ASYMPTOTIC ESTIMATES OF MULTI-DIMENSIONAL STABLE DENSITIES AND THEIR APPLICATIONS

TOSHIRO WATANABE

Dedicated to Minoru Motoo on his 77th birthday

ABSTRACT. The relation between the upper and lower asymptotic estimates of the density and the fractal dimensions on the sphere of the spectral measure for a multivariate stable distribution is discussed. In particular, the problem and the conjecture on the asymptotic estimates of multivariate stable densities in the work of Pruitt and Taylor in 1969 are solved. The proper asymptotic orders of the stable densities in the case where the spectral measure is absolutely continuous on the sphere, or discrete with the support being a finite set, or a mixture of such cases are obtained. Those results are applied to the moment of the last exit time from a ball and the Spitzer type limit theorem involving capacity for a multi-dimensional transient stable process.

1. INTRODUCTION

Since Lévy [20] in 1937, the study of stable processes and their distributions has been an important subject not only in theoretical probability but also in physics and finance. Many properties of univariate stable distributions are found in Zolotarev [43]. However, there are few known results on the asymptotic properties of multivariate stable densities up to now. Only Pruitt and Taylor [28] investigated the asymptotic estimates of multivariate stable densities in a general setting. The others made certain strong assumptions on the spectral measure, for instance, symmetry, absolute continuity or discreteness. In this paper, surprisingly explicit estimates can be obtained for a multivariate α-stable density $p(x)$ on $\mathbb{R}^d$. We first obtain the general upper and lower estimates without any technical assumption. See Theorem 1.1. In particular, we find the best possible and worst possible cases. Then we discuss the upper and lower asymptotic estimates of $p(x + y)$ as $|x| \to \infty$ with y being bounded, from the point of view of their relation to the fractal dimensions of the spectral measure σ on S^{d-1}. See Theorems 1.2 and 1.4. The asymptotic order delicately depends on the direction of $x/|x|$ in relation to the location of the support of σ. Thus we discover that a wide range of decay orders is possible. See Remark 1.3. Moreover, we solve the problem (Theorem 1.1) and the conjecture (Theorem 1.5) on the upper asymptotic estimates of multivariate stable densities in the classical work of Pruitt and Taylor [28] in 1969.

In what follows, we use the terminology in Sato's book [32]. Let $\{X_t : t \geqslant 0\}$ be a Lévy process on $\mathbb{R}^d$, $d \geqslant 1$, with generating triplet (A, ν, γ). Here A is the Gaussian covariance matrix, ν is the Lévy measure, and γ is the location parameter. That is,

$$(1.1) \qquad E \exp(i\langle z, X_t \rangle) = \exp(t\psi(z)), \qquad z \in \mathbb{R}^d,$$

with

$$(1.2) \qquad \psi(z) = \int_{\mathbb{R}^d} (e^{i\langle z, x \rangle} - 1 - 1_{\{|x| \leqslant 1\}}(x) i\langle z, x \rangle) \nu(dx) + i\langle \gamma, z \rangle - \frac{1}{2}\langle Az, z \rangle,$$

where ν is a measure on $\mathbb{R}^d$ satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d}(1 \wedge |x|^2)\nu(dx) < \infty$, $\gamma \in \mathbb{R}^d$, and A is a nonnegative-definite matrix. If $A = 0$, $\{X_t\}$ is said to be purely non-Gaussian. A Lévy process $\{X_t\}$ on $\mathbb{R}^d$ is called a trivial process if there is $c \in \mathbb{R}^d$ such that, for every t, $X_t = tc$ a.s.; otherwise $\{X_t\}$ is said to be nontrivial.

A set B in $\mathbb{R}^d$ is called *one-sided* if there is $c \neq 0$ in $\mathbb{R}^d$ such that $B \subset \{x : \langle c, x \rangle \geqslant 0\}$. A measure ρ on $\mathbb{R}^d$ is called one-sided if the support S_ρ of ρ is one-sided. A measure ρ on $\mathbb{R}^d$ is called *degenerate* if there are $a \in \mathbb{R}^d$ and a proper linear subspace V of $\mathbb{R}^d$ such that $S_\rho \subset a + V$; otherwise ρ is called nondegenerate. A Lévy process $\{X_t\}$ on $\mathbb{R}^d$ is called degenerate if the distribution of X_t is degenerate for every $t > 0$ (equivalently, for some $t > 0$); otherwise $\{X_t\}$ is called nondegenerate. See [32], Proposition 24.17 for conditions for nondegenerateness in terms of the generating triplet.

A Lévy process $\{X_t\}$ on $\mathbb{R}^d$, $d \geqslant 1$, is called *stable* if, for every $a > 0$, there are $b > 0$ and $c \in \mathbb{R}^d$ such that

$$(1.3) \qquad \{X_{at} : t \geqslant 0\} \overset{\mathrm{d}}{=} \{bX_t + tc : t \geqslant 0\}.$$

If $\{X_t\}$ is a nontrivial stable process, then define the index α of $\{X_t\}$ by $\alpha = \log a / \log b$ for $a \neq 1$; this α is uniquely determined and $0 < \alpha \leqslant 2$. In this case the process $\{X_t\}$ is sometimes called an α-stable process. Let $\{X_t\}$ be a nontrivial stable process on $\mathbb{R}^d$. If $c = 0$ in (1.3) for every $a > 0$, we call $\{X_t\}$ a *first-class* stable process. Otherwise we call $\{X_t\}$ a *second-class* stable process. The process is first-class stable if and only if it is strictly stable in the terminology of [32]; it is second-class stable if and only if it is stable but not strictly stable.

Let $\{X_t\}$ be a nontrivial α-stable process on $\mathbb{R}^d$ with $\alpha \neq 2$. Then there is a probability measure σ on $S^{d-1} = \{\xi \in \mathbb{R}^d : |\xi| = 1\}$ such that

$$(1.4) \qquad \nu(B) = c \int_{S^{d-1}} \sigma(d\xi) \int_0^\infty 1_B(r\xi) r^{-1-\alpha} dr \qquad \text{for } B \in \mathcal{B}_{\mathbb{R}^d},$$

where c is a positive constant independent of ξ and r, and $\mathcal{B}_{\mathbb{R}^d}$ is the class of all Borel sets in $\mathbb{R}^d$. This σ is uniquely determined and is called the *spectral measure* of the process $\{X_t\}$. We consider σ as a measure on $\mathbb{R}^d$ with the understanding that $\sigma((S^{d-1})^c) = 0$.

Let $\phi(r)$ be a positive and decreasing function on $(0, \infty)$. We use the words "increase" and "decrease" in the wide sense allowing flatness. Then we say $\phi \in \mathcal{D}$ if there is a constant $c_1 > 0$ such that $\phi(r) \leq c_1 \phi(2r)$ for $r > 0$. A positive and decreasing function $\phi \in \mathcal{D}$ is called of *dominated variation*. Denote by B_a an open ball in $\mathbb{R}^d$ with center 0 and radius a. Thus $\overline{B}_a = \{x : |x| \leqslant a\}$ and $B_a + y = \{x : |x - y| < a\}$. For the spectral measure σ of an α-stable process $\{X_t\}$

on $\mathbb{R}^d$ with $\alpha \neq 2$, define functions $\sigma_\xi(r)$ and $\sigma^*(r)$ for $r > 0$ as

$$(1.5) \qquad \sigma_\xi(r) = \sigma(\xi + B_{1/r}) \qquad \text{and} \qquad \sigma^*(r) = \sup_{\xi \in S^{d-1}} \sigma(\xi + B_{1/r}).$$

Obviously $\sigma^*(\cdot)$ is positive and decreasing and $\sigma^*(\cdot) \in \mathcal{D}$. Thus there is a constant $b \geqslant 0$ such that

$$(1.6) \qquad \sigma^*(sr) \leqslant s^{-b}\sigma^*(r) \qquad \text{for } 0 < s < 1 \text{ and } r > 0.$$

See Bingham et al. [3]. Let S_σ be the support of σ. Taking Carathéodory's theorem into account, we define

$$(1.7) \quad \begin{aligned} C_\sigma^0(n) = \{\xi \in S^{d-1} : \xi = \textstyle\sum_{j=1}^n c_j \xi_j \text{ for some } c_j > 0 \text{ and } \xi_j \in S_\sigma, \\ j = 1, \ldots, n, \text{ such that } \xi_1, \ldots, \xi_n \text{ are linearly independent}\} \end{aligned}$$

for $1 \leqslant n \leqslant d$, $T_\sigma(1) = C_\sigma^0(1) = S_\sigma$, and

$$(1.8) \qquad T_\sigma(n) = C_\sigma^0(n) \setminus \overline{C_\sigma^0(n-1)}$$

for $2 \leqslant n \leqslant d$, where $\overline{C_\sigma^0(n-1)}$ is the closure of $C_\sigma^0(n-1)$. We have $\overline{C_\sigma^0(n)} \supset \overline{C_\sigma^0(n-1)}$ for $2 \leqslant n \leqslant d$, since it is clear that $\overline{C_\sigma^0(n)} \supset C_\sigma^0(n-1)$. We write $C_\sigma^0 = C_\sigma^0(d)$ and $C_\sigma = \overline{C_\sigma^0}$. Let $\text{int } C_\sigma$ be the interior of the set C_σ in the relative topology on S^{d-1}. Note that in general, $C_\sigma \supset \bigcup_{n=1}^d T_\sigma(n)$ and $\text{int } C_\sigma \supset C_\sigma^0$. The set C_σ^0 is nonempty if $\{X_t\}$ is nondegenerate. These sets are introduced by Hiraba [12], when S_σ is a finite set.

In the rest of this section, we assume that $\{X_t\}$ is a nondegenerate α-stable process on $\mathbb{R}^d$ with $0 < \alpha < 2$ and $d \geqslant 1$. Thus $\{X_t\}$ is purely non-Gaussian and X_t has the probability density function $p(t, x)$ for $t > 0$. We write $p(1, x) = p(x)$. We denote by m the uniform probability measure on S^{d-1}. In the case where $0 < \alpha < 1$ and ν is one-sided, $p(x)$ can be zero, and thereby an additional condition $|x| > R$ with sufficiently large $R > 0$ is needed in the lower estimates of $p(x)$. Otherwise, there is no need for that condition. Thus we promise to omit such a condition in the lower estimates of $p(x)$. First we give general asymptotic estimates applicable for all α-stable densities.

Theorem 1.1. (i) *There is a constant $c_1 > 0$ such that*

$$(1.9) \qquad p(x) \leqslant c_1(1 + |x|)^{-(1+\alpha)}\sigma^*(1 + |x|) \qquad \text{for } x \in \mathbb{R}^d.$$

In particular, we have

$$(1.10) \qquad p(x) \leqslant c_1(1 + |x|)^{-(1+\alpha)} \qquad \text{for } x \in \mathbb{R}^d.$$

(ii) *Let $\xi^0 \in S^{d-1}$. In the case where $0 < \alpha < 1$ and ν is one-sided, we make an additional assumption that $\xi^0 \in \text{int } C_\sigma$. (Otherwise we make no additional assumption.) Then for any $\delta_1 > 0$, there is $c_2 > 0$ such that it is independent of ξ^0 and*

$$(1.11) \qquad p(r\xi^0 + y) \geqslant c_2(1 + r)^{-(1+\alpha)}\sigma_{\xi^0}(1 + r) \qquad \text{whenever } |y| \leqslant \delta_1.$$

(iii) *For any compact set K_1 in C_σ^0 and for any $\delta_2 > 0$, there is $c_3 > 0$ such that*

$$(1.12) \qquad p(x + y) \geqslant c_3(1 + |x|)^{-(1+\alpha)d} \qquad \text{whenever } x/|x| \in K_1, \ |y| \leqslant \delta_2.$$

(iv) *For any compact set $K_2 \subset S^{d-1}$ with $K_2 \cap C_\sigma = \emptyset$, for any $\delta_3 > 0$, and for any $c_0 > 0$, there is $c_4 > 0$ such that*

$$(1.13) \quad p(x + y) \leqslant c_4 \exp(-c_0|x| \log |x|) \qquad \text{whenever } x \neq 0, \ x/|x| \in K_2, \ |y| \leqslant \delta_3.$$

Pruitt and Taylor [28] showed the second assertion of (i) of Theorem 1.1 by a complicated Fourier-analytic method except in the case of second-class with $\alpha = 1$; in this case the estimate (1.10) has been unproven up to now. Our method of proof is more probabilistic and simpler. Further Port [24] used (1.10) for the classification of transient stable processes into weakly transient and strongly transient. Thus he could not classify the case of second-class with $\alpha = 1$ and $d = 2$, which is shown to be strongly transient by Sato and Watanabe [34]. Pruitt and Taylor [28] called the order $(1 + |x|)^{-(1+\alpha)}$ of the density $p(x)$ the worst possible case. Thus the order $(1+|x|)^{-(1+\alpha)d}$ should be called the best possible case in the case where $C_\sigma^0 = S^{d-1}$. The following theorem gives a proper asymptotic order of the density, which leads to a remark amazing even to the author.

Theorem 1.2. *Let $\phi \in \mathcal{D}$ and let K be a compact set in S_σ. Suppose that*

$$(1.14) \qquad \limsup_{r \to \infty} \sup_{\xi \in S^{d-1}} \frac{\sigma(\xi + B_{1/r})}{\phi(r)} < \infty \ \text{ and } \ \liminf_{r \to \infty} \inf_{\xi \in K} \frac{\sigma(\xi + B_{1/r})}{\phi(r)} > 0.$$

In the case where $0 < \alpha < 1$ and ν is one-sided, we make an additional assumption that $K \subset \operatorname{int} C_\sigma$. Then given $\delta > 0$, we can find $c_1 > 0$ and $c_2 > 0$ such that

$$(1.15) \qquad c_1(1 + |x|)^{-(1+\alpha)}\phi(1 + |x|) \leqslant p(x + y) \leqslant c_2(1 + |x|)^{-(1+\alpha)}\phi(1 + |x|)$$

whenever $x/|x| \in K$, $|y| \leqslant \delta$.

Remark 1.3. Let $d \geqslant 2$ and fix $\alpha \in (0,2)$ in the above theorem. Let β be an arbitrary real number in $(1 + \alpha, d + \alpha)$. Then we can choose the spectral measure σ in such a way that, for any $\delta > 0$, there is a compact set K in S^{d-1}, $c_1 > 0$, and $c_2 > 0$ such that

$$(1.16) \quad c_1(1 + |x|)^{-\beta} \leqslant p(x + y) \leqslant c_2(1 + |x|)^{-\beta} \quad \text{whenever } x/|x| \in K, \ |y| \leqslant \delta.$$

Indeed we can take σ as a so-called s-measure on S^{d-1} with $0 < s < d - 1$ such that $K = S_\sigma$ and $\beta = 1 + s + \alpha$ and that, for $c_3 > 0$ and $c_4 > 0$ independent of $x \in K$,

$$(1.17) \qquad c_3 a^s \leqslant \sigma(K \cap (x + B_a)) \leqslant c_4 a^s \qquad \text{for } x \in K \text{ and } a \in (0,1).$$

The existence of s-measure on S^{d-1} is clear from that in $\mathbb{R}^{d-1}$.

Now we can give interesting asymptotic estimates of $p(r\xi^0 + y)$ for $\xi^0 \in T_\sigma(n)$ with $1 \leq n \leq d$ as $r \to \infty$ in relation to the fractal dimensions of the spectral measure σ. The upper estimate (1.19) and lower one (1.21) are attained in the cases of Theorems 3.8 and 3.10, respectively.

Theorem 1.4. *Consider $\xi^0 \in T_\sigma(n)$ with $1 \leq n \leq d$. Let ϕ_1, $\phi_2 \in \mathcal{D}$.*
  (i) *Suppose that*

$$(1.18) \qquad\qquad \limsup_{r \to \infty} \sup_{\xi \in S^{d-1}} \frac{\sigma(\xi + B_{1/r})}{\phi_1(r)} < \infty.$$

Then, given $\delta_1 > 0$, we can find $c_1 > 0$ such that

$$(1.19) \qquad p(r\xi^0 + y) \leqslant c_1(1 + r)^{-(1+\alpha n)}\phi_1(1 + r) \qquad \text{whenever } r > 0, \ |y| \leqslant \delta_1.$$

  (ii) *Suppose that*

$$(1.20) \qquad\qquad \liminf_{r \to \infty} \frac{\sigma(\xi + B_{1/r})}{\phi_2(r)} > 0 \qquad \text{for any } \xi \in S_\sigma.$$

In the case where $0 < \alpha < 1$ and ν is one-sided, we make an additional assumption that $\xi^0 \in \operatorname{int} C_\sigma$. Then, given $\delta_2 > 0$, we can find $c_2 > 0$ such that

$$(1.21) \qquad p(r\xi^0 + y) \geqslant c_2 \left((1+r)^{-(1+\alpha)} \phi_2(1+r) \right)^n \qquad \text{whenever } |y| \leqslant \delta_2.$$

Next, we give some new estimates of $p(x)$, which include the well-known rotation-invariant case. The upper estimate (1.23) was conjectured by Pruitt and Taylor [28] under the assumption (1.22). Dziubański [7] showed the existence of $\lim_{r \to \infty} r^{d+\alpha} p(r\xi) = k(\xi)$ with some $k(\xi)$ in some special case. Further related results were obtained by Arkhipov [1], Głowacki and Hebisch [10], and Głowacki [9]. They also obtained (1.23) under certain additional assumptions. Namely, [1] assumed first-class and smoothness of the density of the spectral measure. On the other hand, [7], [9], and [10] assumed symmetry. The results of those papers are interesting in view of ours.

Theorem 1.5. (i) *If, for some $c_1 > 0$,*

$$(1.22) \qquad \sigma(d\xi) \leqslant c_1 m(d\xi) \qquad \text{on } S^{d-1},$$

then, for some $c_2 > 0$

$$(1.23) \qquad p(x) \leqslant c_2 (1 + |x|)^{-(d+\alpha)} \qquad \text{for } x \in \mathbb{R}^d.$$

The converse is also true except in the case where $0 < \alpha < 1$ and ν is one-sided.
  (ii) *Suppose that there is a nonempty compact set K in S^{d-1} such that*

$$(1.24) \qquad \liminf_{r \to \infty} \inf_{\xi \in K} \frac{\sigma(\xi + B_{1/r})}{r^{-(d-1)}} > 0.$$

In the case where $0 < \alpha < 1$ and ν is one-sided, we impose an additional condition that $K \subset \operatorname{int} C_\sigma$. Then, for any $\delta > 0$, there is $c_3 > 0$ such that

$$(1.25) \qquad p(x + y) \geqslant c_3 (1 + |x|)^{-(d+\alpha)} \qquad \text{whenever } x/|x| \in K, \ |y| \leqslant \delta.$$

Remark 1.6. The condition (1.24) is satisfied if there are a nonempty relatively open set U in S^{d-1}, a compact set K in $\overline{U}$ and $c_1 > 0$ such that

$$(1.26) \quad \sigma(d\xi) \geqslant c_1 m(d\xi) \qquad \text{on } U \quad \text{and} \quad \liminf_{a \to 0+} \inf_{\xi \in K} \frac{m((\xi + B_a) \cap U)}{a^{d-1}} > 0.$$

We prove the main results above in Section 3 after some preliminaries in Section 2. In Section 3, we add several other interesting results as follows. First we make a more precise observation of both the best possible and the worst possible cases. See Corollaries 3.4 and 3.5. Next we obtain the proper asymptotic order of the stable density $p(x + y)$ as $|x| \to \infty$ with y being bounded for all or most of all the directions of $x/|x|$ in the case where the spectral measure σ is absolutely continuous with respect to the uniform measure m on S^{d-1}, or discrete with the support S_σ being a finite set, or a mixture of those cases. See Theorems 3.8, 3.10 and 3.13. Then we apply those results to the moment of the last exit time from a ball and also to the Spitzer type limit theorems involving capacity in Section 4 for a second-class

transient stable process with $d \geqslant 2$ and $1 \leqslant \alpha < 2$. The reason why we study the order of $p(r\xi + y)$ not of $p(r\xi)$ for $\xi \in S^{d-1}$ is found in Lemma 4.5.

2. Preliminaries

In this section, we provide some preliminary results for Section 3. The support S_ρ of a measure ρ on $\mathbb{R}^d$ is the smallest closed set that carries the whole measure of ρ. We denote by ρ^{n*} n-th convolution power of the measure ρ. The distribution of an $\mathbb{R}^d$-valued random variable X is denoted by $\mathcal{L}(X)$. We write $\{X_t\} \stackrel{\mathrm{d}}{=} \{Y_t\}$ for two stochastic processes $\{X_t\}$ and $\{Y_t\}$ if they have an identical system of finite-dimensional joint distributions. For a Lévy process $\{X_t\}$, we denote $\mu = \mathcal{L}(X_1)$ and $\mu^t = \mathcal{L}(X_t)$. The support of μ^t is denoted by $S(X_t)$. If μ^t is absolutely continuous with the density $p(t, x)$ being continuous in x, then the set $\{x : p(t, x) > 0\}$ is denoted by $G(X_t)$.

Let $\{X_t\}$ be a nontrivial α-stable process on $\mathbb{R}^d$ with $0 < \alpha < 2$ and Lévy measure ν. Let $\psi(z)$ be the function defined in (1.2). If $0 < \alpha < 1$, then

$$(2.1) \qquad \psi(z) = \int_{\mathbb{R}^d} (e^{i\langle z, x \rangle} - 1)\nu(dx) + i\langle \gamma_0, z \rangle$$

where γ_0 is called the drift of $\{X_t\}$. If $1 < \alpha < 2$, then

$$(2.2) \qquad \psi(z) = \int_{\mathbb{R}^d} (e^{i\langle z, x \rangle} - 1 - i\langle z, x \rangle)\nu(dx) + i\langle \gamma_1, z \rangle$$

where γ_1 is called the center of $\{X_t\}$ and $\gamma_1 = EX_1$. We define $\tau \in \mathbb{R}^d$ as $\tau = \gamma_0$ for $0 < \alpha < 1$, $\tau = \gamma_1$ for $1 < \alpha < 2$, and $\tau = c \int_{S^{d-1}} \xi \sigma(d\xi)$ for $\alpha = 1$. Here c is the constant in (1.4). Note that the stable process $\{X_t\}$ is first-class if and only if $\tau = 0$. If $\{X_t\}$ is nondegenerate, then $\mathcal{L}(X_t)$ has the probability density function $p(t, x)$. We write $p(1, x) = p(x)$. Then $p(x)$ is of class C^∞ and satisfies the relation

$$(2.3) \qquad p(t, x) = \begin{cases} t^{-d/\alpha} p(t^{-1/\alpha} x + (1 - t^{1-1/\alpha})\tau) & (\alpha \neq 1), \\ t^{-d} p(t^{-1} x - (\log t)\tau) & (\alpha = 1). \end{cases}$$

A Lévy process $\{X_t\}$ on $\mathbb{R}^d$, $d \geqslant 1$, is called *semistable* if, for some $a > 0$ with $a \neq 1$, there are $b > 0$ and $c \in \mathbb{R}^d$ satisfying (1.3). We call a semistable Lévy process, briefly, a semistable process. In this case we can choose $a > 1$ without loss of generality. If $\{X_t\}$ is a nontrivial semistable process, then the index α can be defined in the same way as for nontrivial stable processes. See Sato [31], [32] concerning semistable processes.

A subset H of $\mathbb{R}^d$ is said to be a closed additive semigroup if H is a closed set such that $H + H \subset H$. If moreover $-H \subset H$, then H is called a closed additive group. It should be noted that a closed additive semigroup H in $\mathbb{R}^d$ is either one-sided or a closed additive group. See Sato and Watanabe [33]. Let $\{X_t\}$ be a nondegenerate α-stable process on $\mathbb{R}^d$ with $0 < \alpha \leqslant 2$ and Lévy measure ν. The set $\mathrm{Sgp}(\nu)$ is the smallest closed additive semigroup containing S_ν. We define a map Π from $\mathbb{R}^d \setminus \{0\}$ to S^{d-1} as $\Pi x = x/|x|$. Note that $C_\sigma = \Pi(\mathrm{Sgp}(\nu) \setminus \{0\})$. The following lemma is due to Taylor [38] and Port and Vitale [27]. See Tortrat [39], Sharpe [35], and Sato and Watanabe [34] concerning the supports of general Lévy processes and semistable processes.

Lemma 2.1. *Let $\{X_t\}$ be a nondegenerate α-stable process on $\mathbb{R}^d$ with $0 < \alpha \leqslant 2$.*
   (i) *Suppose that $0 < \alpha < 1$ and ν is one-sided. Then $\mathrm{Sgp}(\nu)$ is a one-sided, closed convex cone, and*

$$(2.4) \qquad S(X_t) = t\gamma_0 + \mathrm{Sgp}(\nu) \quad and \quad G(X_t) = t\gamma_0 + \mathrm{int}\,\mathrm{Sgp}(\nu) \quad for\ t > 0,$$

where $\mathrm{int}\,\mathrm{Sgp}(\nu)$ is the interior of the set $\mathrm{Sgp}(\nu)$.
   (ii) *Suppose that $1 \leqslant \alpha \leqslant 2$, or suppose that $0 < \alpha < 1$ and ν is not one-sided. Then*

$$(2.5) \qquad\qquad\qquad S(X_t) = G(X_t) = \mathbb{R}^d \quad for\ t > 0.$$

The following lemma was found by Taylor [38].

Lemma 2.2. *Let $\{X_t\}$ be a nondegenerate α-stable process on $\mathbb{R}^d$ with $0 < \alpha \leqslant 2$. Then the following conditions are equivalent:*

$$(2.6) \qquad\qquad\qquad 0 < \alpha < 1 \ and\ -\gamma_0 \notin \mathrm{int}\,\mathrm{Sgp}(\nu),$$
$$(2.7) \qquad\qquad\qquad p(t,0) = 0\ for\ all\ t > 0,$$
$$(2.8) \qquad\qquad\qquad p(t,0) = 0\ for\ some\ t > 0.$$

For the rest in this section let $\{X_t\}$ be a nondegenerate α-stable process on $\mathbb{R}^d$ with $0 < \alpha < 2$. Our basic technique is to decompose $\{X_t\}$ into the sum of independent Lévy processes $\{Y_t\}$ and $\{Z_t\}$. That is, let $\{Y_t\}$ and $\{Z_t\}$ be independent Lévy processes such that $\{X_t\} \overset{\mathrm{d}}{=} \{Y_t + Z_t\}$ and $\{Z_t\}$ is a compound Poisson process with Lévy measure ν_Z equal to ν restricted to $\{|x| > \theta\}$ for some $\theta > 0$. Thus the Lévy measure ν_Y of $\{Y_t\}$ is the restriction of ν to $\{|x| \leqslant \theta\}$. Let $\mathcal{L}(Y_1) = \mu_Y$ and $\mathcal{L}(Z_1) = \mu_Z$. Since $|\log\widehat{\mu}_Z(z)|$ is bounded, we get $|\widehat{\mu}_Y^t(z)| \leqslant e^{-ct|z|^\alpha + c_1 t}$ with some $c > 0$ and $c_1 > 0$ as in Proposition 24.20 of Sato [32]. Thus $\mathcal{L}(Y_t)$, $t > 0$, has a density $p_Y(t,x)$, which is continuous in (t,x), of class C^∞ in x, and bounded for $(t,x) \in [\varepsilon_1, \varepsilon_2] \times \mathbb{R}^d$ for every $0 < \varepsilon_1 < \varepsilon_2 < \infty$. We write $p_Y(t,x) = q(t,x)$ and $\mathcal{L}(Z_t) = \mu_Z^t = \lambda^t$. Further we write $q(x) = q(1,x)$ and $\lambda = \lambda^1$. As before, $B_r = \{x \colon |x| < r\}$; thus $\overline{B}_r = \{x \colon |x| \leqslant r\}$ and $B_r + y = \{x \colon |x - y| < r\}$. The following fact will be useful. See Sato and Watanabe [34] for the proof.

Lemma 2.3. *Let $\{X_t\}$ be a nondegenerate α-stable process on $\mathbb{R}^d$, $0 < \alpha < 2$, with Lévy measure ν. Then $S(Y_t) = S(X_t)$ and $G(Y_t) = G(X_t)$ for $t > 0$.*

Lemma 2.4. *Let $c_0 > 0$. If θ in the above satisfies that $0 < \theta < 1/(2c_0)$, then there is c_1 such that*

$$(2.9) \qquad\qquad q(x) \leqslant c_1 \exp(-c_0|x|\log|x|) \qquad for\ |x| > 0.$$

Proof. Theorem 26.1 of [32] tells us that

$$\int_{|y|>r} q(1/2,y)dy \leqslant c_3 \exp(-2(c_0 + \delta)r\log r) \qquad for\ r > 0$$

with some positive constants c_3 and δ. Let $c_2 = \sup_{x \in \mathbb{R}^d} q(1/2, x)$. Now,

$$
\begin{aligned}
q(x) &= \int_{|x-y| \leqslant |x|/2} q(1/2, x-y) q(1/2, y) dy + \int_{|x-y| > |x|/2} q(1/2, x-y) q(1/2, y) dy \\
&\leqslant 2c_2 \int_{|y| \geqslant |x|/2} q(1/2, y) dy \\
&\leqslant 2c_2 c_3 \exp(-(c_0 + \delta)|x| \log(|x|/2)) \\
&\leqslant c_4 \exp(-c_0 |x| \log |x|)
\end{aligned}
$$

for large $|x|$ with some positive constant c_4. This gives (2.9). $\square$

The following simple lemma, together with Lemma 3.1, will be a key to prove the main results. All the estimates below are carried over under the condition that θ is given and fixed.

Lemma 2.5. *Let φ_1 and φ_2 be nonnegative decreasing functions on $(0, \infty)$ with $\varphi_1 \in \mathcal{D}$ and let K_1 and K_2 be compact sets in S^{d-1}.*

(i) Asssume that there are $c_1 > 0$ and $\delta_0 > 0$ such that if $x \neq 0$ and z satisfy $x/|x| \in K_1$ and $|z| \leqslant \delta_0 |x|$, then

$$(2.10) \qquad\qquad \lambda(x + z + B_1) \leqslant c_1 \varphi_1(1 + |x|).$$

Then, for any $\delta_1 > 0$, there is $c_2 > 0$ such that if $x \neq 0$ and y satisfy $x/|x| \in K_1$ and $|y| \leqslant \delta_1$, then

$$(2.11) \qquad\qquad p(x + y) \leqslant c_2 \varphi_1(1 + |x|).$$

(ii) In the case where $0 < \alpha < 1$ and ν is one-sided, we impose an additional condition that $K_2 \subset \operatorname{int} C_\sigma$. Otherwise we impose no additional condition on K_2. Assume that there are $a > 0$, $c_3 > 0$ and $R_1 > 0$ such that if $x/|x| \in K_2$ and $|x| \geqslant R_1$, then

$$(2.12) \qquad\qquad \lambda(x + B_a) \geqslant c_3 \varphi_2(1 + |x|).$$

Then, for any $\delta_2 > 0$, there are $c_4 > 0$ and $R_2 > 0$ such that if $x/|x| \in K_2$, $|x| \geqslant R_2$, and $|y| \leqslant \delta_2$, then

$$(2.13) \qquad\qquad p(x + y) \geqslant c_4 \varphi_2(1 + |x|).$$

Proof. (i) Since $p(t, x) = \int q(t, x - z) \lambda^t(dz)$, we have

$$
\begin{aligned}
p(x+y) &= \int q(-z) \lambda(x + y + dz) \\
&= \int_{|z| \leqslant \delta_0 |x|/2} q(-z) \lambda(x + y + dz) + \int_{|z| > \delta_0 |x|/2} q(-z) \lambda(x + y + dz).
\end{aligned}
$$

Let J_1 and J_2 be the first and the second term in the last expression. We can choose x_j, $j = 1, \ldots, N = N(x)$, with $|x_j| \leqslant \delta_0 |x|/2$ such that $\{z : |z| \leqslant \delta_0 |x|/2\} \subset \bigcup_{j=1}^N (B_1 + x_j)$, and there is $c_5 > 0$ independent of N and x satisfying $\sum_{j=1}^N Q_j \leqslant c_5$ for $Q_j = \sup_{z \in B_1 + x_j} q(-z)$. This is possible by Lemma 2.4. Let $R = 2\delta_1/\delta_0$. If $x/|x| \in K_1$, $|x| \geqslant R$, and $|y| \leqslant \delta_1$, then

$$
\begin{aligned}
|y + x_j| &\leqslant |y| + |x_j| \leqslant (|y|/|x|)|x| + (\delta_0/2)|x| \leqslant \delta_0 |x|, \\
\lambda(x + y + x_j + B_1) &\leqslant c_1 \varphi_1(1 + |x|)
\end{aligned}
$$

by (2.10). Hence

$$J_1 \leqslant \sum_{j=1}^{N} Q_j \lambda(x + y + x_j + B_1) \leqslant c_1 c_5 \varphi_1(1 + |x|).$$

On the other hand, by Lemma 2.4 and $\varphi_1 \in \mathcal{D}$,

$$J_2 \leqslant \sup_{|z| > \delta_0 |x|/2} q(-z) \leqslant c_6 \exp(-c_0(\delta_0|x|/2) \log(\delta_0|x|/2)) \leqslant c_7 \varphi_1(1 + |x|)$$

with some c_0, c_6, and c_7. Thus we get (2.11) under the condition $|x| \geqslant R$. Hence it holds without this condition with a suitable choice of c_2.

(ii) First, suppose that $1 \leqslant \alpha < 2$, or suppose that $0 < \alpha < 1$ and ν is not one-sided. We have

$$p(x + y) \geqslant \int_{B_a} q(y - z) \lambda(x + dz).$$

Let $c_8 = \inf_{|y| \leqslant \delta_2, |z| < a} q(y - z)$. Then $c_8 > 0$ by Lemma 2.1, Lemma 2.3, and the continuity of $q(\cdot)$. Thus, if $x/|x| \in K_2$, $|x| \geqslant R_1$, and $|y| \leqslant \delta_2$, then

$$p(x + y) \geqslant c_8 \lambda(x + B_a) \geqslant c_8 c_3 \varphi_2(1 + |x|).$$

Next, suppose that $0 < \alpha < 1$, ν is one-sided and $K_2 \subset \operatorname{int} C_\sigma$. Recall that $q(z) > 0$ for $z \in \gamma_0 + \operatorname{int} \operatorname{Sgp}(\nu)$ (Lemmas 2.1 and 2.3) and that

$$(2.14) \qquad\qquad \operatorname{Sgp}(\nu) = \Pi^{-1} C_\sigma \cup \{0\}$$

(the closedness of $\operatorname{Sgp}(\nu)$ under multiplication by positive reals in Lemma 2.1). Choose $r_0 > 0$ so large that $K' = r_0 K_2 + \overline{B}_{a+\delta_2+|\gamma_0|} \subset \operatorname{int} \operatorname{Sgp}(\nu)$. Let $c_9 = \inf_{z \in \gamma_0 + K'} q(z)$. Then $c_9 > 0$. If $x/|x| \in K_2$, $|x| \geqslant R_1 + r_0$, and $|y| \leqslant \delta_2$, then

$$\begin{aligned}
p(x + y) &\geqslant \int_{z \in -r_0 K_2 + B_a} q(y - z) \lambda(x + dz) \geqslant c_9 \lambda(x - r_0 K_2 + B_a) \\
&\geqslant c_9 \lambda(x - r_0 x/|x| + B_a) = c_9 \lambda((|x| - r_0)x/|x| + B_a) \\
&\geqslant c_9 c_3 \varphi_2(1 + (|x| - r_0)) \geqslant c_{10} \varphi_2(1 + |x|)
\end{aligned}$$

with some c_{10}. $\square$

Remark 2.6. Picard [22] studied the density estimates in small time for jump processes. In the case of first-class stable processes, his results are essentially similar to the statements of Lemma 2.5 because of the relation (2.3) with $\tau = 0$. However, the arguments in Lemma 2.7 below are not therein, and thereby the relation to the spectral measure is not discussed in [22].

Let l be a positive integer. For $\xi^0, \xi_1, \ldots, \xi_l \in S^{d-1}$ (not necessarily distinct), $r > 0$, $y \in \mathbb{R}^d$, let

$$(2.15) \qquad I = I(\xi_1, \ldots, \xi_l; r, \xi^0, y) = \int_{(\theta, \infty)^l} 1_{r\xi^0 + y + B_1} \left(\sum_{j=1}^{l} r_j \xi_j \right) \prod_{j=1}^{l} \frac{dr_j}{r_j^{1+\alpha}}$$

and

$$(2.16) \qquad J = J(l, r, \xi^0, y) = \int_{(S^{d-1})^l} I(\xi_1, \ldots, \xi_l; r, \xi^0, y) \prod_{j=1}^{l} \sigma(d\xi_j).$$

Note that $\nu_Z^{l*}(r\xi^0 + y + B_1) = c^l J(l, r, \xi^0, y)$ with c being the constant in (1.4). Denote the smallest linear subspace containing $\xi_1, \ldots, \xi_l$ by $\text{Span}\{\xi_1, \ldots, \xi_l\}$. Define the set Ξ_n with $1 \le n \le d$ as

$$(2.17) \qquad \begin{aligned} \Xi_n = \{&(\xi_1, \ldots, \xi_n) : \xi_j \in S^{d-1}, \\ &j = 1, \ldots, n, \text{ such that } \xi_1, \ldots, \xi_n \text{ are linearly independent}\}. \end{aligned}$$

For $(\xi_1, \ldots, \xi_n) \in \Xi_n$, let P be a isometric linear mapping from $\text{Span}\{\xi_1, \ldots, \xi_n\}$ to $\mathbb{R}^n$ and define $D(\xi_1, \ldots, \xi_n) = |\det(P\xi_1, \ldots, P\xi_n)|$ where $\det(P\xi_1, \ldots, P\xi_n)$ is the determinant considering each $P\xi_j$ as a column vector in $\mathbb{R}^n$. Obviously $D(\xi_1, \ldots, \xi_n)$ does not depend on the choice of the mapping P. Define

$$(2.18) \qquad\qquad A_n = \int_{\Xi_n} \frac{1}{D(\xi_1, \ldots, \xi_n)} \prod_{j=1}^{n} \sigma(d\xi_j).$$

Next we give some upper estimates of the integral J for $\xi^0 \in T_\sigma(n)$ with $1 \le n \le d$.

Lemma 2.7. *Given $\xi^0 \in T_\sigma(n)$ with $1 \le n \le d$, we can find $0 < \delta < 1$ and $R > 0$ with the following properties:*

(i) If $\xi_1, \ldots, \xi_l \in S_\sigma$ and $\dim \text{Span}\{\xi_1, \ldots, \xi_l\} \le n - 1$, then $I = 0$ whenever $|y| \le \delta r$ and $r \ge R$.

(ii) There are constants $b \ge 0$, $c_1 > 0$ and $c_2 > 0$ determined by α, θ, δ, and n, such that, if $\xi_1, \ldots, \xi_l \in S_\sigma$ and $\dim \text{Span}\{\xi_1, \ldots, \xi_l\} \ge n$, then

$$(2.19) \quad J \le c_1 \binom{l}{n} l^{(1+\alpha)n} \alpha^{-l} \theta^{-\alpha l} r^{-(1+\alpha)n} A_n \quad \text{whenever } |y| \le \delta r/2 \text{ and } r \ge R$$

and

$$(2.20) \quad J \le c_2 \binom{l}{n} l^{\alpha n + b} \alpha^{-l} \theta^{-\alpha l} r^{-(1+\alpha n)} \sigma^*(r) \quad \text{whenever } |y| \le \delta r/2 \text{ and } r \ge R.$$

Proof. Given ξ^0 as above, choose $\delta > 0$ so small and R so large that, if $|y| \le \delta r$ and $r \ge R$, then $(r\xi^0 + y + B_1) \cap \Pi^{-1}\overline{C_\sigma^0(n-1)} = \emptyset$. This is possible because, for $x = r\xi^0 + y + z$ with $|y| \le r\delta$ and $|z| < 1$, we have $r(1-\delta) - 1 \le |x| \le r(1+\delta) + 1$ and

$$\left| \frac{x}{|x|} - \xi_0 \right| = \frac{|(r - |x|)\xi^0 + y + z|}{|x|} \le \frac{2(r\delta + 1)}{r(1-\delta) - 1} \le \frac{2(\delta + R^{-1})}{1 - \delta - R^{-1}},$$

and $\xi^0 \notin \overline{C_\sigma^0(n-1)}$.

(i) Assume that $\dim \text{Span}\{\xi_1, \ldots, \xi_l\} \le n - 1$. We claim that, if $w = \sum_{j=1}^{l} r_j \xi_j$ with $r_1, \ldots, r_l > 0$, then $w \in \Pi^{-1} \bigcup_{k=1}^{n-1} C_\sigma^0(k) \cup \{0\}$. Indeed, if $\xi_1, \ldots, \xi_l$ are linearly independent, then this is clear; if $\xi_1, \ldots, \xi_l$ are linearly dependent, then we can find (after change of numbering if necessary) $b_1, \ldots, b_l$ satisfying $\sum_{j=1}^{l} b_j r_j \xi_j = 0$ and $b_l = 1 \ge b_j$ for $1 \le j \le l - 1$, which implies that w is a linear combination with positive coefficients of some $\{\xi_1', \ldots, \xi_{l'}'\} \subsetneq \{\xi_1, \ldots, \xi_l\}$. We can continue this procedure until w is expressed as a linear combination with positive coefficients of some linearly independent subsystem of $\{\xi_1, \ldots, \xi_l\}$. Now we see that $I = 0$ from the definition of I.

(ii) Let $|y| \le \delta r/2$ and $r \ge R$. Suppose that $w = \sum_{j=1}^{l} r_j \xi_j \in r\xi^0 + y + B_1$. We may assume that there is l_1 such that $r_j \ge \delta r/(2l)$ for $j = 1, \ldots, l_1$ and $r_j < \delta r/(2l)$ for $j = l_1 + 1, \ldots, l$ after rearrangement of $\xi_1, \ldots, \xi_l$ if necessary. Write $y_2 = \sum_{j=l_1+1}^{l} r_j \xi_j$, $y_1 = y - y_2$, and $w' = w - y_2 = \sum_{j=1}^{l_1} r_j \xi_j$. Then

$|y_2| \leqslant \delta r/2$, $|y_1| \leqslant |y| + |y_2| \leqslant \delta r$, and $w' \in r\xi^0 + y + B_1 - y_2 = r\xi^0 + y_1 + B_1$. Thus, $w' \notin \Pi^{-1}\overline{C_\sigma^0(n-1)}$. Now the property shown in the proof of (i) tells us that $\dim \mathrm{Span}\{\xi_1, \ldots, \xi_{l_1}\} \geqslant n$. Hence $l_1 \geqslant n$. It follows that I is not larger than

$$\binom{l}{n}\left(\frac{\delta r}{2l}\right)^{-(1+\alpha)n}\int_{(\theta,\infty)^{l-n}}\left(\prod_{j=n+1}^{l}\frac{dr_j}{r_j^{1+\alpha}}\right)\int_{(\theta,\infty)^n}1_{r\xi^0+y_1+B_1}\left(\sum_{j=1}^{l_1}r_j\xi_j\right)\prod_{j=1}^{n}dr_j.$$

Denote by V_n the volume of the n-dimensional unit ball. We have

$$\int_{(\theta,\infty)^n}1_{r\xi^0+y_1+B_1}\left(\sum_{j=1}^{l_1}r_j\xi_j\right)\prod_{j=1}^{n}dr_j \leqslant \frac{V_n}{D(\xi_1,\ldots,\xi_n)}$$

when r_j $(n+1 \leqslant j \leqslant l)$ are fixed. Therefore, noting $V_n \leqslant 2^n$,

$$J \leqslant \binom{l}{n}\left(\frac{\delta r}{2l}\right)^{-(1+\alpha)n}\alpha^{-(l-n)}\theta^{-\alpha(l-n)}2^n A_n,$$

which is written as (2.19). On the other hand, since $l_1 \geqslant n$ it follows that I is not larger than

$$\binom{l}{n}\alpha^{-(n-1)}\left(\frac{\delta r}{2l}\right)^{-\alpha(n-1)}\int_{(\theta,\infty)^{l-n}}\left(\prod_{j=n+1}^{l}\frac{dr_j}{r_j^{1+\alpha}}\right)\sup_{a\in\mathbb{R}^d}\int_{(\delta r)/(2l)}^{\infty}1_{a+B_1}(r_1\xi_1)\frac{dr_1}{r_1^{1+\alpha}}.$$

We have by (1.6)

$$\sup_{a\in\mathbb{R}^d}\int_{S^{d-1}}\sigma(d\xi_1)\int_{(\delta r)/(2l)}^{\infty}1_{a+B_1}(r_1\xi_1)\frac{dr_1}{r_1^{1+\alpha}}$$

$$=r^{-\alpha}\sup_{a'\in\mathbb{R}^d}\int_{S^{d-1}}\sigma(d\xi_1)\int_{\delta/(2l)}^{\infty}1_{a'+B_{1/r}}(u\xi_1)\frac{du}{u^{1+\alpha}}$$

$$\leqslant r^{-\alpha}\sup_{\xi\in S^{d-1}}\sigma(\xi+B_{\frac{2l}{\delta r}})\int_{\delta/(2l)}^{\delta/(2l)+2/r}\frac{du}{u^{1+\alpha}}$$

$$\leqslant c_3\alpha^{-1}\left(\frac{2l}{\delta}\right)^{\alpha}r^{-\alpha-1}\sigma^*\left(\frac{\delta r}{2l}\right) \leqslant c_4 l^{\alpha+b}r^{-\alpha-1}\sigma^*(r)$$

with positive constants b, c_3 and c_4, which implies (2.20). $\qquad\square$

Remark 2.8. Let K be a compact set in $T_\sigma(n)$. Then we see from the proof above the constants c_1 and c_2 can be taken uniformly with respect to $\xi^0 \in K$.

3. Proof of the main results

In this section, we prove the results mentioned in Section 1 and their corollaries and remarks by using the preliminaries in Section 2. Then we present some additional results which give the proper asymptotic orders of the stable densities. We continue to use the notation of Sections 1 and 2.

Lemma 3.1. (i) *There is a constant $c_1 > 0$ such that*

$$(3.1) \qquad \lambda(x+B_1) \leqslant c_1(1+|x|)^{-(1+\alpha)}\sigma^*(1+|x|) \qquad \text{for } x \in \mathbb{R}^d.$$

(ii) *Let $\xi^0 \in S^{d-1}$. There are constants $c_2 > 0$ and $R > 0$ independent of $\xi^0 \in S^{d-1}$ such that*

$$(3.2) \qquad \lambda(r\xi^0+B_2) \geqslant c_2(1+r)^{-(1+\alpha)}\sigma_{\xi^0}(1+r) \qquad \text{for } r \geqslant R.$$

Proof. (i) Recall (1.4) and see that $\nu_Z(dx) = 1_{\{|x| > \theta\}}(x)\nu(dx)$. Then, for $n \geqslant 1$ and $|x| \geqslant 2$,

$$\nu_Z^{n*}(x + B_1) = c^n \int_{(S^{d-1})^n \times (\theta, \infty)^n} 1_{x + B_1} \left(\sum_{j=1}^n r_j \xi_j \right) \prod_{k=1}^n r_k^{-(1+\alpha)} \sigma(d\xi_k) dr_k.$$

If $\sum_{j=1}^n r_j \xi_j \in x + B_1$, then $r_l \geqslant |x|/(2n)$ for some l with $1 \leqslant l \leqslant n$. Thus we see from (1.6) that

$$\nu_Z^{n*}(x + B_1) \leqslant c^n \alpha^{-(n-1)} \theta^{-\alpha(n-1)} \sup_{a \in \mathbb{R}^d} \int_{S^{d-1}} \sigma(d\xi_l) \int_{|x|/(2n)}^\infty 1_{a + B_1} (r_l \xi_l) \frac{dr_l}{r_l^{1+\alpha}}$$

$$= c^n \alpha^{-(n-1)} \theta^{-\alpha(n-1)} |x|^{-\alpha} \sup_{a' \in \mathbb{R}^d} \int_{S^{d-1}} \sigma(d\xi_l) \int_{1/(2n)}^\infty 1_{a' + B_{1/|x|}} (u\xi_l) \frac{du}{u^{1+\alpha}}$$

$$\leqslant c^n \alpha^{-(n-1)} \theta^{-\alpha(n-1)} |x|^{-\alpha} \sup_{\xi \in S^{d-1}} \sigma(\xi + B_{\frac{2n}{|x|}}) \int_{1/(2n)}^{1/(2n) + 2/|x|} \frac{du}{u^{1+\alpha}}$$

$$\leqslant c_3 c^n \alpha^{-n} \theta^{-\alpha(n-1)} (2n)^\alpha |x|^{-\alpha-1} \sigma^* \left(\frac{|x|}{2n} \right)$$

$$\leqslant c_4 c^n \alpha^{-n} \theta^{-\alpha(n-1)} (2n)^{\alpha+b} (1 + |x|)^{-\alpha-1} \sigma^*(1 + |x|)$$

with some positive constants c_3 and c_4. Hence there is c_5 such that

$$\lambda(x + B_1) \leqslant \sum_{n=1}^\infty e^{-a}(n!)^{-1} \nu_Z^{n*}(x + B_1) \leqslant c_5(1 + |x|)^{-(1+\alpha)} \sigma^*(1 + |x|) \qquad \text{for } |x| \geqslant 2,$$

where $a = \nu_Z(\mathbb{R}^d)$. This proves (3.1) with some c_1.

(ii) Let $r > 2\theta + 2$. There is $c_6 > 0$ independent of $\xi^0 \in S^{d-1}$ such that

$$\lambda(r\xi^0 + B_2) \geqslant e^{-a} \nu_Z(r\xi^0 + B_2)$$

$$= e^{-a} \int_{S^{d-1}} \sigma(d\xi_1) \int_\theta^\infty 1_{r\xi^0 + B_2} (r_1 \xi_1) \frac{dr_1}{r_1^{1+\alpha}}$$

$$= e^{-a} r^{-\alpha} \int_{S^{d-1}} \sigma(d\xi_1) \int_{\theta/r}^\infty 1_{\xi^0 + B_{2/r}} (u\xi_1) \frac{du}{u^{1+\alpha}}$$

$$\geqslant c_6 e^{-a} \alpha^{-1} r^{-\alpha-1} \sigma(\xi^0 + B_{1/r})$$

$$\geqslant c_6 e^{-a} \alpha^{-1} (1 + r)^{-\alpha-1} \sigma_{\xi^0}(1 + r),$$

which implies (3.2). $\qquad \square$

Proof of Theorem 1.1. (i) By Lemma 3.1 and (1.6) we can find $c_5 > 0$ and $c_6 > 0$ such that, for any $x, z \in \mathbb{R}^d$ with $|z| \leqslant |x|/2$,

$$\lambda(x + z + B_1) \leqslant c_5(1 + |x + z|)^{-(1+\alpha)} \sigma^*(1 + |x + z|) \leqslant c_6(1 + |x|)^{-(1+\alpha)} \sigma^*(1 + |x|).$$

Hence, applying Lemma 2.5 (i) to $K_1 = S^{d-1}$ and $y = 0$, we get (1.9). The second assertion is obviously true because $\sigma^*(1 + |x|) \leqslant 1$ for $x \in \mathbb{R}^d$.

(ii) By Lemma 3.1 we can find $c_7 > 0$ and $R_3 > 0$ such that

$$\lambda(r\xi^0 + B_2) \geqslant c_7(1 + r)^{-\alpha-1} \sigma_{\xi^0}(1 + r) \text{ for } r > R_3.$$

Hence, applying Lemma 2.5 (ii) to $K_2 = \{\xi^0\}$ we get (1.11).

(iii) We have $K_1 \subset C_\sigma^0$. Let us check the assumption in Lemma 2.5 (ii) with $\varphi_2(r) = r^{-d(1+\alpha)}$. Namely, we claim that there are $c_8 > 0$ and $R_4 > 0$ such that if $\xi \in K_1$ and $r \geqslant R_4$, then

$$(3.3) \qquad \lambda(r\xi + B_1) \geqslant c_8(1+r)^{-d(1+\alpha)}.$$

Let $\xi^0 \in K_1$. By (1.7) we can find a linearly independent system $\xi_1^0, \ldots, \xi_d^0$ in S_σ and positive reals $a_1^0, \ldots, a_d^0$ such that $\xi^0 = \sum_{j=1}^d a_j^0 \xi_j^0$. Choose a small relatively open neighborhood $U(\xi_j^0)$ of ξ_j^0 in S^{d-1} for each j in such a way that, for every $\xi_j \in U(\xi_j^0)$, $j = 1, \ldots, d$, there uniquely exist $a_j > 0$, $j = 1, \ldots, d$, satisfying $\xi^0 = \sum_{j=1}^d a_j \xi_j$ and that $a_1, \ldots, a_d$ continuously depend on $\{\xi_j\}$. We have $a_j = a_j^0$ for $\{\xi_j\} = \{\xi_j^0\}$. For $r > 2$ let

$$V_\varepsilon(r\xi^0) = \left\{ y = \sum_{j=1}^d s_j \xi_j \colon \xi_j \in U(\xi_j^0) \text{ and } a_j r - \varepsilon < s_j < a_j r + \varepsilon \text{ for } j = 1, \ldots, d \right\}.$$

Choose $\varepsilon > 0$ so small that $V_\varepsilon(r\xi^0) \subset r\xi^0 + B_1$. Then, with $a = \nu_Z(\mathbb{R}^d)$, there are c_9 and R_5 such that

$$\lambda(r\xi^0 + B_1) \geqslant e^{-a}(d!)^{-1}\nu_Z^{d*}(V_\varepsilon(r\xi^0))$$

$$\geqslant e^{-a}(d!)^{-1}c^d \int_{U(\xi_1^0)\times\cdots\times U(\xi_d^0)} \sigma(d\xi_1)\cdots\sigma(d\xi_d) \prod_{j=1}^d \int_{a_j r - \varepsilon}^{a_j r + \varepsilon} s_j^{-(1+\alpha)} ds_j$$

$$\geqslant c_9 r^{-d(1+\alpha)} \geqslant c_9(1+r)^{-d(1+\alpha)}$$

for all $r \geqslant R_5$. The set C_σ^0 is relatively open in S^{d-1}. On reflection we see that there is a small relatively open neighborhood $W(\xi^0)$ of ξ^0 in S^{d-1} such that we can find c_{10} and R_6 satisfying

$$\lambda(r\xi + B_1) \geqslant c_{10}(1+r)^{-d(1+\alpha)}$$

for all $\xi \in W(\xi^0)$ and $r \geqslant R_6$. By compactness of K_1 we see (3.3). Now we have only to apply Lemma 2.5 (ii).

(iv) This is a one-sided case. Given K_2, c_0, and δ_3 with $K_2 \cap C_\sigma = \emptyset$, we have to show (1.13). Let δ be the distance between K_2 and C_σ. We have $0 < \delta < 2$. Choosing $0 < \theta < \delta/(8c_0)$, we will apply Lemma 2.4. Let $x/|x| \in K_2$ and $|y| \leqslant \delta_2$. As in the proof of Lemma 2.5 (i),

$$p(x+y) = \int_{|z|\leqslant\delta|x|/4} q(-z)\lambda(x+y+dz) + \int_{|z|>\delta|x|/4} q(-z)\lambda(x+y+dz).$$

Denote by J_1 and J_2 the first and the second integral. As before, we have $J_1 \leqslant \sum_{j=1}^N Q_j\lambda(x+y+x_j+B_1)$ where $|x_j| \leqslant \delta|x|/4$. But $(x+y+x_j+B_1)\cap\Pi^{-1}C_\sigma = \emptyset$ uniformly in y and x_j when $|x|$ is large. Indeed, letting $|y - y'| < 1$,

$$|\Pi(x+y'+x_j) - \Pi x| \leqslant ||x+y'+x_j|^{-1} - |x|^{-1}| \, |x| + |x+y'+x_j|^{-1}|y'+x_j|$$

$$\leqslant 2|x+y'+x_j|^{-1}|y'+x_j|$$

$$\leqslant (|x| - \delta|x|/4 - |y'|)^{-1}(\delta|x|/2 + 2|y'|) \to (1 - \delta/4)^{-1}\delta/2 < \delta$$

as $|x| \to \infty$. It follows from (2.14) that $J_1 = 0$ when $|x|$ is large. Now we get (1.13) for large $|x|$ from Lemma 2.4. If we take the constant c_4 large enough, it holds without restriction that $|x|$ is large. $\qquad \square$

Remark 3.2. A nondegenerate α-semistable process with $0 < \alpha < 2$ on $\mathbb{R}^d$ does not satisfy (1.10) in general. We give such an example for $d = 1$. However, it is noteworthy that the description of the finiteness of the moment of the last exit time from a ball in Corollary 2.7 of Sato and Watanabe [34] in one-dimensional case is true for all semistable processes as well as stable processes.

Proof. Let $d = 1$ and $0 < \alpha < 2$. We give an example of an α-semistable process on $\mathbb{R}$ which does not satisfy (1.10). Let $a > 1$ and $\nu = \sum_{n=-\infty}^{\infty} a^{-n} \delta_{a^{n/\alpha}}(dx)$ be the process' Lévy measure, where $\delta_{a^{n/\alpha}}$ is the δ-measure at $a^{n/\alpha}$. That is, ν is a discrete one-sided measure. In the case of $0 < \alpha < 1$, we assume that the drift of the process is nonpositive. Then the associated α-semistable process does not satisfy (1.10). Indeed, choose $\theta = 1$ in the same decomposition $\{X_t\} \stackrel{\mathrm{d}}{=} \{Y_t + Z_t\}$ as for the stable case. Then $c = \nu_Z(\mathbb{R}) = 1/(a-1)$ and, letting $0 < \varepsilon < 1$ and $n \geqslant 1$, we have $\lambda((a^{n/\alpha} - \varepsilon, a^{n/\alpha} + \varepsilon/2)) \geqslant e^{-c} c a^{-n}$. Note that the density $q(x) = q(1,x)$ of $\mathcal{L}(Y_1)$ is continous and positive on $[0, \infty)$. See Proposition 3.7 of [34]. Thus

$$\int_{a^{n/\alpha} - \varepsilon}^{a^{n/\alpha} + \varepsilon} p(x) dx \geqslant \int_0^{\varepsilon/2} q(y) dy \lambda((a^{n/\alpha} - \varepsilon, a^{n/\alpha} + \varepsilon/2)) \geqslant c_2 a^{-n}$$

with some $c_2 > 0$. Hence, if (1.10) holds, we have

$$0 < \liminf_{n \to \infty} a^n \int_{a^{n/\alpha} - \varepsilon}^{a^{n/\alpha} + \varepsilon} p(x) dx \leqslant \liminf_{n \to \infty} a^n c_1 (a^{n/\alpha} - \varepsilon + 1)^{-1-\alpha} 2\varepsilon = 0,$$

which is absurd. $\square$

Remark 3.3. Let $d \geqslant 2$.

(i) Suppose that there are $\xi^0 \in S_\sigma$ and $c_1 > 0$ such that

(3.4) $\sigma^*(1 + r) \leqslant c_1 \sigma_{\xi^0}(1 + r)$ for $r > 0$.

Then, given $\delta_1 > 0$ there are $c_2 > 0$ and $c_3 > 0$ such that

(3.5) $c_2 (1 + r)^{-(1+\alpha)} \sigma_{\xi^0}(1 + r) \leqslant p(r\xi^0 + y) \leqslant c_3 (1 + r)^{-(1+\alpha)} \sigma_{\xi^0}(1 + r)$

whenever $|y| \leqslant \delta_1$.

(ii) Fix $s \in (0, d-1)$. Let $\ell(r)$ be slowly varying as $r \to \infty$ and define $\ell^*(r) = \int_r^\infty \ell(u) du/u$ in case $\int_1^\infty \ell(u) du/u < \infty$. Then we can choose σ such that it is absolutely continuous with respect to m, $\sigma_{\xi^0}(r)$ satisfies (3.4), and $\sigma_{\xi^0}(r) \asymp r^{-s} \ell(r)$ or $\sigma_{\xi^0}(r) \asymp \ell^*(r)$. In general, $A(t) \asymp B(t)$ means that there are constants $0 < c_3 \leqslant c_4 < \infty$ such that $c_3 B(t) \leqslant A(t) \leqslant c_4 B(t)$ for all sufficiently large t.

Proof. Assertion (i) is obvious from (i) and (ii) of Theorem 1.1. Next we prove Assertion (ii). Fix $s \in (0, d-1)$ and let $\ell(r)$ be slowly varying as $r \to \infty$. Let $h(u)$ be a nonnegative measurable function on $(0, 2]$ such that $h(0+) = \infty$ and $h(u)$ is bounded on $(\delta, 2]$ for every $\delta \in (0, 2)$. Then we can choose σ by $\sigma(d\xi) = h(|\xi - \xi^0|) m(d\xi)$ satisfying $h(1/r) \asymp r^{d-1-s} \ell(r)$. Then we have $\sigma_{\xi^0}(r) \asymp r^{-s} \ell(r)$. In the same way, if $\int_1^\infty \ell(u) du/u < \infty$, then we can set σ by $\sigma(d\xi) = h(|\xi - \xi^0|) m(d\xi)$ with $h(1/r) \asymp r^{d-1} \ell(r)$. Then we get $\sigma_{\xi^0}(r) \asymp \ell^*(r)$. Moreover, the measures σ fulfill (3.4) because $h(0+) = \infty$ and $h(u)$ is bounded on $(\delta, 2]$ for every $\delta \in (0, 2)$. $\square$

Next we observe more precisely both of the best possible and the worst possible cases as corollaries of Theorem 1.1.

Corollary 3.4. *Let $\xi^0 \in S_\sigma$. In the case where $0 < \alpha < 1$ and ν is one-sided, we make an additional assumption that $\xi^0 \in \operatorname{int} C_\sigma$. Suppose that $\sigma(\{\xi^0\}) > 0$. Then, given $\delta > 0$, we can find $c_1 > 0$ and $c_2 > 0$ such that*

$$(3.6) \qquad c_1(1+r)^{-(1+\alpha)} \leqslant p(r\xi^0 + y) \leqslant c_2(1+r)^{-(1+\alpha)} \qquad \text{whenever } |y| \leqslant \delta.$$

Proof. Since $\lim_{r\to\infty} \sigma_{\xi^0}(r) = \sigma(\{\xi^0\}) > 0$, the corollary is clear from (1.10) and (1.11). $\qquad\square$

Corollary 3.5. *Let $d \geqslant 2$. Consider a compact set $K \subset T_\sigma(d)$. Define the set Ξ_d as $\Xi_d = \{(\xi_1,\ldots,\xi_d) : \det(\xi_1,\ldots,\xi_d) \neq 0, \ \xi_j \in S^{d-1}, \ j = 1,\ldots,d\}$ where $\det(\xi_1,\ldots,\xi_d)$ is the determinant considering each ξ_j as a column vector in $\mathbb{R}^d$. Suppose that $\sigma = \sum_{j=1}^N \sigma_j$ with $N \geqslant d$ where $S_{\sigma_j} = S^{d-1} \cap C_j$ with C_j being a nonempty closed convex cone in $\mathbb{R}^d$ and $S_{\sigma_i} \cap S_{\sigma_j} = \emptyset$ for $i \neq j$. Further suppose that, for any set $\{k(j)\}_{j=1}^d$ of strictly increasing integers with $1 \leqslant k(j) \leqslant N$,*

$$(3.7) \qquad \int_{\Xi_d} \frac{1}{|\det(\xi_1,\ldots,\xi_d)|} \prod_{j=1}^d \sigma_{k(j)}(d\xi_j) < \infty.$$

Then, given $\delta_1 > 0$, we can find $c_1 > 0$ and $c_2 > 0$ such that

$$(3.8) \qquad c_1(1+|x|)^{-(1+\alpha)d} \leqslant p(x+y) \leqslant c_2(1+|x|)^{-(1+\alpha)d}$$

whenever $x/|x| \in K, \ |y| \leqslant \delta_1$.

Proof. The left-hand side inequality in (3.8) is true by (1.12) in Theorem 1.1 with $K_1 = K$. Let δ and R be those discovered in Lemma 2.7. Let $|y| \leqslant \delta r/2$ and $r \geqslant R$. By Lemma 2.7 (i),

$$\lambda(r\xi^0 + y + B_1) = \sum_{l=d}^\infty e^{-a}(l!)^{-1}\nu_Z^{l*}(r\xi^0 + y + B_1).$$

Let $n = d$ in the proof of Lemma 2.7 (ii). Then we can express w' by the convexity of C_j, $1 \leqslant j \leqslant N$, as

$$w' = \sum_{j=1}^{l_1} r_j\xi_j = \sum_{j=1}^{l_2} s_j\zeta_j$$

where $1 \leqslant l_2 \leqslant l_1$ and $s_j > 0$ for $1 \leqslant j \leqslant l_2$ and $\zeta_j \in S_{\sigma_{k(j)}}$ for strictly increasing integers $\{k(j)\}_{j=1}^{l_2}$ with $1 \leqslant k(j) \leqslant N$. Hence, by $w' \notin \Pi^{-1}(\overline{C_\sigma^0(d-1)})$, we have $l_1 \geqslant l_2 \geqslant d$ and we may assume that $\xi_j \in S_{\sigma_{k(j)}}$ for $1 \leqslant j \leqslant d$ with $\{k(j)\}_{j=1}^d$ as above. We have, for $l \geqslant d$,

$$\nu_Z^{l*}(r\xi^0 + y + B_1) = c^l J(l, r, \xi^0, y)$$

$$\leqslant c^l c_1 \binom{l}{d} l^{(1+\alpha)d}\alpha^{-l}\theta^{-\alpha l}r^{-(1+\alpha)d}A,$$

where c_1 is a positive constant independent of $\xi^0 \in K$ and A is the maximum of the integral in (3.7) in all possible $\{k(j)\}_{j=1}^d$. Here c is the constant in (1.4). Summing up in l, we get

$$\sup_{\xi^0 \in K} \lambda(r\xi^0 + y + B_1) \leqslant c_3 r^{-(1+\alpha)d} \leqslant c_2(1+r)^{-(1+\alpha)d}$$

with some c_3 and c_2. Hence, applying Lemma 2.5 (i) for $K_1 = K$ and $\delta_0 = \delta/2$, we get the right-hand side inequality in (3.8). $\qquad\square$

Proof of Theorem 1.2. The right-hand side inequality is true by (i) of Theorem 1.1 and (1.14). We show the left-hand side. Let $\xi^0 = \xi \in K$ in Lemma 3.1. Then we have

$$\liminf_{r \to \infty} \inf_{\xi^0 \in K} \frac{\lambda(r\xi^0 + B_2)}{(1+r)^{-(1+\alpha)}\phi(1+r)} \geqslant c_3 \liminf_{r \to \infty} \inf_{\xi^0 \in K} \frac{\sigma_{\xi^0}(1+r)}{\phi(1+r)}$$

with some $c_3 > 0$. It follows from (1.14) that there are $c_4 > 0$ and $R_1 > 0$ such that

$$\lambda(r\xi^0 + B_2) \geqslant c_4(1+r)^{-(1+\alpha)}\phi(1+r)$$

for $\xi^0 \in K$ and $r \geqslant R_1$. Thus, letting $K_2 = K$ and $\delta_2 = \delta$, we obtain from Lemma 2.5 (ii) that there are $c_1 > 0$ and $c_2 > 0$ such that the left-hand side inequality holds. $\square$

Remark 3.6. Let E be the classical Cantor set with Hausdorff dimension $\log 2/\log 3$ in [0,1]. We identify S^1 with $[0, 2\pi]$ by the usual map $\xi = \xi(\theta)$, $\theta \in [0, 2\pi)$. Let $d = 2$ and set $K = \xi(E)$ and $s = \log 2/\log 3$ in Remark 1.3. Then we have $C_\sigma = \xi([0, 1])$ and $\beta = 1 + \alpha + \log 2/\log 3$ and the following.

(i) In the case where $0 < \alpha < 1$, we make an additional assumption that $0 < \theta < 1$. If $\theta \in E$, then there are some positive constants c_1, c_2, and δ_1 such that

$$(3.9) \qquad c_1(1+r)^{-\beta} \leqslant p(r\xi(\theta) + y) \leqslant c_2(1+r)^{-\beta} \quad \text{whenever } |y| \leqslant \delta_1.$$

(ii) If $\theta \notin E$ and $0 < \theta < 1$, then there are some positive constants c_3, c_4, and δ_2 such that

$$(3.10) \qquad c_3(1+r)^{-2-2\alpha} \leqslant p(r\xi(\theta) + y) \leqslant c_4(1+r)^{-2-2\alpha} \quad \text{whenever } |y| \leqslant \delta_2.$$

the proof of assertion (ii) is similar to that of Corollary 3.5. In this example, the surface $\{(x, y) : y = p(x), \ x \in \mathbb{R}^2\}$ in $\mathbb{R}^3$ is waving infinitely many times when $|x|$ is sufficiently large. Thus it is a delicately difficult problem whether all multivariate stable distributions are unimodal in a certain sense as a natural extension of the univariate unimodality proved by Yamazato [42]. See also Kanter [17], Watanabe [40] and Wolfe [41] for the symmetric case.

Proof of Theorem 1.4. (i) Let δ and R be those discovered in Lemma 2.7. Let $|y| \leqslant \delta r/2$ and $r \geqslant R$. By Lemma 2.7 (i),

$$\lambda(r\xi^0 + y + B_1) = \sum_{l=n}^{\infty} e^{-a}(l!)^{-1} \nu_Z^{l*}(r\xi^0 + y + B_1).$$

We have as in the proof of (3.1), for $l \geqslant n$,

$$\nu_Z^{l*}(r\xi^0 + y + B_1) = c^l J(l, r, \xi^0, y)$$

$$\leqslant c^l c_3 \binom{l}{n} l^{\alpha n + b} \alpha^{-l} \theta^{-\alpha l} r^{-(1+\alpha n)} \sigma^*(r)$$

$$\leqslant c^l c_4 \binom{l}{n} l^{\alpha n + b} \alpha^{-l} \theta^{-\alpha l} (1+r)^{-(1+\alpha n)} \phi_1(1+r)$$

with some c_3 and c_4 by Lemma 2.7 (ii). Here c is the constant in (1.4). Summing up in l, we get

$$\lambda(r\xi^0 + y + B_1) \leqslant c_5(1+r)^{-(1+\alpha n)}\phi_1(1+r)$$

with some c_5. Thus by Lemma 2.5 (i) we have (1.19).

(ii) We express ξ^0 as $\xi^0 = \sum_{j=1}^n a_j \xi_j^0$ with $a_j > 0$ and $\xi_j^0 \in S_\sigma$ for $1 \leqslant j \leqslant n$. Then $\sum_{j=1}^n r_j \xi_j - r\xi^0 \in B_1$ if $|r_j - ra_j| < 1/(3n)$ and $|\xi_j - \xi_j^0| < (2(a_j+1)n(1+r))^{-1}$. Hence

$$\lambda(r\xi^0 + B_1) \geqslant e^{-a}(n!)^{-1}\nu_Z^{n*}(r\xi^0 + B_1)$$

$$\geqslant e^{-a}(n!)^{-1}c^n \left(\prod_{j=1}^n \sigma(\xi_j^0 + B_{\frac{1}{2(a_j+1)n(1+r)}})\right) \prod_{j=1}^n \int_{ra_j-1/(3n)}^{ra_j+1/(3n)} r_j^{-(1+\alpha)} dr_j$$

$$\geqslant c_6 \left((1+r)^{-(1+\alpha)}\phi_2(1+r)\right)^n$$

with some positive c_6 for $r \geqslant a_j^{-1}(\theta + 1/n)$, $1 \leqslant j \leqslant n$. It follows from Lemma 2.5 (ii) that, for some $c_4 > 0$, (1.21) holds.   □

Proof of Theorem 1.5. (i) Suppose that (1.22) holds. Then there are $c_3 > 0$ and $c_4 > 0$ such that

$$\sigma^*(r) = \sup_{\xi \in S^{d-1}} \sigma(\xi + B_{1/r}) \leqslant c_3 m(\xi + B_{1/r}) \leqslant c_4 r^{-(d-1)}.$$

Thus we have by Theorem 1.1 (i)

$$p(x) \leqslant c_2(1+|x|)^{-(1+\alpha)}(1+|x|)^{-(d-1)} = c_2(1+|x|)^{-(d+\alpha)}.$$

Let us prove the converse, excepting the case where $0 < \alpha < 1$ and ν is one-sided. Suppose that (1.23) holds. Then by Theorem 1.1 (ii) we see that, for any $\xi \in S^{d-1}$,

$$\liminf_{r\to\infty} \frac{\sigma(\xi + B_{1/r})}{r^{-(d-1)}} \leqslant c_5 \liminf_{r\to\infty} p(r\xi)r^{d+\alpha} \leqslant c_6$$

with some positive constants c_5 and c_6 independent of $\xi \in S^{d-1}$. Thus we obtain from Theorem 2.12 of Mattila [21] that σ is absolutely continuous with respect to m and the derivative $d\sigma/dm$ is bounded.

(ii) Let $\xi \in K$. For all large r, we have

$$\nu_Z(r\xi + B_1) = \nu(r\xi + B_1) = r^{-\alpha}\nu(\xi + B_{1/r})$$

$$= r^{-\alpha}c \int_{S^{d-1}} \sigma(d\xi') \int_0^\infty 1_{\xi+B_{1/r}}(u\xi')u^{-1-\alpha} du$$

$$\geqslant r^{-\alpha-1}c_7\sigma(\xi + B_{1/(2r)}) \geqslant c_8 r^{-(d+\alpha)}$$

with some $c_7 > 0$ and $c_8 > 0$, using (1.24). These constants do not depend on ξ in K. Hence there are $c_9 > 0$ and $R > 0$ such that

$$\lambda(r\xi + B_1) \geqslant e^{-a}\nu_Z(r\xi + B_1) \geqslant c_9(1+r)^{-(d+\alpha)} \qquad \text{for } \xi \in K \text{ and } r \geqslant R.$$

Therefore we get (1.25) from Lemma 2.5 (ii).   □

The following remark is of some interest.

Remark 3.7. Let $d \geqslant 2$ and $\delta > 0$.

(i) If the measure σ is continuous, that is, if $\sigma(\{\xi\}) = 0$ for any $\xi \in S^{d-1}$, then

$$(3.11) \quad \liminf_{r\to\infty} r^{1+\alpha}p(r\xi + y) = 0 \quad \text{for each } \xi \in S^{d-1} \text{ and every } y \text{ with } |y| \leqslant \delta.$$

The converse is also true except in the case where $0 < \alpha < 1$ and ν is one-sided.

(ii) Suppose that

$$(3.12) \quad \liminf_{r\to\infty} r^{d+\alpha}p(r\xi + y) = 0 \quad \text{for each } \xi \in S^{d-1} \text{ and every } y \text{ with } |y| \leqslant \delta.$$

Then we have $0 < \alpha < 1$, ν is one-sided and $S_\sigma \subset \partial C_\sigma$ where ∂C_σ is the boundary of the set C_σ in the relative topology on S^{d-1}, and thereby σ is singular with respect to m.

Proof. Assertion (i) is proved as follows. Assume that σ is continuous and

$$\lim_{r \to \infty} \sigma^*(r) > 0.$$

Then there are $c > 0$, $\xi_n \in S^{d-1}$ and $r_n > 0$ such that $\xi_n \to \xi \in S^{d-1}$ and $r_n \uparrow \infty$ as $n \to \infty$ and $\sigma_{\xi_n}(r_n) \geqslant c$ for any $n \geqslant 1$. Then we have a contradiction :

$$c \leqslant \lim_{k \to \infty} \limsup_{n \to \infty} \sigma_{\xi_n}(r_k) \leqslant \lim_{k \to \infty} \sigma_\xi(r_k/2) = \sigma(\{\xi\}) = 0.$$

Thus, if σ is continuous, then $\liminf_{r \to \infty} \sigma^*(r) = 0$ and thereby (3.11) holds from (1.9). If $\liminf_{r \to \infty} r^{1+\alpha} p(r\xi + y) = 0$ for each $\xi \in S^{d-1}$ and every y with $|y| \leqslant \delta$ except in the case where $0 < \alpha < 1$ and ν is one-sided, then $\sigma(\{\xi\}) = \liminf_{r \to \infty} \sigma_\xi(r) = 0$ for each $\xi \in S^{d-1}$ by (1.11).

(ii) Suppose that (3.12) holds for $d \geqslant 2$. Let $\phi_2(r) = r^{-(d-1)}$. We have, by using Theorem 2.12 of Mattila [21] as in the proof of (i) of Theorem 1.5, that $\sigma = 0$ except in the case where $0 < \alpha < 1$ and ν is one-sided. Next suppose that $0 < \alpha < 1$ and ν is one-sided. Then by Theorem 1.5 (ii) with $K = \{\xi\} \subset \text{int } C_\sigma$ we see that

$$\liminf_{r \to \infty} \frac{\sigma(\xi + B_{1/r})}{\phi_2(r)} = 0 \qquad \text{for any } \xi \in \text{int } C_\sigma.$$

Thus we again find from Theorem 2.12 of Mattila [21] that $\sigma = 0$ on int C_σ. The last assertion is clear from $m(\partial C_\sigma) = 0$. □

Finally we can give the proper asymptotic order of $p(x + y)$ as $|x| \to \infty$ when σ is absolutely continuous with respect to m, or when S_σ is a finite set, or when σ is the sum of such. Let U be a relatively open set in S^{d-1}. Taking Carathéodory's theorem into account again, we define

(3.13) $\quad C^0(U, n) = \{\xi \in S^{d-1} \colon \xi = \sum_{j=1}^{n} c_j \xi_j \text{ for some } c_j > 0 \text{ and } \xi_j \in U,$
$$j = 1, \ldots, n, \text{ such that } \xi_1, \ldots, \xi_n \text{ are linearly independent}\}$$

for $1 \leqslant n \leqslant d$, $T(U, 1) = C^0(U, 1) = U$, and

(3.14) $$T(U, n) = C^0(U, n) \setminus \overline{C^0(U, n-1)}$$

for $2 \leqslant n \leqslant d$. If $S_\sigma = \overline{U}$, then $\overline{C^0(U, n)} = \overline{C^0_\sigma(n)}$ and hence $T(U, n) \subset T_\sigma(n)$ for $1 \leqslant n \leqslant d$; further in general $\bigcup_{n=1}^{d} T(U, n) \subset \text{int } C_\sigma$.

Theorem 3.8. *Let U be a relatively open set in S^{d-1} and let K be a compact subset of U. Suppose that, for some $c_1 > 0$,*

(3.15) $$\sigma(d\xi) \leqslant c_1 m(d\xi) \qquad \text{on } U, \qquad \text{and } \sigma(U^c) = 0,$$

and

(3.16) $$\liminf_{r \to \infty} \inf_{\xi \in K} \frac{\sigma(\xi + B_{1/r})}{r^{-(d-1)}} > 0 \qquad \text{for any } K \subset U.$$

Assume that $\xi^0 \in T(U, n)$ with $1 \leqslant n \leqslant d$. Then, given $\delta > 0$, we can find $c_2 > 0$ and $c_3 > 0$ such that

(3.17) $\quad c_2(1 + r)^{-(d+\alpha n)} \leqslant p(r\xi^0 + y) \leqslant c_3(1 + r)^{-(d+\alpha n)} \quad \text{whenever } |y| \leqslant \delta.$

Proof. For $n = 1$, the theorem is true by Theorem 1.5. Suppose that $2 \leqslant n \leqslant d$, all assumptions of the theorem hold and that $\xi^0 \in T(U, n)$ with $2 \leqslant n \leqslant d$. Then, by Theorem 1.4 (i), we have the right-hand side inequality with $\phi_1(r) = r^{-(d-1)}$. We can find a linearly independent system $\xi_1^0, \ldots, \xi_n^0$ in U and positive reals $a_1^0, \ldots, a_n^0$ such that $\xi^0 = \sum_{j=1}^n a_j^0 \xi_j^0$. Choose big $R_1 > 0$, small $\varepsilon > 0$, and a small relatively open neighborhood $U(\xi_j^0)$ of ξ_j^0 in U for each j in such a way that with $y = r\xi^0 - \sum_{j=2}^n x_j$

$$\nu(dx_j) \geqslant c_4 \frac{dx_j}{|x_j|^{d+\alpha}} \qquad \text{on } U(\xi_j^0) \times (r(a_j^0 - \varepsilon), r(a_j^0 + \varepsilon)),$$

for $2 \leqslant j \leqslant n$ and

$$\nu(dx_1) \geqslant c_5 \frac{dx_1}{|x_1|^{d+\alpha}} \qquad \text{on } y + B_1$$

with some positive c_4 and c_5. Further

$$ra_1^0/2 \leqslant |y| \leqslant 2ra_1^0 \quad \text{and } y + B_1 \subset U \times (\theta, \infty)$$

for $r \geqslant R_1$ and $x_j \in U(\xi_j^0) \times (r(a_j^0 - \varepsilon), r(a_j^0 + \varepsilon))$ with $2 \leqslant j \leqslant n$. This is possible thanks to (3.16) and Theorem 2.12 of Mattila [21]. Let

$$V_j = U(\xi_j^0) \times (r(a_j^0 - \varepsilon), r(a_j^0 + \varepsilon))$$

for $2 \leqslant j \leqslant n$. Then, with $a = \nu_Z(\mathbb{R}^d)$, there are c_6, c_7 and R_2 such that

$$\lambda(r\xi^0 + B_1) \geqslant e^{-a}(n!)^{-1}\nu_Z^{n*}(r\xi^0 + B_1)$$

$$\geqslant e^{-a}(n!)^{-1}c_6^n \int_{y+B_1} |x_1|^{-(d+\alpha)} dx_1 \int_{V_2 \times \cdots \times V_n} \prod_{j=2}^n |x_j|^{-(d+\alpha)} dx_j$$

$$\geqslant c_7 r^{-d-\alpha} r^{-\alpha(n-1)} \geqslant c_7(1+r)^{-(d+n\alpha)}$$

for all $r \geqslant R_2$. Thus we obtain from Lemma 2.5 (ii) the left-hand side inequality of the theorem with the notice that $\bigcup_{n=1}^d T(U, n) \subset \text{int } C_\sigma$. $\qquad \square$

Lemma 3.9. *Assume that S_σ is a finite set. Let $\xi^0 \in T_\sigma(n)$ with $1 \leqslant n \leqslant d$. Then there are $c_1 > 0$, $c_2 > 0$, $0 < \delta < 1$ and $R > 0$ such that*

$$(3.18) \qquad \lambda(r\xi^0 + y + B_1) \leqslant c_1(1+r)^{-(1+\alpha)n} \quad \text{whenever } |y| \leqslant \delta r/2 \text{ and } r \geqslant R,$$

$$(3.19) \qquad \lambda(r\xi^0 + B_1) \geqslant c_2(1+r)^{-(1+\alpha)n} \quad \text{whenever } r \geqslant R.$$

Proof. Let δ and R be those discovered in Lemma 2.7. Let $|y| \leqslant \delta r/2$ and $r \geqslant R$. By Lemma 2.7 (i),

$$\lambda(r\xi^0 + y + B_1) = \sum_{l=n}^\infty e^{-a}(l!)^{-1}\nu_Z^{l*}(r\xi^0 + y + B_1).$$

We have

$$\nu_Z^{l*}(r\xi^0 + y + B_1) = c^l J(l, r, \xi^0, y)$$

$$\leqslant c^l c_3 \binom{l}{n} l^{(1+\alpha)n} \alpha^{-l} \theta^{-\alpha l} r^{-(1+\alpha)n} A_n,$$

by Lemma 2.7 (ii). Here c is the constant in (1.4) and A_n is finite. Summing up in

l, we get

$$\lambda(r\xi^0 + y + B_1) \leqslant c_4 r^{-(1+\alpha)n} \leqslant c_1(1+r)^{-(1+\alpha)n}$$

with some c_4 and c_1.

To get the lower bound, we have only to use the proof of Theorem 1.4 (ii). □

The following result is related to the works of Hiraba [12], [13], and Byczkowski et al. [5]. Note that $C_\sigma = \bigcup_{n=1}^d T_\sigma(n)$ in the case where S_σ is a finite set.

Theorem 3.10. *Suppose that S_σ is a finite set. Assume that $\xi^0 \in T_\sigma(n)$ with $1 \leqslant n \leqslant d$. In the case where $0 < \alpha < 1$ and ν is one-sided, we assume in addition that $\xi^0 \in \operatorname{int} C_\sigma$. Given $\delta > 0$, we can find $c_1 > 0$ and $c_2 > 0$ such that*

$$(3.20) \qquad c_1(1+r)^{-(1+\alpha)n} \leqslant p(r\xi^0 + y) \leqslant c_2(1+r)^{-(1+\alpha)n} \quad \text{whenever } |y| \leqslant \delta.$$

Proof. Use Lemma 3.9. Then apply Lemma 2.5 for a one-point set $K_1 = K_2 = \{\xi^0\}$. The lower bound is also obtained by Theorem 1.4 (ii). □

Remark 3.11. Let K be a compact set in $T_\sigma(n)$. Then the assertion in Theorem 3.10 is true uniformly with respect to $\xi^0 \in K$. In order to convince ourselves of this, we have only to reexamine the proof of Lemma 3.9 and Remark 2.8.

We conclude this section by considering the two-dimensional case $(d = 2)$.

Lemma 3.12. *Suppose that $d = 2$. Assume that $\sigma = \sum_{j=1}^N \sigma_j$, $1 \leqslant N < \infty$, with $S_{\sigma_1}, \dots, S_{\sigma_N}$ being disjoint, each S_{σ_j} is a closed interval (not a one-point set) in S^1, and there are $0 < a_j \leqslant b_j < \infty$ such that*

$$(3.21) \qquad a_j m(d\xi) \leqslant \sigma_j(d\xi) \leqslant b_j m(d\xi) \qquad \text{on } S_{\sigma_j}.$$

Then, given $\delta_1 > 0$, we can find $c_1 > 0$ and $c_2 > 0$ such that, for any $\xi^0 \in S_\sigma$ (if $1 \leqslant \alpha < 2$ or if $0 < \alpha < 1$ and σ is not one-sided) or for any $\xi^0 \in S_\sigma \cap \operatorname{int} C_\sigma$ (if $0 < \alpha < 1$ and σ is one-sided),

$$(3.22) \qquad c_1(1+r)^{-(2+\alpha)} \leqslant p(r\xi^0 + y) \leqslant c_2(1+r)^{-(2+\alpha)} \quad \text{for } |y| \leqslant \delta_1.$$

Moreover, given a compact set $K \subset C_\sigma^0 \setminus S_\sigma$ and $\delta_2 > 0$, we can find $c_3 > 0$ and $c_4 > 0$ such that

$$(3.23) \quad c_3(1+r)^{-(2+2\alpha)} \leqslant p(r\xi^0 + y) \leqslant c_4(1+r)^{-(2+2\alpha)} \quad \text{for } \xi^0 \in K, \ |y| \leqslant \delta_2.$$

Proof. The condition (1.26) is satisfied for $U = \operatorname{int} S_\sigma$ and $K = S_\sigma$. Hence the estimates in (3.22) are consequences of Theorem 1.5. The estimate from below in (3.23) follows from Theorem 1.1 (iii). Let K be a compact set in $C_\sigma^0 \setminus S_\sigma$. Note that $C_\sigma^0 \setminus S_\sigma = T_\sigma(2)$. Thus we see from Theorem 1.4 (i) that the estimate from above in (3.23) is true with $\phi_1(r) = r^{-1}$ and $n = 2$. The constant c_4 is taken uniformly for $\xi^0 \in K$ thanks to Remark 2.8. □

It is still hard to give the proper asymptotic order in the case where σ is a mixture of an absolutely continuous part and a discrete part. However, we can show them for $d = 2$.

Theorem 3.13. *Suppose that $d = 2$. Assume that there are nonnegative integers N_1 and N_2 such that*

$$(3.24) \qquad \qquad \sigma(d\xi) = \sum_{k=1}^{N_1} c_k \delta_{\xi_k} + \sum_{j=1}^{N_2} \sigma_j(d\xi),$$

where $c_k > 0$ for all k, $\xi_1, \ldots, \xi_{N_1}$ are distinct points in S^1, δ_{ξ_k} is the δ-measure at ξ_k, σ_j is m-absolutely continuous for each j, $S_{\sigma_1}, \ldots, S_{\sigma_{N_2}}$ are disjoint closed intervals (not one-point sets) in S^1, and there are $0 < a_j \leqslant b_j < \infty$ such that

$$(3.25) \qquad a_j m(d\xi) \leqslant \sigma_j(d\xi) \leqslant b_j m(d\xi) \qquad on\ S_{\sigma_j}.$$

Further assume that $N_1 \geqslant 2$ or $N_2 \geqslant 1$. Let $P_\sigma = \{\xi_1, \ldots, \xi_{N_1}\}$. Let $\xi^0 \in S^{d-1}$. If $0 < \alpha < 1$ and σ is one-sided, we assume that $\xi^0 \in \mathrm{int}\, C_\sigma$. Let $\delta > 0$. Note that $C_\sigma = P_\sigma \cup (S_\sigma \setminus P_\sigma) \cup (C_\sigma^0 \setminus S_\sigma)$. If $\xi^0 \in P_\sigma$, then we can find $c_1 > 0$ and $c_2 > 0$ such that

$$(3.26) \qquad c_1(1+r)^{-(1+\alpha)} \leqslant p(r\xi^0 + y) \leqslant c_2(1+r)^{-(1+\alpha)} \quad for\ |y| \leqslant \delta.$$

If $\xi^0 \in S_\sigma \setminus P_\sigma$, then we can find $c_3 > 0$ and $c_4 > 0$ such that

$$(3.27) \qquad c_3(1+r)^{-(2+\alpha)} \leqslant p(r\xi^0 + y) \leqslant c_4(1+r)^{-(2+\alpha)} \quad for\ |y| \leqslant \delta.$$

If $\xi^0 \in C_\sigma^0 \setminus S_\sigma$, then we can find $c_5 > 0$ and $c_6 > 0$ such that

$$(3.28) \qquad c_5(1+r)^{-(2+2\alpha)} \leqslant p(r\xi^0 + y) \leqslant c_6(1+r)^{-(2+2\alpha)} \quad for\ |y| \leqslant \delta.$$

Proof. Hereafter $c_1, c_2, \ldots$ are positive constants. For $\xi^0 \in P_\sigma$, (3.26) is true by Corollary 3.4.

Assume $N_1 \geqslant 2$ and $N_2 \geqslant 1$. Let $\xi^0 \in S_\sigma \setminus P_\sigma$ and $|y| \leqslant \delta$. Let us prove (3.27). Denote the first and the second term in the right-hand side of (3.24) by σ^{d} and σ^{ac}, respectively. Let $\{X_t^{\mathrm{d}}\}$ and $\{X_t^{\mathrm{ac}}\}$ be independent α-stable processes such that $\{X_t\} \stackrel{\mathrm{d}}{=} \{X_t^{\mathrm{d}} + X_t^{\mathrm{ac}}\}$ and the σ-measures of $\{X_t^{\mathrm{d}}\}$ and $\{X_t^{\mathrm{ac}}\}$ are equal to σ^{d} and σ^{ac}, respectively. They are nondegenerate. Let $p^{\mathrm{d}}(x)$ and $p^{\mathrm{ac}}(x)$ be their continuous densities at time 1. Then

$$(3.29) \qquad p(r\xi^0 + y) = \int_{\mathbb{R}^2} p^{\mathrm{ac}}(r\xi^0 + y - z)p^{\mathrm{d}}(z)dz.$$

In the same way, we decompose $\{Z_t\}$ as $\{Z_t\} \stackrel{\mathrm{d}}{=} \{Z_t^{\mathrm{d}} + Z_t^{\mathrm{ac}}\}$. Let λ^{d} and λ^{ac} be the distribution of Z_1^{d} and Z_1^{ac}. Then, as in the proof of Theorem 1.5 (ii), there is $R_1 > 0$ such that

$$\lambda(r\xi^0 + B_1) \geqslant c_1\lambda^{\mathrm{ac}}(r\xi^0 + B_1) \geqslant c_2(1+r)^{-(2+\alpha)} \qquad for\ r > R_1.$$

This gives the lower estimate by Lemma 2.5 (ii) with $K_2 = \{\xi^0\}$. Next, let us show the upper estimate. When $\xi^0 \in T_{\sigma^{\mathrm{d}}}(2)$, choose a small $0 < \delta_1 < 1$ such that if $|r\xi^0 - z| \leqslant \delta_1 r$, then $z/|z| \in T_{\sigma^{\mathrm{d}}}(2)$. This is possible as in the proof of Lemma 2.7. When $\xi^0 \notin T_{\sigma^{\mathrm{d}}}(2)$, choose $0 < \delta_1 < 1$ such that if $|r\xi^0 - z| \leqslant \delta_1 r$, then $z/|z| \notin C_{\sigma^{\mathrm{d}}}$ (note that $\xi^0 \notin C_{\sigma^{\mathrm{d}}}$ since $C_{\sigma^{\mathrm{d}}} \setminus C_{\sigma^{\mathrm{d}}}^0 \subset S_{\sigma^{\mathrm{d}}} = P_\sigma$). Write (3.29) as $p(r\xi^0 + y) = I_1 + I_2$, where

$$I_1 = \int_{|r\xi^0-z|\leqslant\delta_1 r} p^{\mathrm{ac}}(r\xi^0+y-z)p^{\mathrm{d}}(z)dz, \quad I_2 = \int_{|r\xi^0-z|>\delta_1 r} p^{\mathrm{ac}}(r\xi^0+y-z)p^{\mathrm{d}}(z)dz.$$

If $\xi^0 \in T_{\sigma^{\mathrm{d}}}(2)$, then

$$p^{\mathrm{d}}(z) \leqslant c_3(1+|z|)^{-2(1+\alpha)} \leqslant c_4(1+r)^{-2(1+\alpha)}$$

for $|r\xi^0 - z| \leqslant \delta_1 r$ and for large r, by Theorem 3.10 and Remark 3.11. If $\xi^0 \notin T_{\sigma^{\mathrm{d}}}(2)$, then

$$p^{\mathrm{d}}(z) \leqslant c_5 e^{-|z|\log|z|} \leqslant c_6 e^{-r\log r}$$

for $|r\xi^0 - z| \leqslant \delta_1 r$, by Theorem 1.1 (iv). Thus we have $I_1 \leqslant c_7(1+r)^{-2(1+\alpha)}$. On the other hand, $I_2 \leqslant c_8(1+r)^{-(2+\alpha)}$, since

$$p^{\mathrm{ac}}(r\xi^0 + y - z) \leqslant c_9(1+r)^{-(2+\alpha)}$$

for $|r\xi^0 - z| > \delta_1 r$ with large r, by Theorem 1.5 (i). Thus we get (3.27).

Still keeping the assumption that $N_1 \geqslant 2$ and $N_2 \geqslant 1$, let us next prove (3.28) for $\xi^0 \in C_\sigma^0 \setminus S_\sigma$. The lower estimate follows from Theorem 1.1 (iii). To get the upper estimate, let I_1, I_2, and I_3 be the integral in (3.29) with domain of integration changed to $\{z \colon |r\xi^0 - z| \leqslant \delta_1 r\}$, $\{z \colon |r\xi^0 - z| > \delta_1 r$ and $|z| \leqslant \delta_2 r\}$, and $\{z \colon |r\xi^0 - z| > \delta_1 r$ and $|z| > \delta_2 r\}$, respectively, where $\delta_1 > 0$ and $\delta_2 > 0$ are chosen sufficiently small. Then we can check $I_j \leqslant c_j(1+r)^{-(2+2\alpha)}$ for $j = 1, 2, 3$. Indeed, for $j = 1$ this is by the same discussion as in the case where $\xi^0 \in S_\sigma \setminus P_\sigma$. For $j = 2$, we consider two cases, $\xi^0 \in C_{\sigma^{\mathrm{ac}}}^0$ and $\xi^0 \notin C_{\sigma^{\mathrm{ac}}}^0$. In the first case we can use

$$p^{\mathrm{ac}}(r\xi^0 + y - z) \leqslant c_4(1 + |r\xi^0 - z|)^{-(2+2\alpha)} \leqslant c_5(1+r)^{-(2+2\alpha)}$$

for large r by Lemma 3.12, letting $\Pi(r\xi^0 - z) \in C_{\sigma^{\mathrm{ac}}}^0 \setminus S_{\sigma^{\mathrm{ac}}}$. In the second case we can use Theorem 1.1 (iv). For $j = 3$, notice that

$$p^{\mathrm{ac}}(r\xi^0 + y - z) \leqslant c_6(1 + |r\xi^0 - z|)^{-(2+\alpha)} \leqslant c_7(1+r)^{-(2+\alpha)}$$

for large r by Theorem 1.5 (i), and that

$$(3.30) \qquad P(|W_1| > s) \leqslant \mathrm{const}\,(1+s)^{-\alpha} \qquad \text{for } s \geqslant 0$$

for any α-stable process $\{W_t\}$ on $\mathbb{R}^d$, to conclude

$$I_3 \leqslant c_7(1+r)^{-(2+\alpha)} \int_{|z| > \delta_2 r} p^{\mathrm{d}}(z)dz \leqslant c_3(1+r)^{-(2+2\alpha)}.$$

The estimate (3.30) is well-known in one dimension and, for general d, can be reduced to a one-dimensional case by considering components.

It remains to prove (3.27) and (3.28) in the following three cases: (1) $N_1 \geqslant 2$ and $N_2 = 0$, (2) $N_1 = 1$ and $N_2 \geqslant 1$, (3) $N_1 = 0$ and $N_2 \geqslant 1$. Case (1) is treated by Theorem 3.10. The case (3) is done by Lemma 3.12. In case (2), with the choice of $\tau^{\mathrm{d}} = 0$, there is a straight line V through the origin such that the distribution ρ of X_1^{d} is concentrated on V, thus $p(r\xi^0 + y) = \int_V p^{\mathrm{ac}}(r\xi^0 + y - z)\rho(dz)$, and we note that $\mathrm{dis}\,(r\xi^0, V) = c_1 r$ with some $c_1 > 0$ if $\xi^0 \notin P_\sigma$. Thus the proof is similar to the case of $N_1 \geqslant 2$ and $N_2 \geqslant 1$. $\qquad \square$

4. Applications to stable processes

We study implications of our results on the asymptotic estimates in the set $\mathfrak{T}$ below and in the Spitzer type limit theorems mentioned in the last paragraph of Section 1. Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ be the standard (Markov) process in the sense of Blumenthal and Getoor [4] induced by a Lévy process $\{X_t\}$ on $\mathbb{R}^d$. That is, $P^x(X_t \in B) = P(x + X_t \in B)$ for any $x \in \mathbb{R}^d$, $t \geqslant 0$, and Borel set B. The symbol E^x stands for the expectation under P^x. Denote the dual process of X by $\check{X} = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, \widetilde{P}^x)$. Let T_B be the hitting time of a Borel set B defined by $T_B = \inf\{t > 0 \colon X_t \in B\}$, where we understand $T_B = \infty$ if $X_t \notin B$ for all $t > 0$. Let L_B be the last exit time from an open set B, that is, $L_B = \sup\{t \geqslant 0 \colon X_t \in B\}$. Let $B_r = \{x \colon |x| < r\}$. The process is recurrent if and only if $L_{B_r} = \infty$ a.s. for all $r > 0$; it is transient if and only if $L_{B_r} < \infty$ a.s. for all $r > 0$. Recall that, in

general, $A(t) \asymp B(t)$ means that there are constants $0 < c_1 \leqslant c_2 < \infty$ such that $c_1 B(t) \leqslant A(t) \leqslant c_2 B(t)$ for all sufficiently large t.

Lemma 4.1. *Let $\{X_t\}$ be a transient Lévy process on $\mathbb{R}^d$. Let $\eta > 0$. Then one of the following is true:*

$$\text{(4.1)} \qquad\qquad E^0[L_{B_r}{}^\eta] < \infty \quad \text{for all } r > 0;$$

$$\text{(4.2)} \qquad\qquad E^0[L_{B_r}{}^\eta] = \infty \quad \text{for all } r > 0.$$

This is Theorem 2.8 of [33]. Given a transient Lévy process on $\mathbb{R}^d$, denote

$$\text{(4.3)} \qquad\qquad \mathfrak{T} = \{0\} \cup \{\eta > 0 \colon (4.1) \text{ is true}\}.$$

The bigger this set $\mathfrak{T}$, the stronger a degree of transience. The process $\{X_t\}$ is called *strongly transient* if $1 \in \mathfrak{T}$; it is called *weakly transient* if $1 \notin \mathfrak{T}$. Hawkes [11] and Takeuchi [37] are early works on the moment of the last exit time for a symmetric Lévy process. Strong transience is important in the theory of the range of random walks. See Jain and Pruitt [15]. After Sato [29], [30], the set $\mathfrak{T}$ has been studied in detail by Sato and Watanabe [33], [34]. In [33], they obtained a criterion for $\eta \in \mathfrak{T}$ for a general transient Lévy process and discussed the relation with the moment of the last exit time from a half line for a one-dimensional random walk. This had been studied by Janson [16] and Kesten and Maller [18]. Dawson et al. [6] investigated the application of the set $\mathfrak{T}$ in branching systems. The following is Lemma 2.3 of [33].

Lemma 4.2. *Let $\{X_t\}$ be a transient Lévy process on $\mathbb{R}^d$. Let $\eta > 0$. Then $\eta \in \mathfrak{T}$ if and only if*

$$\text{(4.4)} \qquad\qquad \int_0^\infty t^\eta P^0[X_t \in B_1]dt < \infty;$$

$\eta \notin \mathfrak{T}$ if and only if

$$\text{(4.5)} \qquad\qquad \int_0^\infty t^\eta P^0[X_t \in B_1]dt = \infty.$$

Let us remark the following fact ([32] Theorems 37.8, 37.16, 37.18).

Lemma 4.3. *Let $\{X_t\}$ be a nondegenerate α-stable process on $\mathbb{R}^d$. If $d \geqslant 3$, then it is transient. If $d = 1$ or 2, then it is recurrent if and only if it is first-class α-stable with $d \leqslant \alpha \leqslant 2$.*

Define $E_B(t)$ for $t \geqslant 0$ as

$$\text{(4.6)} \qquad\qquad E_B(t) = \int_{\mathbb{R}^d} P^x(T_B \leqslant t)dx.$$

Asymptotic expansion of $E_B(t)$ as $t \to \infty$ for a bounded Borel set B has been a subject of research in many papers since Spitzer [36] in 1964 for a transient Brownian motion. Later Le Gall [19] extended Spitzer's results by determining the more exact order of the asymptotic expansion conjectured by M. Kac in a footnote of Spitzer [36]. Assume that the original Lévy process $\{X_t\}$ on $\mathbb{R}^d$ is transient and nondegenerate. Let $C(B)$ be the capacity for a Borel set B. We have $0 \leqslant C(B) < \infty$ for any bounded Borel set B. See Port and Stone [26], Sato [32], Chapter 8, and

Bertoin [2], Chapter 2. Fix a nonnegative continuous function $f(x)$ with compact support, $f(0) > 0$ and $\int_{\mathbb{R}^d} f(x)dx = 1$. Let

$$(4.7) \qquad\qquad r(t) = \int_t^\infty ds \int_{\mathbb{R}^d} f(x)(E^x f(X_s))dx.$$

We have $r(t) < \infty$ for $t \geqslant 0$ because of the transience. In the following, let B be a bounded Borel set in $\mathbb{R}^d$. In the case where X_t has a purely singular distribution for every $t > 0$, we make an additional assumption that $P^x(T_{\overline{B}} = T_{\mathrm{int}B}) = 1$ for almost every x. Let

$$(4.8) \qquad\qquad \varphi_B(x) = P^x(T_B < \infty), \qquad \widetilde{\varphi}_B(x) = \widetilde{P}^x(T_B < \infty).$$

If $1 \in \mathfrak{T}$, then $\int_{\mathbb{R}^d} \widetilde{\varphi}_B(x)\varphi_B(x)dx < \infty$. Define

$$(4.9) \qquad\qquad \Delta_B^{(1)}(t) = E_B(t) - tC(B)$$

and

$$(4.10) \qquad\qquad \Delta_B^{(2)}(t) = E_B(t) - tC(B) - \int_{\mathbb{R}^d} \widetilde{\varphi}_B(x)\varphi_B(x)dx.$$

The following lemma is a result obtained by a series of works [36], [8], [23], [24], [25], [26]. Assertion (i) is given in Theorem 14.2 of [26] and assertion (ii) is in Lemmas 3.1 and 3.3 and Theorem 1 of [25] with (3.19) and (14.13) of [26], respectively. See also Proposition 6.2 of [34].

Lemma 4.4. (i) *If* $1 \notin \mathfrak{T}$*, then*

$$(4.11) \qquad \Delta_B^{(1)}(t) = (C(B))^2 \int_0^t r(s)ds + o\left(\int_0^t r(s)ds\right), \qquad t \to \infty.$$

(ii) *If* $1 \in \mathfrak{T}$ *and*

$$(4.12) \qquad\qquad r(\cdot) \in \mathcal{D}, \quad \text{that is,} \quad \sup_{t>0} \frac{r(t/2)}{r(t)} < \infty,$$

then

$$(4.13) \qquad \Delta_B^{(2)}(t) = -(C(B))^2 \int_t^\infty r(s)ds + o\left(\int_t^\infty r(s)ds\right), \qquad t \to \infty.$$

Sato and Watanabe [34] obtained the set $\mathfrak{T}$ for all transient α-semistable processes with $d = 1$ and classified all transient α-semistable processes into weakly transient and strongly transient. Port [23], [24] completely studied the asymptotic order of $\Delta_B^{(1)}(t)$ when $1 \notin \mathfrak{T}$ and partially did that of $\Delta_B^{(2)}(t)$ when $1 \in \mathfrak{T}$ for transient α-stable processes. See also Sato and Watanabe [34] for α-semistable processes. Now the case in $1 \in \mathfrak{T}$ where we do not have a proper order of $\Delta_B^{(2)}(t)$ is the same as for the set $\mathfrak{T}$, that is, $d \geqslant 2$, $1 \leqslant \alpha < 2$ and second-class with $\sigma(\{-\tau/|\tau|\}) = 0$. Combining the following lemma and our estimates in Sections 1 and 3, we can find that the set $\mathfrak{T}$ and the proper asymptotic order of $\Delta_B^{(2)}(t)$ as $t \to \infty$ delicately depend on the direction of $-\tau/|\tau|$ in relation to the location of S_σ in this case.

Lemma 4.5. *Let* $\{X_t\}$ *be a nondegenerate transient second-class α-stable process on $\mathbb{R}^d$ with $1 \leqslant \alpha < 2$ and $\sigma(\{-\tau/|\tau|\}) = 0$. Assume that $d \geqslant 2$ and that there are $\beta \geqslant 1 + \alpha$, $c_1 > 0$, $c_2 > 0$ and $R > 0$ such that*

$$(4.14) \qquad c_1(1+r)^{-\beta} \leqslant p(-r\tau + y) \leqslant c_2(1+r)^{-\beta} \qquad \text{for } r \geqslant R,\ |y| \leqslant 1.$$

(i) *If $1 < \alpha < 2$, then*

$$(4.15) \qquad \mathfrak{T} = [0, d/\alpha + (1 - 1/\alpha)\beta - 1) \quad and \quad -\Delta_B^{(2)}(t) \asymp t^{2-d/\alpha-(1-1/\alpha)\beta}.$$

(ii) *If $\alpha = 1$, then*

$$(4.16) \qquad \mathfrak{T} = [0, d-1] \quad and \quad -\Delta_B^{(2)}(t) \asymp \begin{cases} (\log t)^{-\beta+1} & for\ d = 2, \\ t^{2-d}(\log t)^{-\beta} & for\ d \geqslant 3. \end{cases}$$

Proof. For any $\eta > 0$, it follows from Lemma 4.2 that $\int_0^\infty t^{\eta-1}r(t)dt < \infty$ if and only if $\eta \in \mathfrak{T}$. We prove (i). Suppose that $1 < \alpha < 2$. We have by (2.3)

$$r(t) \asymp \int_{|x| \leqslant 1} dx \int_t^\infty p(s, x)ds \asymp t^{1-d/\alpha-(1-1/\alpha)\beta}, \qquad t \to \infty,$$

and hence $\mathfrak{T} = [0, d/\alpha + (1 - 1/\alpha)\beta - 1)$ and

$$\int_t^\infty r(s)ds \asymp t^{2-d/\alpha-(1-1/\alpha)\beta}, \qquad t \to \infty.$$

Now use Lemma 4.4. Next we prove (ii). Suppose that $\alpha = 1$. We have by (2.3)

$$r(t) \asymp \int_{|x| \leqslant 1} dx \int_t^\infty p(s, x)ds \asymp (\log t)^{-\beta}t^{-(d-1)}, \qquad t \to \infty.$$

Hence $\mathfrak{T} = [0, d-1]$ and

$$\int_t^\infty r(s)ds \asymp \begin{cases} (\log t)^{-\beta+1} & for\ d = 2, \\ (\log t)^{-\beta}t^{-(d-2)} & for\ d \geqslant 3. \end{cases}$$

Then use Lemma 4.4. $\qquad\square$

In the following propositions, assume $d \geqslant 2$, $1 \leqslant \alpha < 2$ and that our stable process $\{X_t\}$ is second-class.

Proposition 4.6. *Suppose that the assumptions in Theorem 3.8 are satisfied for our process $\{X_t\}$.*
   (i) *If $1 < \alpha < 2$, and $-\tau/|\tau| \in T(U, n)$ with $1 \leqslant n \leqslant d$, then*

$$(4.17) \qquad \mathfrak{T} = [0, d + n(\alpha - 1) - 1) \quad and \quad -\Delta_B^{(2)}(t) \asymp t^{2-d-n(\alpha-1)}.$$

   (ii) *If $\alpha = 1$, and $-\tau/|\tau| \in T(U, n)$ with $1 \leqslant n \leqslant d$, then*

$$(4.18) \qquad \mathfrak{T} = [0, d-1] \quad and \quad -\Delta_B^{(2)}(t) \asymp \begin{cases} (\log t)^{-1-n} & for\ d = 2, \\ t^{2-d}(\log t)^{-d-n} & for\ d \geqslant 3. \end{cases}$$

Proposition 4.7. *Assume that S_σ is a finite set. By Theorem 3.10, we have the following:*
   (i) *If $1 < \alpha < 2$ and $-\tau/|\tau| \in T_\sigma(n)$ with $1 \leqslant n \leqslant d$, then*

$$(4.19) \qquad \mathfrak{T} = [0, d/\alpha + (\alpha - 1/\alpha)n - 1) \quad and \quad -\Delta_B^{(2)}(t) \asymp t^{2-d/\alpha-(\alpha-1/\alpha)n}.$$

   (ii) *If $\alpha = 1$ and $-\tau/|\tau| \in T_\sigma(n)$ with $1 \leqslant n \leqslant d$, then*

$$(4.20) \qquad \mathfrak{T} = [0, d-1] \quad and \quad -\Delta_B^{(2)}(t) \asymp \begin{cases} (\log t)^{-2n+1} & for\ d = 2, \\ t^{2-d}(\log t)^{-2n} & for\ d \geqslant 3. \end{cases}$$

Proposition 4.8. *Suppose that the assumptions in Theorem 3.13 are satisfied for our process $\{X_t\}$ with $d = 2$. Note that $C_\sigma^0 \setminus S_\sigma = C_\sigma \setminus S_\sigma$.*
  (i) *Assume $1 < \alpha < 2$.*
    (1) *If $-\tau/|\tau| \in P_\sigma$, then*

$$(4.21) \qquad \mathfrak{T} = [0, \alpha + 1/\alpha - 1) \quad and \quad -\Delta_B^{(2)}(t) \asymp t^{2-\alpha-1/\alpha}.$$

    (2) *If $-\tau/|\tau| \in S_\sigma \setminus P_\sigma$, then*

$$(4.22) \qquad \mathfrak{T} = [0, \alpha) \quad and \quad -\Delta_B^{(2)}(t) \asymp t^{1-\alpha}.$$

    (3) *If $-\tau/|\tau| \in C_\sigma^0 \setminus S_\sigma$, then*

$$(4.23) \qquad \mathfrak{T} = [0, 2\alpha - 1) \quad and \quad -\Delta_B^{(2)}(t) \asymp t^{2-2\alpha}.$$

  (ii) *Assume $\alpha = 1$.*
    (1) *If $-\tau/|\tau| \in P_\sigma$, then*

$$(4.24) \qquad \mathfrak{T} = [0, 1] \quad and \quad -\Delta_B^{(2)}(t) \asymp (\log t)^{-1}.$$

    (2) *If $-\tau/|\tau| \in S_\sigma \setminus P_\sigma$, then*

$$(4.25) \qquad \mathfrak{T} = [0, 1] \quad and \quad -\Delta_B^{(2)}(t) \asymp (\log t)^{-2}.$$

    (3) *If $-\tau/|\tau| \in C_\sigma^0 \setminus S_\sigma$, then*

$$(4.26) \qquad \mathfrak{T} = [0, 1] \quad and \quad -\Delta_B^{(2)}(t) \asymp (\log t)^{-3}.$$

The final proposition shows that the asymptotics of $\Delta_B^{(2)}(t)$ can be more delicate than those of Lemma 4.5 in a certain special case.

Proposition 4.9. *Let $\ell(r)$ and $\ell^*(r)$ be slowly varying as in Remark 3.3. Suppose that $d \geqslant 2$, $1 \leqslant \alpha < 2$ and second-class, and further $\xi^0 = -\tau/|\tau|$ satisfies (3.4) and (3.5) with $\sigma_{\xi^0}(r) \asymp r^{-s}\ell(r)$ and $0 < s < d - 1$ or with $\sigma_{\xi^0}(r) \asymp \ell^*(r)$.*
  (i) *Assume $1 < \alpha < 2$. Then $\eta \in \mathfrak{T}$ if and only if $\int_1^\infty t^{\eta-(d-1)/\alpha-\alpha}\sigma_{\xi^0}(t^{1-1/\alpha})dt < \infty$.*
    (1) *If $\sigma_{\xi^0}(r) \asymp r^{-s}\ell(r)$ with $0 < s < d - 1$, then*

$$(4.27) \qquad -\Delta_B^{(2)}(t) \asymp t^{2-(d-1)/\alpha-\alpha-(1-1/\alpha)s}\ell(t^{1-1/\alpha}).$$

    (2) *If $\sigma_{\xi^0}(r) \asymp \ell^*(r)$, then*

$$(4.28) \qquad -\Delta_B^{(2)}(t) \asymp t^{2-(d-1)/\alpha-\alpha}\ell^*(t^{1-1/\alpha}).$$

  (ii) *Assume $\alpha = 1$. Then $\mathfrak{T} = [0, d-1]$.*
    (1) *If $\sigma_{\xi^0}(r) \asymp r^{-s}\ell(r)$ with $0 < s < d - 1$, then*

$$(4.29) \qquad -\Delta_B^{(2)}(t) \asymp \begin{cases} (\log t)^{-1-s}\ell(\log t) & for\ d = 2, \\ t^{2-d}(\log t)^{-2-s}\ell(\log t) & for\ d \geqslant 3. \end{cases}$$

    (2) *If $\sigma_{\xi^0}(r) \asymp \ell^*(r)$, then*

$$(4.30) \qquad -\Delta_B^{(2)}(t) \asymp \begin{cases} (\log t)^{-1}\ell^*(\log t) & for\ d = 2, \\ t^{2-d}(\log t)^{-2}\ell^*(\log t) & for\ d \geqslant 3. \end{cases}$$

Proof. Proof is similar to Lemma 4.5 by using (2.3). $\qquad\square$

We add some final remarks on the set $\mathfrak{T}$.

Remark 4.10. Let $\{X_t\}$ be a nondegenerate second-class transient α-stable process with $d \geqslant 2$ and $1 \leqslant \alpha < 2$.

(i) Suppose that $-\tau/|\tau| \in C^0_\sigma$. Then $\mathfrak{T} \subset [0, d\alpha - 1)$ for $1 < \alpha < 2$ and $\mathfrak{T} = [0, d - 1]$ for $\alpha = 1$.

(ii) Suppose that there is $\phi \in \mathcal{D}$ such that

$$\liminf_{r \to \infty} \frac{\sigma(\xi + B_{1/r})}{\phi(r)} > 0 \qquad \text{for any } \xi \in S_\sigma.$$

If $-\tau/|\tau| \in \bigcup_{n=1}^d T_\sigma(n)$, then $\mathfrak{T} = [0, d - 1]$ for $\alpha = 1$ and $\mathfrak{T}$ is bounded for $1 < \alpha < 2$.

(iii) If $-\tau/|\tau| \notin C_\sigma$, then $\mathfrak{T} = [0, \infty)$.

Proof. (i) If $-\tau/|\tau| \in C^0_\sigma$ for $1 < \alpha < 2$, then we see from (iii) of Theorem 1.1 that, for $\eta \geqslant d\alpha - 1$,

$$\int_{|x| \leqslant 1} dx \int_1^\infty t^\eta p(t, x) dt \geqslant c_1 \int_1^\infty t^{\eta - d/\alpha - (1 - 1/\alpha)d(1+\alpha)} dt = \infty$$

with some $c_1 > 0$. Thus $\mathfrak{T} \subset [0, d\alpha - 1)$ for $1 < \alpha < 2$ by Lemma 4.2. The proof of $\mathfrak{T} = [0, d - 1]$ for $\alpha = 1$ is similar by using Theorem B (i) of [34].

(ii) Let $\alpha = 1$ and $-\tau/|\tau| \in \bigcup_{n=1}^d T_\sigma(n)$. By Theorem B (i) of [34], we have $\mathfrak{T} \supset [0, d - 1]$. Since $\phi \in \mathcal{D}$, there are $B \geqslant 0$ and $c_2 > 0$ such that

$$\phi(1 + r) \geqslant c_2 (1 + r)^{-B} \qquad \text{for } r \geqslant 0.$$

By Theorem 1.4 (ii) we have, for $\eta > d - 1$,

$$\int_{|x| \leqslant 1} dx \int_1^\infty t^\eta p(t, x) dt = \int_{|x| \leqslant 1} dx \int_1^\infty t^{\eta - d} p(t^{-1}x - (\log t)\tau) dt$$

$$\geqslant c_3 \int_1^\infty t^{\eta - d} (\log t)^{-d(2+B)} dt = \infty$$

with some $c_3 > 0$. Hence $\mathfrak{T} \subset [0, d - 1]$ by Lemma 4.2. Let $1 < \alpha < 2$ and $-\tau/|\tau| \in \cup_{n=1}^d T_\sigma(n)$. Then we see that $\mathfrak{T} \subset [0, \alpha d + (1 - 1/\alpha)Bd - 1)$ as above.

The proof of (iii) is similar by using (iv) of Theorem 1.1. $\square$

The complete study of determining the set $\mathfrak{T}$ and the proper order of $\Delta_B^{(2)}(t)$ for the transient stable process is still far from us. The situation is more difficult for transient semistable processes. A result of Ishikawa [14] suggests that there might be an essential difference of the asymptotics of $\Delta_B^{(2)}(t)$ between stable and semistable processes. We finish this article by posing a natural conjecture related to Proposition 6.3 of [34]. Let $\{X_t\}$ be a transient Lévy process. Define two indices a_1 for $1 \notin \mathfrak{T}$ and a_2 for $1 \in \mathfrak{T}$ as

$$a_1 = \inf\{c \geqslant 0 : \limsup_{t \to \infty} \frac{\Delta_B^{(1)}(t)}{t^c} < \infty\}$$

and

$$a_2 = \sup\{c \geqslant 0 : \limsup_{t \to \infty} \frac{-\Delta_B^{(2)}(t)}{t^{-c}} < \infty\}.$$

Then we conjecture that if $\bar{\mathfrak{T}} = [0, b]$ with $0 \leqslant b < \infty$, then $a_1 = 1 - b$ for $1 \notin \mathfrak{T}$ and $a_2 = b - 1$ for $1 \in \mathfrak{T}$.

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Center for Mathematical Sciences, The University of Aizu, Aizu-Wakamatsu Fukushima, 965-8580 Japan

E-mail address: `t-watanb@u-aizu.ac.jp`