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ORIGINAL RESEARCH

Asymptotic equilibrium of integro-differential equations with infinite delay

Le Anh Minh[1] · Dang Dinh Chau[2]

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Abstract The asymptotic equilibrium problems of ordinary differential equations in a Banach space have been considered by several authors. In this paper, we investigate the asymptotic equilibrium of the integro-differential equations with infinite delay in a Hilbert space.

Keywords Asymptotic equilibrium · Integro-differential equations · Infinite delay

Introduction

The asymptotic equilibrium problems of ordinary differential equations in a Banach space have been considered by several authors, Mitchell and Mitchell [3], Bay et al. [1], but the results for the asymptotic equilibrium of integro-differential equations with infinite delay still is not presented. In this paper, we extend the results in [1] to a class of integro-differential equations with infinite delay in a Hilbert space H which has the following form:

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = A(t)\left(x(t) + \displaystyle\int_{-\infty}^{t} k(t-\theta)x(\theta)\mathrm{d}\theta\right), & t \geqslant 0, \\ x(t) = \varphi(t), & t \leqslant 0 \end{cases} \tag{1}$$

where $A(t): H \to H$, φ in the phase space $\mathscr{B}$, and x_t is defined as

$$x_t(\theta) = x(t+\theta), \quad -\infty < \theta \leqslant 0.$$

Preliminaries

We assume that the phase space $(\mathscr{B}, ||.||_{\mathscr{B}})$ is a seminormed linear space of functions mapping $(-\infty, 0]$ into H satisfying the following fundamental axioms (we refer reader to [2])

(A_1) For $a > 0$, if x is a function mapping $(-\infty, a]$ into H, such that $x \in \mathscr{B}$ and x is continuous on [0, a], then for every $t \in [0, a]$ the following conditions hold:

- (i) x_t belongs to $\mathscr{B}$;
- (ii) $||x(t)|| \leqslant G||x_t||_{\mathscr{B}}$;
- (iii) $||x_t||_{\mathscr{B}} \leqslant K(t)\sup_{s\in[0,t]}||x(s)|| + M(t)||x_0||_{\mathscr{B}}$

where G is a possitive constant, $K, M: [0,\infty) \to [0,\infty)$, K is continuous, M is locally bounded, and they are independent of x.

(A_2) For the function x in (A_1), x_t is a $\mathscr{B}$-valued continuous function for t in [0, a].

(A_3) The space $\mathscr{B}$ is complete.

✉ Le Anh Minh
leanhminh@hdu.edu.vn

Dang Dinh Chau
chaudida@gmail.com

1 Department of Mathematical Analysis, Hong Duc University, Thanh Hóa, Vietnam

2 Department of Mathematics, Hanoi University of Science, VNU, Hanoi, Vietnam

Example 1

(i) Let BC be the space of all bounded continuous functions from $(-\infty, 0]$ to H, we define $C^0 := \{\varphi \in BC : \lim_{\theta\to-\infty}\varphi(\theta) = 0\}$ and $C^\infty := \{\varphi \in BC : \lim_{\theta\to-\infty}\varphi(\theta) \text{ exists in } H\}$ endowed with the norm

$$||\varphi||_{\mathscr{B}} = \sup_{\theta\in(-\infty,0]} ||\varphi(\theta)||$$

then C^0, C^∞ satisfies (A_1)–(A_3). However, BC satisfies (A_1) and (A_3), but (A_2) is not satisfied.

(ii) For any real constant γ, we define the functional spaces C_γ by

$$C_\gamma = \left\{\varphi \in C((-\infty, 0], X) : \lim_{\theta\to-\infty} e^{\gamma\theta}\varphi(\theta) \text{ exists in } H\right\}$$

endowed with the norm

$$||\varphi||_{\mathscr{B}} = \sup_{\theta\in(-\infty,0]} e^{\gamma\theta}||\varphi(\theta)||.$$

Then conditions (A_1)–(A_3) are satisfied in C_γ.

Remark 1 In this paper, we use the following acceptable hypotheses on $K(t)$, $M(t)$ in (A_1)(iii) which were introduced by Hale and Kato [2] to estimate solutions as $t \to \infty$,

(γ_1) $K = K(t)$ is a constant for all $t \geqslant 0$;
(γ_2) $M(t) \leqslant M$ for all $t \geqslant 0$ and some M.

Example 2 For the functional space C_γ in Example 1, the hypotheses (γ_1) and (γ_2) are satisfied if $\gamma \geqslant 0$.

Definition 1 Equation (1) has an asymptotic equilibrium if every solution of it has a finite limit at infinity and, for every $h_0 \in H$, there exists a solution $x(t)$ of it such that $x(t) \to h_0$ as $t \to \infty$.

Main results

Now, we consider the asymptotic equilibrium of Eq. (1) which satisfies the following assumptions:

(M_1) $A(t)$ is a strongly continuous bounded linear operator for each $t \in \mathbb{R}^+$;
(M_2) $A(t)$ is a self-adjoint operator for each $t \in \mathbb{R}^+$;
(M_3) k satisfies

$$\int_0^{+\infty} |k(\theta)|\mathrm{d}\theta = L < +\infty;$$

and

(M_4) There exists a constant $T > 0$ such that

$$\sup_{h\in S(0,1)} \int_T^\infty ||A(t)h||\mathrm{d}t < q < \frac{1}{\kappa}, \tag{2}$$

herein $S(0, 1)$ is a unit ball in H, $\kappa = L(K + M) + 1$, where K, M, L are given in (γ_1), (γ_2) and (M_3).

Theorem 1 *If* (M_1), (M_2), (M_3) *and* (M_4) *are satisfied, then Eq.* (1) *has an asymptotic equilibrium.*

Proof We shall begin with showing that all solutions of (1) has a finite limit at infinity. Indeed, Eq. (1) may be rewritten as

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = A(t)\left(x(t) + \int_{-\infty}^{0} k(-\theta)x_t(\theta)\mathrm{d}\theta\right),$$

then for $t \geqslant s \geqslant T$ we have

$$x(t) = x(s) + \int_s^t A(\tau)\left(x(\tau) + \int_{-\infty}^{0} k(-\theta)x_\tau(\theta)\mathrm{d}\theta\right)\mathrm{d}\tau$$

and

$$\begin{aligned} &||x(t)|| \\ &= \sup_{h\in S(0,1)}\left|\left\langle x(s) + \int_s^t A(\tau)\left(x(\tau) + \int_{-\infty}^{0} k(-\theta)x_\tau(\theta)\mathrm{d}\theta\right)\mathrm{d}\tau, h\right\rangle\right| \\ &\leqslant ||x(s)|| + \sup_{h\in S(0,1)}\int_s^t\left|\left\langle x(\tau) + \int_{-\infty}^{0} k(-\theta)x_\tau(\theta)\mathrm{d}\theta, A(\tau)h\right\rangle\right| d\tau \\ &\leqslant ||x(s)|| + q\left((LK+1)\sup_{\xi\in[0,t]}||x(\xi)|| + LM||\varphi||_{\mathscr{B}}\right) \end{aligned} \tag{3}$$

implies

$$|||x(t)||| \leqslant ||x(s)|| + q\big((LK+1)|||x(t)||| + LM||\varphi||_{\mathscr{B}}\big)$$

or

$$|||x(t)||| \leqslant \frac{||x(s)|| + qLM||\varphi||_{\mathscr{B}}}{1 - q(LK+1)} \tag{4}$$

where

$$|||x(t)||| = \sup_{0\leqslant\xi\leqslant t} ||x(\xi)||.$$

Now, we conclude that $x(t)$ is bounded since

$$0<q<\frac{1}{\kappa}=\frac{1}{L(K+M)+1}<\frac{1}{LK+1}\Rightarrow q(LK+1)<1$$

and by (4).

Putting

$$M^*=\sup_{t\in\mathbb{R}}||x(t)||,$$

we have

$$||x(t)-x(s)||=\sup_{h\in S(0,1)}|<x(t)-x(s),h>|$$

$$\leqslant\sup_{h\in S(0,1)}\int_s^t\left|<A(\tau)\left(x(\tau)+\int_{-\infty}^{0}k(-\theta)x_\tau(\theta)\mathrm{d}\theta\right),h>\right|\mathrm{d}\tau,$$

$$\leqslant[M^*(LK+1)+LM||\varphi||_{\mathscr{B}}]\sup_{h\in S(0,1)}\int_s^t||A(\tau)h||d\tau\to 0$$

as $t\geqslant s\to+\infty$. That means all solutions of (1) have a finite limit at infinity. To complete the proof, it remains to show that for any $h_0\in H$, there exists a solution $x(t)$ of (1) such that

$$\lim_{t\to+\infty}x(t)=h_0.$$

Indeed, let h_0 be an arbitrary fixed element of H; we choose the initial function φ belongs to $\mathscr{B}$ such that $\varphi(0)=h_0$ and $||\varphi||_{\mathscr{B}}\leqslant||h_0||$ and consider the functional

$$g_1(t,h)=\langle h_0,h\rangle-\int_t^{\infty}\left\langle A(\tau)\left(h_0+\int_{-\infty}^{\tau}k(\tau-\theta)x_0(\theta)\mathrm{d}\theta\right),h\right\rangle\mathrm{d}\tau$$

We have

$$|g_1(t,h)|\leqslant||h_0||||h||+\int_t^{+\infty}\|x_0(\tau)+\int_{-\infty}^{\tau}k(\tau-\theta)x_0(\theta)\mathrm{d}\theta\|||A(\tau)h||\mathrm{d}\tau.$$

Since $x_0(\tau)\equiv h_0$, then

$$|g_1(t,h)|\leqslant||h_0||(||h||+q\kappa).$$

It follows from Riesz representation theorem that there exists an element $x_1(t)$ in H, such that

$$g_1(t,h)=\langle x_1(t),h\rangle$$

and

$$||x_1(t)||\leqslant||h_0||(1+q\kappa).$$

Now, we consider the functional

$$g_2(t,h)=\langle h_0,h\rangle-\int_t^{+\infty}\left\langle A(\tau)\left(x_1(t)+\int_{-\infty}^{\tau}k(\tau-\theta)x_1(\theta)\mathrm{d}\theta\right),h\right\rangle\mathrm{d}\tau.$$

By an argument analogous to the previous one, we get

$$|g_2(t,h)|\leqslant||h_0||[||h||+q\kappa+(q\kappa)^2]$$

and there exists an element $x_2(t)$ in H, such that

$$g_2(t,h)=\langle x_2(t),h\rangle$$

with

$$||x_2(t)||\leqslant||h_0||(1+q\kappa+(q\kappa)^2).$$

Continuing this process, we obtain the linear continuous functional

$$g_n(t,h)=\langle h_0,h\rangle-\int_t^{+\infty}\left\langle A(\tau)\left(x_{n-1}(t)+\int_{-\infty}^{\tau}k(\tau-\theta)x_{n-1}(\theta)\mathrm{d}\theta\right),h\right\rangle\mathrm{d}\tau \tag{5}$$

and $x_n(t)\in H$ such that

$$g_n(t,h)=\langle x_n(t),h\rangle$$

satisfies the following estimate

$$||x_n(t)||\leqslant(1+q\kappa+(q\kappa)^2+\cdots+(q\kappa)^n)||h_0||\leqslant\frac{||h_0||}{1-q\kappa}.$$

Futhermore,

$$||x_n(t)-x_{n-1}(t)||\leqslant||h_0||(q\kappa)^n.$$

This inequality shows that $\{x_n(t)\}$ is uniformly convergent on $[T,+\infty)$ since $q\kappa<1$. Put

$$x(t)=\lim_{n\to+\infty}x_n(t).$$

In (5), let $n\to+\infty$, we have

$$\langle x(t),h\rangle=\langle h_0,h\rangle-\int_t^{+\infty}\left\langle A(\tau)\left(x(t)+\int_{-\infty}^{\tau}k(\tau-\theta)x(\theta)\mathrm{d}\theta\right),h\right\rangle\mathrm{d}\tau \tag{6}$$

and since

$$|\langle x_n(t), h_0\rangle| < \int_T^{+\infty} \|x_{n-1}(\tau) + \int_{-\infty}^{\tau} k(\tau-\theta)x_{n-1}(\theta)\mathrm{d}\theta\| \|A(\tau)h\|\mathrm{d}\tau$$

or

$$|\langle x_n(t), h_0\rangle| \leqslant \frac{||h_0||q}{1-q\kappa},$$

we have $x_n(t) \to h_0$ as $q \to 0$, which means that there exists a solution of (1) converging to h_0. The theorem is proved.

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