_u}(\mathbf{y}_{\text{loc}}(s), s)\, \mathrm{d}s + \epsilon \int_\delta^\tau \boldsymbol{\eta}(s) + \mathbf{A_u}(\boldsymbol{\xi}(s), s)\, \mathrm{d}s, \\ \boldsymbol{\xi}(\tau) &= \psi(\tau)\mathbf{w_0} + \epsilon \int_0^\delta \psi(\tau - s)\left[-\mathbf{M_u}(\mathbf{y}_{\text{loc}}(s), s)\mathbf{w}_{\text{loc}}(s) + \mathbf{B_u}(\mathbf{y}_{\text{loc}}(s), s)\right] \mathrm{d}s \\ &\quad + \epsilon \int_\delta^\tau \psi(\tau - s)\left[-\mathbf{M_u}(\boldsymbol{\xi}(s), s)\boldsymbol{\eta}(s) + \mathbf{B_u}(\boldsymbol{\xi}(s), s)\right] \mathrm{d}s, \end{aligned} \tag{24}$$

for $\tau \in [\delta, \delta + h)$.

Therefore, $(\mathbf{y}, \mathbf{w})$ solves the integral equation (17) and hence is a mild solution of the MR equation if and only if the integral equation (24) has a solution. To show that such a solution exists, we solve the following fixed point problem. Let $\boldsymbol{\Phi} = (\boldsymbol{\xi}, \boldsymbol{\eta}) \in X_K^{\delta,h}$. Define the operator $\mathbf{F} : X_K^{\delta,h} \to C([\delta, \delta + h); \mathbb{R}^{2n})$ by

$$(\mathbf{F}\boldsymbol{\Phi})(\tau) = \boldsymbol{\Phi_0}(\tau) + \begin{pmatrix} \epsilon \int_\delta^\tau \boldsymbol{\eta}(s) + \mathbf{A_u}(\boldsymbol{\xi}(s), s)\, \mathrm{d}s \\ \epsilon \int_\delta^\tau \psi(\tau - s)\left[-\mathbf{M_u}(\boldsymbol{\xi}(s), s)\boldsymbol{\eta}(s) + \mathbf{B_u}(\boldsymbol{\xi}(s), s)\right] \mathrm{d}s \end{pmatrix}, \tag{25}$$

where

$$\mathbf{\Phi_0}(\tau) = \begin{pmatrix} \mathbf{y_0} + \epsilon \int_0^\delta \mathbf{w}_{\mathrm{loc}}(s) + \mathbf{A_u}(\mathbf{y}_{\mathrm{loc}}(s), s)\, \mathrm{d}s \\ \psi(\tau)\mathbf{w_0} + \epsilon \int_0^\delta \psi(\tau - s)\left[-\mathbf{M_u}(\mathbf{y}_{\mathrm{loc}}(s), s)\mathbf{w}_{\mathrm{loc}}(s) + \mathbf{B_u}(\mathbf{y}_{\mathrm{loc}}(s), s)\right] \mathrm{d}s \end{pmatrix}. \tag{26}$$

Note that $\mathbf{\Phi}_0$ depends only on the local solution $(\mathbf{y}_{\mathrm{loc}}, \mathbf{w}_{\mathrm{loc}})$ of the Maxey–Riley equation and hence is independent of $\mathbf{\Phi}$. We show that the operator $\mathbf{F}$ maps $X_K^{\delta,h}$ to itself (with K and h to be determined) and has a unique fixed point.

4.2 Existence of the Continuation

Proposition 1 *Assume that (H1) holds. There exist constants $h, K > 0$ such that the operator $\mathbf{F}$ defined in (25) maps $X_K^{\delta,h}$ to itself and has at least one fixed point.*

Proof For any $h, K > 0$ and $\mathbf{\Phi} \in X_K^{\delta,h}$ the function $\mathbf{F\Phi} : [\delta, \delta + h) \to \mathbb{R}^{2n}$ is clearly continuous, that is, $\mathbf{F\Phi} \in C\left([\delta, \delta + h); \mathbb{R}^{2n}\right)$. We choose $h, K > 0$ such that $\mathbf{F\Phi} \in X_K^{\delta,h}$, i.e., $\|\mathbf{F\Phi}\|_\infty \le K$. To this end, note that for any $h > 0$ and $\tau \in [\delta, \delta + h)$, we have

$$\begin{aligned} |(\mathbf{F\Phi})(\tau)| \le |\mathbf{\Phi_0}(\tau)| &+ \epsilon \int_\delta^{\delta+h} |\boldsymbol{\eta}(s)| + |\mathbf{A_u}(\boldsymbol{\xi}(s), s)|\, \mathrm{d}s \\ &+ \epsilon \int_\delta^{\delta+h} \psi(\tau - s)\left[|\mathbf{M_u}(\boldsymbol{\xi}(s), s)\boldsymbol{\eta}(s)| + |\mathbf{B_u}(\boldsymbol{\xi}(s), s)|\right] \mathrm{d}s. \end{aligned}$$

Take the supremum over $\tau \in [\delta, \delta + h)$ and use the bounds on $\|\mathbf{M_u}\|_\infty$, $\|\mathbf{B_u}\|_\infty$, $\|\mathbf{A_u}\|_\infty$, $|\mathbf{w}(\tau)|$, $\|\psi\|_\infty$ and $\|\boldsymbol{\eta}\|_\infty$ to get

$$\begin{aligned} \|\mathbf{F\Phi}\|_\infty &\le \|\mathbf{\Phi_0}\|_\infty + \epsilon \int_\delta^{\delta+h} \|\boldsymbol{\eta}\|_\infty + \|\mathbf{A_u}\|_\infty\, \mathrm{d}s \\ &\quad + \epsilon \int_\delta^{\delta+h} \left[\|\mathbf{M_u}\|_\infty \|\boldsymbol{\eta}\|_\infty + \|\mathbf{B_u}\|_\infty\right] \mathrm{d}s \\ &\le \|\mathbf{\Phi_0}\|_\infty + \epsilon h\,(K + L_A) + \epsilon h\,(L_M K + L_B). \end{aligned}$$

For

$$h \le \frac{1}{2\epsilon\,(L_M + 1)},$$

we have

$$\|\mathbf{F\Phi}\|_\infty \le \|\mathbf{\Phi_0}\|_\infty + \frac{K}{2} + \frac{L_B + L_A}{2\,(L_M + 1)}.$$

Since $\mathbf{\Phi}_0 : [0, \infty) \to \mathbb{R}^{2n}$ is a continuous function, there exists $0 < K' < \infty$ such that

$$\|\mathbf{\Phi_0}\|_\infty := \sup_{\delta \le \tau < \delta + h} |\mathbf{\Phi}_0(\tau)| = K'$$

Choosing

$$K \ge \left[K' + \frac{L_B + L_A}{2\,(L_M + 1)} \right],$$

we have $\|\mathbf{F\Phi}\|_\infty \le K$.

In short, with any $h, K > 0$ satisfying

$$h \le \frac{1}{2\epsilon\,(L_M + 1)}, \quad K = K' + \frac{L_B + L_A}{2\,(L_M + 1)}, \tag{27}$$

the operator $\mathbf{F}$ maps $X_K^{\delta,h}$ to itself.

To prove the existence of a fixed point for the operator $\mathbf{F} : X_K^{\delta,h} \to X_K^{\delta,h}$, we use Schauder's fixed point theorem:

Theorem 5 (Schauder's Fixed Point Theorem) *Let X be a real space, $D \subset X$ nonempty, closed, bounded and convex. Let $\mathcal{F} : D \to D$ be a continuous, compact operator. Then $\mathcal{F}$ has a fixed point.*

The space $X_K^{\delta,h}$ is nonempty, closed, bounded and convex. Therefore, to apply Schauder's fixed point theorem, it remains to show that $\mathbf{F} : X_K^{\delta,h} \to X_K^{\delta,h}$ is continuous and compact. For this, we need the following lemma.

Lemma 2 *The operator* $\mathbf{F}$ *is continuous and maps $X_K^{\delta,h}$ to a family of equicontinuous functions in $X_K^{\delta,h}$.*

Proof The proof of the continuity of $\mathbf{F} : X_K^{\delta,h} \to X_K^{\delta,h}$ is straightforward and is therefore omitted here. We prove the equicontinuity of its range in "Appendix 4". □

By Arzela–Ascoli theorem, therefore, the operator $\mathbf{F} : X_K^{\delta,h} \to X_K^{\delta,h}$ is compact. Hence, $\mathbf{F}$ satisfies all the conditions of Schauder's theorem and has at least one fixed point. This concludes the proof of Proposition 1. □

4.3 Uniqueness of the Continuation

We now show that the continuation constructed in Sects. 4.1 and 4.2 is unique.

Proposition 2 *Assume that (H1) and (H2) hold and $\epsilon < 1/L_M$. There exists $h > 0$ such that the continuation (24) of the local solution of the MR equation is unique.*

Proof Suppose $(\mathbf{y}_1, \mathbf{w}_1)$ and $(\mathbf{y}_2, \mathbf{w}_2)$ are two different continuations of the local solution of (17) from $[0, \delta)$ to $[\delta, \delta + h)$. That is

$$\begin{aligned} \mathbf{y}_1(\tau) &= \mathbb{1}_{[0,\delta)}(\tau)\mathbf{y}_{\mathrm{loc}}(\tau) + \mathbb{1}_{[\delta,\delta+h)}(\tau)\boldsymbol{\xi}_1(\tau), \ \mathbf{w}_1(\tau) \\ &= \mathbb{1}_{[0,\delta)}(\tau)\mathbf{w}_{\mathrm{loc}}(\tau) + \mathbb{1}_{[\delta,\delta+h)}(\tau)\boldsymbol{\eta}_1(\tau), \end{aligned}$$

and

$$\begin{aligned} \mathbf{y}_2(\tau) &= \mathbb{1}_{[0,\delta)}(\tau)\mathbf{y}_{\text{loc}}(\tau) + \mathbb{1}_{[\delta,\delta+h)}(\tau)\boldsymbol{\xi}_2(\tau), \ \mathbf{w}_2(\tau) \\ &= \mathbb{1}_{[0,\delta)}(\tau)\mathbf{w}_{\text{loc}}(\tau) + \mathbb{1}_{[\delta,\delta+h)}(\tau)\boldsymbol{\eta}_2(\tau), \end{aligned}$$

where, as discussed in Sect. 4.1, $(\boldsymbol{\xi}_i, \boldsymbol{\eta}_i)$ solves the integral equations

$$\begin{pmatrix} \boldsymbol{\xi}_i(\tau) \\ \boldsymbol{\eta}_i(\tau) \end{pmatrix} = \boldsymbol{\Phi}_0(\tau) + \epsilon \begin{pmatrix} \int_\delta^\tau \boldsymbol{\eta}_i(s) + \mathbf{A}_\mathbf{u}(\boldsymbol{\xi}_i(s), s)\, \mathrm{d}s \\ \int_\delta^\tau \psi(\tau - s)\,[-\mathbf{M}_\mathbf{u}(\boldsymbol{\xi}_i(s), s)\boldsymbol{\eta}_i(s) + \mathbf{B}_\mathbf{u}(\boldsymbol{\xi}_i(s), s)]\ \mathrm{d}s \end{pmatrix}, \tag{28}$$

for $i \in \{1, 2\}$.

Define $\boldsymbol{\Phi}_i = (\boldsymbol{\xi}_i, \boldsymbol{\eta}_i)$ and bound $|\boldsymbol{\Phi}_1 - \boldsymbol{\Phi}_2|$ by

$$\begin{aligned} |\boldsymbol{\Phi}_1(\tau) - \boldsymbol{\Phi}_2(\tau)| \le \epsilon \int_\delta^{\delta+h} & |\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)| + |\mathbf{A}_\mathbf{u}(\boldsymbol{\xi}_1(s), s) - \mathbf{A}_\mathbf{u}(\boldsymbol{\xi}_2(s), s)|\, \mathrm{d}s \\ & + \epsilon \int_\delta^{\delta+h} |\psi(\tau - s)|\,(|\mathbf{M}_\mathbf{u}(\boldsymbol{\xi}_1(s), s)(\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s))| \\ & + |\boldsymbol{\eta}_2(s)||\mathbf{M}_\mathbf{u}(\boldsymbol{\xi}_1(s), s) - \mathbf{M}_\mathbf{u}(\boldsymbol{\xi}_2(s), s)|)\ \mathrm{d}s, \\ & + \epsilon \int_\delta^{\delta+h} |\psi(\tau - s)||B_u(\boldsymbol{\xi}_1(s), s) - \mathbf{B}_\mathbf{u}(\boldsymbol{\xi}_2(s), s)|\, \mathrm{d}s, \end{aligned} \tag{29}$$

where we wrote $|\mathbf{M}_\mathbf{u}(\boldsymbol{\xi}_1(s), s)\boldsymbol{\eta}_1(s) - \mathbf{M}_\mathbf{u}(\boldsymbol{\xi}_2(s), s)\boldsymbol{\eta}_2(s)|$ as

$$|\mathbf{M}_\mathbf{u}(\boldsymbol{\xi}_1(s), s)(\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)) + (\mathbf{M}_\mathbf{u}(\boldsymbol{\xi}_1(s), s) - \mathbf{M}_\mathbf{u}(\boldsymbol{\xi}_2(s), s))\,\boldsymbol{\eta}_2(s)|.$$

Since $(\mathbf{y}_i, \mathbf{w}_i)$ solves the MR equation $[0, \delta + h)$, inequality (19) applies and we have

$$\|\boldsymbol{\eta}_i\|_\infty := \sup_{\delta \le \tau < \delta+h} |\boldsymbol{\eta}_i(\tau)| \le \sup_{0 \le \tau < \delta+h} |\mathbf{w}_i(\tau)| \le \frac{|\mathbf{w_0}| + \epsilon L_B}{1 - \epsilon L_M}, \quad i \in \{0, 1\}.$$

Taking the supremum over $\tau \in [\delta, \delta + h)$ on both sides of (29) and using the above upper bound on $\|\boldsymbol{\eta}_i\|_\infty$, we get

$$\begin{aligned} \|\boldsymbol{\Phi}_1 - \boldsymbol{\Phi}_2\|_\infty &\le \epsilon h\,[L_c\|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2\|_\infty + L_c\|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|_\infty] \\ &\quad + \epsilon h\,\bigg[L_M\|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2\|_\infty \\ &\quad + L_c\left(\frac{|\mathbf{w_0}| + \epsilon L_B}{1 - \epsilon L_M}\right)\|\boldsymbol{\xi}_1 - \xi_2\|_\infty + L_c\|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|_\infty\bigg], \\ &\le 2h\epsilon\left[3L_c + L_M + L_c\left(\frac{|\mathbf{w_0}| + \epsilon L_B}{1 - \epsilon L_M}\right)\right]\|\boldsymbol{\Phi}_1 - \boldsymbol{\Phi}_2\|_\infty. \end{aligned}$$

Taking $h > 0$ small enough, one obtains $\|\boldsymbol{\Phi}_1 - \boldsymbol{\Phi}_2\|_\infty \le \frac{1}{2}\|\boldsymbol{\Phi}_1 - \boldsymbol{\Phi}_2\|_\infty$ which, in turn, implies the uniqueness of the solution: $\boldsymbol{\Phi}_1 = \boldsymbol{\Phi}_2$. The time window h can for instance be chosen as

$$h = \frac{1}{2} \min \left(\frac{1}{\epsilon(L_M + 1)}, \frac{1}{2\epsilon \left[3L_c + L_M + L_c \left(\frac{|\mathbf{w}_0| + \epsilon L_B}{1 - \epsilon L_M} \right) \right]} \right), \tag{30}$$

which also respects the inequality (27). With this h, therefore, the continuation (24) is unique. □

Remark The above analysis is a contraction mapping argument. It is, therefore, tempting to use the Banach fixed point theorem (instead of the Schauder's fixed point theorem) in order to obtain the existence and uniqueness of the continuation (24) at once. The Banach fixed point theorem, however, does not apply here. This is because in proving the above contraction property, we made use of inequality (19) which applies to the mild solutions of the MR equation. As a result, it was necessary to show the existence of continuation (24) first. Otherwise, inequality (19) does not apply and the estimates used in the above contraction argument fail.

So far we have proved the existence of a unique mild solution to the MR equation over the time interval $[0, \delta + h)$ with h given in (30). The steps taken in Sects. 4.1, 4.2 and 4.3 can be applied to this extended local solution to prove the existence and uniqueness of a mild solution over the time interval $[0, \delta + 2h)$. This is because the continuation window h is independent of the constants K and δ from the complete metric space $X_K^{\delta,h}$.

Applying this argument repeatedly extends the mild solution of the Maxey–Riley equation from its local interval of existence and uniqueness $[0, \delta)$ to $[0, \delta + nh]$, for any $n \in \mathbb{N}$. Thus the solution can be extended uniquely to $[0, \infty)$. This proves Theorem 4.

5 Summary and Discussion

Motivated by the recent observations on the relevance of the memory effects on inertial particle dynamics, we have derived global existence and asymptotic decay results for the Maxey–Riley equation in the presence of the Basset–Boussinesq memory term. This memory term, a fractional derivative of order $1/2$ (Daitche (2013), Farazmand and Haller (2014)), greatly complicates the analytical and numerical treatment of the equation. While the behavior of the solutions has been well understood in the absence of the memory term (Rubin et al. 1995; Mograbi and Bar-Ziv 2006; Haller and Sapsis 2008; Sapsis and Haller 2010), no global analytic results have been available for the full equation with memory.

We have proved that the solutions converge asymptotically to a trapping region where the particle velocity is $\mathcal{O}(\epsilon)$-close to the fluid velocity. Here, ϵ is proportional to $(a/L)^2$ where a is the particle radius and L is the characteristic length scale of the fluid flow. This result holds for $0 < \epsilon \ll 1$ small enough which translates into $a \ll L$ (see Theorem 3 for the exact statement of the assumption). This assumption is not restrictive since the MR equation is only valid under the very same condition $a \ll L$ (Maxey and Riley 1983).

We also derived an upper envelope for the transient dynamics. This envelope exhibits an algebraic decay to the asymptotic state, hence confirming the numerical observations of Daitche and Tél (2011), Guseva et al. (2013) and Daitche and Tél (2014) in a more general framework. We showed with an example that this envelope can be saturated and therefore our upper estimates are sharp.

Upon neglecting the memory term, the convergence to the asymptotic limit is exponential (Rubin et al. 1995; Mograbi and Bar-Ziv 2006; Haller and Sapsis 2008). Therefore, the Basset–Boussinesq memory fundamentally alters the behavior of the inertial particles and cannot be readily neglected. From a mathematical point of view, the memory term also fundamentally changes the structure of the equation. In the absence of memory, the Maxey–Riley equation is an ordinary differential equation, generating a dynamical system. The memory term turns the equation into an integro-differential equation that does not generate a dynamical system.

Our asymptotic results are only applicable if the Maxey–Riley equation possesses global solutions. Because of the particular coupling and nonlinearity of the equation, available results on fractional differential equations do not guarantee the existence and uniqueness of global solutions to the Maxey–Riley equation. To this end, we have included here the first proof of the existence and uniqueness of global solutions to the Maxey–Riley equation. As already pointed out by Farazmand and Haller (2014), the particle velocity is not differentiable at the initial time but is continuous for all times.

Acknowledgments We would like to thank Anton Daitche for his help with implementing the numerical scheme of Daitche (2013).

Appendix 1: Proof of Theorem 2

Consider the fractional differential equation

$$\frac{d\mathbf{w}}{d\tau} + \kappa \frac{d^{1/2}\mathbf{w}}{d\tau^{1/2}} + \mathbf{w} = 0, \tag{31}$$

with $\mathbf{w}(0) = \mathbf{w}_0$ as initial condition. Let $\mathbf{W}(s) = (\mathcal{L}[\mathbf{w}])(s)$ denote the Laplace transform of $\mathbf{w}(\tau)$. Since

$$\left(\mathcal{L}\left[\frac{d\mathbf{w}}{d\tau}\right]\right)(s) = s\mathbf{W}(s) - \mathbf{w}_0$$

and

$$\left(\mathcal{L}\left[\frac{1}{\sqrt{\tau}}\right]\right)(s) = \sqrt{\frac{\pi}{s}},$$

the Laplace transform of the Riemann–Liouville derivative in (31) has the expression

$$\begin{aligned}\left(\mathcal{L}\left[\frac{d^{1/2}\mathbf{w}}{d\tau^{1/2}}\right]\right)(s) &= \frac{1}{\sqrt{\pi}}\left(\mathcal{L}\left[\int_0^\tau \frac{d\mathbf{w}}{d\tau}\frac{1}{\sqrt{\tau-\xi}}\,\mathrm{d}\xi\right]\right)(s) + \frac{1}{\sqrt{\pi}}\left(\mathcal{L}\left[\frac{\mathbf{w}_0}{\sqrt{\tau}}\right]\right)(s),\\ &= \frac{1}{\sqrt{\pi}}\left(\mathcal{L}\left[\frac{d\mathbf{w}}{d\tau}\right]\right)(s)\left(\mathcal{L}\left[\frac{1}{\sqrt{\tau}}\right]\right)(s) + \frac{\mathbf{w}_0}{\sqrt{s}},\\ &= (s\mathbf{W}(s) - \mathbf{w}_0)\frac{1}{\sqrt{s}} + \frac{\mathbf{w}_0}{\sqrt{s}},\end{aligned}$$

$$= \sqrt{s}\mathbf{W}(s),$$

where we used the identity

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^\tau \frac{\mathbf{w}(s)}{\sqrt{\tau - s}}\,\mathrm{d}s = \int_0^\tau \frac{\dot{\mathbf{w}}(s)}{\sqrt{\tau - s}}\,\mathrm{d}s + \frac{\mathbf{w}(0)}{\sqrt{\tau}}.$$

Now we use the Laplace transform on (31) and solve for $\mathbf{W}(s)$ to get

$$\mathbf{W}(s) = \frac{\mathbf{w}_0}{s + \kappa\sqrt{s} + 1}.$$

The denominator can be factorized as

$$\mathbf{W}(s) = \frac{\mathbf{w}_0}{\left(\sqrt{s} + \lambda_+\right)\left(\sqrt{s} + \lambda_-\right)},$$

where

$$\lambda_\pm = \frac{\left(\kappa \pm \sqrt{\kappa^2 - 4}\right)}{2}.$$

Hence the general solution of (31) is

$$\mathbf{w}(\tau; \mathbf{w}_0) = \mathbf{w}_0\left(\mathcal{L}^{-1}\left[\frac{1}{\left(\sqrt{s} + \lambda_+\right)\left(\sqrt{s} + \lambda_-\right)}\right]\right)(\tau) \tag{32}$$

The function $\mathbf{w}(\tau; \mathbf{w}_0)$ is proportional to the Mittag-Leffler function of order 1/2, which is defined as

$$E_{1/2}(-z) = e^{z^2}\operatorname{erfc} z \tag{33}$$

for any complex number $z \in \mathbb{C}$ (see, e.g., Bateman et al. 1955, Section 18.1). The Laplace transform of $E_{1/2}$ is given by (see Haubold et al. 2011, Eq. 11.13):

$$\left(\mathcal{L}\left[E_{1/2}\left(-a\sqrt{z}\right)\right]\right)(s) = \frac{1}{\sqrt{s}\left(\sqrt{s} + a\right)} \tag{34}$$

for any $a \in \mathbb{C}$.

To study the behavior of $E_{1/2}(-z)$ as $z \to \infty$, we will make use of the asymptotic expansion of the complementary error function (Abramowitz and Stegun 1972, Eq. 7.1.23):

$$\operatorname{erfc} z \sim \frac{e^{-z^2}}{z\sqrt{\pi}}\left(1 - \frac{1}{2z^2} + \frac{3}{4z^4} + \mathcal{O}\left(\frac{1}{z^6}\right)\right). \tag{35}$$

Substituting in (33), we obtain

$$E_{1/2}(-z) \sim \frac{1}{z\sqrt{\pi}}\left(1 - \frac{1}{2z^2} + \frac{3}{4z^4} + \mathcal{O}\left(\frac{1}{z^6}\right)\right). \tag{36}$$

The asymptotic expansion of erfc z is valid only if $|\arg(z)| < \frac{3\pi}{4}$ (Abramowitz and Stegun 1972). It also diverges for any finite value of z; its sole purpose is to give the rate of decay as $z \to \infty$.

The general solution will depend on whether the discriminant of $\lambda_{\pm}$, i.e., $\kappa^2 - 4$, is positive, zero, or negative.

Case 1: $\kappa > 2$ ($R > 16/9$)

We have

$$\mathbf{W}(s) = \frac{\mathbf{w}_0}{\left(\sqrt{s} + \lambda_+\right)\left(\sqrt{s} + \lambda_-\right)}$$

or, after some algebra,

$$\mathbf{W}(s) = \frac{\mathbf{w}_0}{\lambda_+ - \lambda_-}\left[\frac{\lambda_+}{\sqrt{s}\left(\sqrt{s} + \lambda_+\right)} - \frac{\lambda_-}{\sqrt{s}\left(\sqrt{s} + \lambda_-\right)}\right].$$

Invert the two terms in the above expression with the rule (34) to get

$$\mathbf{w}(\tau; \mathbf{w}_0) = \frac{\mathbf{w}_0}{\lambda_+ - \lambda_-}\left[\lambda_+ E_{1/2}\left(-\lambda_+\sqrt{\tau}\right) - \lambda_- E_{1/2}\left(-\lambda_-\sqrt{\tau}\right)\right]. \tag{37}$$

Since $\kappa - \sqrt{\kappa^2 - 4}$ is always greater than zero, we can use the asymptotic expansion (36) to find that in the limit $\tau \to \infty$,

$$\begin{aligned}
\mathbf{w}(\tau; \mathbf{w}_0) &\sim \frac{\mathbf{w}_0}{\lambda_+ - \lambda_-}\left[\frac{1}{\sqrt{\pi\tau}}\left(1 - \frac{1}{2\lambda_+^2\tau}\right)\right.\\
&\quad \left. - \frac{1}{\sqrt{\pi\tau}}\left(1 - \frac{1}{2\lambda_-^2\tau}\right) + \mathcal{O}\left(\tau^{-5/2}\right)\right],\\
&\sim \frac{\mathbf{w}_0}{2\sqrt{\pi}\left(\lambda_+ - \lambda_-\right)}\left(\frac{\lambda_+^2 - \lambda_-^2}{\lambda_+^2\lambda_-^2}\right)\tau^{-3/2} + \mathcal{O}\left(\tau^{-5/2}\right),\\
&\sim \left(\frac{\kappa\mathbf{w}_0}{2\sqrt{\pi}}\right)\tau^{-3/2} + \mathcal{O}\left(\tau^{-5/2}\right),
\end{aligned} \tag{38}$$

where we used that $\lambda_+ + \lambda_- = \kappa$ and $\lambda_+\lambda_- = 1$.

Case 2: $\kappa = 2$ ($R = 16/9$)

We have

$$\mathbf{W}(s) = \frac{\mathbf{w}_0}{\left(\sqrt{s} + 1\right)^2} \tag{39}$$

or, after a bit of algebra,

$$\begin{aligned}\mathbf{W}(s) &= \mathbf{w}_0\left(\frac{1}{\sqrt{s}\left(\sqrt{s}+1\right)} - \frac{1}{\sqrt{s}\left(\sqrt{s}+1\right)^2}\right)\\ &= \mathbf{w}_0\left(\frac{1}{\sqrt{s}\left(\sqrt{s}+1\right)} + 2\frac{d}{ds}\left(\frac{1}{\sqrt{s}+1}\right)\right).\end{aligned} \tag{40}$$

We can invert the first term in (40) with (34). The second term can be inverted by using the Laplace transforms (Gorenflo and Mainardi 1997, Equations A.27, A.28 and A.35.)

$$\left(\mathcal{L}\left[\frac{1}{\sqrt{\pi\tau}} - E_{1/2}\left(-\sqrt{\tau}\right)\right]\right)(s) = \frac{1}{\sqrt{s}+1} \tag{41}$$

and

$$\left(\mathcal{L}\left[-\tau f(\tau)\right]\right)(s) = \frac{d}{ds}(\mathcal{L}\left[f(\tau)\right])(s). \tag{42}$$

Thus the inverse Laplace transform of (39) is

$$\mathbf{w}(\tau;\mathbf{w}_0) = \mathbf{w}_0\left[E_{1/2}\left(-\sqrt{\tau}\right)(1+2\tau) - \frac{2\sqrt{\tau}}{\sqrt{\pi}}\right]. \tag{43}$$

With the asymptotic expansion (36), we find that in the limit $\tau \to \infty$,

$$\begin{aligned}\mathbf{w}(\tau;\mathbf{w}_0) &\sim \mathbf{w}_0\left[\frac{1}{\sqrt{\pi\tau}}\left(1 - \frac{1}{2\tau} + \frac{3}{4\tau^2} + \mathcal{O}\left(\tau^{-3}\right)\right)\right.\\ &\quad\left. + \frac{2\sqrt{\tau}}{\sqrt{\pi}}\left(1 - \frac{1}{2\tau} + \frac{3}{4\tau^2} + \mathcal{O}\left(\tau^{-3}\right)\right) - \frac{2\sqrt{\tau}}{\sqrt{\pi}}\right]\\ &\sim \left(\frac{\mathbf{w}_0}{\sqrt{\pi}}\right)\tau^{-3/2} + \mathcal{O}\left(\tau^{-5/2}\right)\end{aligned} \tag{44}$$

Case 3: $0 < \kappa < 2$ ($R < 16/9$)

We have

$$\mathbf{W}(s) = \frac{\mathbf{w}_0}{\left(\sqrt{s}+\lambda_+\right)\left(\sqrt{s}+\lambda_-\right)}$$

This is the same Laplace transform as in the case $\kappa > 2$, except that λ_+ and λ_- are now complex conjugate numbers. The inverse Laplace transform is the same as (37):

$$\mathbf{w}(\tau;\mathbf{w}_0) = \frac{\mathbf{w}_0}{\lambda_+ - \lambda_-}\left[\lambda_+ E_{1/2}\left(-\lambda_+\sqrt{\tau}\right) - \lambda_- E_{1/2}\left(-\lambda_-\sqrt{\tau}\right)\right]. \tag{45}$$

The quotients

$$\frac{\lambda_+}{\lambda_+ - \lambda_-} = \frac{1}{2}\left(1 - i\frac{\kappa}{\sqrt{4-\kappa^2}}\right)$$

and

$$-\frac{\lambda_-}{\lambda_+ - \lambda_-} = \frac{1}{2}\left(1 + i\frac{\kappa}{\sqrt{4-\kappa^2}}\right)$$

in (45) are also complex conjugates. Since $\left(e^{\overline{w}}\right) = \overline{(e^w)}$ and erfc $\overline{w} = \overline{\text{erfc}\, w}$ for every $w \in \mathbb{C}$, it follows also that $E_{1/2}\left(\overline{w}\right) = \overline{E_{1/2}\left(w\right)}$. Thus

$$\mathbf{w}(\tau; \mathbf{w_0}) = \mathbf{w_0}\left[\left(\frac{\lambda_+}{\lambda_+ - \lambda_-}E_{1/2}\left(-\lambda_+\sqrt{\tau}\right)\right) + \overline{\left(\frac{\lambda_+}{\lambda_+ - \lambda_-}E_{1/2}\left(-\lambda_+\sqrt{\tau}\right)\right)}\right]$$

or simply twice the real part of $w(\tau; w_0)$.

$$\begin{aligned}\mathbf{w}(\tau; \mathbf{w}_0) &= 2\mathbf{w}_0 \operatorname{Re}\left(\frac{\lambda_+}{\lambda_+ - \lambda_-}E_{1/2}(-\lambda_+\sqrt{\tau})\right),\\ &= 2\mathbf{w}_0\left[\operatorname{Re}\left(\frac{\lambda_+}{\lambda_+ - \lambda_-}\right)\operatorname{Re}\left(E_{1/2}\left(-\lambda_+\sqrt{\tau}\right)\right)\right. \\ &\quad \left. + \operatorname{Im}\left(\frac{\lambda_+}{\lambda_+ - \lambda_-}\right)\operatorname{Im}\left(E_{1/2}\left(-\lambda_+\sqrt{\tau}\right)\right)\right].\end{aligned} \tag{46}$$

It is possible to further simplify (45). It turns out that the Mittag-Leffler function $E_{1/2}\left(-z\right)$ may be written as (DLMF, Section 7.19)

$$E_{1/2}\left(-z\right) = \sqrt{\frac{4t}{\pi}}\left[U\left(x, t\right) + iV\left(x, t\right)\right], \tag{47}$$

where

$$U\left(x, t\right) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}\frac{e^{-(x+s)^2/(4t)}}{1+s^2}\,\mathrm{d}s, \tag{48}$$

$$V\left(x, t\right) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}\frac{s e^{-(x+s)^2/(4t)}}{1+s^2}\,\mathrm{d}s, \tag{49}$$

$z = \frac{1-ix}{2\sqrt{t}}$, $x \in \mathbb{R}$, and $t > 0$. The functions $U\left(x, t\right)$ and $V\left(x, t\right)$ are known as the Voigt functions (DLMF, Section 7.19; Olver et al. 2010). If we set

$$z = \frac{1-ix}{2\sqrt{t}} = \lambda_+\sqrt{\tau} = \left(\frac{\kappa}{2} + i\frac{\sqrt{4-\kappa^2}}{2}\right)\sqrt{\tau},$$

then we can solve for x and t to get

$$t = \frac{1}{\kappa^2\tau}$$

and

$$x = -\frac{\sqrt{4-\kappa^2}}{\kappa}.$$

Thus

$$\begin{aligned} E_{1/2}\left(-\lambda_+\sqrt{\tau}\right) = \frac{2}{\kappa\sqrt{\pi\tau}} \left[U\left(-\frac{\sqrt{4-\kappa^2}}{\kappa}, \frac{1}{\kappa^2\tau}\right) \right. \\ \left. - iV\left(-\frac{\sqrt{4-\kappa^2}}{\kappa}, \frac{1}{\kappa^2\tau}\right)\right]. \end{aligned} \tag{50}$$

Hence (46) can be written as

$$\begin{aligned} \mathbf{w}\left(\tau; \mathbf{w}_0\right) = \frac{2\mathbf{w}_0}{\kappa\sqrt{\pi\tau}} \left[U\left(-\frac{\sqrt{4-\kappa^2}}{a}, \frac{1}{\kappa^2\tau}\right) \right. \\ \left. - \frac{\kappa}{\sqrt{4-\kappa^2}} V\left(-\frac{\sqrt{4-\kappa^2}}{\kappa}, \frac{1}{\kappa^2\tau}\right)\right]. \end{aligned} \tag{51}$$

For the asymptotic behavior of $\mathbf{w}(\tau; \mathbf{w}_0)$ as $\tau \to \infty$, we can repeat the steps as in the case $\kappa > 2$ and obtain

$$\mathbf{w}(\tau; \mathbf{w}_0) \sim \left(\frac{\kappa\mathbf{w}_0}{2\sqrt{\pi}}\right)\tau^{-3/2} + \mathcal{O}\left(\tau^{-5/2}\right). \tag{52}$$

This asymptotic expansion, however, is justified only if $|\arg\left(\lambda_+\sqrt{\tau}\right)|$ and $|\arg\left(\lambda_+\sqrt{\tau}\right)|$ are smaller than $\frac{3\pi}{4}$. Since $\lambda_\pm = \left(\kappa \pm i\sqrt{4-\kappa^2}\right)/2$, we see that this will be the case whenever $\kappa > 0$, since then $0 < \arg\left(\lambda_+\sqrt{\tau}\right) < \frac{\pi}{2}$ and $-\frac{\pi}{2} < \arg\left(\lambda_-\sqrt{\tau}\right) < 0$ (to see this, note that the two complex numbers λ_+ and λ_- lie to the right of the imaginary axis, so that the argument cannot be greater than $\pi/2$). Note that since $\kappa = \sqrt{9R/2}$, the required condition $\kappa > 0$ is always satisfied.

Appendix 2: Proof of Theorem 3

We will use the following Gronwall-type inequality.

Lemma 3 (Chu and Metcalf 1967) *Let the functions $\alpha, \beta : \mathbb{R}^+ \to \mathbb{R}$ be continuous and the function $K(\tau, s)$ be continuous and nonnegative for $0 \le s \le \tau$. If*

$$\alpha(\tau) \le \beta(\tau) + \int_0^\tau K(\tau, s)\alpha(s)\, ds,$$

then

$$\alpha(\tau) \le \beta(\tau) + \int_0^\tau H(\tau, s)\beta(s)\, ds,$$

where $H(\tau, s) = \sum_{j=1}^{\infty} K_j(\tau, s)$, $K_1(\tau, s) = K(\tau, s)$ *and*

$$K_j(\tau, s) = \int_s^{\tau} K_{j-1}(\tau, \xi) K(\xi, s)\, d\xi, \quad j \geq 2.$$

Corollary 1 *If* $K(\tau, s) = k(\tau - s)$*, then one can show that* $K_j(\tau, s) = k_j(\tau - s)$ *where*

$$k_j(\tau) = (k * k * \cdots * k)(\tau),$$

where the convolution is j*-fold. As a result,* $H(\tau, s) = h(\tau - s)$ *where*

$$h(\tau) = \sum_{j=1}^{\infty} k_j(\tau).$$

Proof We prove $K_2(\tau, s) = k * k(\tau - s)$. The rest follows similarly by induction.

$$\begin{aligned} K_2(\tau, s) &:= \int_s^{\tau} K(\tau, \xi) K(\xi, s)\, \mathrm{d}\xi \\ &= \int_s^{\tau} k(\tau - \xi) k(\xi - s)\, \mathrm{d}\xi \\ &= \int_0^{\tau - s} k(\tau - s - \eta) k(\eta)\, \mathrm{d}\eta \\ &= k * k(\tau - s) =: k_2(\tau - s), \end{aligned}$$

where we used the change of variable $\eta = \xi - s$. □

Proof of Theorem 3 It follows from the integral equation (17) that

$$|\mathbf{w}(\tau; \mathbf{y}_0, \mathbf{w}_0)| \leq \psi(\tau)|\mathbf{w}_0| + \epsilon L_B (1 - \phi(\tau)) + \epsilon L_M \int_0^{\tau} \psi(\tau - s)|\mathbf{w}(s; \mathbf{y}_0, \mathbf{w}_0)|\, \mathrm{d}s \tag{53}$$

where $\tau \in [0, \delta)$. Using Lemma 3 with $\alpha(\tau) = |\mathbf{w}(\tau; \mathbf{y}_0, \mathbf{w}_0)|$, $\beta(\tau) = \psi(\tau)|\mathbf{w}_0| + \epsilon L_B (1 - \phi(\tau))$ and $K(\tau, s) = \epsilon L_M \psi(\tau - s)$, we get

$$\begin{aligned} |\mathbf{w}(\tau; \mathbf{y}_0, \mathbf{w}_0)| \leq{}& \psi(\tau)|\mathbf{w}_0| + \epsilon L_B (1 - \phi(\tau)) \\ &+ \int_0^{\tau} h(\tau - s) \left[\psi(s)|\mathbf{w}_0| + \epsilon L_B (1 - \phi(\tau))\right] \mathrm{d}s \\ ={}& \left[\psi(\tau) + \int_0^{\tau} h(\tau - s)\psi(s)\, \mathrm{d}s\right] |\mathbf{w}_0| + \epsilon L_B (1 - \phi(\tau)) \\ &+ \epsilon L_B \int_0^{\tau} h(\tau - s) (1 - \phi(s))\, \mathrm{d}s, \end{aligned} \tag{54}$$

where $h(\tau;\epsilon)=\sum_{j=1}^{\infty}k_j(\tau)$ with $k_1=\epsilon L_M\psi$ and $k_j=k_{j-1}*k_1$. Induction on j leads to the expression

$$k_j=(\epsilon L_M)^j\psi^{*j}.$$

Therefore we have the identity

$$\begin{aligned}\psi(\tau)+\int_0^\tau h(\tau-s)\psi(s)\,\mathrm{d}s &= \frac{k_1(\tau)}{\epsilon L_M}+\int_0^\tau\sum_{j=1}^{\infty}k_j(\tau-s)\frac{k_1(s)}{\epsilon L_M}\,\mathrm{d}s\\ &= \frac{k_1(\tau)}{\epsilon L_M}+\frac{1}{\epsilon L_M}\sum_{j=1}^{\infty}\int_0^\tau k_j(\tau-s)k_1(s)\,\mathrm{d}s\\ &= \frac{k_1(\tau)}{\epsilon L_M}+\frac{1}{\epsilon L_M}\sum_{j=1}^{\infty}k_{j+1}(\tau)\\ &= \frac{1}{\epsilon L_M}\sum_{j=1}^{\infty}k_j(\tau)\\ &= \frac{1}{\epsilon L_M}h(\tau),\end{aligned}$$

where we omitted the dependence of h on the parameter ϵ for notational simplicity.

This shows that

$$|\mathbf{w}(\tau;\mathbf{y}_0,\mathbf{w}_0)|\le\frac{|\mathbf{w}_0|}{\epsilon L_M}h(\tau)+\epsilon L_B\,(1-\phi(\tau))+\epsilon L_B\int_0^\tau h(\tau-s)\,(1-\phi(s))\,\mathrm{d}s. \tag{55}$$

Since $0\le\phi(\tau)\le1$, we have that $(1-\phi(\tau))\le1$ and therefore the inequality can be further simplified to

$$|\mathbf{w}(\tau;\mathbf{y}_0,\mathbf{w}_0)|\le\frac{|\mathbf{w}_0|}{\epsilon L_M}h(\tau)+\epsilon L_B\,[1-\phi(\tau)]+\epsilon L_B\int_0^\tau h(s)\,\mathrm{d}s. \tag{56}$$

So far we have assumed that the series $\sum_{j=1}^{\infty}k_j=\sum_{j=1}^{\infty}(\epsilon L_M)^j\psi^{*j}$ converges uniformly to a limit h. To prove this, we first show that for any j and $\tau\ge0$, $0\le\psi^{*j}(\tau)\le1$. For $j=1$, this property holds since $0\le\psi\le1$. For $j=2$, we have

$$0\le\psi^{*2}(\tau):=\int_0^\tau\psi(\tau-s)\psi(s)\,\mathrm{d}s\le\int_0^\tau\psi(s)\,\mathrm{d}s=1-\phi(\tau)\le1.$$

By induction on j, we get $0\le\psi^{*j}(\tau)\le1$. As a result, $(\epsilon L_M)^j\psi^{*j}\le(\epsilon L_M)^j$. Since $\epsilon L_M<1$, the series $\sum_{j=1}^{\infty}(\epsilon L_M)^j$ converges. It follows that

$$|h(\tau)|\le\sum_{j=1}^{\infty}(\epsilon L_M)^j=\frac{\epsilon L_M}{1-\epsilon L_M} \tag{57}$$

by summing up the geometric series. By the dominated convergence theorem, the sequence $\sum_{j=1}^{n}(\epsilon L_M)^j \psi^{*j}$ converges uniformly to a function h as $n \to \infty$. Since for any n, the series $\sum_{j=1}^{n}(\epsilon L_M)^j \psi^{*j}$ is continuous, so is the limiting function h. This shows that $h : [0, \infty) \to \mathbb{R}$ is continuous and $h \geq 0$.

Now, observe that

$$\begin{aligned} \int_0^\tau h(\xi)\, \mathrm{d}\xi &= \int_0^\tau \sum_{j=1}^{\infty}(\epsilon L_M)^j \psi^{*j}(\xi)\, \mathrm{d}\xi \\ &= \sum_{j=1}^{\infty}(\epsilon L_M)^j \int_0^\tau \psi^{*j}(\xi)\, \mathrm{d}\xi \\ &\leq \sum_{j=1}^{\infty}(\epsilon L_M)^j = \frac{\epsilon L_M}{1-\epsilon L_M}, \end{aligned} \tag{58}$$

where we used the uniform convergence of the series and the fact that, for any j,

$$\begin{aligned} 0 \leq \int_0^\tau \psi^{*j}(\xi)\, \mathrm{d}\xi &\leq \left(\int_0^\tau \psi^{*(j-1)}(\xi)\, \mathrm{d}\xi\right)\left(\int_0^\tau \psi(\xi)\, \mathrm{d}\xi\right) \\ &\leq \cdots \leq \left(\int_0^\tau \psi(\xi)\, \mathrm{d}\xi\right)^j = (1-\phi(\tau))^j \leq 1, \end{aligned}$$

by repeated application of Young's inequality for convolutions. This also shows that $h(\tau) \to 0$ as $\tau \to \infty$, since $|h|_1 < \infty$ and h is uniformly continuous.

Using inequality (58) in (56) and the definition of h, we get

$$|\mathbf{w}(\tau; \mathbf{y}_0, \mathbf{w}_0)| \leq \frac{|\mathbf{w}_0|}{\epsilon L_M} h(\tau) + \epsilon L_B\,(1-\phi(\tau)) + \frac{\epsilon^2 L_M L_B}{1-\epsilon L_M} \tag{59}$$

$$= |\mathbf{w}_0| \left[\sum_{j=1}^{\infty}(\epsilon L_M)^{j-1}\psi^{*j}(\tau)\right] + \epsilon L_B\,(1-\phi(\tau)) + \frac{\epsilon^2 L_M L_B}{1-\epsilon L_M}. \tag{60}$$

This proves part (i) of the theorem.

Taking the sup of $|\mathbf{w}(\tau; \mathbf{y}_0, \mathbf{w}_0)|$ over $[0, \delta)$, we get

$$\sup_{0 \leq \tau < \delta} |\mathbf{w}(\tau; \mathbf{y}_0, \mathbf{w}_0)| \leq \frac{|\mathbf{w}_0| + \epsilon L_B}{1-\epsilon L_M}, \tag{61}$$

which proves part (ii) of Theorem 3.

If $\delta = \infty$, then we can take the limitsup of $|\mathbf{w}|$. Using inequality (59), we get the asymptotic estimate

$$\limsup_{\tau \to \infty} |\mathbf{w}(\tau; \mathbf{y}_0, \mathbf{w}_0)| \leq \frac{\epsilon L_B}{1-\epsilon L_M}, \tag{62}$$

which proves part (iii) of Theorem 3. Here, we used the fact that $\lim_{\tau\to\infty} h(\tau) = 0$ and $\lim_{\tau\to\infty} \phi(\tau) = 0$. $\square$

Appendix 3: Proof of Lemma 1

Let $\tau_1, \tau_2 \in [0, \delta)$. Bound $|\mathbf{z}_{\text{loc}}(\tau_2) - \mathbf{z}_{\text{loc}}(\tau_1)|$ by

$$\begin{aligned}
|\mathbf{z}_{\text{loc}}(\tau_2) - \mathbf{z}_{\text{loc}}(\tau_1)| &\leq |\mathbf{y}_{\text{loc}}(\tau_2) - \mathbf{y}_{\text{loc}}(\tau_1)| + |\mathbf{w}_{\text{loc}}(\tau_2) - \mathbf{w}_{\text{loc}}(\tau_1)| \\
&\leq |\mathbf{w}_0||\psi(\tau_2) - \psi(\tau_1)| + \epsilon \int_{\tau_1}^{\tau_2} |\mathbf{w}_{\text{loc}}(s)| + |\mathbf{A}_{\mathbf{u}}(\mathbf{y}_{\text{loc}}(s), s)| \, \mathrm{d}s \\
&\quad + \epsilon \int_{\tau_1}^{\tau_2} \psi(\tau_2 - s) \Big[|\mathbf{M}_{\mathbf{u}}(\mathbf{y}_{\text{loc}}(s), s)||\mathbf{w}_{\text{loc}}(s)| \\
&\quad + |\mathbf{B}_{\mathbf{u}}(\mathbf{y}_{\text{loc}}(s), s)| \Big] \, \mathrm{d}s \\
&\quad + \epsilon \int_0^{\tau_1} |\psi(\tau_2 - s) - \psi(\tau_1 - s)| \Big[|\mathbf{M}_{\mathbf{u}}(\mathbf{y}_{\text{loc}}(s), s)||\mathbf{w}_{\text{loc}}(s)| \\
&\quad + |\mathbf{B}_{\mathbf{u}}(\mathbf{y}_{\text{loc}}(s), s)| \Big] \, \mathrm{d}s
\end{aligned}$$

Without loss of generality, suppose $\tau_1 \leq \tau_2$, so that $|\psi(\tau_2 - s) - \psi(\tau_1 - s)| = \psi(\tau_2 - s) - \psi(\tau_1 - s)$. Taking the infinity norm over $[0, \delta)$ to bound $\|\mathbf{M}_{\mathbf{u}}(\mathbf{y}_{\text{loc}}(s), s)\|_\infty$, $\|\mathbf{B}_{\mathbf{u}}(\mathbf{y}_{\text{loc}}(s), s)\|_\infty$, $\|\mathbf{A}_{\mathbf{u}}(\mathbf{y}_{\text{loc}}(s), s)\|_\infty$ and $|\mathbf{w}_{\text{loc}}(s)|$ by Theorem 3, we get

$$\begin{aligned}
|\mathbf{z}_{\text{loc}}(\tau_2) - \mathbf{z}_{\text{loc}}(\tau_1)| &\leq |\mathbf{w}_0||\psi(\tau_2) - \psi(\tau_1)| + \epsilon \left(L_A + \frac{|\mathbf{w}_0| + \epsilon L_B}{1 - \epsilon L_M} \right) |\tau_2 - \tau_1| \\
&+ \epsilon \left[\frac{L_M |\mathbf{w}_0| + L_B}{1 - \epsilon L_M} \right] \left(|\tau_2 - \tau_1| + \int_0^{\tau_1} \psi(\tau_1 - s) - \psi(\tau_2 - s) \, \mathrm{d}s \right).
\end{aligned}$$

By the results of Theorem 2, $\psi(\tau_1 - s) - \psi(\tau_2 - s) = \phi'(\tau_2 - s) - \phi'(\tau_1 - s) \geq 0$. Integrate and rearrange to obtain

$$\begin{aligned}
|\mathbf{z}_{\text{loc}}(\tau_2) - \mathbf{z}_{\text{loc}}(\tau_1)| &\leq |\mathbf{w}_0||\psi(\tau_2) - \psi(\tau_1)| + \epsilon \left(L_A + \frac{|\mathbf{w}_0| + \epsilon L_B}{1 - \epsilon L_M} \right) |\tau_2 - \tau_1| \\
&\quad + \epsilon \left[\frac{L_M |\mathbf{w}_0| + L_B}{1 - \epsilon L_M} \right] (|\tau_2 - \tau_1| + \phi(\tau_2) - \phi(\tau_1)) \\
&\quad + \epsilon \left[\frac{L_1 |\mathbf{w}_0| + L_2}{1 - \epsilon L_1} \right] (\phi(0) - \phi(\tau_2 - \tau_1)).
\end{aligned}$$

Since both ψ and ϕ are uniformly continuous over $[0, \infty)$ by Theorem 2, each of $|\psi(\tau_2) - \psi(\tau_1)|$, $|\phi(\tau_2) - \phi(\tau_1)|$ and $|\phi(0) - \phi(\tau_2 - \tau_1)| \to 0$ as $|\tau_2 - \tau_1| \to 0$. Hence $|\mathbf{z}_{\text{loc}}(\tau_2) - \mathbf{z}_{\text{loc}}(\tau_1)| \to 0$ as $\tau_1, \tau_2 \to \delta_-$.

Now, if we take a sequence $\{t_n\}$ $t_n \in [0, \delta)$ such that $\lim_{n\to\infty} t_n \to \delta$, then it follows that $\{\mathbf{z}_{\text{loc}}(t_n)\}$ is a Cauchy sequence. The sequence is convergent in $\mathbb{R}^{2n}$ since $\mathbb{R}^{2n}$ is a complete metric space. The limit is given by the integral equation (16) evaluated at $\tau = \delta$:

$$\mathbf{z}_{\text{loc}}(\delta) = \begin{pmatrix} \mathbf{y}_0 + \epsilon \int_0^\delta \mathbf{w}_{\text{loc}}(s) + \mathbf{A}_\mathbf{u}(\mathbf{y}_{\text{loc}}(s), s)\, \mathrm{d}s \\ \psi(\delta)\mathbf{w}_0 + \epsilon \int_0^\delta \psi(\tau - s)\left[-\mathbf{M}_\mathbf{u}(\mathbf{y}_{\text{loc}}(s), s)\mathbf{w}_{\text{loc}}(s) + \mathbf{B}_\mathbf{u}(\mathbf{y}_{\text{loc}}(s), s)\right]\, \mathrm{d}s \end{pmatrix}.$$

This ends the proof.

Appendix 4: Proof of Lemma 2

Let $\mathbf{\Phi} = (\boldsymbol{\xi}, \boldsymbol{\eta}) \in X_K^{\delta,h}$, and $\tau_1, \tau_2 \in [\delta, \delta + h)$. Bound $|(\mathbf{F\Phi})(\tau_2) - (\mathbf{F\Phi})(\tau_1)|$ by

$$\begin{aligned}|(\mathbf{F\Phi})(\tau_2) - (\mathbf{F\Phi})(\tau_1)| \le |\mathbf{\Phi_0}(\tau_2) - \mathbf{\Phi_0}(\tau_1)| &+ \epsilon \int_{\tau_1}^{\tau_2} |\boldsymbol{\eta}(s)| + |\mathbf{A}_\mathbf{u}(\boldsymbol{\xi}(s), s)|\, \mathrm{d}s \\ &+ \epsilon \int_{\tau_1}^{\tau_2} \psi(\tau_2 - s)\, [|\mathbf{M}_\mathbf{u}(\boldsymbol{\xi}(s), s)||\boldsymbol{\eta}(s)| + |\mathbf{B}_\mathbf{u}(\boldsymbol{\xi}(s), s)|]\, \mathrm{d}s \\ &+ \epsilon \int_{\delta}^{\tau_1} (\psi(\tau_2 - s) - \psi(\tau_1 - s))\, [|\mathbf{M}_\mathbf{u}(\boldsymbol{\xi}(s), s)||\boldsymbol{\eta}(s)| \\ &+ |\mathbf{B}_\mathbf{u}(\boldsymbol{\xi}(s), s)|]\, \mathrm{d}s,\end{aligned}$$

where

$$\begin{aligned}|\mathbf{\Phi_0}(\tau_2) - \mathbf{\Phi_0}(\tau_1)| \le |\mathbf{w}_0||\psi(\tau_2) - \psi(\tau_1)| & \\ + \epsilon \int_0^\delta |\psi(\tau_2 - s) - \psi(\tau_1 - s)| & \left[|\mathbf{M}_\mathbf{u}(\mathbf{y}_{\text{loc}}(s), s)||\mathbf{w}_{\text{loc}}(s)| \right. \\ & \left. + |\mathbf{B}_\mathbf{u}(\mathbf{y}_{\text{loc}}(s), s)|\right]\, \mathrm{d}s.\end{aligned}$$

Without loss of generality, suppose $\tau_1 \le \tau_2$, so that $|\psi(\tau_2 - s) - \psi(\tau_1 - s)| = \psi(\tau_2 - s) - \psi(\tau_1 - s)$. Taking the infinity norm over $[\delta, \delta + h)$ to bound $\|\mathbf{M}_\mathbf{u}(\boldsymbol{\xi}(s), s)\|_\infty$, $\|\mathbf{B}_\mathbf{u}(\boldsymbol{\xi}(s), s)\|_\infty$, $\|\mathbf{A}_\mathbf{u}(\boldsymbol{\xi}(s), s)\|_\infty$, $\|\boldsymbol{\eta}(s)\|_\infty$ and $|\mathbf{w}_{\text{loc}}(s)|$ by inequality (19), we get

$$\begin{aligned}|(\mathbf{F\Phi})(\tau_2) - (\mathbf{F\Phi})(\tau_1)| \le |\mathbf{w}_0||\psi(\tau_2) - \psi(\tau_1)| & \\ + \epsilon \left(\frac{L_M|\mathbf{w}_0| + L_B}{1 - \epsilon L_M}\right) & \int_0^\delta \psi(\tau_1 - s) - \psi(\tau_2 - s)\, \mathrm{d}s \\ + \epsilon (K + L_A)|\tau_2 - \tau_1| & + \epsilon\, (L_M K + L_B)\, |\tau_2 - \tau_1| \\ + \epsilon\, (L_M K + L_B) & \int_\delta^{\tau_1} \psi(\tau_1 - s) - \psi(\tau_2 - s)\, \mathrm{d}s.\end{aligned}$$

By the results of Theorem 2, $\psi(\tau_1 - s) - \psi(\tau_2 - s) = \phi'(\tau_2 - s) - \phi'(\tau_1 - s) \ge 0$. Finally, integrate and rearrange to obtain

$$\begin{aligned}|(\mathbf{F\Phi})(\tau_2) - (\mathbf{F\Phi})(\tau_1)| \ &\le |\mathbf{w}_0||\psi(\tau_2) - \psi(\tau_1)| \\ &\quad + \epsilon \left(\frac{L_M|\mathbf{w}_0| + L_B}{1 - \epsilon L_M}\right) [(\phi(\tau_1 - \delta) - \phi(\tau_2 - \delta)) \\ &\quad + (\phi(\tau_2) - \phi(\tau_1))]\end{aligned}$$

$$+\epsilon(K+L_A)|\tau_2-\tau_1| + \epsilon\,(L_M K + L_B)\,|\tau_2-\tau_1|$$
$$+\epsilon\,(L_M K + L_B)\,[(\phi(0)-\phi(\tau_2-\tau_1))$$
$$+\,(\phi(\tau_2-\delta)-\phi(\tau_1-\delta))]\,.$$

Since both ψ and ϕ are uniformly continuous over $[0,\infty)$ by Theorem 2, each of $|\psi(\tau_2)-\psi(\tau_1)|$, $|\phi(\tau_2)-\phi(\tau_1)|$, $|\phi(\tau_1-\delta)-\phi(\tau_2-\delta)|$ and $|\phi(0)-\phi(\tau_2-\tau_1)|\to 0$ as $|\tau_2-\tau_1|\to 0$. Hence $|(\mathbf{F}\boldsymbol{\Phi})(\tau_2)-(\mathbf{F}\boldsymbol{\Phi})(\tau_1)|\to 0$ as $|\tau_2-\tau_1|\to 0$. This shows that $\mathbf{F}$ maps $X_K^{\delta,h}$ to a family of uniformly equicontinuous functions in $C([\delta,\delta+h)\,;\mathbb{R}^{2n})$.

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