

# ASYMPTOTIC EFFICIENCY OF A CLASS OF NON-PARAMETRIC TESTS FOR REGRESSION PARAMETERS<sup>1</sup>

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**0. Introduction and summary.** For testing hypotheses about  $\alpha$  and  $\beta$  in the linear regression model  $Y_j = \alpha + \beta x_j + Z_j$ , Brown and Mood [18] have proposed distribution-free tests, based on their median estimates. Daniels [6] has also given a distribution-free test for the hypothesis that the regression parameters have specified values. This latter test is an improvement on the Brown and Mood median procedure, although both are based on the signs of the observations. Recently Hájek [10] constructed rank tests, which are asymptotically most powerful, for testing the hypothesis that  $\beta = 0$ , while  $\alpha$  is regarded as a nuisance parameter.

In this paper, a class of rank score tests for the hypothesis  $H: \alpha = \beta = 0$ , is proposed in Section 2. This class includes as special cases, the Wilcoxon and the normal scores type of tests. In Sections 3 and 4 the limiting distribution of the test statistics is shown to be central  $\chi^2$ , under  $H$ , and non-central  $\chi^2$ , under a sequence of alternatives tending to the hypothesis at a suitable rate. In Section 5, the Pitman efficiency of the proposed tests relative to the classical  $F$ -test, is proved to be the same as the efficiency of the corresponding rank score tests relative to the  $t$ -test in the two sample problem.

**1. Assumptions and notations.** Let  $(Y_{n1}, \dots, Y_{nn})$  be a sequence of random vectors, where  $Y_{nj}, j = 1, \dots, n$ , are independent with distributions

$$(1.1) \quad P_{\alpha\beta}(Y_{nj} \leq y) = F(y - \alpha - \beta x_{nj})$$

where  $P_{\alpha\beta}$  denotes that the probability is being computed for the parameter values  $\alpha$  and  $\beta$ .

The  $x_{nj}$  are known constants depending on  $n$ ; and we shall suppress this dependence in our notation, whenever this causes no confusion. The problem here is to construct rank score tests for the hypothesis  $H: \alpha = \beta = 0$ . The form of  $F$  is not known but we shall assume only that  $F \in \mathcal{F}$ , where

$$(1.2) \quad \mathcal{F} = \{ \text{absolutely continuous } F: \begin{array}{l} \text{(i) } F'(y) = f(y) \text{ is absolutely continuous,} \\ \text{(ii) } \int_{-\infty}^{\infty} (f'(y)/f(y))^2 f(y) dy \text{ is finite,} \\ \text{(iii) } f(-y) = f(y) \} \end{array} \}$$

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We shall also assume that the constants  $x_j, j = 1, \dots, n$ , satisfy the following conditions:

$$(1.3) \quad \lim [\{\max_j (x_j - \bar{x}_n)^2\} / \{\sum_j (x_j - \bar{x}_n)^2\}] = 0,$$

$$(1.4) \quad \lim [n^{-1} \sum_j (x_j - \bar{x}_n)^2] < \infty, \quad |\lim \bar{x}_n| < \infty,$$

$$(1.5) \quad \lim n^{-1} \sum_j (x_j - \bar{x}_n)^2 > 0,$$

where  $\bar{x}_n = n^{-1} \sum_j x_j$ , and the summation goes from 1 to  $n$ .

All limits in this paper unless otherwise stated are taken as  $n \rightarrow \infty$ . We shall write  $\mathcal{L}(X_n | P) \rightarrow N(a, b^2)$  to denote that the distribution law of  $(X_n - a)/b$  tends to the standard normal distribution under  $P$ . The following class of functions shall be used in the sequel.

$$(1.6) \quad \psi(u) = -[g'(G^{-1}(\frac{1}{2}u + \frac{1}{2}))/g(G^{-1}(\frac{1}{2}u + \frac{1}{2}))], \quad 0 < u < 1,$$

where  $G^{-1}$  is the inverse of  $G$ , and  $G$  is any known distribution function belonging to the class  $\mathcal{F}$ . The (1.6)-function that corresponds to  $F$  is

$$(1.7) \quad \varphi(u) = -[f'(F^{-1}(\frac{1}{2}u + \frac{1}{2}))/f(F^{-1}(\frac{1}{2}u + \frac{1}{2}))], \quad 0 < u < 1.$$

Observe that unlike  $\psi(u)$  of (1.6),  $\varphi(u)$  is not known since it is defined through the unknown  $F$ .

**2. A class of test statistics.** Let  $R_j$  be the rank of  $|Y_j|$  in the sequence of absolute values  $|Y_1|, \dots, |Y_n|$  of the  $n$  observations. Consider a pair of statistics  $T_1$  and  $T_2$  defined by

$$(2.1) \quad T_1(Y) = n^{-1} \sum_j \psi_n(R_j/n + 1) \text{Sign } Y_j,$$

$$(2.2) \quad T_2(Y) = n^{-1} \sum_j x_j \psi_n(R_j/n + 1) \text{Sign } Y_j,$$

where

$$(2.3) \quad \psi_n(u) = \psi(j/n + 1), \quad (j - 1)/n < u \leq j/n,$$

and

$$(2.4) \quad \lim \int_0^1 [\psi_n(u) - \psi(u)]^2 du = 0$$

by [9]. Let a symmetric  $2 \times 2$  matrix  $\|\gamma_{ki}\|_n$  be given by

$$(2.5) \quad \begin{aligned} \gamma_{11} &= \int_0^1 \psi^2(u) du; & \gamma_{22,n} &= n^{-1} \sum_j x_j^2 \int_0^1 \psi^2(u) du; \\ \gamma_{21,n} &= n^{-1} \sum_j x_j \int_0^1 \psi^2(u) du. \end{aligned}$$

Define

$$(2.6) \quad M(\psi) = n(T_1, T_2) \|\gamma_{ki}\|_n^{-1} (T_1, T_2)'$$

where  $(\mathbf{V})'$  denotes the transpose of  $(\mathbf{V})$ , and  $\|\gamma_{ki}\|_n^{-1}$  is the inverse of  $\|\gamma_{ki}\|_n$ . We propose  $M(\psi)$  as the class of test statistics for the hypothesis  $H: \alpha = \beta = 0$ .

Observe that  $M$  is well defined since both  $\|\gamma_{ki}\|_n^{-1}$  and its limit as  $n \rightarrow \infty$ , exist by (1.5).

To every  $G \in \mathcal{F}$ , corresponds one test statistic  $M$ . In particular, if  $G$  is the logistic distribution function  $G(y) = \{1 + \exp(-y)\}^{-1}$ , it can easily be checked that  $\psi(u) = u$ . The corresponding test statistic  $M$ , defined through  $T_1(Y) = n^{-1} \sum_j (R_j/n + 1) \text{Sign } Y_j$  and  $T_2(Y) = n^{-1} \sum_j x_j (R_j/n + 1) \text{Sign } Y_j$  is said to be of the Wilcoxon type. If  $G$  is chosen to be the normal distribution function  $\Phi$ , then  $\psi(u) = \Phi^{-1}(\frac{1}{2}u + \frac{1}{2})$  and the corresponding  $M$ -statistic is said to be of the Van der Waerden (normal scores) type. On taking  $G$  to be the double exponential distribution function,  $\psi(u)$  reduces to unity, and the corresponding  $M$ -statistic defined through  $T_1(Y) = n^{-1} \sum_j \text{Sign } Y_j$ , and  $T_2(Y) = n^{-1} \sum_j x_j \text{Sign } Y_j$  is said to be of the sign type. We remark that the components  $T_1$  and  $T_2$  of  $M$ , are familiar for we recognize  $T_1$  as being equivalent to the usual rank score statistic for the one sample problem [8], while  $T_2$  is similar to Hájek's statistic for the test of symmetry [10].

**3. Limiting distribution of  $M$  under the hypothesis.** We note that under  $H$ , the joint distribution of  $T_1$  and  $T_2$  is independent of  $F$ , but depends only on the function  $\psi$  and hence on  $G$ , through which  $\psi$  is defined. The following theorem gives the limiting null distribution of  $M$ .

**THEOREM 3.1.** *Under the assumptions of Section 2,*

$$\lim P_0(M \leq y) = P(\chi_2^2 \leq y)$$

where  $\chi_2^2$  denotes the central chi-square random variable, with 2 degrees of freedom, and  $P_0$  denotes that the probability is computed under  $H: \alpha = \beta = 0$ .

**PROOF.** It suffices to prove that  $\mathcal{L}(n^{\frac{1}{2}}(T_1, T_2) | P_0)$  tends to the bivariate normal distribution with covariance matrix  $\|\gamma_{ki}\| = \lim \|\gamma_{ki}\|_n$ . The idea of the proof is as in [10] to replace  $T_i, i = 1, 2$ , by sums of independent random variables and apply the central limit theorem.

Now introduce two statistics,

$$(3.1) \quad S_1^{(0)} = n^{-1} \sum_j \psi_n(F^*(|Y_j|)) \text{Sign } Y_j$$

and

$$(3.2) \quad S_2^{(0)} = n^{-1} \sum_j x_j \psi_n(F^*(|Y_j|)) \text{Sign } Y_j$$

where  $F^*$  is the distribution function of  $|Y_j|$ , i.e.  $P_0\{|Y_j| \leq y\} = F^*(y) = 2F(y) - 1$  for  $y > 0$ . Because the vectors  $(R_1, \dots, R_n), (|Y_1|, \dots, |Y_n|)$  and  $(\text{Sign } Y_1, \dots, \text{Sign } Y_n)$  are mutually independent, and  $E_0(\text{sign } Y_j) = 0$  for all  $j$ , we have

$$\begin{aligned} E_0\{n^{\frac{1}{2}}(T_2 - S_2^{(0)})\}^2 &= \text{Var}_0\{n^{\frac{1}{2}}(T_2 - S_2^{(0)})\} \\ &= n^{-1} \sum_j x_j^2 E_0[\psi_n(R_j/n + 1) - \psi_n(U_j)]^2 \end{aligned}$$

where  $U_j = F^*(|Y_j|)$  are independent random variables uniformly distributed on  $(0, 1)$ . Clearly

$$E_0[\psi_n(R_j/n + 1) - \psi_n(U_j)]^2 = E_0[\psi_n(R_1/n + 1) - \psi_n(U_1)]^2$$

for every  $j = 1, \dots, n$ . Using Lemma 2.1 of [9], and (1.4), it is seen that

$$(3.3) \quad n^{\frac{1}{2}}(T_i - S_i^{(0)}) \rightarrow 0 \text{ in } P_0\text{-probability,} \quad i = 1, 2.$$

Let

$$(3.4) \quad S_1 = n^{-1} \sum_j \psi(U_j) \text{ Sign } Y_j$$

and

$$(3.5) \quad S_2 = n^{-1} \sum_j x_j \psi(U_j) \text{ Sign } Y_j,$$

where  $U_j = F^*(|Y_j|)$ . Due to the independence of  $|Y_j|$  and  $\text{Sign } Y_j$ , we have

$$\begin{aligned} E_0[n^{\frac{1}{2}}(S_2^{(0)} - S_2)]^2 &= \text{Var}_0 \{n^{\frac{1}{2}}(S_2^{(0)} - S_2)\} \\ &= n^{-1} \sum_j x_j^2 E_0[\psi_n(F^*(|Y_j|)) - \psi(F^*(|Y_j|))]^2 \\ &= n^{-\frac{1}{2}} \sum_j x_j^2 \int_0^1 [\psi_n(u) - \psi(u)]^2 du \rightarrow 0. \end{aligned}$$

Hence

$$(3.6) \quad n^{\frac{1}{2}}(S_i^{(0)} - S_i) \rightarrow 0 \text{ in } P_0\text{-probability,} \quad i = 1, 2.$$

On combining (3.3) and (3.6), we obtain that for  $i = 1, 2$ ,

$$(3.7) \quad n^{\frac{1}{2}}(T_i - S_i) \rightarrow 0 \text{ in } P_0\text{-probability.}$$

Because the  $\psi(U_j), j = 1, \dots, n$ , are independent identically distributed random variables, which are also independent of  $\text{sign } Y_j$ , the central limit theorem gives immediately

$$(3.8) \quad \mathcal{L}(n^{\frac{1}{2}}T_1 | P_0) \rightarrow N(0, \gamma_{11}) = N(0, \gamma_1^2).$$

On the other hand, it is clear that the general central limit theorem (Loève [17], Theorem B, p. 280) applies to  $n^{\frac{1}{2}}S_2$ , and on account of (3.7), we also obtain that,

$$(3.9) \quad \mathcal{L}(n^{\frac{1}{2}}T_2 | P_0) \rightarrow N(0, \gamma_{22}) = N(0, \gamma_2^2)$$

where  $\gamma_{22} = \lim \gamma_{22,n}$  defined in (2.5). To prove the joint asymptotic normality of  $n^{\frac{1}{2}}T_1$  and  $n^{\frac{1}{2}}T_2$ , it is sufficient, because of (3.7) and a well known theorem (Cramér [4], p, 299), to establish the joint asymptotic normality of  $n^{\frac{1}{2}}S_1$  and  $n^{\frac{1}{2}}S_2$ .

But for any arbitrary  $\lambda_1$  and  $\lambda_2$ ,  $n^{\frac{1}{2}}(\lambda_1 S_1 + \lambda_2 S_2) = n^{-\frac{1}{2}} \sum_j x_j^* \psi(U_j) \text{ Sign } Y_j$  where  $x_j^*$  satisfy conditions (1.3) and (1.4). It follows that

$$\mathcal{L}(n^{\frac{1}{2}}(\lambda_1 T_1 + \lambda_2 T_2) | P_0)$$

tends to a normal distribution, and hence that  $\mathcal{L}(n^{\frac{1}{2}}(T_1, T_2) | P_0)$  tends to the bivariate normal distribution with zero means, and covariance matrix  $\|\gamma_{kl}\|$ . The assertion of the theorem follows by the usual transformation, see for example Sverdrup [21].

As a direct consequence of Theorem 3.1, it follows that the critical function

$$\begin{aligned} \varphi(M) &= 1 \text{ if } M > \chi_{2,\epsilon}^2 \\ &= 0 \text{ otherwise} \end{aligned}$$

where  $\chi_{2,\epsilon}^2$  is the 100(1 -  $\epsilon$ )% point of the  $\chi^2$  distribution with 2 degrees of freedom, provides an asymptotic level  $\epsilon$  test of  $H$ .

**4. Limiting distribution of  $M$  under near alternatives.** In order to determine the efficiency of the class  $M(\psi)$  of test statistics, it is necessary to find its distribution under a sequence of alternatives tending to the hypothesis, at a suitable rate. In this section we discuss the distribution of  $M$  for alternatives tending to  $H$  at the rate of  $n^{-\frac{1}{2}}$ ; and for this, we shall follow the method based on Le Cam's contiguity lemma [10] and [15]. First we give the set-up under which the contiguity principle is applicable.

Let  $P_n = \prod_{j=1}^n P_j$  be the distributions of  $(Y_1, \dots, Y_n)$  under a sequence  $K_n$  of alternatives defined by

$$(4.1) \quad K_n : \alpha_n = an^{-\frac{1}{2}}; \quad \beta_n = bn^{-\frac{1}{2}}$$

and let

$$(4.2) \quad r_j = p_j(Y_j)/p_0(Y_j) \quad \text{for } p_0(y) > 0$$

where  $p_j, j = 1, \dots, n$ , are densities corresponding to  $P_j$ , and  $p_0$  corresponds to the distribution  $P_0$ , under the hypothesis.

Define

$$(4.3) \quad W_n = 2 \sum_j (r_j^{\frac{1}{2}} - 1).$$

With the above notation, we shall prove the following:

**LEMMA 4.1.** *If (1.2), (1.3) and (1.4) hold, and  $P_n$  are the distributions under  $K_n$  given in (4.1), then  $P_n$  are contiguous to  $P_0$ .*

**PROOF.** The lemma will be proved if we show that

- (i)  $\lim \max_j P_0(|r_j - 1| > \epsilon) = 0$  for every  $\epsilon > 0$ , and
- (ii)  $\mathcal{L}(W_n | P_0) \rightarrow N(-\frac{1}{4}\sigma^2, \sigma^2)$ .

For (i), write  $r_j = f(Y_j - h_j)/f(Y_j)$  where  $h_j = n^{-\frac{1}{2}}(a + bx_j)$  and we may assume  $h_j \neq 0$ . Again the dependence on  $n$  of  $h_j$  is suppressed. Then

$$\begin{aligned} \max_j P_0(|r_j - 1| > \epsilon) &\leq \max_j \epsilon^{-1} E_0 |r_j - 1| \\ &= \max_j \epsilon^{-1} |h_j| \int_{-\infty}^{\infty} |h_j^{-1}\{f(y - h_j) - f(y)\}| dy. \end{aligned}$$

Now

$$|h_j^{-1}\{f(y - h_j) - f(y)\}| \leq |h_j^{-1}| \int_{y-h_j}^y |f'(x)| dx,$$

and

$$\int_{-\infty}^{\infty} |h_j^{-1}\{f(y - h_j) - f(y)\}| dy \leq \int_{-\infty}^{\infty} |f'(y)| dy \quad \text{for all } j = 1, \dots, n.$$

Hence we have that

$$\max_j P_0\{|r_j - 1| > \epsilon\} \leq \max_j \epsilon^{-1} |h_j| \int_{-\infty}^{\infty} |f'(y)| dy \rightarrow 0.$$

To prove (ii), define

$$(4.4) \quad S_n^* = n^{-1} \sum_j (a + bx_j) f'(Y_j)/f(Y_j),$$

where  $f'(x) = d/dxf(x)$ , and write  $W_n$  in the form

$$(4.5) \quad W_n = 2 \sum_j [ \{s(Y_j - h_j)/s(Y_j)\} - 1 ],$$

where  $s(x) = f^{\frac{1}{3}}(x)$ . It can be seen that

$$(4.6) \quad E_0 W_n \sim -\sigma_n^2/4$$

and that

$$(4.7) \quad \text{Var}_0 (n^{\frac{1}{3}} S_n^*) = \sigma_n^2$$

where  $\sigma_n^2 = \sum_j h_j^2 \int_{-\infty}^{\infty} (f'(y)/f(y))^2 f(y) dy = \sum_j h_j^2 \int_0^1 \varphi^2(u) du$ . Furthermore,

$$(4.8) \quad E_0(W_n - E_0 W_n - n^{\frac{1}{3}} S_n^*) \leq 4 \sum_j h_j^2 \int_{-\infty}^{\infty} [h_j^{-1} \{s(y - h_j) - s(y)\} - s'(y)]^2 dy$$

and the right hand side of (4.8) tends to zero by (1.4) and Lemma 4.3 of [10]. The limiting distribution of  $n^{\frac{1}{3}} S_n^*$  is of course normal with zero mean and variance  $\sigma^2 = \lim \sigma_n^2$ . It follows from (4.6) and (4.8) that  $\mathcal{L}(W_n | P_0) \rightarrow N(-\frac{1}{4}\sigma^2, \sigma^2)$ , and the proof of the lemma is therefore complete.

We shall now apply the contiguity principle to obtain the limit distribution of  $(T_1, T_2)$  under the sequence  $K_n$  of alternatives. The main result of this section is the following:

**THEOREM 4.1.** *Under the assumptions of Lemma 4.1,  $\mathcal{L}(n^{\frac{1}{3}}(T_1, T_2) | P_n)$  tends to the bivariate normal distribution with mean vector  $(\mu_1, \mu_2)$  and covariance matrix  $\|\gamma_{ki}\|$ , where the means are defined by:*

$$(4.9) \quad \begin{aligned} \mu_1 &= \lim n^{-1} \sum_j (a + bx_j) \int_0^1 \psi(u) \varphi(u) du, \\ \mu_2 &= \lim n^{-1} \sum_j x_j (a + bx_j) \int_0^1 \psi(u) \varphi(u) du \end{aligned}$$

and the functions  $\psi$  and  $\varphi$  are defined in (1.6) and (1.7) respectively.

**PROOF.** We shall first prove that  $\mathcal{L}(n^{\frac{1}{3}} T_i | P_n) \rightarrow N(\mu_i, \gamma_i^2)$ ,  $i = 1, 2$ , then show that  $(T_1, T_2)$  has a joint asymptotic normal distribution under  $K_n$ . By the contiguity principle the first one will follow if we prove that

(a)  $\mathcal{L}(n^{\frac{1}{3}} T_i, W_n | P_0)$ ,  $i = 1, 2$ , tend to bivariate normal distributions with certain correlation coefficients  $\rho_i$ , and  $W_n$  is as defined in (4.3);

(b)  $\mu_i = \rho_i \sigma \gamma_i$ , where  $\sigma^2$  is the limit of  $\sigma_n^2$  defined in (4.7).

By (3.7) and (4.8) it is sufficient to consider  $\mathcal{L}(n^{\frac{1}{3}}(S_i, S_n^*) | P_0)$ . Also due to the symmetry of the density function  $f$ , we may write  $S_n^*$  of (4.4), as

$$\begin{aligned} S_n^* &= n^{-1} \sum_j (a + bx_j) f^{*'}(|Y_j|) / f^*(|Y_j|) \text{Sign } Y_j \\ &= n^{-1} \sum_j (a + bx_j) \varphi(U_j) \text{Sign } Y_j \end{aligned}$$

where  $f^*$  is the density function corresponding to the distribution function  $F^*$  of  $|Y_j|$ . Now

$$n(S_2, S_n^*) = [ \sum_j x_j \psi(U_j), \sum_j (a + bx_j) \varphi(U_j) ] \text{Sign } Y_j,$$

so that

$$\text{Cov}_0 [n^{\frac{1}{2}}(S_2, S_n^*)] = n^{-1} \sum_j x_j(a + bx_j) \int_0^1 \psi(u)\varphi(u) du$$

which, by (1.4) tends to a finite limit. Furthermore it can be seen that under conditions (1.3) and (1.4) the bivariate central limit theorem (Cramér [5], p. 114, Theorem 21a) applies to  $n^{\frac{1}{2}}(S_i, S_n^*)$ ,  $i = 1, 2$ , giving that  $\mathcal{L}(n^{\frac{1}{2}}(S_i, S_n^*) | P_0)$  tend to the bivariate normal distributions with correlation coefficients  $\rho_i$  where

$$\begin{aligned} \rho_1 &= \lim \{n^{-1} \sum_j (a + bx_j) \int_0^1 \psi(u)\varphi(u) du \\ &\quad \cdot [n^{-1} \sum_j (a + bx_j)^2 \int_0^1 \psi^2(u) du \int_0^1 \varphi(u) du]^{-\frac{1}{2}}\}, \\ \rho_2 &= \lim \{n^{-1} \sum_j x_j(a + bx_j) \int_0^1 \psi(u)\varphi(u) du \\ &\quad \cdot [n^{-1} \sum_j (a + bx_j)^2 \int_0^1 \psi^2(u) du \int_0^1 \varphi^2(u) du]^{-\frac{1}{2}}\}. \end{aligned}$$

Finally it is immediate that  $\mu_i = \rho_i\sigma\gamma_i$ , hence we have proved that  $\mathcal{L}(n^{\frac{1}{2}}T_i | P_n) \rightarrow N(\mu_i, \gamma_i^2)$ ,  $i = 1, 2$ .

For the joint asymptotic distribution of  $T_1$  and  $T_2$  under  $K_n$ , note that

$$(4.10) \quad \lim \mathcal{L}(n^{\frac{1}{2}}(\lambda_1 T_1 + \lambda_2 T_2), W_n | P_0) = \lim \mathcal{L}(n^{\frac{1}{2}}(\lambda_1 S_1 + \lambda_2 S_2), S_n^* | P_0)$$

which by Theorem 21a of [5], is a bivariate normal. We have also shown that  $\mathcal{L}(n^{\frac{1}{2}}(\lambda_1 S_1 + \lambda_2 S_2) | P_0)$  is asymptotically normal. This fact, together with (4.10) implies that  $\mathcal{L}(n^{\frac{1}{2}}(\lambda_1 S_1 + \lambda_2 S_2) | P_n)$  is also asymptotically normal. Since  $\lambda_1$  and  $\lambda_2$  are arbitrary, it follows that  $n^{\frac{1}{2}}(T_1, T_2)$  has a limiting bivariate normal under  $K_n$ . This completes the proof of Theorem 4.1.

We are now in the position to give the limit distribution of  $M$ , and this is stated in the following:

**THEOREM 4.2.** *Under the assumptions of Lemma 4.1,  $\mathcal{L}(M | P_n) \rightarrow \mathcal{L}(\chi_2^2(\Delta^2))$  where  $\chi_2^2(\Delta^2)$  denotes the non-central chi-square random variable with 2 degrees of freedom and non-centrality parameter  $\Delta^2$  given by*

$$(4.11) \quad \Delta^2 = \lim (a^2 + 2ab\bar{x}_n + b^2n^{-1} \sum_j x_j^2) (\int \psi\varphi)^2 / \int \psi^2.$$

**PROOF.** The proof follows directly from Theorem 4.1, and  $\Delta^2$  is obtained by straightforward computation.

**5. Asymptotic efficiency of  $M$ -tests.** We employ a measure of relative efficiency of two test statistics due to Pitman (see Noether [19]). If under the same sequence of alternatives, the two test statistics have non-central chi square limit distributions, with the same degrees of freedom, it has been shown by Andrews [2] and Hannan [11], that their relative asymptotic efficiency is given by the ratio of their non-centrality parameters. To find the asymptotic efficiency of the  $M$ -tests relative to the classical  $F$ -test, we need therefore, to compute the non-centrality parameter of the latter.

The classical test statistic  $\tilde{M}$  for  $H$  is based on a quadratic function in the least squares estimates  $\tilde{\alpha}$  and  $\tilde{\beta}$  of  $\alpha$  and  $\beta$ .  $H$  is rejected if

$$(5.1) \quad \tilde{M} = n(\tilde{\alpha}, \tilde{\beta}) \|\tau_{kl}\|_n^{-1} (\tilde{\alpha}, \tilde{\beta})'$$

is too large, where  $\|\tau_{ki}\|_n^{-1}$  is the inverse of  $\|\tau_{ki}\|_n$  defined by

$$(5.2) \quad \tau_{11} = \tau_1^2 = (\sum_j x_j^2) / \sum_j (x_j - \bar{x}_n)^2; \quad \tau_2^2 = \{n^{-1} \sum_j (x_j - \bar{x}_n)^2\}^{-1};$$

$$\tau_{21} = \tau_{12} = -\bar{x}_n / n^{-1} \sum_j (x_j - \bar{x}_n)^2.$$

If, as is usual in classical tests, the distribution function  $F$  is assumed to be  $\Phi$ , the normal df, then the estimates  $\tilde{\alpha}$  and  $\tilde{\beta}$  being linear functions of normal random variables, are themselves normal random variables, so that, under  $H$ ,  $\tilde{M}$  has a central  $\chi^2$  distribution with 2 degrees of freedom. And under any given alternative of the form  $\alpha = a, \beta = b$ ,  $\tilde{M}$  has a non-central  $\chi^2$  distribution with 2 degrees of freedom, and non-centrality parameter given by (see e.g. Lehmann [16], p. 284)

$$\tilde{\Delta}_n^2 = n(a + b\bar{x}_n)^2 + b \sum_j (x_j - \bar{x}_n)^2.$$

It follows that under the sequence of alternatives  $K_n$ , the non-centrality parameter reduces to

$$(5.3) \quad \tilde{\Delta}^2 = \lim (a^2 + 2ab\bar{x}_n + b^2 n^{-1} \sum_j x_j^2).$$

For any distribution function  $F \in \mathcal{F}$ , the limit distribution of  $\tilde{M}$  will still be  $\chi^2$ , this is a consequence of theorem of Eicker [7], which gives general conditions for the least squares estimates to be asymptotically normally distributed. It can be verified [1] that Eicker's conditions are implied by our general assumptions of Section 1. We summarize the above facts in

LEMMA 5.1 *Under the assumptions of Lemma 4.1,  $\mathcal{L}(\tilde{M} | P_n) \rightarrow \mathcal{L}(\chi_2^2(\tilde{\Delta}^2))$  where  $\chi_2^2(\tilde{\Delta}^2)$  denotes the non-central chi-square random variable with 2 degrees of freedom and non-centrality parameter  $\tilde{\Delta}^2$  given by (5.3).*

From (4.11) and (5.3), it follows that the asymptotic efficiency of the  $M$ -tests relative to the classical  $\tilde{M}$ -test is given by

$$(\int_0^1 \psi(u)\varphi(u) du)^2 / (\int_0^1 \psi^2(u) du).$$

If the common variance of the  $Y_j$  is  $\sigma_0^2$ , instead of unity, as we assumed, the efficiency becomes

$$(5.4) \quad e_{M, \tilde{M}}(\psi) = \sigma_0^2 (\int_0^1 \psi(u)\varphi(u) du)^2 / (\int_0^1 \psi^2(u) du).$$

On taking  $\psi(u) = u$  (Wilcoxon), the efficiency expression on the right hand side of (5.4) reduces, after integration by parts, to

$$(5.5) \quad e_{Mw, \tilde{M}}(F) = 12\sigma_0^2 (\int_{-\infty}^{\infty} f^2(y) dy)^2,$$

and on choosing  $\psi(u) = \Phi^{-1}(\frac{1}{2}u + \frac{1}{2})$  (Van der Waerden) the efficiency simplifies to

$$(5.6) \quad e_{Mn.sc, \tilde{M}}(F) = \sigma_0^2 \int_{-\infty}^{\infty} f^2(y) dy / \Phi'(\Phi^{-1}(F(y)))$$

where  $\Phi'$  is the density of the standard normal distribution. Another special case of interest is obtained by taking  $\psi(u) = \text{sign } u = 1$ . The efficiency, in this case, becomes

$$(5.7) \quad e_{Ms, \tilde{M}}(F) = 4\sigma_0^2 f^2(0).$$



We recognize (5.5) and (5.6) as the efficiencies of the Wilcoxon and the normal scores tests for shift relative to the  $t$ -test. Efficiency (5.5) has been studied in detail by Hodges and Lehmann [13], and (5.6) by Chernoff and Savage [3]. The question of choice between  $M_w$  and  $M_{n.sc}$  as test statistics has been fully discussed in [14]. In general, provided the functions  $\psi$  and  $\varphi$  are non-decreasing on  $(0, 1)$ , the efficiency expression in (5.4) is the same as that of the corresponding rank score tests relative to the student's  $t$ -test or the  $F$ -test in the one sample [8] or the  $c$ -sample [20] problem. This is an immediate consequence of the similarity between the Chernoff-Savage  $J(u)$  function used in [20], and Hájek's  $\varphi(u)$  and  $\psi(u)$  functions.

In [6] Daniels proposed a distribution-free test for  $H$  which is related to the Hodges' bivariate sign test [12] for symmetry. It would be of interest to compare Daniels' test and the  $M$ -tests with respect to their efficiency behaviour, but such a comparison does not seem to be readily feasible because the asymptotic distribution of Daniels' test under a sequence of alternatives is not known.

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#### REFERENCES

- [1] ADICHIE, J. N. (1966) Unpublished Ph.D. Dissertation. University of California, Berkeley.
- [2] ANDREWS, F. C. (1954) Asymptotic behaviour of some rank tests for analysis of variance. *Ann. Math. Statist.* **25** 724-735.
- [3] CHERNOFF, H. and SAVAGE, I. R. (1958) Asymptotic normality and efficiency of certain non-parametric test statistics. *Ann. Math. Statist.* **29** 972-994.
- [4] CRAMÉR, H. (1945) *Mathematical Methods of Statistics*. Princeton Univ. Press.
- [5] CRAMÉR, H. (1963) *Random Variables and Probability Distributions* (2nd edition). Cambridge Univ. Press.
- [6] DANIELS, H. E. (1954) A distribution-free test for regression parameters. *Ann. Math. Statist.* **25** 499-513.
- [7] EICKER, F. (1963) Asymptotic normality and consistency of the least squares estimators for families of linear regressions. *Ann. Math. Statist.* **34** 447-456.
- [8] GOVINDARAJULU, Z. (1960) Central limits theorems and asymptotic efficiency for one sample non-parametric procedures. Technical Report No. 11, Univ. of Minnesota
- [9] HÁJEK, J. (1961) Some extensions of the Wald-Wolfowitz-Noether theorem. *Ann. Math. Statist.* **32** 506-523.
- [10] HÁJEK, J. (1962) Asymptotically most powerful rank order tests. *Ann. Math. Statist.* **33** 1124-1147.
- [11] HANNAN, E. J. (1956) The asymptotic power of tests based on multiple correlation. *J. Roy. Statist. Soc. Ser. B* **18** 227-233.
- [12] HILL, B. M. (1960) A relation between Hodges bivariate sign test and a non-parametric test of Daniels. *Ann. Math. Statist.* **31** 1190-1196.
- [13] HODGES, J. L., JR. and LEHMANN, E. L. (1956) Efficiency of some non-parametric competitors of the  $t$ -test. *Ann. Math. Statist.* **27** 324-335.

- [14] HODGES, J. L., JR. and LEHMANN, E. L. (1961) Comparison of the normal scores and Wilcoxon tests. *Proc. Fourth Berk. Symp. Math. Statist. Prob.* **1** 307-317. Univ. of California Press.
- [15] LECAM, L. (1960) Locally asymptotic normal families of distribution. *Univ. of California Publ. Statist.* **3** No. 2 37-98.
- [16] LEHMANN, E. L. (1959) *Testing Statistical Hypotheses*. Wiley, New York.
- [17] LOÈVE, M. (1962) *Probability Theory* (3rd edition). Van Nostrand, New York.
- [18] MOOD, A. M. (1950) *Introduction to the Theory of Statistics*. McGraw-Hill, New York.
- [19] NOETHER, G. E. (1954) On a theorem of Pitman. *Ann. Math. Statist.* **25** 514-522.
- [20] PURI, M. L. (1964) Asymptotic efficiency of a class of  $c$ -Sample tests. *Ann. Math. Statist.* **35** 102-121.
- [21] SVERDRUP, E. (1952) The limit distribution of a continuous function of random variables. *Skand. Aktuarietidskrift.* **35** 1-10.