Journal of Applied Mathematics and Physics (ZAMP)
Vol. 41, January 1990

0044-2275/90/010114-13 $ 1.50 + 0.20

Asymptotic estimates for Laguerre polynomials

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1. Introduction

The asymptotic behaviour of Laguerre polynomials

$$L_n^{(\alpha)}(x) = \sum_{m=0}^{n} \binom{n+\alpha}{n-m} \frac{(-x)^m}{m!} \tag{1.1}$$

has been the subject of several investigations. The earlier results are summarized in Szegö [12]. For $x = 4n + \mathcal{O}(\sqrt{n})$ an Airy function is used, and it describes the transition of the oscillatory to the monotonic region in the x-interval. Later, a more complete description has been given by Tricomi, summarized in [5]. Let

$$\kappa = n + (\alpha + 1)/2. \tag{1.2}$$

Tricomi distinguished four cases:

(*i*)	x near 0,	Hilb's formula,
(*ii*)	$0 < x < 4\kappa$,	oscillatory region,
(*iii*)	x near 4κ,	turning point region,
(*iv*)	$x > 4\kappa$,	monotonic region.

In (*i*) a Bessel function is used for the transition of $x < 0$ (a monotonic region) to the oscillatory region. The early zeros of $L_n^{(\alpha)}(x)$ can be approximated in terms of those of the Bessel function $J_\alpha(z)$. The transition at the turning point in (*iii*) is described by an Airy function.

The x-regions of validity in Tricomi's results do not overlap. Erdélyi [6] showed that the whole of the real x-axis could in fact be covered by just two asymptotic approximations. He obtained the leading terms of the approximations by using the differential equation of the Laguerre polynomial. These forms are substantially extended by Olver [10], who gave complete asymptotic expansions of the Whittaker functions, which, as special cases yield Erdélyi's results; moreover, Olver supplied error bounds for the remainders in the expansions.

In a recent paper, Frenzen and Wong [7] also derived complete expansions, together with explicit integral representations of the error terms and order estimates. Their approach was based on two integral representations of the Laguerre polynomials.

When we allow α to grow (with n or independently), the zeros of the Laguerre polynomial increase. It appears that the oscillatory region of $L_n^{(\alpha)}(4\kappa x)$ is located in an interval $(x_1, x_2) \subset [0, 1]$; $x_{1,2}$ will be given in §3. The J-Bessel function can still be used to describe the behaviour in a domain including the interval $(-\infty, x_1]$, but the argument of the Bessel function is more complicated than in the case with α fixed. In a domain including $[x_1, \infty)$ a Hermite polynomial can be used for describing the transition in both points $x_{1,2}$.

In §2 we summarize the results of Frenzen and Wong [7], in which n is large and $\alpha > -1$ is fixed. In §3 we discuss three asymptotic approximations in which κ of (1.2) is large, that is, α or n or both are large. These forms are available in the literature for Whittaker functions, and are now interpreted for Laguerre polynomials. Two results follow from Dunster [4], in which a J-Bessel and an Airy function as comparison functions are used. The other interpretation is obtained from Olver [11], which originally contains a parabolic cylinder function. Here we can use a Hermite polynomial. This is especially of interest for estimating the zeros of $L_n^{(\alpha)}(x)$ for the case that n and α are large. In §4 we show the results of a numerical evaluation of the approximation to the zeros.

Information on the special functions used in this paper can be found in Olver [10].

2. Asymptotic forms with n large and $\alpha > -1$ fixed

In this section we summarize the results of [7], which are obtained by using two integral representations of the Laguerre polynomials. The cited reference gives a detailed discussion of the conformal mappings that are needed to transform the integrals in the complex plane to standard forms.

Let a and b be fixed numbers, $0 < b < a < 1$, and let $v = 4\kappa$, where κ is given in (1.2).

2.1. Approximation in terms of a Bessel function

Let

$$A(x) = \begin{cases} \frac{1}{2}i[\sqrt{x^2 - x} - \operatorname{arcsinh}\sqrt{-x}], & \text{if } x \le 0; \\ \frac{1}{2}[\sqrt{x - x^2} + \arcsin\sqrt{x}], & \text{if } 0 \le x < 1. \end{cases}$$

Then for $p = 0, 1, 2, \ldots$

$$2^\alpha e^{-2\kappa x} L_n^{(\alpha)}(4\kappa x) = \frac{J_\alpha(\nu A)}{A^\alpha} \sum_{k=0}^{[(p-1)/2]} \frac{\alpha_{2k}}{(2\kappa)^{2k}} - \frac{J_{\alpha+1}(\nu A)}{A^{\alpha+1}} \sum_{k=0}^{[p/2]-1} \frac{\beta_{2k+1}}{(2\kappa)^{2k+1}} + \varepsilon_p, \tag{2.1}$$

where ε_p is a remainder, and $J_\alpha(z)$ is the familiar Bessel function. The coefficients $\alpha_{2k}, \beta_{2k+1}$ follow from the following scheme. Let

$$h(u) = \left[\frac{u}{\sinh z(u)}\right]^{\alpha+1} \frac{dz}{du},$$

where the relation between z and u is given by the equation

$$z - x \coth z = u - A^2(x)/u.$$

Then we define recursively (with $h_0(u) = h(u)$) a set of functions $\{h_n\}, \{g_n\}$ and coefficients $\{\alpha_n\}, \{\beta_n\}$ by writing

$$h_n(u) = \alpha_n + \beta_n/u + (1 + \alpha^2/u^2) g_n(u), \quad h_{n+1}(u) = g_n'(u) - \frac{\alpha+1}{u} g_n(u).$$

In [7] explicit expressions for α_0 and β_1 are given, and an explicit integral representation for ε_p. By constructing order estimates for ε_p it is shown that (2.1) is an asymptotic expansion as $n \to \infty$, $\alpha > -1$ fixed, and $x \in (-\infty, a]$. When $x < 0$ the expression $J_\alpha(\nu A)/A^\alpha$ can be replaced with a similar expression containing the modified Bessel function.

Expansion (2.1) also follows from [10, p. 446]. Olver supplied explicit error bounds, which are obtained from his theory on differential equations.

2.2. Approximation in terms of an Airy function

Let

$$B(x) = \begin{cases} i[3\beta(x)/2]^{1/3}, & \text{if } 0 < x \le 1; \\ [3\gamma(x)/2]^{1/3}, & \text{if } x \ge 1, \end{cases}$$

$$\beta(x) = 1/4\pi - A(x) = 1/2[\arccos\sqrt{x} - \sqrt{x - x^2}],$$

$$\gamma(x) = 1/2\sqrt{x^2 - x} - \operatorname{arccosh}\sqrt{x}].$$

Then for $p = 0, 1, 2, \ldots$

$$(-1)^n 2^\alpha e^{-2\kappa x} L_n^{(\alpha)}(4\kappa x) = Ai(\nu^{2/3} B^2) \sum_{k=0}^{[(p-1)/2]} \alpha_{2k} \nu^{-2k-1/3} - Ai'(\nu^{2/3} B^2) \sum_{k=0}^{[p/2]-1} \beta_{2k+1} \nu^{-2k-5/3} + \varepsilon_p, \tag{2.2}$$

where ε_p is a remainder, and $Ai(z)$ is the Airy function. The coefficients α_{2k}, β_{2k+1} follow from the recursion

$$h_n(u) = \alpha_n + \beta_n u + (u^2 - B^2)g_n(u), \quad h_{n+1}(u) = g_n'(u),$$

with

$$h_0(u) = h(u) = [1 - z^2(u)]^{(\alpha-1)/2} \frac{dz}{du},$$

where the relation between z and u is given by

$$1/2[\operatorname{arctanh} z - xz] = 1/3u^3 - B^2(x)u.$$

Again, the first coefficients α_0 and β_1 are given explicitly, and an estimate of the remainder ε_p is given. It is shown that (2.2) is an asymptotic expansion as $n \to \infty$, $\alpha > -1$ fixed, and $x \in [b, \infty)$.

A similar expansion follows from Olver [10, p. 412], where the methods are based on differential equations, and explicit error bounds are given for the remainder.

3. Asymptotic forms with κ large

In this section we consider the Laguerre polynomial for large values of κ defined in (1.2). Without giving all technical details we give the main steps that yield the leading terms of the approximations. The technique is based on the Liouville-Green (LG) transformation for differential equations, which is extensively investigated in [10].

The relation between Laguerre polynomials and Whittaker functions is

$$L_n^{(\alpha)}(z) = \frac{(-1)^n}{n!} z^{-(\alpha+1)/2} e^{z/2} W_{\kappa,\mu}(z)$$

$$= \frac{\Gamma(\alpha+n+1)}{n!\,\Gamma(\alpha+1)} z^{-(\alpha+1)/2} e^{z/2} M_{\kappa,\mu}(z), \tag{3.1}$$

with κ as in (1.2) and $\mu = \alpha/2$. The functions $M_{\kappa,\mu}(z)$, $W_{\kappa,\mu}(z)$ are solutions of Whittaker's equation

$$\frac{d^2y}{dz^2} = \left[\frac{1}{4} - \frac{\kappa}{z} + \frac{\mu^2 - \frac{1}{4}}{z^2}\right] y.$$

A first transformation $z = 4\kappa x$ yields

$$\frac{d^2W}{dx^2} = \left[\kappa^2 \frac{4(x - x_1)(x - x_2)}{x^2} - \frac{1}{4x^2}\right] W, \tag{3.2}$$

where

$$x_1 = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{1-\tau^2}, \quad x_2 = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{1-\tau^2}, \quad \tau = \frac{\alpha}{2\kappa}. \tag{3.3}$$

Solutions of (3.2) are $M_{\kappa,\mu}(4\kappa x)$, $W_{\kappa,\mu}(4\kappa x)$.

When $\tau \in [0, \tau_0]$, fixed in $(0, 1)$, the quantities x_1, x_2 are well separated. In that case the analysis concentrates on the points $x = 0$, $x = x_1$, a pole and a transition point of the differential equation (3.2), or on the transition point x_2. Alternatively, when $\tau \in [\tau_0, 1]$ the turning points $x_{1,2}$ may coalesce at $\frac{1}{2}$, and the domain of interest is an interval of positive x-values that contains both points $x_{1,2}$.

3.1. Approximation in terms of a Bessel function

The results of this subsection follow from the recent paper [4] of Dunster, where complete expansions for the Whittaker functions are given for a complex x-domain, and error bounds for the remainders. Earlier results of this kind, but rather limited compared with Dunster's, are given by Baumgartner [2].

We apply the LG transformation to (3.2) by introducing $\zeta = \zeta(x)$ and $w(\zeta)$, writing for $x < x_2$

$$W(x) = \sqrt{\dot{x}}\, w(\zeta), \quad \frac{\gamma^2 - \zeta}{4\zeta^2}\left(\frac{d\zeta}{dx}\right)^2 = \frac{4(x - x_1)(x - x_2)}{x^2}; \tag{3.4}$$

the dot indicates differentiation with respect to ζ, and

$$\gamma = \frac{\alpha}{\kappa} = 2\tau.$$

Solving the differential equation for ζ with the conditions $\zeta(-\infty) = -\infty$, $\zeta(0) = 0$, $\zeta(x_1) = \gamma^2$, one obtains the following relations

(*i*) $0 < \zeta \le \gamma^2, \quad 0 < x \le x_1,$

$$-\sqrt{\gamma^2 - \zeta} + \gamma \operatorname{arctanh} \frac{\sqrt{\gamma^2 - \zeta}}{\gamma}$$
$$= -2R + \tau \ln\left|\frac{\tau^2 - 2x + 2\tau R}{2x\sqrt{1-\tau^2}}\right| + \ln\left|\frac{2R + 2x - 1}{\sqrt{1-\tau^2}}\right|; \tag{3.5}$$

(*ii*) $\gamma^2 \le \zeta, \quad x_1 \le x < x_2,$

$$\sqrt{\zeta - \gamma^2} - \gamma \arctan \frac{\sqrt{\zeta - \gamma^2}}{\gamma}$$
$$= 2R - \tau \arctan \frac{2x - \tau^2}{2\tau R} - \arctan \frac{1 - 2x}{2R} + \frac{\pi}{2}(1 - \tau), \tag{3.6}$$

where $R = \sqrt{|(x - x_1)(x - x_2)|}$. When $\zeta < 0, x < 0$, relation (3.5) can be used with arctanh replaced with arccoth.

The LG transformation brings (3.2) in the form

$$\ddot{w} = \left[\kappa^2 \frac{\gamma^2 - \zeta}{4\zeta^2} - \frac{1}{4\zeta^2}\right] w + \phi(\zeta) w, \tag{3.7}$$

where

$$\phi(\zeta) = -\frac{\dot{x}^2}{4x^2} + \frac{1}{4\zeta^2} + \sqrt{\dot{x}} \frac{d^2}{d\zeta^2} \frac{1}{\sqrt{\dot{x}}}$$
$$= \frac{16\zeta(\zeta + 4\gamma^2)R^6 - x(\gamma^2 - \zeta)^3[x(1 - 4\tau^2) + \tau^2 + 4x^3]}{256\zeta^2 R^6 (\gamma^2 - \zeta)^2}$$

The function $\zeta(x)$ is analytic for $x < x_2$; ϕ is analytic in a corresponding ζ-domain.

For identifying the solutions of (3.7) in terms of the Laguerre polynomial, one needs the coefficient c_1 in the expansion

$$\zeta = c_1 x + \mathcal{O}(x^2), \quad \text{as} \quad x \to 0.$$

From (3.5) it follows straightforwardly

$$c_1 = \frac{16}{e} \sqrt{1 - \tau^2} \left(\frac{1 + \tau}{1 - \tau}\right)^{1/(2\tau)}. \tag{3.8}$$

When in (3.7) the function ϕ is neglected, then the Bessel function $w_0(\zeta) = \sqrt{\zeta} J_\alpha(\kappa\sqrt{\zeta})$ is a solution. A second form containing the he Y-Bessel function cannot be used for approximating the Laguerre polynomial. This can be verified by comparing the behaviour of the choice $w_0(\zeta)$ and that of $w(\zeta)$ (for the case of the Laguerre polynomial) at $\zeta = 0$. The complete solution is put in the form $w(\zeta) = w_0(\zeta) + \varepsilon$. When we use the M-function in (3.1), we obtain

$$L_n^{(\alpha)}(4\kappa x) = A_n^{(\alpha)} x^{-(\alpha+1)/2} e^{2\kappa x} \sqrt{\dot{x}} [\sqrt{\zeta} J_\alpha(\kappa\sqrt{\zeta}) + \varepsilon], \tag{3.9}$$

where $A_n^{(\alpha)}$ does not depend on x. This quantity follows from the behaviour of the Bessel function at the origin, and

$$L_n^{(\alpha)}(0) = \binom{n + \alpha}{n} = \frac{\Gamma(n + \alpha + 1)}{\Gamma(\alpha + 1) n!}.$$

Thus we obtain

$$A_n^{(\alpha)} = \frac{\Gamma(n + \alpha + 1)}{n!} [2/(\kappa\sqrt{c_1})]^\alpha,$$

where c_1 is given in (3.8).

From [4] it follows that, if $\kappa \to \infty$, the remainder ε in (3.9) is small compared with $\sqrt{\zeta} J_\alpha(\kappa\sqrt{\zeta})$, except in the neighbourhoods of the zeros of this function where ε is small compared with the envelope of $\sqrt{\zeta}J_\alpha(\kappa\sqrt{\zeta})$. In other words, when ε is neglected, (3.9) is an asymptotic approximation as $\kappa \to \infty$. This holds uniformly with respect to x and τ in the domains $(-\infty, \xi_1]$, $[0, \tau_0]$, respectively, where τ_0 is a fixed number in $(0, 1)$, and $\xi_1 = \theta_1 x_1 + (1-\theta_1)x_2$, θ_1 fixed in $(0, 1)$. The condition $0 \le \tau \le \tau_0$ can be written as

$$0 \le \alpha \le \frac{\tau_0}{1-\tau_0}(2n+1).$$

We may write this as

$$0 \le \alpha \le \alpha_1 n,$$

where α_1 is any fixed positive number.

3.2. *Approximation in terms of a Hermite polynomial*

In this subsection we take into account that, if $\tau \to 1$, the turning points $x_{1,2}$ of (3.2) coalesce at $\frac{1}{2}$. Considering the methods of Olver [11], we use the LG transformation

$$W(x) = \sqrt{x}w(\eta), \quad (\eta^2 - \varrho^2)\left(\frac{d\eta}{dx}\right)^2 = \frac{4(x-x_1)(x-x_2)}{x^2}, \tag{3.10}$$

where $x > 0$ and ϱ is a non-negative number defined by

$$\int_{-\varrho}^{\varrho} \sqrt{\varrho^2 - \eta^2}\,d\eta = 2\int_{x_1}^{x_2} \sqrt{(x_2 - x)(x - x_1)}\,\frac{dx}{x}.$$

Evaluation of the integrals yields

$$\varrho = \sqrt{2(1-\tau)}. \tag{3.11}$$

The relation between η and x is one-to-one, with

$$\eta(0) = -\infty, \quad \eta(x_1) = -\varrho, \quad \eta(x_2) = \varrho, \quad \eta(+\infty) = +\infty. \tag{3.12}$$

Solving the differential equation for η in (3.10) with the above boundary conditions, we obtain the following relations. Let R be as in (3.5), (3.6), and $0 < \tau < 1$.

(*i*) $\varrho \le \eta < \infty, \quad x_2 \le x < \infty$:

$$\tfrac{1}{2}\eta\sqrt{\eta^2 - \varrho^2} - \tfrac{1}{2}\varrho^2 \operatorname{arccosh}\frac{\eta}{\varrho}$$
$$= 2R - \tau \ln\left[\frac{2x - \tau^2 + 2\tau R}{2x\sqrt{1-\tau^2}}\right] - \ln\left[\frac{2R + 2x - 1}{\sqrt{1-\tau^2}}\right]; \tag{3.13}$$

(*ii*) $-\varrho \le \eta \le \varrho, \quad x_1 \le x \le x_2$:

$$\tfrac{1}{2}\eta\sqrt{\varrho^2-\eta^2}+\tfrac{1}{2}\varrho^2\arcsin\frac{\eta}{\varrho}$$

$$= 2R - \tau\arctan\frac{2x-\tau^2}{2\tau R} - \arctan\frac{1-2x}{2R}; \tag{3.14}$$

(*iii*) $-\infty < \eta \le -\varrho, \quad 0 < x \le x_1$:

$$-\tfrac{1}{2}\eta\sqrt{\varrho^2-\eta^2}-\tfrac{1}{2}\varrho^2\operatorname{arccosh}\frac{-\eta}{\varrho}$$

$$= -2R + \tau\ln\left[\frac{\tau^2-2x+2\tau R}{2x\sqrt{1-\tau^2}}\right] + \ln\left[\frac{1-2R-2x}{\sqrt{1-\tau^2}}\right]. \tag{3.15}$$

If $\tau = 1$ we have

$$\tfrac{1}{2}\eta^2 = 2x - \ln(2x) - 1, \quad \operatorname{sign}(\eta) = \operatorname{sign}(x-\tfrac{1}{2}). \tag{3.16}$$

The differential equation (3.2) transforms into

$$\frac{d^2w}{d\eta^2} = [\kappa^2(\eta^2-\varrho^2)+\psi(\eta)]w$$

in which

$$\psi(\eta) = \frac{16(3\eta^2+2\varrho^2)R^6 - x(\eta^2-\varrho^2)^3[4x^3+(1-4\tau^2)x+\tau^2]}{64(\eta^2-\varrho^2)^2R^6}.$$

By investigating the behaviour of $x(\eta)$ at $\pm\infty$ it can be shown that

$$\psi(\eta) = \mathcal{O}[1/(\eta^2+1)]$$

uniformly with respect to $\tau \in [\tau_0, 1]$, with τ_0 fixed in $(0, 1)$.

The above results can be interpreted for $\tau > 1$, but for the Laguerre polynomial these values do not make sense.

From Olver's theory it follows that the Whittaker function $W_{\kappa,\mu}(z)$ can be written as

$$W_{\kappa,\mu}(4\kappa x) = (8\kappa)^{1/4}\left[\frac{2\kappa(1-\varrho^2/4)}{e}\right]^{\kappa(1-\varrho^2/4)}\sqrt{\dot{x}}[U(-\tfrac{1}{2}\kappa\varrho^2, \eta\sqrt{2\kappa})+\varepsilon], \tag{3.17}$$

in which $U(a, z)$ is a parabolic cylinder function (see [1, Chapter 19]). Again, when $\kappa \to \infty$, the remainder ε is small in the same sense as described for ε in (3.9). As in [11] an upper bound for ε can be constructed.

By using (3.1), it follows that for the Laguerre polynomials the quantity $-\frac{1}{2}\kappa\varrho^2$ in the U-function in (3.7) can be written as (see (1.2), (3.3) and (3.11)) $-\frac{1}{2}\kappa\varrho^2 = -(\kappa-\mu) = n - \frac{1}{2}$. Hence, the parabolic cylinder function reduces to a Hermite polynomial:

$$U(-n-\tfrac{1}{2}, z) = D_n(z) = 2^{-n/2} e^{-z^2/4} H_n(z/\sqrt{2}), \tag{3.18}$$

where $D_n(z)$ is another notation for parabolic cylinder functions.

By combining (3.1), (3.17), it follows that

$$L_n^{(\alpha)}(4\kappa x) = \frac{(-1)^n}{n!} 2^{-\alpha-n/2-3/4} \kappa^{-\alpha/2-1/4} x^{-\alpha/2} e^{2\kappa x - \kappa\eta^2/2}$$
$$\times \left[\frac{n+\alpha+1/2}{e}\right]^{n+\alpha+1/2} \left[\frac{\eta^2-\varrho^2}{(x-x_1)(x-x_2)}\right]^{1/4} [H_n(\eta\sqrt{\kappa})+\bar{\varepsilon}], \tag{3.19}$$

where

$$\kappa = n + (\alpha+1)/2, \quad \tau = \frac{\alpha}{2\kappa}, \quad \varrho = \sqrt{2(1-\tau)}.$$

When $\bar{\varepsilon}$ is neglected, this form can be viewed as an asymptotic estimate with κ as the large parameter. This asymptotic property holds uniformly with respect to $x \in [0, \infty)$, $\tau \in [\tau_0, 1]$.

The latter gives for α the condition

$$\alpha \geq \frac{\tau_0}{1-\tau_0}(2n+1),$$

which may be written as $\alpha \geq \alpha_2 n$, where α_2 is any fixed positive number.

In Olver [11] the role of the large parameter differs slightly from that in the above analysis. We consider κ as the large parameter. Olver's asymptotic estimate of the Whittaker function $W_{\kappa,\mu}(z)$ is valid for $\alpha \to \infty$, uniformly with respect to $x \in [0, \infty)$ and $n \in [0, n_0\alpha)$, where n_0 is positive and fixed. It follows that in our version (3.19) of Olver's result these conditions can be used also.

The asymptotic representation (3.19) of $L_n^{(\alpha)}(z)$ in terms of the Hermite polynomial seems to be new. In [9, p. 251] the limit

$$\lim_{\alpha\to\infty} [\alpha^{-n/2} L_n^{(\alpha)}(\alpha + t\sqrt{\alpha})] = \frac{(-1)^n}{n!} 2^{-n/2} H_n(t/\sqrt{2}) \tag{3.20}$$

is given, without reference to a source. This relation was also given by Calogero [3]. We verify it by using special values of the parameters in (3.19). When α is large with respect to n, we have $\tau \to 1$. In the limit $\tau = 1$, the relation between η and x is given by (3.16), and $\varrho = 0$. So we have if $\alpha \gg n$

$$L_n^{(\alpha)}(4\kappa x) \sim \frac{(-1)^n}{n!} 2^{-\alpha-n/2-3/4} \kappa^{-\alpha/2-1/4} x^{-1/2} (2ex)^{\kappa}$$
$$\times \left[\frac{n+\alpha+1/2}{e}\right]^{n+\alpha+1/2} \sqrt{\frac{\eta}{x-1/2}}\, H_n(\eta\sqrt{\kappa}),$$

Writing $4\kappa x = \alpha + t\sqrt{\alpha}$, we observe that $x \to 1/2$. In this limit, η can be replaced with $x - 1/2$. A few further calculations give indeed (3.20). It is valid for fixed values of t and n, although (3.20) can be replaced with an asymptotic relation in which $t = o(\sqrt{\alpha})$, $n = o(\alpha)$, as $\alpha \to \infty$.

We cannot claim that (3.19) holds uniformly in the (x, τ)-domain $[0, \infty) \times [0, 1]$, that is, inclusive the origin in both intervals. The reason is that for $\tau \to 0$ the mapping $x \mapsto \eta(x)$ tends to a limit mapping in a non-uniform way. For instance, $\tau = 0$ gives in (3.3) $x_1 = 0$, and in (3.12) $\eta(0) = -\infty$, as well as $\eta(0) = -\sqrt{2}$.

Recall that Dunster's result (3.9) is valid for x bounded away from x_2, $x < x_2$ and for $0 \le \alpha \le \alpha_1 n$. It follows that (3.9) and (3.19) describe the asymptotic behaviour of $L_n^{(\alpha)}(4\kappa x)$ in overlapping domains of the (α, x)-quarter plane, but not in the complete quarter plane $[0, \infty) \times [0, \infty)$. The missing piece is considered in the next subsection.

3.3. Approximation in terms of an Airy function

When α is restricted to an interval $[0, \alpha_3 n]$, α_3 fixed and positive, the parameter τ may tend to zero if $\kappa \to \infty$; this happens, for instance, when α is fixed. When, in addition, x is large, this case is not covered in the previous two subsections; (α fixed is covered in §2.2, but we want to include the above indicated α-domain). In this subsection we consider (3.2) for $x \in [\xi_3, \infty)$ and $\alpha \in [0, \alpha_3 n]$, where $\xi_3 = \theta_3 x_1 + (1 - \theta_3) x_2$, θ_3 fixed in $(0, 1)$. This x-domain contains one critical point: the turning point x_2.

The appropriate LG transformation is

$$W(x) = \sqrt{x}\, w(\xi), \quad \xi \left(\frac{d\xi}{dx}\right)^2 = \frac{4(x - x_1)(x - x_2)}{x^2}$$

with conditions $\xi(x_2) = 0$, $\xi(+\infty) = +\infty$. The relation between ξ and x follows from (3.13) and (3.14), with the left-hand sides replaced with $\frac{2}{3}\xi^{3/2}$. The differential equation (3.2) is transformed into

$$\frac{d^2 w}{d\xi^2} = [\kappa^2 \xi + \chi(\xi)] w,$$

in which

$$\chi(\xi) = -\frac{\dot{x}^2}{4x^2} + \sqrt{\dot{x}} \frac{d^2}{d\xi^2} \frac{1}{\sqrt{\dot{x}}}.$$

The comparison function is an airy function, and we write

$$w(\xi) = Ai(\kappa^{2/3}\xi) + \varepsilon.$$

By considering the function $\xi(x)$ as $x \to +\infty$, we can identify the solution w with the Laguerre polynomial. We have

$$\tfrac{2}{3}\xi^{3/2} = 2x - \ln x + \mathcal{O}(x^{-1}), \quad \text{as } x \to +\infty,$$

and we finally arrive at

$$L_n^{(\alpha)}(4\kappa x) = 2^{3/2}\sqrt{\pi}\kappa^{1/6}(4\kappa)^n \frac{(-1)^n}{n!} x^{-(\alpha+1)/2} e^{2\kappa x}\sqrt{x}[Ai(\kappa^{2/3}\xi) + \bar{\varepsilon}].$$

Further details can be found in Dunster [4], where complete expansions are given for Whittaker functions, with error bounds for the remainders in a complex x-domain.

3.4. Three forms are not enough to cover the real x-axis

A fourth (α, x)-domain can be introduced denoted by $[\alpha_4 n, \infty) \times (-\infty, 0]$, where α_4 is fixed and positive. In this case τ is bounded away from unity. The x-interval contains a double pole at its boundary, and no turning points. This case is less complicated than the previous ones, and details will not be given. In fact an asymptotic expansion in terms of elementary functions can be obtained. Confer [10, p. 362] and the expansion of the modified Bessel functions in [10, pp. 374–378].

4. Computation of zeros of Laguerre polynomials

The asymptotic estimate (3.19) has the Hermite polynomial as approximant. This polynomial has the same number of zeros as $L_n^{(\alpha)}(4\kappa x)$ itself. The zeros of $L_n^{(\alpha)}(4\kappa x)$ occur in the region $x_1 < x < x_2$. Let $l_{n,m}^{(\alpha)}$, $h_{n,m}$ be the m-th zeros of $L_n^{(\alpha)}(z)$, $H_n(z)$, $m = 1, 2, \dots, n$. For given α and n, we can compute

$$\eta_{n,m} = \frac{h_{n,m}}{\sqrt{\kappa}}, \quad m = 1, 2, \dots, n. \tag{4.1}$$

Upon inverting (3.14) we can obtain $x_{n,m}$, giving the estimate

$$l_{n,m}^{(\alpha)} \sim 4\kappa x_{n,m}, \quad m = 1, 2, \dots, n. \tag{4.2}$$

From properties of the Hermite polynomials (see, for instance, Hochstadt [8, p. 50]) it follows that all zeros $h_{n,m}$ are located in the interval $[-\sqrt{2n+1}, \sqrt{2n+1}]$. It follows that the numbers $\eta_{n,m}$ of (4.1) belong to the interval $[-\varrho, \varrho]$, when n is large, $\alpha \geq 0$.

The estimate (4.2) is valid for $\kappa \to \infty$, uniformly with respect to $m \in \{1, 2, \dots, n\}$ when $\alpha \geq \alpha_2 n$, where α_2 is a fixed positive number.

In Table 4.1 we show for $n = 10$ the correct number of decimal digits in the approximation (4.2). That is, we show

$$\log_{10}\left|\frac{l_{10,m}^{(\alpha)} - \tilde{l}_{10,m}^{(\alpha)}}{l_{10,m}^{(\alpha)}}\right|, \quad m = 1, 2, \dots, 10$$

where $\tilde{l}_{10,m}^{(\alpha)}$ is the approximation obtained by the procedure described in

Table 4.1
Correct number of decimal digits in the approximations of zeros of $L_{10}^{(\alpha)}(x)$

α	0	1	5	10	25	50	75	100
m								
1	1.7	2.3	3.2	3.7	4.4	5.0	5.3	5.6
2	2.4	2.7	3.4	3.8	4.5	5.0	5.4	5.6
3	2.8	3.0	3.5	3.9	4.5	5.1	5.4	5.6
4	3.0	3.2	3.6	4.0	4.6	5.1	5.4	5.7
5	3.2	3.4	3.8	4.1	4.6	5.1	5.5	5.7
6	3.4	3.5	3.9	4.2	4.7	5.2	5.5	5.7
7	3.5	3.6	4.0	4.2	4.7	5.2	5.5	5.8
8	3.7	3.8	4.1	4.3	4.8	5.3	5.6	5.8
9	3.8	3.9	4.1	4.4	4.9	5.3	5.6	5.8
10	3.9	4.0	4.2	4.5	4.9	5.4	5.6	5.8

(4.1), (4.2). It follows that the large zeros are better approximated than the small zeros. Furthermore, large values of α give better approximations, and the approximations are uniform with respect to m.

Acknowledgment

The author would like to thank the referee for helpful suggestions on an earlier version of the paper.

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Abstract

We give a brief summary of recent results concerning the asymptotic behaviour of the Laguerre polynomials $L_n^{(\alpha)}(x)$. First we summarize the results of a paper of Frenzen and Wong in which $n \to \infty$ and $\alpha > -1$ is fixed. Two different expansions are needed in that case, one with a J-Bessel function and one with an Airy function as main approximant. Second, three other forms are given in which α is not necessarily fixed. These results follow from papers of Dunster and Olver, who considered the expansion of Whittaker functions. Again Bessel and Airy functions are used, and in another form the comparison function is a Hermite polynomial. A numerical verification of the new expansion in terms of the Hermite polynomial is given by comparing the zeros of the approximant with the related zeros of the Laguerre polynomial.

(Received: January 20, 1989)