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ASYMPTOTIC ESTIMATES FOR TWO-DIMENSIONAL AREA-PRESERVING MAPPINGS

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ABSTRACT

We consider the perturbation series for the formal integral of two-dimensional area-preserving mappings and obtain the asymptotic formulas for large order terms of this expansion.

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1. - INTRODUCTION

A general method of calculating the large orders of perturbation series in classical mechanics has been proposed in Ref. 1). The detailed calculations in Ref. 1) have been done only for the simple case of the two-dimensional area-preserving mapping of the following form:

$$\begin{cases} x' = -y \\ y' = x + y^3 \end{cases} \tag{1}$$

Here we continue the investigation of the structure of large orders of perturbation series for the two-dimensional area-preserving mappings $T(x,y) = (x'y')$:

$$T \quad \begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases} \tag{2}$$

where

$$\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} = 1 \tag{3}$$

As a specific example the polynomial mapping will be considered

$$T \quad \begin{cases} x' = \cos\alpha\,(x+y^3) - y\sin\alpha \\ y' = \sin\alpha\,(x+y^3) + y\cos\alpha \end{cases} \tag{4}$$

where $\alpha = 2\pi\mu$.

At $\mu = 1/4$ one obtains the mapping (1). The coupling constant for this mapping is $r^2 = x^2 + y^2$ and the unperturbed part describes a rotation by α. In this paper we will concentrate primarily on giving the peculiarities of the application of the method, proposed in Ref. 1), to the mapping (4) with different μ, both rational and irrational.

As in Ref. 1) we will be interested in the large orders of the perturbation series for the formal integral of this mapping. Let us recall that the formal integral for the mapping (2) is the function of two variables $H(x,y)$ such that

$$H\,(x',y') = H(x,y) \tag{5}$$

where x', y' are connected to x, y by Eqs. (2).

It is known[2] that for arbitrary polynomial mappings $H(x,y)$ can be written as a power series in x and y:

$$H\,(x,y) = \sum_n H_n\,(x,y) \tag{6}$$

where

$$H_n\,(x,y) = \sum_{k+\ell=n} b_{k\ell}\, x^k y^\ell \tag{7}$$

are the homogeneous polynomials of degree n.

It is not difficult to find the algorithm for recursive calculation of H_n. But it is also well known[2),3)] that this series is only a formal one and the coefficients of H_n blow up at $n \to \infty$. It was the purpose of Ref. 1) to investigate the method of obtaining the asymptotic formulas for H_n at large n. As usual[4)] the behaviour of large orders of the perturbation series is closely connected with the singularity structure of $H(x,y)$ at $r \to 0$. In Ref. 1) it has been shown that $H(x,y)$ has square-root singularities near all periodic points of a given mapping.

The periodic point with the period equal to N is the point x_p, y_p such that

$$T^N\,(x_p, y_p) = (x_p, y_p) \tag{8}$$

where T^N is the result of the N fold application of our mapping:

$$T^N \begin{cases} x_N = f_N\,(x,y) \\ y_N = g_N\,(x,y) \end{cases} \tag{9}$$

For the polynomial mappings f_N and g_N are polynomials of large degree.

As the set of periodic points is dense at $r \to 0$[2),1)], the function $H(x,y)$ has singularities at a dense set of points, which lead to the divergence of series (6).

To find the explicit character of a singularity of $H(x,y)$ near the given periodic point it is necessary to know the matrix of the first derivatives of the mapping (9) calculated at this point:

$$M_N(x_p, y_p) = \begin{pmatrix} \frac{\partial f_N}{\partial x} & \frac{\partial f_N}{\partial y} \\ \frac{\partial g_N}{\partial x} & \frac{\partial g_N}{\partial y} \end{pmatrix}_{\substack{x=x_p \\ y=y_p}} \tag{10}$$

In Ref. 1) it has been noted that the quantity

$$\Delta_N = \mathrm{Sp}\, M_N - 2 \tag{11}$$

decreases at $N \to \infty$ faster than any fixed power of $1/N$ and for mapping (1) it has been found that at $N \to \infty$ and N odd

$$\Delta_{4N} = \mathrm{const} \cdot N^3 \exp\left(-\frac{\pi N}{2}\right) \sin\left(\frac{\pi N}{2}\right) \tag{12}$$

where const. ≈ 563

But the dependence of the pre-exponent factor on N (N^3 term) has been obtained only from numerical calculations.

In Section II of this paper an analytical expression for Δ_N up to a constant factor will be obtained for the general case with rational μ. For the mapping (1) it coincides with Eq. (12). The constant in analogous formulas is a measure of non-integrability of the theory considered and cannot be obtained analytically.

In Section III the positions of the periodic points and the explicit asymptotic formulas for Δ_N for the mapping (4) with rational and irrational μ will be investigated.

In Section IV the asymptotics of the perturbation series for $H(x,y)$ will be found. As specific examples of rational and irrational rotation angles the cases of $\mu = 1/5$ and $\mu = (\sqrt{5} + 1)/2$ will be considered.

2. - CALCULATION OF THE MONODROMY MATRIX TRACE

If x_0, y_0 is a periodic point with a period equal to N then $T^k(x_0,y_0)$, $1 \leq k \leq N-1$ is another periodic point with the same period. So we always have a complete sequence of periodic points x_p, y_p connected by the transformation $(x_{p+1}, y_{p+1}) = T(x_p, y_p)$ and $T(x_N, y_N) = (x_0, y_0)$. We will refer to this sequence as the periodic cycle.

Let us introduce instead of the N points (x_p, y_p), their Fourier components f_n, g_n

$$
\begin{aligned}
x_p &= \sum_{\{n\}} \exp(i\omega p n)\, f_n \\
y_p &= \sum_{\{n\}} \exp(i\omega p n)\, g_n
\end{aligned}
\tag{13}
$$

where $\omega = 2\pi M/N$ (for different mappings it is convenient to choose different M).

Using the identity

$$
\sum_{p=0}^{N-1} \exp(i\omega p n) = \begin{cases} N & \text{when} \quad n \equiv 0 \pmod N \\ 0 & \text{when} \quad n \not\equiv 0 \pmod N \end{cases}
\tag{14}
$$

one obtains

$$
f_n = \frac{1}{N} \sum_{p=0}^{N-1} \exp(-i\omega p n)\, x_p
\tag{15}
$$

and the analogous formula for g_n.

If $N = 2Q$ is even then we choose $-Q+1 \leq n \leq Q$, if $N = 2Q+1$ is odd we choose $-Q \leq n \leq Q$. We will call this interval the main zone and will use the symbol $\{n\}$ to denote the summation over all n in the main zone. If n_1 and n_2 are both in the main zone, then from equality $n_1 \equiv n_2 \pmod N$ it follows that $n_1 = n_2$. From Eq. (15) it is clear that if we know all N periodic points we can find f_n for all n in the main zone.

The direct determination of the matrix M_N from Eqs. (10) and (9) is difficult. It is easy to see that

$$M_N = \prod_{i=0}^{N-1} m_i \qquad (16)$$

where m_i is the matrix of derivatives of the mapping (2) calculated at the i^{th} periodic point of the given cycle

$$m_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix} \qquad (17)$$

where $\alpha_i = \partial f/\partial x$, $\beta_i = \partial f/\partial y$, $\gamma_i = \partial g/\partial x$, $\delta_i = \partial g/\partial y$ and all derivatives are calculated at the periodic point (x_i, y_i) of the given cycle.

Thus to obtain M_N one has to linearize Eqs. (2) near the given periodic cycle and then solve the corresponding linearized equations. Equation (16) means that the matrix M_N is the monodromy matrix of their solutions.

Let $x_n = x_n^{(0)} + u_n, y_n = y_n^{(0)} + d_n$, $x' = x_{n+1}$, $y' = y_{n+1}$ and $x_n^{(0)}$, $y_n^{(0)}$ be the co-ordinates of the points of our periodic cycle. From Eqs. (2) one finds

$$\begin{aligned} u_{n+1} &= \alpha_n u_n + \beta_n d_n \\ d_{n+1} &= \gamma_n u_n + \delta_n d_n \end{aligned} \qquad (18)$$

where α_n, β_n, γ_n, δ_n are the corresponding derivatives calculated at the point $[x_n^{(0)}, y_n^{(0)}]$

Let (u_n, d_n) and (u_n', d_n') $n = 0,1,\ldots, N-1$ be the two linearly independent solutions of these equations. Using Eq. (16) one obtains

$$Sp\, M_N = \frac{1}{c} \left(u_N d_0' - u_N' d_0 + d_N' u_0 - d_N u_0' \right) \qquad (19)$$

where c is the analogue of the Wronskian of Eqs. (18):

$$c = u_k d_k' - u_k' d_k \qquad (20)$$

Using the area-preserving conditions (3) it is easy to show that c is indedendent of k. As usual, if one solution of the given linear system is known one can find the explicit form of the other one.

Let

$$u_k' = r_k\, u_k$$
$$d_k' = s_k\, d_k \tag{21}$$

From Eqs. (18) one has

$$r_k = \frac{u_0'}{u_0} + c\sum_{n=0}^{k-1} \frac{\beta_n}{u_n u_{n+1}}$$
$$s_k = \frac{d_0'}{d_0} - c\sum_{n=0}^{k-1} \frac{\gamma_n}{d_n d_{n+1}} \tag{22}$$

Using these values one obtains

$$\Delta_N = \mathrm{Sp}\, M_N - 2 = \frac{1}{2}\left\{ -K_N\left((u_N-u_0)d_0 - (d_N-d_0)u_0\right) + \frac{(u_N-u_0)^2}{u_N u_0} + \frac{(d_N-d_0)^2}{d_N d_0} \right\} \tag{23}$$

where

$$K_N = \sum_{n=0}^{N-1} \left(\frac{\beta_n}{u_n u_{n+1}} - \frac{\gamma_n}{d_n d_{n+1}} \right)$$

Let us discuss the question of choosing the solution (u_n, d_n). If it were possible to embed x_k and y_k into continuous functions $x(t)$ and $y(t)$ in such a manner that $x_k = x(k)$ and $y_k = y(k)$, then one of the solutions of the linearized equations (18) would be known. It is evident that if $x(t)$ and $y(t)$ obey the equations (2):

$$x(t+1) = f\,(x(t), y(t))$$
$$y(t+1) = g\,(x(t), y(t)) \tag{24}$$

then

$$u(t) = \dot{x}(t)$$
$$d(t) = \dot{y}(t) \tag{25}$$

are the solutions of Eqs. (18).

But it is known[2)] that such an embedding does not exist in the vicinity of periodic points. Let us assume that we know the positions of all periodic points. Let us try to construct the difference analogue of the derivative which uses the positions of the periodic points only and gives the correct answer for as large a number of Fourier components as possible. If f_n is the value of some periodic function $f(t)$ at $t = n$, then we introduce the following operator

$$D_j f = \sum_{n=0}^{N-1} D(j-n) f_n \tag{26}$$

where $\omega = 2\pi M/N$ and

$$D(P) = \sum_{\{n\}} \frac{i\omega n}{N} \exp\,(i\omega n P)$$

[Note that the summation in $D(P)$ is done over all n in the main zone.] It is easy to prove that for $f_n = \exp(i\omega nk)$ $D_j f = i\omega\{k\} f_j$ where $k = N_m + \{k\}$ and $\{k\}$ is in the main zone. This means that the operator D_J gives the correct derivative for functions whose Fourier components are in the main zone [e.g., for x_n and y_n from Eq. (13)]. For the arbitrary function $f(t) = \Sigma_{k=-\infty}^{+\infty} \exp(i\omega kt)\phi_k$ with $\omega = 2\pi M/N$ one has

$$D_j f = \frac{df}{dt}\Big|_{t=j} - 2\pi i M \sum_{p=-\infty}^{+\infty} p \sum_{\{r\}} f_{pN+r} \exp\,(i\omega r j) \tag{27}$$

and the inner summation is done over all r in the main zone.

In the next section it will be shown that for the quantities that interested us the large Fourier components are exponentially small. This means that the operator D_j correctly differentiates the functions up to exponentially small terms. Taking these terms into account is our main problem. Let

$$\begin{aligned} u_n &= D_n x + \xi_n \\ d_n &= D_n y + \eta_n \end{aligned} \tag{28}$$

It is natural to expect that ξ_n and η_n are exponentially small. To find the equations for ξ_n and η_n, note firstly that the transformations (13) are the identity transformations. One simply replaces the $2N$ quantities x_n and y_n by f_n and g_n. But, as will be seen in the next section, f_n and g_n with large n are exponentially small. It is this property which leads to the possibility of finding explicit expressions for ξ_n and η_n.

Let us substitute the Fourier expansions (13) into the right-hand side of our mapping (2) [or (24) with the substitution $n \to t$]. This gives the formal expansions of the right-hand sides into Fourier series

$$f(x_n, y_n) = \sum_p \exp(i\omega p n)\, \mathcal{F}_p$$
$$g(x_n, y_n) = \sum_p \exp(i\omega p n)\, G_p \tag{29}$$

As far as the polynomial mapping is concerned this procedure can easily be done. The important fact is that due to the non-linearity of our mapping the expansions (29) have Fourier components beyond the main zone. For example, for the non-linear part of the mapping (4) one has

$$\mathcal{F}_p = -\sin\alpha \sum_{k_1+k_2+k_3=p} g_{k_1} g_{k_2} g_{k_3} \tag{30}$$

where all k_i are in the main zone, but p is not.

Using Eqs. (27) and (29) one can find the following non-homogeneous equations for ξ_n and η_n:

$$\xi_{n+1} = \alpha_n \xi_n + \beta_n \eta_n + R_n$$
$$\eta_{n+1} = \gamma_n \xi_n + \delta_n \eta_n + T_n \tag{31}$$

where

$$R_n = -2\pi i M \sum_p p \sum_{\{r\}} \mathcal{F}_{pN+r} \exp(i\omega r n)$$
$$T_n = -2\pi i M \sum_p p \sum_{\{r\}} G_{pN+r} \exp(i\omega r n) \tag{32}$$

Taking into account terms with $p = \pm 1$ only one obtains

$$R_n = -2\pi i M \sum_{\{r\}} \left(\mathcal{F}_{N+r} - \mathcal{F}_{-N+r}\right) \exp(i\omega r n)$$
$$T_n = -2\pi i M \sum_{\{r\}} \left(G_{N+r} - G_{-N+r}\right) \exp(i\omega r n) \tag{33}$$

If (u_n, d_n) and (u'_n, d'_n) are two linear independent solutions of the homogeneous equations (18) then,

$$\xi_n = C_n^{(1)} u_n + C_n^{(2)} u'_n$$
$$\eta_n = C_n^{(1)} d_n + C_n^{(2)} d'_n \tag{34}$$

where the coefficients $C_n^{(i)}$ have the following form

$$C_n^{(1)} = \frac{1}{c}\sum_{k=0}^{n-1}\left(R_k d'_{k+1} - T_k u'_{k+1}\right)$$
$$C_n^{(2)} = \frac{1}{c}\sum_{k=0}^{n-1}\left(R_k d_{k+1} - T_k u_{k+1}\right) \tag{35}$$

and c was defined in Eq. (20).

Substituting these expressions into Eq. (28), using Eq. (23) and keeping in mind that $D_n x$ and $D_n y$ are periodic functions with the same period as x_n and y_n one obtains to first order in R_n and T_n:

$$\Delta_N = \frac{1}{2} K_N \sum_{n=0}^{N-1}\left(T_n u_{n+1} - R_n d_{n+1}\right) \tag{36}$$

On the right-hand side of this equation the equalities $u_n = D_n x$ and $d_m = D_n y$ can be used. Using Eq. (33) one has (for even N)

$$\Delta_N = \frac{1}{2} K_N (2\pi M)^2 \Big\{ \sum_{\{r\}} \left(G_{N-r} - G_{-N-r}\right)\cdot f_r \cdot r \cdot \exp(i\omega r) - $$
$$- \sum_{\{r\}} \left(\mathcal{F}_{N-r} - \mathcal{F}_{-N-r}\right)\cdot g_r \cdot r \cdot \exp(i\omega r) +$$
$$+(-)^M \cdot \frac{N}{2}\left[\left(G_{3N/2} - G_{-N/2}\right) f_{N/2} - \left(\mathcal{F}_{3N/2} - \mathcal{F}_{-N/2}\right) g_{N/2}\right]\Big\} \tag{37}$$

At odd N the terms in square brackets must be omitted. So if the exact positions of all points of the given cycle are know, one can find f_n and g_n from Eqs. (13) and then T_n and R_n from Eq. (37). But, of course, for an arbitrary mapping this is impossible. So we try to obtain an asymptotic formula for Δ_N at large N (and, correspondingly, small r).

3. - THE POSITIONS OF PERIODIC POINTS AND THE BEHAVIOUR OF Δ_N AT $N \to \infty$

Let us find the first few terms in the perturbation series expansion of the solutions of Eqs. (24). For clarity we consider the mapping (4). It is convenient to use the complex function $Z(t) = x(t) + iy(t)$. Then Eqs. (24) become

$$T: \quad Z(t+1) = e^{i\alpha}\left(Z(t) + \frac{i}{8}\left(Z(t) - \bar{Z}(t)\right)^3\right) \tag{38}$$

If μ is an irrational number, $Z(t)$ can be expanded into a Fourier series[2)]

$$Z(t) = z e^{i\omega t} + \sum_{k\neq 1} z_k \exp(i\omega k t) \tag{39}$$

and

$$\omega = \alpha + \sum_{k \neq 0} \omega_k (Z\bar{Z})^k$$

Substituting these expressions in Eq. (38) and considering Z as a coupling constant, it is not difficult to obtain successively all the coefficients Z_k and ω_k. At the first order of perturbation theory we have

$$Z(t) = Ze^{i\omega t} - \frac{i}{8}\left(\frac{3Z\bar{Z}^2}{1-e^{-2i\alpha}} e^{-i\omega t} + \frac{Z^3}{1-e^{2i\alpha}} e^{3i\omega t} - \frac{\bar{Z}^3}{1-e^{-4i\alpha}} e^{-3i\omega t} \right) + O(|Z|^5) \quad (40)$$

and

$$\omega = \alpha - \frac{3}{8} Z\bar{Z} - \frac{3}{128}(Z\bar{Z})^2 (8\,\mathrm{Ctg}\,\alpha + \mathrm{Ctg}\,2\alpha) + O(|Z|^6)$$

The parameter Z is connected to the initial point $Z(0) = Z_0$ by the following relation

$$Z = Z_0 + \frac{i}{8}\left(\frac{3Z_0\bar{Z}_0^2}{1-e^{-2i\alpha}} + \frac{Z_0^3}{1-e^{2i\alpha}} - \frac{\bar{Z}_0^3}{1-e^{-4i\alpha}} \right) + O(|Z_0|^5) \qquad (41)$$

If $Z_0 = re^{i\phi}$, then $Z = |Z|e^{i\Psi}$ where

$$h^2 = Z\bar{Z} = r^2 \left(1 - \frac{r^2}{8}\left(\frac{\cos(4\varphi - 2\alpha)}{\sin(2\alpha)} - 2\,\frac{\cos(2\varphi - \alpha)}{\sin\alpha} \right)\right) + O(r^6)$$

$$\Psi = \varphi + \frac{r^2}{16}\left(\frac{\sin(4\varphi - 2\alpha)}{\sin(2\alpha)} - 4\,\frac{\sin(2\varphi - \alpha)}{\sin\alpha} \right) + O(r^4) \qquad (42)$$

Z as a function of Z_0 is the so called normal co-ordinate for the mapping (38)[2]. In these co-ordinates the mapping (38) is a multiplication by $e^{i\omega}$, e.g., if Z_0 is governed by the mapping (38), then

$$Z' = e^{i\omega} Z \qquad (43)$$

and any formal integral is a function of $h^2 = \bar{Z}Z$.

For irrational μ this method gives all terms of the perturbation expansions both for $Z(t)$ and for $H(x,y,) = \bar{Z}Z$. At rational $\mu = m/n$ these series can be used only up to the occurrence of the first resonant term $1 - \exp(2\pi i n\mu) = 0$. In particular, Eqs. (40) - (43) are valid for all $|\mu| \neq 0, 1/4, 1/2$. To obtain the perturbation series at rational μ one can use another method[2].

Note that when $\mu = m/n$ the linear (unperturbed) part of the mapping (4) is a rotation by $2\pi m/n$. So, the linear part of the mapping T^n is the identity transformation and in complex

$$z' = z + \phi(z,\bar{z}) \tag{44}$$

where $\phi(Z,\bar{Z})$ is a polynomial. For example, for the mapping (4) and $\mu = 1/n$ $(n \neq 1,2,4)$ one obtains

$$\begin{aligned}\phi(z,\bar{z}) = &-\frac{3in}{8} z^2\bar{z} - \frac{3}{64}n\Big[\frac{4}{1-e^{2i\alpha}} z^4\bar{z} - \frac{1}{1-e^{4i\alpha}} z^5 + \\ &+\Big(-\frac{3}{1-e^{-2i\alpha}} + \frac{3}{2}(n-1) + \frac{5}{1-e^{2i\alpha}} + \frac{1}{1-e^{4i\alpha}}\Big) z^3\bar{z}^2 - \\ &-\frac{8}{1-e^{-2i\alpha}} z^2\bar{z}^3 + \frac{5}{1-e^{-4i\alpha}} z\bar{z}^4\Big] + O(|z|^7)\end{aligned} \tag{45}$$

Let us try to embed this discrete mapping into a continuous one in such a manner that: (i) the following relation is satisfied

$$z(t+1) = z(t) + \phi(z(t), \bar{z}(t)) \tag{46}$$

and (ii) the evolution of $Z(t)$ is described by Hamilton equations with a certain Hamiltonian $\tilde{H}(x,y)$

$$\dot{z} = -i\frac{\partial \tilde{H}}{\partial \bar{z}} \tag{47}$$

Then, evidently, $\tilde{H}(x,y)$ is an integral of the mapping (44) and one can prove[2] that it will also be an integral of the mapping (4). This embedding can easily be done within the perturbation series by expanding the difference $Z(t+1) - Z(t)$ as a Taylor series

$$z(t+1) - z(t) = \dot{z}(t) + \frac{1}{2}\ddot{z}(t) + \ldots \tag{48}$$

and finding all derivatives successively from Eq. (46). For the mapping (3) using Eq. (45) one obtains that $H = 3/(16n)h^4$ where

$$\begin{aligned}h^4 = (z\bar{z})^2 &- \frac{i}{4}(z\bar{z})\Big(\frac{\bar{z}^4}{1-e^{-4i\alpha}} + \frac{2z^3\bar{z}}{1-e^{2i\alpha}} - c.c.\Big) + \\ &+\frac{1}{24}(z\bar{z})^4\big(8\,\mathrm{Ctg}\,\alpha + \mathrm{Ctg}\,2\alpha\big) + O(|z|^8)\end{aligned} \tag{49}$$

For the mapping (1) $\mu = 1/4$ and Eqs. (45) and (49) are not applicable. In Ref. 1) it has been shown that in this case

$$\tilde{H} = \frac{1}{4}\left(x^4 + y^4 + 2x^3y^3\right) + O(r^8) \tag{50}$$

It is not difficult to obtain an arbitrary number of terms by this method. In practical calculation it is convenient to keep in mind that the mapping (38) can be rewritten as the product of two inversions

$$\begin{aligned} &T = I_1 \circ I_2 \\ &I_1 : \quad z' = e^{i\alpha}\bar{z} \\ &I_2 : \quad z' = \bar{z} + \frac{i}{8}(z - \bar{z})^3 \\ &I_1^2 = 1 \ , \quad I_2^2 = 1 \end{aligned} \tag{51}$$

Using these relations one can prove that for the mapping (3) at $N \geq 3$ there is always a periodic point at the abscissa axis. It is not difficult to find the approximate position of this point (and the positions at the other points of the given cycle). If some point is a periodic point with period N then the following condition must be satisfied

$$\omega N = 2\pi k \tag{52}$$

where ω is the rotation frequency. At small r and $|\mu| \neq 0, 1/4, 1/2$ the first terms in the expansion of $\omega(Z)$ are given by Eqs. (40) and (41), from which one has in the leading order

$$x_p^2 \approx \frac{16\pi}{3N}\eta \tag{53}$$

and $\eta = \mu N - k$.

Let us consider the case of rational μ, e.g., $\mu = 1/n$. Then $\eta = m/n$ and $m = N - nk$ or $N = m + nk$. At fixed N one has different families of periodic points corresponding to different k. The point with minimal x_p^2 corresponds to minimal $|\bar{m}|$. If $N \equiv m_0 \pmod n$ and $0 \leq m_0 \leq n - 1$ then $m_{MIN} = m_o$ when $m_o \leq n/2$ and $m_{MIN} = -(n-m_o)$ when $m_o \geq n/2$. The first case gives real x_p, y_p, while the second one implies that the periodic point co-ordinates are purely imaginary. In the latter case it is convenient to use the substitution $x_p = i\tilde{x}_p$ and $y_p = i\tilde{y}_p$ which corresponds to the mapping (4) with the sign of the y^3 terms reversed. At irrational μ there are also different families of periodic points. If $\eta_o = \mu N - [\mu N]$, where $[\beta]$ denotes the integer part of β, then $\eta_{MIN} = \eta_o$ when $\eta_o \leq 1/2$ and $\eta_{MIN} = \eta_o - 1$ when $\eta_o \geq 1/2$. But if for rational $\mu = m/n$ $|\eta_{MIN}| \geq 1/n$ at all N, then at irrational μ there are certain sequences for which $\eta_{MIN} \to 0$ at $N \to \infty$. Their specific form is deeply dependent on the arithmetic nature of the given irrational number[5]. Let us consider, for example, $\mu = (\sqrt{5}+1)/2$ for which the best approximations can be obtained from the Fibonacci numbers

$$F_{n+1} = F_n + F_{n+1} \tag{54}$$

$F_o = 0,\ F_1 = 1$

$F_n = (0,1,1,2,3,5,8,13,21,34,55,89,144,\ldots,)$

If $N = F_n$, $k = F_{n+1}$ then

$$\eta_N \approx (-)^{n+1} \frac{c}{\sqrt{5}\,N} \tag{55}$$

and $c = 1$. For the Fibonacci numbers with arbitrary initial conditions F_o and F_1 η_N is given by Eq. (55), but $c = F_1^2 - F_0^2 - F_0 F_1$. For example, for the sequence F'_n: $(1,3,4,7\ldots)$ $c = 5$. There are irrational μ for which η_N decreases faster than (55)[5]. Using the approximate expressions for $Z(t)$ and for the positions of the periodic points, one can obtain an asymptotic formula for Δ_N at large N. Let us first calculate K_N in Eqs. (36) and (23). At small $|Z|$ one can take into account the first terms in Eq. (40) only:

$$z(t) = z\, e^{i\omega t} \quad \text{and} \quad \omega \approx \alpha - \frac{3}{8} z\bar{z} \tag{56}$$

Then it is evident that the two independent solutions of the linear Eqs. (18) have the following forms:

$$z_1 = \dot{z}(t) \quad \text{and} \quad z_2 = t\,\dot{z}(t) + \frac{1}{2} z(t) \tag{57}$$

Comparing these with Eqs. (21) and (22), one finds

$$K_N = \frac{N}{h^4} \tag{58}$$

where $h^4 = (x^2+y^2)^2$ is the first term of the expansion (6). For the mapping (1) one must use $h^4 = (x^4+y^4)$. For rational μ, $h_p^2 = a/N$ and $K_N \approx N^3$, which is in agreement with Eq. (18) obtained in Ref. 1) by comparison with numerical calculations. By taking into account further terms in the perturbation series for $Z(t)$ it is possible to find K_N with any power accuracy.

To estimate the large Fourier componenets in Eq. (37) we assume that there are functions $x(t)$ and $y(t)$ which are good approximations to x_n and y_n. The Fourier components of these functions can easily be estimated from the usual formula:

$$f_k = \frac{1}{T} \int_0^T \exp(i\omega k t)\, x(t)\, dt \tag{59}$$

where T is the period of the functions considered and $\omega = 2\pi/T$.

Let $i\lambda$ the position of the singularity nearest to the real axis. Deforming the integration contour, one obtains

$$f_k = \frac{1}{T} \exp(-\omega \lambda |k|)\, \Psi(\omega k) \tag{60}$$

The function $\Psi(Z)$ is defined by the behaviour of $x(t)$ near the singularity. Analogous expressions can be obtained for the functions T_n and R_n. The quantity $i\lambda$ is, roughly speaking, the "time" of motion from the initial point to the singular points of the functions $x(t)$ and $y(t)$ and, of course, it is very dependent on the initial point. But, at least for rational μ, it is natural to assume that the character of the singularity and, consequently, the form of $\Psi(Z)$, is independent of the initial point. Let us take into account a finite number of terms of the perturbation series for the Hamiltonians (47) and (49) and let us solve Eqs. (47). As $\tilde{H}(x,y)$ is a polynomial the singularity points are the points where $x(t)$, $y(t) \to \infty$. The character of these singularities is defined by the coefficients in $\tilde{H}(x,y)$ and does not depend on the choice of the initial point $x(0)$ and $y(0)$. Substituting expressions like Eq. (60) to Eq. (37) and changing the summation over r to an integration one obtains

$$\Delta_N \approx K_N \exp(-\lambda\omega N) \int_{-\frac{N}{2}}^{\frac{N}{2}} dr \cdot \frac{1}{r} \Psi_1(\omega r) \cdot \frac{r}{N} \Psi_2(\omega r) + c.c. \tag{61}$$

setting $r = NZ$ and taking into account Eq. (58) we have

$$\Delta_N \approx c \frac{N}{h_p^4} \exp(-\lambda\omega N) \tag{62}$$

where c is constant and $h_p^4 = H(x_p, y_p)$.

For the mapping (1) $\mu = 1/4$ and the Hamiltonian begins with the term $x^4 + y^4$, which fixes $\mathrm{Re}\,\lambda = N/8$. All other terms give $O(1)$ contributions to $\mathrm{Re}\,\lambda$, which are not important because they redefine the constant term only. As a result one obtains Eq. (12). But for other $|\mu| \neq 0, 1/4, 1/2$ the Hamiltonian begins with the term $(x^2+y^2)^2$, which does not define λ. To find it, all the other terms are essential. Let us assume that Eq. (56) is correct up to $Z = a \sim 1$. Then the following estimate can be obtained

$$\lambda = \frac{1}{\omega} \ln \frac{a}{Z} \tag{63}$$

The constant a in this formula is dependent on the higher order terms in the perturbation series of the Hamiltonian. The results of numerical calculations (see below, Tables 1, 2 and 3) strongly suggest $a \approx 1$ and we will use this value below. But it must be emphasized that we know of no good arguments to fix this specific value of a.

Using (63) with $a = 1$ one obtains

$$\Delta_N = c \frac{N}{Z_p^4} |Z_p|^N \tag{64}$$

(Note that at the periodic point $\omega = 2\pi M/N$ and $Z_p^N = |Z_p|^N$.)

Let us give some other arguments to clarify this formula. The direct calculation of T^N in Eq. (9) is very difficult at large N. Let us consider the T^N mapping near the periodic points. Then T^N: $Z_N = Z + \Psi(Z,\bar{Z})$ and the function $\Psi(Z,\bar{Z})$ must have at least N zeros on the circle $Z = |Z_p|$, corresponding to the N periodic points of the given cycles [really, it must have $2N$ zeros corresponding to interchanging elliptic and hyperbolic points[2)]]. So, roughly speaking, $\Psi(Z,\bar{Z})$ must be proportional to $\sin(N\omega\phi)$ and because $\Psi(Z,\bar{Z})$ is some polynomial

$$\Psi(Z,\bar{Z}) \approx R_N |Z_p|^N \sin(N\omega\varphi) \tag{65}$$

Starting from this formula an expression analogous to Eq. (64) can be obtained for matrix M_N (10). These considerations are valid if on the circle $|Z| = \mathrm{const.}$ there are no other periodic points except those which can be obtained from the periodic point on the abscissa axis by the mapping T (and their counterparts with the opposite sign of Δ_N). But the transformation $Z' = -Z$ commutes with the mapping (13). So if Z_p is some periodic point

At odd N this point cannot coincide with any point of a given cycle, so N in Eq. (63) must be replaced by $2N$. For even $N = 2Q$, $T^Q = -1$, and this transformation does not give new periodic points. Thus Eq. (64) is applicable for even N. When N is odd Δ_N is, roughly speaking, equal to the square of Eq. (64).

Unfortunately at irrational μ we do not know of any arguments which give information about c in Eq. (63). But the results of numerical calculations show that c is a slowly varying function of N and for periodic points which are of interest to us c_N will be found numerically.

In the next section it will be shown that the main contribution to large order estimates is given by the periodic points with minimal $|\eta|$ in Eq. (53). As specific examples we consider the mapping (4) with $\mu = 1/5$ and $\mu = (\sqrt{5}+1)/2$. From the above-mentioned facts it is clear that at $\mu = 1/5$ the most important periodic points have $N = 6, 16, 26, \ldots,$ for which $\eta = 1/5$ and $N = 4, 14, 4, \ldots,$ for which $\eta = -1/5$. At $\mu = (\sqrt{5}+1)/2$ one must take into account the fractions whose denominators are equal to the even terms of the Fibonacci numbers: $N = 2, 8, 34, 144, \ldots,$ and for which $|\eta| = 1/(\sqrt{5}N)$. To do the numerical calculations it is convenient to find the point on the abscissa axis which, after the N fold application of our mapping, returns to its original position and then one can easily obtain M_N from Eq. (16). It can be shown that for periodic points on the abscissa axis M_N has the following form:

$$M_N = \begin{pmatrix} 1 + a_N & b_N \\ -c_N & 1 + a_N \end{pmatrix} \tag{66}$$

and c_N can easily be obtained from Eq. (56)

$$c_N = -\frac{dy_N}{dx} = \frac{3}{4} x_p^2 N \tag{67}$$

The quantities b_N and a_N are exponentially small and connected with each other by the relations $\Delta_N = 2a_N = -b_N c_N$. Using Eqs. (64), (53) and (40) one finds

$$\Delta_N^{(as)} = \delta N \left(\frac{a}{N}\right)^{\frac{N}{2}-2} \exp\left(-a \cdot b\right) \tag{68}$$

where for $\mu = 1/5$ and $N = 6, 16, 26, \ldots,$ $a = 16\pi/15$ and for $N = 4, 14, 26, \ldots,$ $a = -16\pi/15$

$$b = \frac{1}{32}\left(8\,\mathrm{Ctg}\,\alpha + \mathrm{Ctg}\,2\alpha\right)$$

To obtain δ we calculated Δ_N for some values of N and found

$$\delta \approx 10^{-2} \tag{69}$$

A comparison of this asymptotic formula with the results of numerical calculations is made in Tables 1 and 2.

The analogous formula can be used when μ is an irrational number. But in this case $a = 16\pi/3(\alpha N-M)$ and δ in Eq. (68) is some (unknown) function of N. For Fibonacci sequences $a = c/N$ and at $N \to \infty$ one has

$$\Delta_N^{(as)} = \delta_N\, N \left(\frac{c}{N^2}\right)^{\frac{N}{2}-2} \exp\left(-\frac{cb}{N}\right) \tag{70}$$

Using this formula for the periodic point with $N = 8$ for which $x_p^2 = -0.1280158$ and $\Delta_8 = 3.78.10^{-4}$ gives $\delta \approx 3.10^{-3}$.

A comparison of the asymptotic formula (68) with $\mu = (\sqrt{5}+1)/2$ and $\delta = 3.10^{-3}$ with the results of numerical calculations is presented in Table 3. It is seen that the function δ_N is some oscillating function which varies very slowly in comparison with Δ_N.

4. - ASYMPTOTIC ESTIMATES FOR $H(x,y)$

Up to now the formal integral has been considered only within the perturbation series. For it to make sense to speak about the singularities of $H(x,y)$, one needs a more general definition. First of all, note that if $H(x,y)$ is an integral of some mapping, then $f(H)$ where f is any function is also an integral. In recursive relations this non-uniqueness manifests itself as an arbitrariness in the coefficients of the $(x^2+y^2)^k$ terms for irrational μ and of the $(x^2+y^2)^{2k}$ terms for rational μ. So one can define $H(x,y)$ along some line in the (x,y) plane by any convenient condition (but which must be consistent with the perturbation series) and then one can extend the definition of $H(x,y)$ to all points which are images of our chosen line. Usually we will assume that $H(x,y)$ is a known function at the abscissa axis

$$H(x,0) = \bar{H}(x) \tag{71}$$

where $\bar{H}(x)$ is a fixed function. [To obtain the large order estimates only the first two coefficients of $\bar{H}(x)$ are important, so this condition does not lead to any difficulties.]

If (x_k,y_k) is the result of the application of some power of our mapping to the point $(x_o,0)$ then according to our definition

$$H(x_k, y_k) = \bar{H}(x_o) \tag{72}$$

There are some points in the (x,y) plane which are not images of the abscissa axis. To define $H(x,y)$ at these points one has to think that $H(x,y)$ is known not only at the line $y = 0$ but also in the corresponding complex plane and to consider x_o in Eq. (72) as a point in this complex plane.

It is evident that the function constructed in this manner is an integral of a given mapping and, if $H(x)$ is consistent with the perturbation series, the expansion of $H(x,y)$ coincides with that obtained from the usual recursive scheme. Sometimes, it is convenient to use other definitions of $H(x,y)$. For example, at irrational μ we will use the condition

$$\int_0^{2\pi} H_n(r\cos\varphi, r\sin\varphi)\,d\varphi = 0 \quad \text{when} \quad n > 1 \tag{73}$$

i.e., we simply set all the coefficients of the $(x^2+y^2)^k$ terms $(k > 1)$ equal zero.

It has been shown in Ref. 1) that the function $H(x,y)$ defined as in Eq. (72) has singularities of the square-root type near all periodic points of a given mapping. Let us briefly review the arguments of Ref. 1).

If $H(x,y)$ is an integral of the mapping T then it will be, of course, an integral of the mapping T^N also. In the near vicinity of any periodic point the mapping T^N is a linear mapping and has the following form:

$$\begin{pmatrix} \delta x' \\ \delta y' \end{pmatrix} = M_N \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} \tag{74}$$

where $\delta x = x - x_p$, $\delta y = y - y_p$ and M_N is the monodromy matrix defined in Eq. (10).

Consequently, $H(x,y)$ must be the integral of this transformation. But the following quadratic form is the only invariant of this linear mapping*):

$$h(x,y) = m_{12}\,(\delta y)^2 + (m_{11} - m_{22})\,\delta x\,\delta y - m_{21}(\delta x)^2 \tag{75}$$

where m_{ij} are the corresponding matrix elements of M_N. For periodic point on the abscissa axis M_N has the form (66) and

$$h(x,y) = (\delta x)^2 + \beta_N\, y^2 \tag{76}$$

where $\beta_N = b_N/c_N = -\Delta_N/c_N^2$, and Δ_N and c_N at large N have the forms (68) and (67) correspondingly. So if $H_p = H(x_p)$ is the value of $H(x,y)$ at the given periodic point then, in a small vicinity of it, H must be a function of h only:

$$H = f\left((\delta x)^2 + \beta_N\, y^2 \right) \tag{77}$$

But according to our definition $H(x,0) = \bar{H}(x)$ and at any point on the abscissa axis one must have

$$H = H_p + \frac{\partial \bar{H}}{\partial x_p}\,\delta x \qquad + \text{ regular terms} \tag{78}$$

This relation defines the unknown function in Eq. (77) and one has

$$H(x,y) = \frac{\partial \bar{H}}{\partial x_p}\sqrt{(\delta x)^2 + \beta_N\, y^2} \qquad + \text{ regular terms} \tag{79}$$

So $H(x,y)$ near the fixed point on the abscissa axis has a square-root singularity of known form. Using the explicit form of M_N near other periodic points, or simply taking into account the invariance of $H(x,y)$ under our mapping, one can find the singularity of $H(x,y)$ near any periodic point.

$$H(x,y) = \sqrt{(H - H_p)^2 + \beta_N(\delta S)^2} \qquad + \text{ regular terms} \tag{80}$$

where δS is the distance calculated along the curve $H = \text{const.}$ and normalized in such a way that for periodic points on the abscissa axis $\delta S = (\partial\bar{H}/\partial x)y$. There is some correction to δS proportional to $H - H_p$ but it is inessential for us.

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*) We assume that $M^n \neq 1$, or that $\lambda_{1,2} \neq \sqrt[k]{1}$ for some integer k. For the matrix under consideration this statement can easily be proved from the results of the last section.

Taking into account only the linear term in β_N (note that β_N is an exponentially small quantity) one obtains

$$H(x,y) = \frac{1}{2}\beta_N \frac{(\delta S)^2}{H - H_p} + \text{regular terms} \tag{81}$$

To find large order estimates of the perturbation series coefficients one must construct some function which has this behaviour near all periodic points of a given mapping. Using Eqs. (64) and (42) it is easy to understand that for rational μ for which the first terms of $H(x,y)$ are given by Eq. (49)

$$H(x,y) = -\frac{128}{9}\sum_N \frac{\delta}{N^3(h^4 - h_p^4)}\left\{ f_N - \left[Z^N e^{-i\phi_0} + \bar{Z}^N e^{i\phi_0} \right] \right\} \tag{82}$$

where $Z(x,y)$ is the normal co-ordinate defined in Eq. (42). $\phi_0 = 3Nh_h^2/16$ is the phase chosen in such a manner that the expression in the square brackets has an expansion near the abscissa axis beginning with the y^2 term.

The function f_N is connected with our definition of the formal integral and is of minor interest to us.

h_p^4 is the value of $H(x,y)$ at the given periodic cycle whose period equals N.

Let us emphasize that with any desired accuracy h^4 does not change under the mapping (4) [or (38)] and the quantity $Z(x,y)$ is multiplied by $\exp(i\omega)$ [see Eq. (43)] but at a periodic point $\omega = 2\pi M/N$ and Z^N (and $\bar{Z}^N$) does not change either. This means that $H(x,y)$ in Eq. (82) is really an invariant of the given mapping. [Of course, it is easy to find an expression which is an invariant not only at the periodic points but at all other points too. To do this it is enough to use instead of the term Z^N in Eq. (82) the term Z^T where $T = 2\pi/\omega$ and ω is defined by a perturbation expansion analogous to Eq. (40).]

Taking into account the pole terms only and omitting the term f_N we have

$$H(x,y) = \frac{128}{9}\sum_N \delta \frac{|Z_p|^N}{N^3 h_p^2 (h^2 - h_p^2)} \cos(N\Psi - \Phi_0) \tag{83}$$

where $|Z|$ and Ψ are given by Eq. (42).

Expanding this expression as a power series in h^2 one finds

$$H(x,y) = \sum_k h^{2k} R_k(\varphi) \tag{84}$$

where

$$R_k = \sum_N R_k^{(N)}(\varphi) \tag{85}$$

and

$$R_k^{(N)} = -\frac{8\Delta_N}{N c_N^2}\left(\frac{1}{h_N^2}\right)^{k-2} \cos(N\varphi + \theta) \tag{86}$$

or using the asymptotic formulas for Δ_N and c_N one has

$$R_k(\varphi) = -\frac{128}{9}\sum_N \frac{\delta}{N^3}|z_p|^N \frac{1}{(h_p^2)^{k+2}} \cos(N\varphi + \theta) \tag{87}$$

where

$$\theta = \frac{N|z_p|^2}{16}\left(\frac{\sin(4\varphi - 2\alpha)}{\sin 2\alpha} - 4\,\frac{\sin(2\varphi - \alpha)}{\sin\alpha} - 3\right) \tag{88}$$

$$z_p^2 = \rho_p^2\,(1 + b_z \rho_p^2)$$

$$b_z = -\frac{1}{16}\,(8\,\mathrm{ctg}\,\alpha + \mathrm{ctg}\,2\alpha)$$

$$\rho_p^2 = \frac{16\pi}{3N}\,(\mu N - M)$$

The expansion of h^2 depends on our definition of $H(x,y)$. For irrational μ one can use

$$h^2 = z\bar{z} = r^2(1 + b_h r^2)\,, \quad h_p^2 = z_p^2 \tag{89}$$

where

$$b_h(\varphi) = -\frac{1}{8}\left(\frac{\cos(4\varphi - 2\alpha)}{\sin 2\alpha} - 2\,\frac{\cos(2\varphi - \alpha)}{\sin\alpha}\right),$$

For rational μ the first terms of $h^2 = \sqrt{H(x,y)}$ can be obtained from Eq. (49):

$$h^2 = r^2(1 + \tilde{b}_h r^2)\,, \quad h_p^2 = z_p^2\,(1 + b' z_p^2) \tag{90}$$

where

$$\tilde{b}_h = b_h - \frac{1}{3} b_z \quad , \quad b' = -\frac{1}{3} b_z$$

The difference between rational and irrational μ lies in the fact that for irrational μ all coefficients of $(Z,\bar{Z})^k$ are arbitrary, but for rational μ the $(Z,\bar{Z})^{2k+1}$ terms cannot be changed.

Finally the following expression for $R_k(\phi)$ can be found

$$R_k(\varphi) = -\frac{128}{9} \sum_N \frac{\delta}{N^3} (\rho_p^2)^{\frac{N}{2}-k-2} \exp(s) \cos(N\varphi + \theta) \tag{91}$$

where s is the correction term, $s = (N/2 - k - 2)\rho_p^Z b_Z - (k+2)\rho_p^2 b$.

For irrational μ one has the same formula with $b' = 0$ and an additional factor of $1/2$. To obtain asymptotic estimates for H_n it is necessary to expand $(h^2)^k$ in powers of $r^2 = (x^2+y^2)$ in Eq. (84). If $R_k(\phi) \sim k!a^k$ then

$$H_n^{(as)} = r^{2n} \exp\left(b_h(\varphi)\right) R_n(\varphi) \tag{92}$$

If $R_k \sim a^k$ then

$$H_n^{(as)} = r^{2n} \exp\left(n \frac{b_h(\varphi)}{a}\right) R_n(\varphi) \tag{93}$$

where $b_h(\phi)$ is the second term in the expansion of h^2 in powers of r^2. For irrational μ, $b_n(\phi)$ is defined in Eq. (89); for rational μ, it is defined in Eq. (90).

In principle, the summation in Eq. (91) is done over all periodic points of the given mapping with an even N. (For odd N β_N is proportional to $|Z|^{2N}$ and, roughly speaking, one must change $N \to 2N$ in this equation). Let us estimate the sum in this equation by the saddle point method. For rational μ, $\rho^2 = a/N$ and the main contribution comes from N which is the solution of the saddle point equation: $U'(N) = 0$ where

$$U = \left(\frac{N}{2} - k - 2\right) \ln \frac{a}{N} \tag{94}$$

This gives

$$2(k+2) = N_{s.p.} \ln\left(e N_{s.p.}/a\right) \tag{95}$$

or

$$N_{s.p.} \approx \frac{2(k+2)}{\ln\left(2a(k+2)\right)-1} \tag{96}$$

For irrational μ, $\rho_p^Z = \tilde{a}/N^2$ and $N_{s.p.}$ is given by the same formula with the substitution $a \to \sqrt{\tilde{a}}$.

A very crude estimate of the saddle point term for rational μ gives

$$(\rho_p)^{\frac{N_{s.p.}}{2}-k-2} \approx k!\left[\frac{2e}{a\ln k}\right]^k \tag{97}$$

For irrational μ one must substitute $k \to 2k$ and $a \to 2\sqrt{\tilde{a}}$.

For odd N one must change $a \to 2a$ in Eq. (97). So, from these formulas one sees that the main contribution to the large order estimates is given by the periodic points with the smallest values of a. All other points give exponentially small terms and can be neglected.

However, it is necessary to keep in mind that values of N which correspond to minimal $|a|$ are rare. For example, it has been shown in the last section that for $\mu = 1/5$ only two sequences of N, $N = 6 + 10k$ and $N = 4 + 10k$, give the smallest $|a|$. For $\mu = (\sqrt{5}+1)/2$ only the even terms of the Fibonacci sequence correspond to smallest $|a|$. This means that when k is not extremely large ($k \lesssim 100$) it is possible that one can obtain $N_{s.p.}$ from Eq. (96), but in reality there is no such term corresponding to the smallest $|a|$. In that case one must take into account either the nearest of such terms or the terms with larger $|a|$. It is most important at irrational μ when the necessary N are exponentially rare. For example, for $\mu = (\sqrt{5}+1)/2$ the periodic point with $N = 8$ is the saddle point for $k \approx 6$ and the point with $N = 34$ only when $k \sim 60$.

Let us take into account for $\mu = (\sqrt{5}+1)/2$ only the one periodic point with $N = 8$. The point with $N = 2$ is important at $k \lesssim 4$, and the point with $N = 34$ becomes important when $k \gtrsim 50$. In this case, of course, there is no necessity to have an asymptotic formula for Δ_N. We will use the following computed values:

$$x_8^2 = -.1280158 \ , \ \Delta_8 = 3.78 \cdot 10^{-4} \ , \ c_8 = -.605572 . \tag{98}$$

Using Eqs. (86) and (93) one obtains (N = 8):

$$H_n^{(as)}(r\cos\varphi, r\sin\varphi) = -r^{2n} \, 4 \cdot \frac{\Delta_N}{c_N^2 \cdot N^2} \left[A \right]^{n-1} \cos\,(N\varphi + \theta) \tag{99}$$

where

$$A = \frac{1}{x_p^2} \exp\left(-x_p^2 \left(b_h(0) - b_h(\varphi)\right)\right)$$

and $b_h(\phi)$ and θ are defined in Eqs. (89) and (88).

x_p in this formula is the position of the periodic point on the abscissa axis. This value can be taken directly from Eq. (98), or it can be obtained by comparing the two expressions for h_p^2

$$h_p^2 = x_p^2 \left(1 + b_h(0)\, x_p^2\right)$$

and

$$h_p^2 = \rho_p^2 \left(1 + b_2 \rho_p^2\right) \tag{100}$$

where for $N = 8$ and $M = 13$ $\rho_p^2 = -0.1167165$.

Let us emphasize that, when one takes into account only one periodic cycle, $H_n^{(as)}$ grows only exponentially. The factorial growth appears when a large number of periodic cycles is considered.

A comparison of the asymptotic formula (99) with the results of numerical calculations is made in Table 4. It is seen that the accuracy of Eq. (99) is high. The small discrepancies are of the order of x_p^2 and can be attributed to the following :

i) In Eq. (99) the corrections of order of const x_p^2 to the pre-exponential term are not included.

ii) We do not take into account the corrections of order (kx_p^4) to A. When $k \gtrsim 8$ they are also important.

iii) The results of numerical calculations are presented when $H(x,y)$

obeys the additional condition (73). But Eq. (99) does not satisfy it. So we must subtract from Eq. (99) its mean value $< H >$. It is not difficult to calculate this integral. When k is not very large $< H >$ has the order $(k\, x_p^2)^2$ and gives only a small correction.

iv) We take into account only one periodic cycle with $N = 8$, so it is difficult to argue that the resulting asymptotic formula will be very accurate at all k.

Taking into account all these corrections connected with the fact that for the periodic cycle considered x_p^2 is not very small, one can improve the comparison with numerical calculations but this is not connected with the problem of large order estimates and it is beyond the scope of this paper.

5. - SUMMARY

The main result of this paper is that the method of large order estimates of perturbation series in classical mechanics proposed in Ref. 1) is applicable and gives good results even at irrational μ, when the existence of small denominators greatly complicates the application of other methods.

It would be interesting:

i) To estimate a in Eq. (63) at rational μ and to explain why numerically $a \approx 1$.

ii) To obtain a more accurate asymptotic formula for Δ_N at irrational μ.

iii) To find the large order estimates for additional integral in a real Hamiltonian system, e.g. for the Henon-Heiles Hamiltonian[6)] where a few terms of the perturbation series for the additional integral are known[7)].

6. - ACKNOWLEDGEMENTS

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N	x_p	Δ_N	$\Delta_N/\Delta_N^{(as)}$
6	0.666351	0.9413	31.9
16	0.442462	$0.3339.10^{-4}$	2.810
26	0.351535	$0.4237.10^{-10}$	1.135
36	0.300480	$0.9872.10^{-17}$	1.006

Table 1 - Δ_N for $N = 10k + 6$

N	ix_p	Δ_N	$\Delta_N/\Delta_N^{(as)}$
4	0.847611	3.639	80.0
14	0.508800	$-0.5139.10^{-3}$	4.110
24	0.382380	$1.053\ \ 10^{-9}$	1.335
34	0.319096	$-0.3218.10^{-15}$	1.009

Table 2 - Δ_N for $N = 10k + 4$

N	x_p	Δ_N	$\delta_N = \Delta_N/\Delta_N^{(as)}$
12	0.588357	$4.67.10^{-2}$	77
16	0.630977	0.621	80
18	0.324544	$-0.279.10^{-8}$	- 0.33
22	0.552207	$-0.745.10^{-6}$	- 6.1
28	0.395609	$0.123.10^{-12}$	0.53
32	0.534135	$0.420.10^{-7}$	0.96
42	0.523475	$-0.203.10^{-11}$	-11.7
58	0.445590	$-0.132.10^{-20}$	- 1.3

Table 3 - Δ_N at $\mu = (\sqrt{s}+1)/2$ and $\delta = 3.10^{-3}$

	$\phi = 0$		$\phi = \pi/8$		$\phi = \pi/4$		$\phi = 3\pi/8$		$\phi = \pi/2$	
N	H_N/r^{2N}	$H_N/H_N^{(as)}$	H_N/r^{2N}	$H_N/H_N^{(as)}$	H_N/r^{2N}	$H_N/H_N^{(as)}$	H_N/r^{2N}	$H_N/H_N^{(as)}$	H_N/r^{2N}	$H_N/H_N^{(as)}$
4	0.3330E-01	1.0844	-0.3374E-02	0.1114	0.3195E-01	1.1328	-0.1835E-01	0.6718	0.1188E-01	0.3599
5	-0.2197E+00	0.9159	0.1878E+00	0.7923	-0.2058E+00	0.9341	0.1675E+00	0.7698	-0.2192E+00	0.7926
6	0.1519E+01	0.8104	-0.1517E+01	0.8176	0.1415E+01	0.8223	-0.1452E+01	0.8378	0.1970E+01	0.8503
7	-0.1234E+02	0.8434	0.1218E+02	0.8387	-0.1151E+02	0.8561	0.1186E+02	0.8593	-0.1632E+02	0.8410
8	0.9507E+02	0.8315	-0.9380E+02	0.8247	0.8870E+02	0.8442	-0.9335E+02	0.8489	0.1361E+03	0.8372
9	-0.7495E+03	0.8392	0.7389E+03	0.8299	-0.6994E+03	0.8520	0.7458E+03	0.8513	-0.1136E+04	0.8345
10	0.5837E+04	0.8367	-0.5749E+04	0.8248	0.5448E+04	0.8496	-0.5911E+04	0.8470	0.9488E+04	0.8318
11	-0.4578E+05	0.8399	0.4501E+05	0.8248	-0.4274E+05	0.8531	0.4702E+05	0.8458	-0.7924E+05	0.8293
12	0.3578E+06	0.8403	-0.3511E+06	0.8218	0.3341E+06	0.8536	-0.3732E+06	0.8426	0.6619E+06	0.8269
13	-0.2802E+07	0.8426	0.2743E+07	0.8202	-0.2618E+07	0.8560	0.2964E+07	0.8402	-0.5530E+07	0.8247
14	0.2193E+08	0.8441	-0.2141E+08	0.8175	0.2049E+08	0.8577	-0.2353E+08	0.8371	0.4621E+08	0.8227
15	-0.1717E+09	0.8463	0.1671E+09	0.8152	-0.1605E+09	0.8602	0.1867E+09	0.8340	-0.3862E+09	0.8209
16	0.1345E+10	0.8484	-0.1304E+10	0.8123	0.1258E+10	0.8625	-0.1481E+10	0.8305	0.3229E+10	0.8193
17	-0.1054E+11	0.8509	0.1017E+11	0.8094	-0.9856E+10	0.8652	0.1175E+11	0.8269	-0.2700E+11	0.8178
18	0.8255E+11	0.8534	-0.7930E+11	0.8062	0.7725E+11	0.8681	-0.9315E+11	0.8230	0.2258E+12	0.8166
19	-0.6470E+12	0.8562	0.6182E+12	0.8029	-0.6057E+12	0.8712	0.7384E+12	0.8190	-0.1889E+13	0.8155
20	0.5071E+13	0.8592	-0.4819E+13	0.7994	0.4750E+13	0.8744	-0.5851E+13	0.8146	0.1581E+14	0.8145
21	-0.3976E+14	0.8623	0.3755E+14	0.7957	-0.3725E+14	0.8779	0.4635E+14	0.8101	-0.1323E+15	0.8138
22	0.3118E+15	0.8656	-0.2925E+15	0.7917	0.2923E+15	0.8816	-0.3671E+15	0.8054	0.1107E+16	0.8132
23	-0.2445E+16	0.8691	0.2278E+16	0.7876	-0.2293E+16	0.8854	0.2906E+16	0.8004	-0.9272E+16	0.8127
24	0.1918E+17	0.8728	-0.1774E+17	0.7833	0.1800E+17	0.8895	-0.2300E+17	0.7953	0.7764E+17	0.8124

Table 4 - Comparison of the asymptotic formula for H_N ($r \cos\phi$, $r \sin\phi$) at $\mu = (\sqrt{5}+1)/2$ with numerical calculations.

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