ASYMPTOTIC EXPANSION FOR DAMPED WAVE EQUATIONS WITH PERIODIC COEFFICIENTS

R. ORIVE and E. ZUAZUA
Departamento de Matemática Aplicada, Facultad de Ciencias Matemáticas
Universidad Complutense de Madrid, 28040 Madrid, ESPAÑA.
orive@sunma4.mat.ucm.es, zuazua@eucmax.sim.ucm.es

A. F. PAZOTO
Departamento de Métodos Matemáticos, Instituto de Matemática
Universidade Federal do Rio de Janeiro, P.O.Box 68530, 21945-979 Rio de Janeiro, BRASIL.
ademir@acd.ufrj.br

We consider a linear dissipative wave equation in $\mathbb{R}^N$ with periodic coefficients. By means of Bloch wave decomposition, we obtain an expansion of solutions as $t \to \infty$ and conclude that, in a first approximation, the solutions behave as the homogenized heat kernel.

1. Introduction

1.1. *Setting of the problem*

This paper is concerned with the analysis of the asymptotic behavior, as $t \to \infty$, of the solutions of

$$\begin{cases} \rho(x)u_{tt} - \frac{\partial}{\partial x_k}\left(a_{k\ell}(x)\frac{\partial u}{\partial x_\ell}\right) + a_0\rho(x)u_t = 0 \quad \text{in } \mathbb{R}^N \times (0,\infty) \\ \quad u(x,0) = \varphi^0(x) \\ \quad u_t(x,0) = \varphi^1(x) \end{cases} \tag{1.1}$$

where the coefficients satisfy

$$\begin{cases} a_{k\ell} \in L^\infty_\#(Y) \quad \text{where} \quad Y =]0,2\pi[^N, \text{ i.e., each } a_{k\ell} \text{ is a} \\ Y\text{-periodic bounded measurable function defined on } \mathbb{R}^N, \\ \exists \alpha > 0 \quad \text{such that} \quad a_{k\ell}(x)\eta_k\eta_\ell \geq \alpha|\eta|^2 \quad \forall \eta \in \mathbb{R}^N, \text{ a.e. } x \in \mathbb{R}^N, \\ a_{k\ell} = a_{\ell k} \quad \forall k,\ell = 1,\dots,N, \text{ and} \\ a_0 \text{ is a positive constant,} \end{cases} \tag{1.2}$$

and

$$\begin{cases} \rho \in L^\infty_\#(Y), \text{ i.e., } \rho \text{ is } Y\text{-periodic, and} \\ \exists \rho_0, \rho_1 \in \mathbb{R}_+, \text{ such that } 0 < \rho_0 \leq \rho(x) \leq \rho_1, \text{ a.e. } x \in Y. \end{cases} \tag{1.3}$$

The equation (1.1) has a dissipative nature. Indeed, the energy associated to (1.1), given by

$$E(t) = \frac{1}{2} \int\limits_{R^N} \left[\rho(x) |u_t|^2 + a_{k\ell}(x) \frac{\partial u}{\partial x_\ell} \frac{\partial u}{\partial x_k} \right] dx,$$

is decreasing

$$\frac{\partial E}{\partial t} = -a_0 \int\limits_{R^N} \rho(x) |u_t|^2 dx.$$

Moreover, solutions of (1.1) satisfy the conservation law

$$\frac{\partial}{\partial t} \int\limits_{R^N} (u_t + a_o u) \rho(x) dx = \int\limits_{R^N} \frac{\partial}{\partial x_k} \left(a_{k\ell}(x) \frac{\partial u}{\partial x_\ell} \right) dx = 0.$$

That is, the mass of $u_t + a_o u$ with respect to the weight $\rho(\cdot)$ is conserved along time:

$$m_\rho(u_t + a_o u) = m_\rho(\varphi^1 + a_0 \varphi^0) = \int\limits_{R^N} (\varphi^1 + a_0 \varphi^0) \rho(x) dx. \tag{1.4}$$

Using Bloch waves decomposition together with a choice of a convenient Lyapunov function we obtain the asymptotic expansion of the solutions of (1.1). In particular we conclude that solutions behave as the homogenized heat kernel as $t \to \infty$. In fact, equation (1.1) can be viewed as a heat equation "perturbed" by the second order term $\rho(x) u_{tt}$, that introduces oscillations that, according to our analysis, are not strong enough to change the behavior of solutions as $t \to \infty$ in a first approximation.

To be more precise, equation (1.1) can be viewed as a perturbation of the parabolic equation

$$\begin{cases} \rho(x) a_0 u_t - \frac{\partial}{\partial x_k} \left(a_{k\ell}(x) \frac{\partial u}{\partial x_\ell} \right) = 0 & \text{in } \mathbb{R}^N \times (0, \infty) \\ u(x, 0) = \varphi^0(x). \end{cases} \tag{1.5}$$

The asymptotic behavior of solutions of (1.5) is well known when $\rho = 1$, $a_0 = 1$. The first term in the asymptotic expansion was obtained in [6]. It was shown that

$$t^{\frac{N}{2}(1-\frac{1}{p})} \| u(t) - m(\varphi) G_h(\cdot, t) \|_p \longrightarrow 0, \text{ as } t \to \infty, \ 1 \le p \le \infty,$$

where

$$m(\varphi) = \int\limits_{R^N} \varphi \, dx$$

and G_h is the fundamental solution of the homogenized system

$$\begin{cases} u_t - q_{k\ell} \frac{\partial^2 u}{\partial x_\ell x_k} = 0 & \text{in } \mathbb{R}^N \times (0, \infty) \\ u(x, 0) = \delta_0(x). \end{cases}$$

Here and in the sequel we denote by δ_0 the Dirac delta at the origin and by $\{q_{jk}\}_{j,k=1}^N$ the homogenized coefficients associated to the periodic matrix with coefficients (1.2). The constant $m(\varphi)$ is the mass of the solution of (1.5) which is conserved along time, i.e.,

$$\frac{\partial}{\partial t}[m(u(\cdot,t))] = 0, \quad \forall t \in (0,\infty).$$

Later on, in J.H. Ortega and E. Zuazua [7], the asymptotic expansion as $t \to \infty$ of the solutions of (1.5) with $L^1(\boldsymbol{R}^N) \cap L^2(\boldsymbol{R}^N)$ initial data was obtained by means of the Bloch wave decomposition (see [3] and [4] for an introduction to Bloch waves). Their first result was stablished in the $L^2(\boldsymbol{R}^N)$-setting and then, thanks to the parabolic regularizing effect, a converge result in $L^\infty(\boldsymbol{R}^N)$ was derived. The results in [7] are, to some extent, an extension of those of [5] on the constant coefficient heat equation. The analysis in [7] can be easily adapted to the general parabolic equation (1.5) with variable, periodic density ρ.

This work is devoted to adapt the analysis in [7] to the case of the dissipative wave equation (1.1) under consideration.

Our main result will only be given for the solution u of (1.1) in the $L^2(\boldsymbol{R}^N)$-setting with $L^2 \cap L^1(\boldsymbol{R}^N) \times H^{-1} \cap L^1(\boldsymbol{R}^N)$ initial data but, as it will become clear during the proofs, a similar analysis allows to obtain the asymptotic expansion of (u, u_t) in $H^s(\boldsymbol{R}^N) \times H^{s-1}(\boldsymbol{R}^N)$ with initial data in $H^s \cap L^1(\boldsymbol{R}^N) \times H^{s-1} \cap L^1(\boldsymbol{R}^N)$. An asymptotic expansion in $L^\infty(\boldsymbol{R}^N)$ can also be given, but here, in the absence of the parabolic regularizing effect used in [7], we should consider sufficiently smooth initial data so that the asymptotic expansion holds in H^s with $s > 0$ large enough so that $H^s \hookrightarrow C^0$.

1.2. *Main results*

The well-posedness of the equation (1.1) under the conditions (1.2) and (1.3) can be easily obtained writing (1.1) as an abstract evolution equation in the space of finite energy $H = H^1(\boldsymbol{R}^N) \times L^2(\boldsymbol{R}^N)$, with the inner product

$$\big((u,v),(\widetilde{u},\widetilde{v})\big)_H = \int_{\boldsymbol{R}^N} u\widetilde{u}\,dx \;+ \int_{\boldsymbol{R}^N} a_{k\ell}(x)\frac{\partial u}{\partial x_k}\frac{\partial \widetilde{u}}{\partial x_\ell}\,dx \;+ \int_{\boldsymbol{R}^N} v\widetilde{v}\rho(x)dx,$$

with $\{a_{k\ell}\}$ as in (1.2) and ρ as in (1.3), whenever $(u,v),(\widetilde{u},\widetilde{v}) \in H$. Under these conditions the operator associated to (1.1) is maximal and dissipative on H. Then, Lummer-Phillip's theorem guarantees that the operator associated to (1.1) is the infinitesimal generator of a continuous semigroup. Thus, we deduce that for any initial data $(\varphi^0, \varphi^1) \in L^2(\boldsymbol{R}^N) \times H^{-1}(\boldsymbol{R}^N)$ the equation (1.1) has a unique weak solution $u = u(x,t)$ such that

$$u \in C^0(\boldsymbol{R}^+, L^2(\boldsymbol{R}^N)) \cap C^1(\boldsymbol{R}^+, H^{-1}(\boldsymbol{R}^N)).$$

Let us now state the main result.

Theorem 1 *Assume that $\varphi^0 \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ and $\varphi^1 \in L^1(\mathbb{R}^N) \cap H^{-1}(\mathbb{R}^N)$ with $|x|^{k+1}\varphi^0(x)$, $|x|^{k+1}\varphi^1(x) \in L^1(\mathbb{R}^N)$ for some $k \geq 0$. Let $u = u(x,t)$ be the solution of (1.1). Then, there exist periodic functions $c_\alpha \in L^\infty_\#(Y)$, with $|\alpha| \leq k$, and constants $c_{\beta,n}$, with $n \leq \frac{k}{2}$ and $4 \leq |\beta| \leq 2k$, depending on the initial data, the coefficients $\{a_{k\ell}\}$ and ρ, such that the solution u satisfies*

$$\Big\| u(\cdot,t) - \sum_{|\alpha|\leq k} c_\alpha(\cdot)[G^*_\alpha(\cdot,t) + \sum_{n=1}^{p(\alpha)} \frac{t^n}{n!} \sum_{m=0}^{a(\alpha,n)} \sum_{|\beta|=4n+2m} c_{\beta,n}, G^*_{\alpha+\beta}(\cdot,t)] \Big\| \leq c_k t^{-\frac{2k+2+N}{4}} \tag{1.6}$$

as $t \to \infty$. Here $p(\alpha) = \left[\frac{k-|\alpha|}{2}\right]$, $a(\alpha,n) = p(\alpha) - n$,

$$G^*_\alpha(x,t) = \frac{1}{(2\pi)^N} \int_{R^N} \xi^\alpha e^{-\frac{q_{k\ell}}{a_0\bar\rho}\xi_k\xi_\ell t} e^{ix\cdot\xi} d\xi, \tag{1.7}$$

and $\bar\rho$ is the averaged density:

$$\bar\rho = \frac{1}{(2\pi)^N} \int_Y \rho(x)dx. \tag{1.8}$$

We observe that $G^*_\alpha = (-i)^{|\alpha|}(\partial^\alpha G^*/\partial x_\alpha)$ where $G^* = G^*(x,t)$ is the fundamental solution of the underlying parabolic homogenized system

$$\begin{cases} \bar\rho a_0 G^*_t - q_{k\ell}\frac{\partial^2 G^*}{\partial x_\ell x_k} = 0 & \text{in } \mathbb{R}^N \times (0,\infty) \\ G^*(x,0) = \delta_0(x). \end{cases} \tag{1.9}$$

The convergence result in (1.6) indicates that, roughly, the solution u of the equation (1.1) may be approximated at any order by a linear combination of the derivatives of the fundamental solution of the heat equation, modulated by the periodic functions $c_\alpha(\cdot)$. The role of these coefficients is to adapt the gaussian asymptotic profiles to the periodicity of the medium where the solution u evolves.

We consider $v(x,t) = c_0(x)G^*(x,t)$, the first term of the asymptotic expansion of the solution of (1.1). As we shall see below (see Section 5), $c_0(\cdot)$ turns out be a constant and more precisely

$$c_0 = \frac{1}{a_0\bar\rho} m_\rho(u_t + a_0 u). \tag{1.10}$$

It is important to observe that the mass of $v(x,t)$ with respect to the weight $\rho(\cdot)$, as $t \to \infty$, is the same as the mass of $u_t + a_0 u$ associated to (1.1) which is constant in time as seen in (1.4). In fact, we have

$$\begin{aligned} m_\rho(v_t) &= -\frac{c_0}{(2\pi)^N} \int_{R^N}\int_{R^N} \frac{q_{k\ell}}{a_0\bar\rho}\xi_k\xi_\ell e^{-\frac{q_{k\ell}}{a_0\bar\rho}\xi_k\xi_\ell t} e^{ix\cdot\xi} d\xi\, \rho(x)dx \;(x = y\sqrt{t}, \xi = \frac{\eta}{\sqrt{t}}) \\ &= -\frac{c_0}{t}\int_{R^N} \rho(y\sqrt{t}) \frac{1}{(2\pi)^N} \int_{R^N} \frac{q_{k\ell}}{a_0\bar\rho}\eta_k\eta_\ell e^{-\frac{q_{k\ell}}{a_0\bar\rho}\eta_k\eta_\ell} e^{iy\cdot\eta} d\eta\, dy, \end{aligned}$$

and

$$m_\rho(v) = c_0 \int\limits_{R^N} \rho(y\sqrt{t}) \frac{1}{(2\pi)^N} \int\limits_{R^N} e^{-\frac{q_{k\ell}}{a_0\bar\rho}\eta_k\eta_\ell} e^{iy\cdot\eta} d\eta\, dy.$$

Since $\rho(y\sqrt{t}) \rightharpoonup \bar\rho$ weakly-$*$ in $L^\infty(\mathbb{R}^N)$ as $t\to\infty$, then

$$m_\rho(v_t) \quad \to \quad 0 \qquad \text{as } t\to\infty,$$

and, thanks to

$$\frac{1}{(2\pi)^N} \int\limits_{R^N}\int\limits_{R^N} e^{-\frac{q_{k\ell}}{a_0\bar\rho}\eta_k\eta_\ell} e^{iy\cdot\eta} d\eta\, dy = 1$$

we have

$$m_\rho(v) \to \bar\rho\, c_0, \qquad \text{as } t\to\infty.$$

Thus, we obtain as $t\to\infty$,

$$m_\rho(v_t + a_0 v) \longrightarrow a_o\, \bar\rho\, c_0 = m_\rho(u_t + a_0 u).$$

Therefore, as expected, the total mass of the solution is captured by the first term in the asymptotic expansion.

The rest of the paper is organized as follows. First, in Section 2 and 3, we study the simpler problem in which $\rho \equiv 1$:

$$\begin{cases} u_{tt} - \frac{\partial}{\partial x_k}\left(a_{k\ell}(x)\frac{\partial u}{\partial x_\ell}\right) + a_0 u_t = 0 & \text{in } \mathbb{R}^N \times (0,\infty) \\ u(x,0) = \varphi^0(x) \\ u_t(x,0) = \varphi^1(x). \end{cases} \tag{1.11}$$

In Section 2 we recall some basic results on Bloch wave decomposition. In Section 3 we prove some basic lemmas and obtain the asymptotic behavior of the solutions of (1.11). In Section 4 we prove the main result, Theorem 1, in the general case. Finally, in Section 5, we analyze the periodic functions c_α and the constants $c_{\beta,n}$ entering in the asymptotic expansion.

2. Bloch wave decomposition

All along this section we assume that $\rho \equiv 1$. The general case will be discussed in Section 4.

In this section we recall some basic results on Bloch wave decompositions. We refer to [3] and to [4] for the notations and the proofs.

Let us condider the following spectral problem parametrized by $\xi \in \mathbb{R}^N$: To find $\lambda = \lambda(\xi) \in \mathbb{R}$ and $\psi = \psi(x;\xi)$ (no identically zero) such that

$$\begin{cases} A\psi(\cdot;\xi) = \lambda(\xi)\psi(\cdot;\xi) \quad \text{in} \quad \mathbb{R}^N, \\ \psi(\cdot;\xi) \text{ is } (\xi, Y)\text{-periodic, i.e.,} \\ \psi(y+2\pi m;\xi) = e^{2\pi i m\cdot\xi}\psi(y) \quad \forall m \in Z^N,\ y \in \mathbb{R}^N, \end{cases} \tag{2.12}$$

where A is the elliptic operator in divergence form

$$A \stackrel{\text{def}}{=} -\frac{\partial}{\partial x_k}\left(a_{k\ell}(x)\frac{\partial}{\partial x_\ell}\right), \quad y \in \boldsymbol{R}^N. \tag{2.13}$$

We can write $\psi(x;\xi) = e^{ix\cdot\xi}\phi(x,\xi)$, ϕ being Y-periodic in the variable x. It is clear from (2.12) that the (ξ, Y)-periodicity is unaltered if we replace ξ by $(\xi + m)$ with $m \in Z^N$. Therefore, ξ can be confined to the dual cell $\xi \in Y' = [-\frac{1}{2}, \frac{1}{2}[^N$. Under these conditions, it is known (see [4]) that the above spectral problem admits a discrete sequence of eigenvalues with the following properties:

$$\begin{cases} 0 \le \lambda_1(\xi) \le \cdots \le \lambda_m(\xi) \le \cdots \to \infty, \\ \lambda_m(\xi) \text{ is a Lipschitz function of } \xi \in Y', \forall m \ge 1. \end{cases} \tag{2.14}$$

Besides, the corresponding eigenfunctions denoted by $\psi_m(\cdot;\xi)$ and $\phi_m(\cdot;\xi)$, form orthonormal basis in the subspaces of $L^2_{loc}(\boldsymbol{R}^N)$ which are (ξ, Y)-periodic and Y-periodic, respectively. Moreover, as a consequence of the min-max principle, it follows that (see [4])

$$\lambda_2(\xi) \ge \lambda_2^{(N)} > 0, \quad \forall \xi \in Y', \tag{2.15}$$

where $\lambda_2^{(N)}$ is the second eigenvalue of A in the cell Y with Neumann boundary conditions on ∂Y.

The Bloch waves introduced above enable us to describe the spectral resolution of the unbounded self-adjoint operator A in $L^2(\boldsymbol{R}^N)$, in the orthogonal basis of Bloch waves

$$\{\psi_m(x;\xi) = e^{ix\cdot\xi}\phi_m(x;\xi) : \, m \ge 1, \, \xi \in Y'\}.$$

Thus, we have

Proposition 1 *Let $g \in L^2(\boldsymbol{R}^N)$. The m^{th} Bloch coefficient of g is defined as follows:*

$$\widehat{g}_m(\xi) = \int_{R^N} g(x) e^{-ix\cdot\xi}\overline{\phi_m}(x;\xi)dx \quad \forall m \ge 1, \ \xi \in Y'.$$

Then the following inverse formula holds:

$$g(x) = \int_{Y'} \sum_{m=1}^{\infty} \widehat{g}_m(\xi) e^{ix\cdot\xi}\phi_m(x;\xi)d\xi.$$

Further, we have Parseval's identity:

$$\int_{R^N} |g(x)|^2 dx = \int_{Y'} \sum_{m=1}^{\infty} |\widehat{g}_m(\xi)|^2 d\xi.$$

Finally, for all g in the domain of A, we have

$$Ag(x) = \int_{Y'} \sum_{=1}^{\infty} \lambda_m(\xi)\widehat{g}_m(\xi) e^{ix\cdot\xi}\phi_m(x;\xi)d\xi,$$

and, consequently, the equivalence of norms in $H^1(\mathbb{R}^N)$ and in $H^{-1}(\mathbb{R}^N)$:

$$\|g\|^2_{H^1(\mathbb{R}^N)} = \int_{Y'} \sum_{m=1}^{\infty} (1+\lambda_m(\xi))|\widehat{g}_m(\xi)|^2 d\xi,$$
$$\|g\|^2_{H^{-1}(\mathbb{R}^N)} = \int_{Y'} \sum_{m=1}^{\infty} \frac{|\widehat{g}_m(\xi)|^2}{1+\lambda_m(\xi)} d\xi.$$

Using Proposition 1, the solution of (1.11) can be written as follows.

Lemma 1 *Let $u(x,t)$ be the solution of (1.11). Then*

$$u(x,t) = \int_{Y'} \sum_{m=1}^{\infty} \left(\beta^1_m(\xi)e^{-\alpha^1_m(\xi)t} + \beta^2_m(\xi)e^{-\alpha^2_m(\xi)t}\right) e^{ix\cdot\xi}\phi_m(x,\xi)d\xi, \tag{2.16}$$

where

$$\alpha^1_m(\xi) = \frac{1}{2}(a_0 - \sqrt{a_0^2 - 4\lambda_m(\xi)}), \tag{2.17}$$
$$\alpha^2_m(\xi) = \frac{1}{2}(a_0 + \sqrt{a_0^2 - 4\lambda_m(\xi)}), \tag{2.18}$$

and

$$\beta^1_m(\xi) = \frac{\alpha^2_m(\xi)}{\sqrt{a_0^2-4\lambda_m(\xi)}}\widehat{\varphi}^0_m(\xi) + \frac{1}{\sqrt{a_0^2-4\lambda_m(\xi)}}\widehat{\varphi}^1_m(\xi), \tag{2.19}$$
$$\beta^2_m(\xi) = -\frac{\alpha^1_m(\xi)}{\sqrt{a_0^2-4\lambda_m(\xi)}}\widehat{\varphi}^0_m(\xi) - \frac{1}{\sqrt{a_0^2-4\lambda_m(\xi)}}\widehat{\varphi}^1_m(\xi), \tag{2.20}$$

where $\widehat{\varphi}^0_m(\xi)$ and $\widehat{\varphi}^1_m(\xi)$ are the Bloch coefficients of the initial data φ^0 and φ^1.

Proof. Since $u(x,t) \in L^2(\mathbb{R}^N)$ for all $t > 0$, we have that

$$u(x,t) = \int_{Y'} \sum_{m=1}^{+\infty} \widehat{u}_m(\xi,t)e^{ix\cdot\xi}\phi_m(x,\xi)d\xi, \tag{2.21}$$

where $\widehat{u}_m(\xi,t)$ is defined by Proposition 1 and satisfies for any $m \geq 1$ the following differential equation

$$\begin{cases} \partial_t^2\widehat{u}_m + a_0\partial_t\widehat{u}_m + \lambda_m(\xi)\widehat{u}_m = 0 \text{ in } Y' \times (0,+\infty) \\ \widehat{u}_m(\cdot,0) = \widehat{\varphi}^0_m,\ \frac{\partial\widehat{u}_m}{\partial t}(\cdot,0) = \widehat{\varphi}^1_m \text{ in } Y'. \end{cases} \tag{2.22}$$

Here, ∂_t denotes the derivative with respect to t. Solving the differential equation (2.22) we find

$$\widehat{u}_m(\xi,t) = \beta^1_m(\xi)e^{-\alpha^1_m(\xi)t} + \beta^2_m(\xi)e^{-\alpha^2_m(\xi)t}, \tag{2.23}$$

where $\{\alpha_m^i(\xi),\, i=1,2\}$ are defined by (2.17) and (2.18), and they are the two roots of the characteric equation

$$y^2 + a_0 y + \lambda_m(\xi) = 0.$$

The constants $\{\beta_m^i(\xi),\, i=1,2\}$ as in (2.19)-(2.20) are obtained in order to meet the initial data in (1.11)$\square$.

We also have the following result on the dependence of λ_1 and ϕ_1 with respect to the parameter ξ (see [4] and [2]).

Proposition 2 *Assume that the coefficients $a_{k\ell}$ satisfy (1.2). Then there exists $\delta_1 > 0$ such that the first eigenvalue λ_1 is an analytic function on $B_{\delta_1} = \{\xi : |\xi| < \delta_1\}$ and satisfies*

$$c_1|\xi|^2 \le \lambda_1(\xi) \le c_2|\xi|^2, \quad \forall \xi \in Y', \tag{2.24}$$

and

$$\begin{aligned} \lambda_1(0) = \partial_k \lambda_1(0) = 0, &\qquad k = 1, \ldots, N, \\ \partial_{k\ell}^2 \lambda_1(0) = 2q_{k\ell}, &\qquad k, \ell = 1, \ldots, N, \\ \partial^\alpha \lambda_1(0) = 0 &\qquad \forall \alpha \text{ such that } |\alpha| \text{ is odd.} \end{aligned} \tag{2.25}$$

Futhermore, there is a choice of the first eigenfunction $\phi_1(x,\xi)$ satisfying

$$\begin{cases} \xi \to \phi_1(x,\xi) \in L^\infty \cap H^1_\#(Y) \text{ is analytic on } B_{\delta_1} \\ \\ \phi_1(x,0) = (2\pi)^{-\frac{N}{2}}. \end{cases}$$

The coefficients $q_{k\ell}$ are those of the homogenized matrix associated with the family $(a_{k\ell}^\varepsilon)$, where $a_{k\ell}^\varepsilon(x) = a_{k\ell}(x/\varepsilon)$ as $\varepsilon \to 0$. Since $\alpha_1^1(\xi)$ and $\beta_1^1(\xi)$ is defined by (2.19), respectively, we have:

Proposition 3 *Assume the same hypotheses as in Proposition 2. Then there exists $\delta > 0$, with $\delta \le \delta_1$, such that $\alpha_1^1(\xi)$ and $\beta_1^1(\xi)$ are analytic functions on B_δ. Futhermore, $\alpha_1^1(\xi)$ satisfies*

$$c_3|\xi|^2 \le \alpha_1^1(\xi) \le c_4|\xi|^2, \quad \forall \xi \in B_\delta, \tag{2.26}$$

and

$$\begin{aligned} \alpha_1^1(0) = \partial_k \alpha_1^1(0) = 0 &\qquad k = 1, \cdots, N, \\ \partial_{k\ell}^2 \alpha_1^1(0) = 2\frac{q_{k\ell}}{a_0}, &\qquad k, \ell = 1, \cdots, N, \\ \partial^\beta \alpha_1^1(0) = 0 &\qquad \forall \beta \text{ such that } |\beta| \text{ is odd.} \end{aligned} \tag{2.27}$$

3. Asymptotic expansion when $\rho \equiv 1$

3.1. *Bloch component of u with exponential decay*

We start this section proving that, in (2.16), the terms corresponding to the eigenvalues $\lambda_m(\xi)$, $m \geq 2$, decay exponentially as $t \to \infty$. Further, we also prove that the term corresponding to $\lambda_1(\xi)$ goes to zero exponentially, a $t \to \infty$, whenever $\xi \in U = \{\xi \in Y' : |\xi| > \delta\}$, with $\delta > 0$.

Lemma 2 *Let $\widehat{u}_m = \widehat{u}_m(\xi, t)$, $m \geq 1$, be the Bloch coefficients associated to the solution $u = u(x,t)$ of (1.11) given in (2.16). Then, there exists positive constants α and β, such that*

$$\int_{Y'} \sum_{m=2}^{\infty} |\widehat{u}_m(\xi,t)|^2 \, d\xi \leq \alpha e^{-\beta t} \left(\|\varphi^0\|^2 + \|\varphi^1\|^2_{H^{-1}} \right). \tag{3.28}$$

Proof. We consider the Lyapunov function

$$L_m(\xi,t) = E_m(\xi,t) + \varepsilon F_m(\xi,t),$$

where

$$\begin{aligned} E_m(\xi,t) &= \frac{1}{2}\left(|\partial_t \widehat{u}_m(\xi,t)|^2 + \lambda_m(\xi) |\widehat{u}_m(\xi,t)|^2 \right), \\ F_m(\xi,t) &= \partial_t \widehat{u}_m(\xi,t) \overline{\widehat{u}_m}(\xi,t) + \frac{a_0}{2} \left| \widehat{u}_m(\xi,t) \right|^2, \end{aligned}$$

and ε is a suitable constant to be chosen later. Here, $\bar{\cdot}$ denotes the complex conjugate. It follows from (2.15) that for $m \geq 2$,

$$\begin{aligned} |L_m(\xi,t) - E_m(\xi,t)| &\leq \varepsilon\Big[\frac{1}{2\lambda_2^{(N)}} |\partial_t \widehat{u}_m(\xi,t)|^2 + \Big(\frac{\lambda_m(\xi)}{2} + \frac{a_0 \lambda_m(\xi)}{2\lambda_2^{(N)}}\Big) |\widehat{u}_m(\xi,t)|^2\Big] \\ &\leq \varepsilon \, c_0 \, E_m(\xi,t), \quad \text{ with } \; c_0 = \max\left(\frac{1}{\lambda_2^{(N)}}, \frac{a_0}{\lambda_2^{(N)}} + 1 \right). \end{aligned}$$

Consequently, we have for $\varepsilon < 1/c_0$ that

$$(1 - \varepsilon c_0) \, E_m(\xi,t) \leq L_m(\xi,t) \leq (1 + \varepsilon c_0) \, E_m(\xi,t). \tag{3.29}$$

Now, we claim that

$$\partial_t L_m(\xi,t) \leq -c L_m(\xi,t) \tag{3.30}$$

holds for some positive constant independent of ξ whenever $m \geq 2$. Clearly, from (3.29) and (3.30) we conclude the proof of (3.28) for all $m \geq 2$.

In order to prove the claim, we proceed as follows: Multiplying the equation (2.22) by $\overline{\partial_t \widehat{u}_m}(\xi,t)$, we obtain

$$\partial_t E_m(\xi,t) = -a_0 |\partial_t \widehat{u}_m(\xi,t)|^2.$$

Next, we multiply the equation (2.22) by $\overline{\widehat{u}_m}(\xi,t)$ to obtain

$$\partial_t \left(\partial_t \widehat{u}_m(\xi,t)\overline{\widehat{u}_m}(\xi,t) + \frac{a_0}{2}|\widehat{u}_m(\xi,t)|^2\right) = -\lambda_m(\xi)|\widehat{u}_m(\xi,t)|^2 + |\partial_t \widehat{u}_m(\xi,t)|^2$$

Adding the identities above and choosing ε small we deduce that

$$\begin{aligned}\partial_t L_m(\xi,t) &= -a_0|\partial_t \widehat{u}_m(\xi,t)|^2 - \varepsilon\lambda_m(\xi)|\widehat{u}_m(\xi,t)|^2 + \varepsilon|\partial_t \widehat{u}_m(\xi,t)|^2 \\ &\leq -c_1(\varepsilon)\, E_m(\xi,t) \quad \text{ with } 0 < c_1(\varepsilon) = 2\min(\varepsilon, a_0 - \varepsilon). \quad (3.31)\end{aligned}$$

Now, using (3.29) with ε sufficiently small and satisfying (3.31), in particular $\varepsilon < a_0$, there exits $c_2 = c_2(\varepsilon) > 0$ such that

$$\partial_t L_m(\xi,t) \leq -c_2 L_m(\xi,t), \text{ where } c_2 = \frac{c_1}{1-\varepsilon(c_0)^{-1}} > 0.$$

Therefore,

$$L_m(\xi,t) \leq L_m(\xi,0)e^{-c_2 t},$$

and using again (3.29) we obtain

$$E_m(\xi,t) \leq cE_m(\xi,0)e^{-c_2 t}, \tag{3.32}$$

where $c = 1 + \varepsilon(c_0)^{-1}$. Recalling the definition of $E_m(\xi,t)$ we have from (3.32) and (2.14) that

$$\lambda_m(\xi)|\widehat{u}_m(\xi,t)|^2 \leq ce^{-c_2 t}\{|\partial_t \widehat{u}_m(\xi,0)|^2 + \lambda_m(\xi)|\widehat{u}_m(\xi,0)|^2\}$$

and

$$|\widehat{u}_m(\xi,t)|^2 \leq ce^{-c_2 t}\{\frac{1}{\lambda_2^{(N)}}|\partial_t \widehat{u}_m(\xi,0)|^2 + |\widehat{u}_m(\xi,0)|^2\}.$$

Adding the inequalities above it follows that

$$(1+\lambda_m(\xi))|\widehat{u}_m(\xi,t)|^2 \leq ce^{-c_2 t}(1 + \frac{1}{\lambda_2^{(N)}})\{|\partial_t \widehat{u}_m(\xi,0)|^2 + (1+\lambda_m(\xi))|\widehat{u}_m(\xi,0)|^2\},$$

and consequently

$$|\widehat{u}_m(\xi,t)|^2 \leq ce^{-c_2 t}(1 + \frac{1}{\lambda_2^{(N)}})\{|\widehat{u}_m(\xi,0)|^2 + \frac{|\partial_t \widehat{u}_m(\xi,0)|^2}{1+\lambda_m(\xi)}\}.$$

Therefore,

$$\int_{Y'} \sum_{m=2}^{\infty} |\widehat{u}_m(\xi,t)|^2 \, d\xi \leq c_3 e^{-c_2 t} \int_{Y'} \sum_{m=2}^{\infty} \{|\widehat{\varphi}_m^0(\xi)|^2 + \frac{|\widehat{\varphi}_m^1(\xi)|^2}{1+\lambda_m(\xi)}\} d\xi,$$

where $\widehat{\varphi}_m^0(\xi)$ and $\widehat{\varphi}_m^1(\xi)$ are the Bloch coefficients of the initial data φ^0 and φ^1, respectively. This completes the proof of (3.28)$\square$.

Remark 1 *We choose $\delta > 0$ in Proposition 3 such that $\delta \leq \delta_1$ (δ_1 being the radius of the ball where λ_1 and ϕ_1 are analytic as in Proposition 2) and satisfying that*

$$a_0^2 - 4\lambda_1(\xi) \geq a_0^2 - 4c_2\delta^2 = c_\delta > 0, \, \forall \xi \in B_\delta. \tag{3.33}$$

Obviously this can be done since $\lambda_1(0) = \nabla_\xi \lambda(0) = 0$.

Lemma 3 *Let $\widehat{u}_1 = \widehat{u}_1(\xi, t)$ be the first Bloch coefficient of the solution u of (1.11) given in (2.16). Then, there exist positive constants α and β, such that*

$$\int_{Y'-B_\delta} |\widehat{u}_1(\xi,t)|^2 \, d\xi \leq \alpha e^{-\beta t} \left(\|\varphi^0\|^2 + \|\varphi^1\|^2_{H^{-1}} \right), \tag{3.34}$$

with δ satisfying (3.33). On the other hand, the second component $\beta_1^2(\xi) e^{-\alpha_1^2(\xi)t}$ of $\widehat{u}_1(\xi,t)$ as in (2.23) satisfies

$$\int_{B_\delta} |\beta_1^2(\xi)|^2 e^{-2\alpha_1^2(\xi)t} \, d\xi \leq C e^{-a_0 t} \left(\|\varphi^0\|^2 + \|\varphi^1\|^2_{H^{-1}} \right). \tag{3.35}$$

Proof. In order to prove (3.34) we argue as in Lemma 2. We consider the Lyapunov fuction $L_1(\xi, t)$ and, instead of (2.15), we use Proposition 2 and Remark 1 to obtain (3.29) for $\xi \in Y' - B_\delta$ and $m = 1$. We observe that (2.24) gives us

$$\frac{1}{\lambda_1(\xi)} < \frac{1}{c_1\delta^2}, \quad \forall \xi \in Y' - B_\delta,$$

and (3.34) is obtained following the same steps of Lemma 2.

To prove (3.35) it is enough to observe that, thanks to (3.33), and since, according to (2.18) $\alpha_1^2(\xi) \geq a_0/2$ for all $\xi \in B_\delta$, we have

$$\begin{aligned}
(1 + \lambda_1(\xi))|\beta_1^2(\xi)|^2 e^{-2\alpha_1^2(\xi)t} &\leq (1 + c_2|\xi|^2)|\beta_1^2(\xi)|^2 e^{-2\alpha_1^1(\xi)t} \\
&\leq (1 + c_2\delta^2)\frac{4 + (a_0 - \sqrt{c_\delta})^2}{2c_\delta} e^{-a_0 t}(|\widehat{\varphi}_0^1(\xi)|^2 + |\widehat{\varphi}_1^1(\xi)|^2) \\
&\leq (1 + c_2\delta^2)\frac{4 + (a_0 - \sqrt{c_\delta})^2}{2c_\delta} e^{-a_0 t}((1 + \lambda_1(\xi))|\widehat{\varphi}_0^1|^2 + |\widehat{\varphi}_1^1|^2).
\end{aligned}$$

Consequently,

$$|\beta_1^2(\xi)|^2 e^{-2\alpha_1^2(\xi)t} \leq C(\delta) e^{-a_0 t} \left(|\widehat{\varphi}_1^0(\xi)|^2 + \frac{|\widehat{\varphi}_1^1(\xi)|^2}{1 + \lambda_1(\xi)} \right),$$

and the result follows $\square$.

3.2. *Bloch component of u with polynomial decay*

Thanks to Lemmas 2 and 3, to conclude the proof of the Theorem 1 it is sufficient to analyse

$$I(x,t) = \int_{B_\delta} \beta_1^1(\xi) e^{-\alpha_1^1(\xi)t} e^{ix\cdot\xi} \phi_1(x;\xi) d\xi, \tag{3.36}$$

since the other components of u have an exponentially decay in $L^2(\mathbb{R}^N)$. To do that we make use of classical asymptotic lemmas (see Lemma 5 and Lemma 6 below) and assume that the initial data $\varphi^0 \in L^1(\mathbb{R}^N)$ and $\varphi^1 \in L^1(\mathbb{R}^N) \cap H^{-1}(\mathbb{R}^N)$ are such that $|x|^{k+1}\varphi^0(x)$, $|x|^{k+1}\varphi^1(x) \in L^1(\mathbb{R}^N)$. Under these conditions the first Bloch coefficients $\widehat{\varphi}_1^0(\xi)$ and $\widehat{\varphi}_1^1(\xi)$ of the initial data belong to $C^{k+1}(B_\delta)$ (see Lemma 4), what is crucial in the proof of the asymptotic expansion (see the definitions of $\beta_1^1(\xi)$ and $\alpha_1^1(\xi)$ in Lemma 1).

Lemma 4 *Let $\varphi \in L^1(\mathbb{R}^N)$ be a function such that $|x|^k\varphi \in L^1(\mathbb{R}^N)$. Then , its first Bloch coefficient $\widehat{\varphi}_1(\xi)$, belongs to $C^k(B_\delta)$, where B_δ is a neighborhood of $\xi = 0$ where the first Bloch wave $\phi_1(x;\xi)$ is analytic.*

Proof. Since

$$\widehat{\varphi}_1(\xi) = \int_{R^N} \varphi(x) e^{-i\cdot\xi} \overline{\phi_1}(x;\xi) dx,$$

for all $\alpha \in (\mathbb{N} \cup \{0\})^N$ with $|\alpha| \leq k$, we have

$$\frac{\partial \widehat{\varphi}_1}{\partial \xi_\alpha}(\xi) = \sum_{\beta \leq \alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \int_{R^N} \varphi(x)(-i)^{|\beta|} x^\beta e^{-i\cdot\xi} \frac{\partial \overline{\phi_1}}{\partial \xi_{\alpha-\beta}}(x;\xi) dx,$$

where $\beta \leq \alpha$ if and only if $\beta_k \leq \alpha_k$ for all $j = 1, \ldots, N$, and

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \prod_{k=1}^{N} \begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix}.$$

Moreover, Proposition 2 gives us that the function $\xi \to \phi_1(x;\xi)$ is analytic with values in $L^\infty_\#(Y)$, what guarantees that

$$\begin{aligned} \left| \frac{\partial \widehat{\varphi}_1}{\partial \xi_\alpha}(\xi) \right| &\leq \sum_{\beta \leq \alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} c_\beta \int_{R^N} |\varphi(x) x^\beta| dx, \\ &\leq \sum_{\beta \leq \alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} c_\beta \int_{R^N} (1 + |x|^k)|\varphi(x)| dx, \end{aligned}$$

for $0 \leq |\alpha| \leq k$, where c_β is a positive constant. Thus, using again Proposition 2, we have that the map $\xi \to e^{-i\xi\cdot x}\partial_\xi^\alpha \phi_1(x;\xi)$ is continuous, and the result follows $\square$.

Now, we are going to present some basic asymptotic results. The following definition will be useful to simplify the notation.

Definition 1 *Given $f, g \in C(\mathbb{R};\mathbb{R})$, we say that f and g are of the same order as $t \to \infty$ and we denote it by $f \sim g$ when*

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = 1.$$

The following basic lemmas on asymptotic analysis are needed (see [1], p. 263):

Lemma 5 *Let $f : [0, b] \to \mathbb{R}$ be a continuous function such that it has the uniform asymtotic series expansion*

$$f(x) = x^{\alpha} \sum_{n=1}^{\infty} a_n x^{\beta_n},\ x \in [0, b],$$

with $\alpha > 1$ and $\beta_n > 0$. Then

$$\int_0^b e^{-tx} f(x)dx \sim \sum_{n=1}^{\infty} a_n \frac{\Gamma(\alpha+\beta_n+1)}{t^{(\alpha+\beta_n+1)}}, \quad as\ t \to \infty.$$

When $f(x) = x^{\alpha}$ we have that

$$\int_0^b e^{-tx} x^{\alpha} dx \sim \frac{\Gamma(\alpha+1)}{t^{(\alpha+1)}}, \quad as\ t \to \infty,$$

where

$$\Gamma(z) = \int_0^{\infty} e^{-x} x^{z-1} dx,\ z > 0.$$

As a consequence of Lemma 5, the following holds in [7]:

Lemma 6 *Let $c > 0$. Then*

$$\int_{b_\delta} e^{-c|\xi|^2 t} |\xi|^k \, d\xi \sim c_k t^{-\frac{k+N}{2}}, \quad as\ t \to \infty,$$

for all $k \in \mathbb{N}$, where c_k is a positive constant that may be computed explicitly. On the other hand , if $q = (q_{ij})$ is a symmetric positive matrix, we also have

$$\int_{Y'} e^{-\frac{q_{ij}}{a_0}\xi_i \xi_j t} \xi^{\beta} \, d\xi \sim c_{|\beta|} t^{-\frac{(|\beta|+N)}{2}}, \quad as\ t \to \infty,$$

for all multi-index $\beta \in (\mathbb{N} \cup \{0\})^N$ and for a suitable constant $c_{|\beta|}$ that may computed as well.

In the sequel we prove three lemmas which are related to the asymptotic behavior of the terms $\beta_1^1(\xi)$, $\phi_1(x;\xi)$ and $e^{-\alpha_1^1(\xi)t}$, respectively, that appear in (3.36).

We recall that the idea we have in mind is to prove that the solutions of (1.1) may be approximated by a linear combination of the derivatives of the fundamental solution of the homogenized heat equation and, as we said in the begining of this section, in view of Lemma 5 and 6, our analysis may be restricted to consider (3.36). Thus, our first step in this direction is to prove that (3.36) can be replaced by

$$J(x,t) = \int_{B_\delta} \sum_{|\alpha|\le k} d_\alpha \xi^\alpha e^{-\alpha_1^1(\xi)t} e^{ix\cdot\xi}\phi_1(x;\xi)d\xi, \quad (x,t)\in \mathbb{R}^N\times\mathbb{R}^+, \tag{3.37}$$

where

$$d_\alpha = \frac{1}{\alpha!}\partial^\alpha \beta_1^1(0), \tag{3.38}$$

which are obtained by means of the Taylor expansion of $\beta_1^1(\xi)$ in $\xi = 0$.

Lemma 7 *Let $\varphi^0 \in L^1(\mathbb{R}^N)\cap L^2(\mathbb{R}^N)$ and $\varphi^1 \in L^1(\mathbb{R}^N)\cap H^{-1}(\mathbb{R}^N)$ be such that $|x|^{k+1}\varphi^0(x)$, $|x|^{k+1}\varphi^1(x) \in L^1(\mathbb{R}^N)$. Consider $I(x,t)$ as in (3.36) and $J(x,t)$ as in (3.37). Then, there exist $c_k > 0$ such that*

$$\|I(\cdot,t) - J(\cdot,t)\| \le c_k t^{-\frac{2k+2+N}{4}} \text{ as } t\to\infty.$$

Proof. Since $\varphi^0 \in L^1(\mathbb{R}^N)\cap L^2(\mathbb{R}^N)$ and $\varphi^1 \in L^1(\mathbb{R}^N)\cap H^{-1}(\mathbb{R}^N)$ with $|x|^{k+1}\varphi^0$, $|x|^{k+1}\varphi^1 \in L^1(\mathbb{R}^N)$ then, from Lemma 4, we have that $\widehat{\varphi}_1^0, \widehat{\varphi}_1^1 \in C^{k+1}(B_\delta)$, and, according to (2.19), $\beta_1^1 \in C^{k+1}(B_\delta)$ as well. Thus, thanks to (3.38), from the Taylor expansion we have that for all $\xi\in B_\delta$,

$$|\beta_1^1(\xi) - \sum_{|\alpha|\le k} d_\alpha\xi^\alpha| \le C_k|\xi|^{k+1}, \quad C_k > 0. \tag{3.39}$$

These constants can be computed explicitly in terms of $\partial^\alpha\lambda_1(0)$ and $\partial^\alpha\phi_1(0)$, and the mass of the initial datum (see Section 5). Indeed, from Parseval's identity, we have

$$\|I(\cdot,t) - J(\cdot,t)\|^2 = \int_{B_\delta} |\beta_1^1(\xi) - \sum_{|\alpha|\le k} d_\alpha\xi^\alpha|^2 e^{-2\alpha_1^1(\xi)t}d\xi.$$

Then, thanks to estimate (2.26) in Proposition 3 and Lemma 6, we obtain

$$\begin{aligned}\|I(\cdot,t) - J(\cdot,t)\|^2 &= \int_{B_\delta} |\beta_1^1(\xi) - \sum_{|\alpha|\le k} d_\alpha\xi^\alpha|^2 e^{-2c_3|\xi|^2 t}d\xi \\ &\le c_k \int_{B_\delta} |\xi|^{2(k+1)} e^{-2c_3|\xi|^2 t}d\xi \sim c_k t^{-\frac{2k+2+N}{2}}, \text{ as } t\to\infty.\end{aligned}$$

This conclues the proof of Lemma 7 □ .

In a second step, we compute the Taylor expansion of $\phi_1(x,\xi)$ around $\xi=0$, and prove that all the terms entering in the definition (3.37) of J and that we denote by J_α with $\alpha \in (\mathbb{N}\cup 0)^N$,

$$J_\alpha(x,t) = \frac{1}{(2\pi)^N}\int_{B_\delta} \xi^\alpha e^{-\alpha_1^1(\xi)t} e^{ix\cdot\xi}\phi_1(x;\xi)d\xi, \quad (x,t)\in \mathbb{R}^N\times\mathbb{R}^+, \tag{3.40}$$

may be approximated in L^2-setting by a linear combination of the form

$$\frac{1}{(2\pi)^N}\sum_{|\gamma|\le k-|\alpha|} \widetilde{d}_\gamma(x)\int_{B_\delta} \xi^\alpha e^{-\alpha_1^1(\xi)t} e^{ix\cdot\xi},$$

where $\widetilde{d}_\gamma$ are periodic functions defined by

$$\widetilde{d}_\alpha(\cdot) = \frac{1}{\alpha!}\partial_\xi^\alpha \phi_1(\cdot;0) \in L^\infty_\#(\mathbb{R}^N). \tag{3.41}$$

This way be done obtaining the same rate of decay as in Lemma 7.

Now we present a result from [2] that will be needed when stating and proving in Lemma 9 the facts metioned above.

Lemma 8 *Let us introduce*

$$G(x) = \int_{Y'} g(\xi)e^{ix\cdot\xi}\omega(x;\xi)d\xi, \quad x\in\mathbb{R}^N, \tag{3.42}$$

where $g\in L^2(Y')$ *and* $\omega\in L^\infty(Y';L^2_\#(Y))$. *Then we have*

$$\|G\|^2 = \int_{Y'} |g(\xi)|^2 \|\omega(\cdot;\xi)\|^2_{L^2(Y)}d\xi.$$

Proof. To check this result we expand $\omega(x;\xi)$ as a function of x in the orthonormal basis $\{\phi_m(x;\xi)\}_{m=1}^\infty$ where $\xi\in Y'$ is a parameter:

$$\omega(x;\xi) = \sum_{m=1}^\infty a_m(\xi)\phi_m(x;\xi).$$

Introducing this expression in (3.42), we get

$$G(x) = \int_{Y'} g(\xi)\sum_{m=1}^\infty a_m(\xi)e^{ix\cdot\xi}\phi_m(x;\xi)d\xi.$$

Applying the Parseval's identity of Proposition 1, it follows that

$$\|G\|^2 = \int_{Y'} |g(\xi)|^2\sum_{m=1}^\infty |a_m(\xi)|^2 d\xi.$$

This completes the proof of the lemma if we use the Parseval's identity in $L^2(Y)$:

$$\|\omega(\cdot;\xi)\|^2_{L^2(Y)} = \sum_{m=1}^{\infty} |a_m(\xi)|^2 \quad \forall \xi \in Y' \, \square .$$

Lemma 9 *We consider $J_\alpha(x,t)$ defined in (3.40), with $|\alpha| \le k$, and*

$$I_\alpha(x,t) = \frac{1}{(2\pi)^N} \int_{B_\delta} \xi^\alpha e^{-\alpha_1^1(\xi)t} e^{ix\cdot\xi} d\xi, \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}^+, \tag{3.43}$$

Then there exits periodic functions $\widetilde{d}_\gamma = \widetilde{d}_\gamma(x)$ defined in (3.41), such that

$$\Big\| J_\alpha(\cdot,t) - \sum_{|\gamma| \le k-|\alpha|} \widetilde{d}_\gamma(\cdot) I_{\gamma+\alpha}(\cdot,t) \Big\| \le c_{k,|\alpha|}\, t^{-\frac{2k+2+N}{4}}, \ \text{as } t \to \infty.$$

Proof. We set

$$R_k(x;\xi) = \phi_1(x;\xi) - \sum_{|\alpha| \le k} \widetilde{d}_\alpha(x) \xi^\alpha,$$

where $\widetilde{d}_\alpha(\cdot)$ is defined in (3.41). Since ϕ_1 is an analytic function with respect to ξ in B_δ and values in $L^2(Y)$ we have, for all $\xi \in B_\delta$,

$$\|R_k(\cdot;\xi)\|_{L^2(Y)} \le C_k |\xi|^{k+1}, \ C_k > 0.$$

Thus, $R_k \in L^\infty(Y'; L^2_\#(Y))$. Then, for $\alpha \in (\mathbb{N} \cup \{0\})^N$ with $|\alpha| \le k$, we have

$$J_\alpha(x,t) - \sum_{|\gamma| \le k-|\alpha|} \widetilde{d}_\gamma(x) I_{\gamma+\alpha}(x,t) = \frac{1}{(2\pi)^N} \int_{B_\delta} \xi^\alpha e^{-\alpha_1^1(\xi)t} e^{ix\cdot\xi} R_k(x;\xi) d\xi,$$

and since $R_k(y;\xi)$ is Y-periodic in the variable y, it follows from Lemma 8 that

$$\begin{aligned} \Big\| J_\alpha(\cdot,t) - \sum_{|\gamma|=0}^{k-|\alpha|} \widetilde{d}_\gamma(\cdot) I_{\gamma+\alpha}(\cdot,t) \Big\|^2 &\le \int_{B_\delta} \frac{|\xi|^{2|\alpha|}}{(2\pi)^{2N}} e^{-2\alpha_1^1(\xi)t} \big\| R_{k-|\alpha|}(\cdot;\xi) \big\|^2_{L^2(Y)} d\xi \\ &\le C_{k-|\alpha|} \int_{B_\delta} |\xi|^{2k+2} e^{-2\alpha_1^1(\xi)t} d\xi, \end{aligned}$$

and we conclude as in Lemma 7, using the Proposition 2 and Lemma 6 $\square$.

The proofs of Lemmas 7 and 9, as well as that of Lemma 10 that we present below, provide a systematic way of computing the coefficients that appear in the statement of Theorem 1. We note that the functions $c_\alpha(\cdot)$ are related to the derivatives of the first Bloch eigenfunction $\phi_1(x;\xi)$ with respect to ξ in $\xi = 0$. We shall describe how to compute them explicitly in Section 5.

Now, we are going to study the asymptotic behavior of the integral $I_\alpha(x,t)$ defined in Lemma 9. We observe that, according to Proposition 3,

$$\alpha_1^1(0) = \partial_k \alpha_1^1(0) = 0, \qquad k = 1, \dots, N,$$
$$\partial^2_{k\ell}\alpha_1^1(0) = \frac{2q_{k\ell}}{a_0}, \qquad \ell, k = 1, \dots, N,$$

where the coefficients $q_{k\ell}$ are those of the homogenized matrix associated with the family $(a^\varepsilon_{k\ell})$, where $a^\varepsilon_{k\ell}(x) = a_{k\ell}(x/\varepsilon)$, as $\varepsilon \to 0$. Then, for all $\xi \in B_\delta$, we have

$$e^{-\alpha_1^1(\xi)t} \sim e^{-\frac{q_{k\ell}}{a_0}\xi_k\xi_\ell t}, \qquad \text{as } t \to \infty.$$

This fact provides a first idea of the behavior of $I_\alpha(x,t)$, defined in (3.43), as $t \to \infty$. Futhermore, in Lemma 10 it turns evident that solutions of (1.1) behave as a linear combination of functions $G^*_\alpha(x,t)$ introduced in (1.7), which are the derivatives $(-i)^{|\alpha|}\partial^\alpha_x$ of the fundamental solution G^* of the homogenized heat equation.

Lemma 10 *We consider the function $I_\alpha(x,t)$ defined in (3.43) with $|\alpha| \le k$. Then, there exist constans $c_{\beta,n}$, with $4 \le |\beta| \le 4p(\alpha)$, such that*

$$\left\| I_\alpha(\cdot,t) - I^*_\alpha(\cdot,t) - \sum_{n=1}^{p(\alpha)} \frac{t^n}{n!} \sum_{m=0}^{a(\alpha,n)} \sum_{|\beta|=4n+2m} c_{\beta,n} I^*_{\alpha+\beta}(\cdot,t) \right\| \le c_{k,|\alpha|} t^{-\frac{2k+2+N}{4}},$$

as $t \to \infty$, with $p(\alpha) = \left[\frac{k-|\alpha|}{2}\right]$, $a(\alpha,n) = p(\alpha) - n$, and where for $\alpha \in (\mathbb{N} \cup 0)^N$

$$I^*_\alpha(x,t) = \frac{1}{(2\pi)^N} \int_{B_\delta} \xi^\alpha e^{-\frac{q_{k\ell}}{a_0}\xi_k\xi_\ell t} e^{ix\cdot\xi} d\xi, \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}^+. \tag{3.44}$$

Here and in the sequel $[\cdot]$ denotes the integer part.

Remark 2 *In view of (1.7) with $\bar\rho = 1$ and (3.44) the analogy between I^*_α and G^*_α is clear. I^*_α is in fact obtained by integrating the same quantity as in G^*_α but this time in B_δ instead of $\mathbb{R}^N$. In particular, $I^*_\alpha(x,t)$ has the same polynomial decay as $G^*_\alpha(x,t)$, because*

$$\|G^*_\alpha(\cdot,t) - I^*_\alpha(\cdot,t)\| \le ce^{-c(\delta)t}, \qquad \textit{with } c(\delta) > 0. \tag{3.45}$$

Indeed, by Parseval's identity and the coercitivity of coefficients $\{q_{k\ell}\}$ we have

$$\begin{aligned} \|G^*_\alpha(\cdot,t) - I^*_\alpha(\cdot,t)\|^2 &= \frac{1}{(2\pi)^{2N}} \int_{R^N - B_\delta} |\xi|^{2|\alpha|} e^{-2\frac{q_{k\ell}}{a_0}\xi_k\xi_\ell t} d\xi \\ &\le \frac{1}{(2\pi)^{2N}} \int_{R^N - B_\delta} |\xi|^{2|\alpha|} d\xi e^{-2\frac{c}{a_0}|\xi|^2 t} d\xi \le ce^{-c(\delta)t}. \end{aligned}$$

Proof of Lemma 10. We consider the map $\xi \in B_\delta \to \nu(\xi)$ given by

$$\nu(\xi) = \alpha_1^1(\xi) - \frac{1}{2}\partial^2_{k\ell}\alpha_1^1(0)\xi_k\xi_\ell = \alpha_1^1(\xi) - \frac{q_{k\ell}}{a_0}\xi_k\xi_\ell. \tag{3.46}$$

Thanks to Proposition 3, the map $\nu = \nu(\xi)$ is analytic in B_δ. Moreover, it follows from (2.27) that for $\xi \in B_\delta$

$$|\nu(\xi)| = \left|\alpha_1^1(\xi) - \frac{q_{k\ell}}{a_0}\xi_k\xi_\ell\right| \le c|\xi|^4. \tag{3.47}$$

The function $(e^{-\nu(\xi)t} - 1)$ is also analytic, and by (3.47) we have

$$\left|e^{-\nu(\xi)t} - \sum_{n=0}^{p}\frac{t^n}{n!}(-\nu(\xi))^n\right| \le C_p\,(\nu(\xi)t)^{p+1} \le c_p|\xi|^{4p+4}t^{p+1}. \tag{3.48}$$

Thus, defining for $p \ge 1$

$$\nu_{\alpha,p}(x,t) = \frac{1}{(2\pi)^N}\int_{B_\delta}\xi^\alpha[\sum_{n=0}^{p}\frac{t^n}{n!}(-\nu(\xi))^n]e^{-\frac{q_{k\ell}}{a_0}\xi_k\xi_\ell t}e^{ix\cdot\xi}d\xi,\ (x,t) \in \mathbb{R}^N \times \mathbb{R}^+,$$

and replacing (3.46) in $I_\alpha(x,t)$, we get

$$I_\alpha(x,t) - \nu_{\alpha,p}(x,t) = \frac{1}{(2\pi)^N}\int_{B_\delta}\xi^\alpha e^{-\frac{q_{k\ell}}{a_0}\xi_k\xi_\ell t}[e^{-\nu(\xi)t} - \sum_{n=0}^{p}\frac{t^n}{n!}(-\nu(\xi))^n]e^{ix\cdot\xi}d\xi.$$

Then, from Parseval's identity and (3.48) it follows that

$$\begin{aligned}\|I_\alpha(\cdot,t) - \nu_{\alpha,p}(\cdot,t)\| &= \int_{B_\delta}\frac{|\xi|^{2|\alpha|}}{(2\pi)^{2N}}\left|e^{-\nu(\xi)t} - \sum_{n=0}^{p}\frac{t^n}{n!}(-\nu(\xi))^n\right|^2 e^{-2\frac{q_{k\ell}}{a_0}\xi_k\xi_\ell t}d\xi \\ &\le c\,t^{2p+2}\int_{B_\delta}|\xi|^{2|\alpha|+8p+8}e^{-2\frac{q_{k\ell}}{a_0}\xi_k\xi_\ell t}d\xi \\ &\sim \tilde{c}\,t^{2p+2}t^{-\frac{2|\alpha|+8p+8+N}{2}} = \tilde{c}\,t^{-\frac{2|\alpha|+4p+4+N}{2}} \quad \text{as } t \to \infty. \end{aligned} \tag{3.49}$$

Now we choose p such that the above decay rate is of the order of $(2k+2+N)/2$, i.e., p satisfies $(2|\alpha| + 4p + 4 + N \ge 2k + N + 2)$, or, equivalently,

$$2p \quad \ge \quad k - |\alpha| - 1. \tag{3.50}$$

Thus, we have

$$\|I_\alpha(\cdot,t) - \nu_{\alpha,p}(\cdot,t)\| \quad \le \quad \tilde{c}\,t^{-\frac{2k+n+2}{4}}, \quad \text{as } t \to \infty. \tag{3.51}$$

To conclude the proof we are going to study the asymptotic behavior of the integral $\nu_{\alpha,p}(x,t)$ defined above. But before doing it, we note that if we consider the Taylor expansion of $\nu(\xi)$ around $\xi = 0$, we obtain

$$\nu_{\alpha,p}(x,t) \quad = \quad \frac{1}{(2\pi)^N}\int_{B_\delta} \xi^\alpha \{1 + \sum_{n=1}^{p} \frac{t^n}{n!} \left(\sum_{|\beta|=0}^{\infty} \frac{1}{\beta!} \partial_\xi^\beta \nu(0) \xi^\beta \right)^n \} e^{-\frac{q_{k\ell}}{a_0}\xi_k \xi_\ell t} e^{ix\cdot\xi} d\xi.$$

Indeed, since $\partial^\beta \nu(0) = 0$ for $|\beta| < 4$ and $|\beta|$ odd (see (3.46) and Proposition 3) we have

$$\nu(\xi) = \sum_{m=0}^{\infty} \sum_{|\beta|=4+2m} \frac{1}{\beta!} \xi^\beta \partial_\xi^\beta \alpha_1^1(0),$$

and, consequently

$$(-\nu(\xi))^n = (-1)^n \left(\sum_{m=0}^{\infty} \sum_{|\beta|=4+2m} \frac{1}{\beta!} \xi^\beta \partial_\xi^\beta \alpha_1^1(0) \right)^n = \sum_{m=0}^{\infty} \sum_{|\beta|=4n+2m} c_{\beta,n}\, \xi^\beta, \quad (3.52)$$

for suitable constants $c_{\beta,n}$ that will be computed in Section 5. This fact suggests the following approximation

$$\nu_{\alpha,p}(x,t) \sim \frac{1}{(2\pi)^N}\int_{B_\delta} \xi^\alpha \{1 + \sum_{n=1}^{p} \frac{t^n}{n!} \sum_{m=0}^{a(n)} \sum_{|\beta|=4n+2m} c_{\beta,n}\, \xi^\beta \} e^{-\frac{q_{k\ell}}{a_0}\xi_k \xi_\ell t} e^{ix\cdot\xi} d\xi,$$

where $a(n)$ is an index to be chosen for any $n = 1, \ldots, p$. Thus, let us define

$$\begin{aligned} \omega_{\alpha,p}(x,t) \quad &= \quad \frac{1}{(2\pi)^N}\int_{B_\delta} \xi^\alpha \{1 + \sum_{n=1}^{p} \frac{t^n}{n!} \sum_{m=0}^{a(n)} \sum_{|\beta|=4n+2m} c_{\beta,n}\, \xi^\beta \} e^{-\frac{q_{k\ell}}{a_0}\xi_k \xi_\ell t} e^{ix\cdot\xi} d\xi \\ &= \quad I_\alpha^*(x,t) + \sum_{n=1}^{p} \frac{t^n}{n!} \sum_{m=0}^{a(n)} \sum_{|\beta|=4n+2m} c_{\beta,n}\, I_{\beta+\alpha}^*(x,t), \end{aligned}$$

with $a(n)$ to be chosen later, and consider the difference

$$\begin{aligned} &\nu_{\alpha,p}(x,t) - \omega_{\alpha,p}(x,t) = \\ &\quad = \frac{1}{(2\pi)^N} \sum_{n=1}^{p} \frac{t^n}{n!} \int_{B_\delta} \xi^\alpha \big[(-\nu(\xi))^n - \sum_{m=0}^{a(n)} \sum_{|\beta|=4n+2m} \xi^\beta c_{\beta,n}\big] e^{-\frac{q_{k\ell}}{a_0}\xi_k \xi_\ell t} e^{ix\cdot\xi} d\xi. \end{aligned}$$

Now, computing in (3.52) the Taylor expansion of order $(4n + 2a(n))$ of $(-\nu(\xi))^n$, we obtain the existence of a positive constant $C_{a(n)} > 0$ satisfying, for all $\xi \in B_\delta$,

$$\left| (-\nu(\xi))^n - \sum_{m=0}^{a(n)} \sum_{|\beta|=4n+2m} \xi^\beta c_{\beta,n} \right| \le C_{a(n)}\, |\xi|^{4n+2a(n)+2}.$$

Hence,

$$\|\nu_{\alpha,p}(\cdot,t)-\omega_{\alpha,p}(\cdot,t)\|^2 \le$$

$$\le \sum_{n=1}^{p}\frac{t^{2n}}{(n!)^2}\int\limits_{B_\delta}\frac{|\xi|^{2|\alpha|}}{(2\pi)^{2N}}|(-\nu(\xi))^n-\sum_{m=0}^{a(n)}\sum_{|\beta|=4n+2m}\xi^\beta c_{\beta,n}|^2 e^{-2\frac{q_{k\ell}}{a_0}\xi_k\xi_\ell t}d\xi$$

$$\le \frac{1}{(2\pi)^{2N}}\sum_{n=1}^{p}\frac{t^{2n}}{(n!)^2}C_\nu^2\int\limits_{B_\delta}|\xi|^{2|\alpha|}|\xi|^{2(4n+2a(n)+2)}e^{-2\frac{q_{k\ell}}{a_0}\xi_k\xi_\ell t}d\xi$$

$$\le c\sum_{n=1}^{p}t^{-\frac{N+4n+4+4a(n)+2|\alpha|}{2}}, \text{ as } t\to\infty.$$

Finally, in order to obtain the same decay rate as in Lemma 9 we take into account (3.49), and choose $a(n)$ satisfying $(N+4n+4+4a(n)+2|\alpha|\ge 2|\alpha|+4p+4+N)$, i.e.,

$$a(n)\ge p-n.$$

Recalling that p satisfies (3.50), we choose

$$p(\alpha)=\left[\frac{k-|\alpha|}{2}\right] \text{ and } a(\alpha,n)=\left[\frac{k-|\alpha|}{2}\right]-n.$$

This gives us that

$$\left\|\nu_{\alpha,p}(\cdot,t)-I^*_\alpha(\cdot,t)-\sum_{n=1}^{p(\alpha)}\frac{t^n}{n!}\sum_{m=0}^{a(n,\alpha)}\sum_{|\beta|=4n+2m}c_{\beta,n}\,I^*_{\beta+\alpha}(\cdot,t)\right\|\le c\,t^{-\frac{2k+n+2}{4}},$$

as $t\to\infty$, and, since (3.51) is satisfied, we conclude the proof of Lemma 10□ .

In the sequel, we are going to prove the main result of this section, i.e., we obtain the complete asymptotic expansion of the solution of (1.11) when $\rho\equiv 1$.

Proof of Theorem 1 with $\rho\equiv 1$.

Firstly, for $(x,t)\in \mathbb{R}^N\times\mathbb{R}^+$, we denote by

$$H(x,t) = \sum_{|\alpha|\le k}c_\alpha(x)[G^*_\alpha(x,t)+\sum_{n=1}^{p(\alpha)}\frac{t^n}{n!}\sum_{m=0}^{a(\alpha,n)}\sum_{|\beta|=4n+2m}c_{\beta,n}G^*_{\alpha+\beta}(x,t)],$$

the asymptotic expansion presented in Theorem 1, where $G^*_\alpha(x,t)$ was defined in (1.7).

From Lemma 2 and Lemma 3 it follows that

$$\begin{aligned}\|u(\cdot,t)-H(\cdot,t)\| &\le \|u(\cdot,t)-I(\cdot,t)\|+\|I(\cdot,t)-H(\cdot,t)\|\\ &\le c\,e^{-at}+\|I(\cdot,t)-H(\cdot,t)\|,\end{aligned}$$

where $I(x,t)$ is defined in (3.36). Then, to conclude the proof it is enough to prove that

$$\|I(\cdot,t)-H(\cdot,t)\| \quad \leq \quad c_k t^{-\frac{2k+2+N}{4}}, \text{ as } t\to\infty.$$

In fact, from Lemma 7 we have

$$\begin{aligned}\|I(\cdot,t)-H(\cdot,t)\| &\leq \|I(\cdot,t)-J(\cdot,t)\|+\|J(\cdot,t)-H(\cdot,t)\| \\ &\leq c_k t^{-\frac{2k+2+N}{4}}+\|J(\cdot,t)-H(\cdot,t)\|, \quad \text{as } t\to\infty, \end{aligned} \tag{3.53}$$

and recalling that

$$J(x,t)=\sum_{|\alpha|\leq k}(2\pi)^N d_\alpha J_\alpha(x,t),$$

where both, $J(x,t)$ and $J_\alpha(x,t)$, were defined in (3.37) and (3.40), respectly, we obtain from Lemma 9,

$$\begin{aligned}\|J(\cdot,t)-H(\cdot,t)\| &= \|\sum_{|\alpha|\leq k}(2\pi)^N d_\alpha\{J_\alpha(\cdot,t)-\sum_{|\gamma|\leq k-|\alpha|}\widetilde{d}_\gamma(\cdot)I_{\gamma+\alpha}(\cdot,t)\}\| \\ &\leq c'_k t^{-\frac{2k+2+N}{4}}, \quad \text{as } t\to\infty. \end{aligned} \tag{3.54}$$

Thus, if we define the periodic functions as

$$c_\alpha(x)=(2\pi)^N\sum_{\gamma\leq\alpha}\widetilde{d}_\gamma(x)d_{\alpha-\gamma}, \tag{3.55}$$

thanks to (3.53) and (3.54), it follows that

$$\|I(\cdot,t)-H(\cdot,t)\| \quad \leq \quad c't^{-\frac{2k+2+N}{4}}+\left\|\sum_{|\alpha|\leq k}c_\alpha(\cdot)I_\alpha(\cdot,t)-H(\cdot,t)\right\|, \text{ as } t\to\infty.$$

Now, we are going to prove that the following holds:

$$\left\|\sum_{|\alpha|\leq k}c_\alpha(\cdot)I_\alpha(\cdot,t)-H(\cdot,t)\right\| \quad \leq \quad c_k t^{-\frac{2k+2+N}{4}}, \quad \text{as } t\to\infty.$$

Using Lemma 10, we obtain, as $t\to\infty$,

$$\|\sum_{|\alpha|\leq k}c_\alpha(\cdot)\big[I_\alpha(\cdot,t)-I^*_\alpha(\cdot,t)-\sum_{n=1}^{p(\alpha)}\frac{t^n}{n!}\sum_{m=0}^{a(\alpha,n)}\sum_{|\beta|=4n+2m}c_{\beta,n}I^*_{\alpha+\beta}(\cdot,t)\big]\|\leq C_k t^{-\frac{2k+2+N}{4}},$$

where C_k depends on the L^∞-norm of the functions $c_\alpha(\cdot)$ and the function $I^*_\alpha(x,t)$ defined in (3.44). Finally, Remark 8 gives us that

$$\|\sum_{|\alpha|\leq k}c_\alpha(\cdot)\big(I^*_\alpha(\cdot,t)+\sum_{n=1}^{p(\alpha)}\frac{t^n}{n!}\sum_{m=0}^{a(\alpha,n)}\sum_{|\beta|=4n+2m}c_{\beta,n}I^*_{\alpha+\beta}(\cdot,t)\big)-H(\cdot,t)\|\leq ce^{-c(\delta)t},$$

as $t \to \infty$, and, returning to (3.53) we obtain

$$\Big\| \sum_{|\alpha| \leq k} c_\alpha(\cdot) I_\alpha(\cdot, t) - H(\cdot, t) \Big\| \quad \leq \quad C_k t^{-\frac{2k+2+N}{4}} + ce^{-c(\delta)t}, \qquad \text{as } t \to \infty,$$

what concludes the proof □ .

4. Proof of the general case

Theorem 1 is proved following the same steps of Section 2 and 3 for the case $\rho \equiv 1$. However, the Bloch wave decomposition used for the equation (1.11) in Section 2 can not be applied for the problem (1.1), due to variable density ρ. Consequently, we need to introduce a different spectral problem.

Given $\xi \in Y'$ we consider the spectral problem of finding numbers $\lambda = \lambda(\xi) \in \mathbb{R}$ and functions $\psi = \psi(x; \xi)$ (no identically zero) such that

$$\begin{cases} A\psi(\cdot;\xi) = \lambda(\xi)\psi(\cdot;\xi)\rho(\cdot) \quad \text{in} \quad \mathbb{R}^N, \\ \quad \psi(\cdot;\xi) \text{ is } (\xi, Y)\text{-periodic, i.e.,} \\ \psi(y + 2\pi m; \xi) = e^{2\pi i m \cdot \xi}\psi(y) \quad \forall m \in Z^N, \ y \in \mathbb{R}^N, \end{cases} \tag{4.56}$$

where A is the elliptic operator in divergence form defined in (2.13) and ρ satisfies (1.3). If we consider $\psi(x;\xi) = e^{ix\cdot\xi}\phi(x;\xi)$, the variational formulation obtained for (4.56) for any $\varphi \in H^1_\#(Y)$ is given by

$$\langle A(\xi)\phi, \varphi\rangle = \int_Y a_{k\ell}(x) \left(\frac{\partial \phi}{\partial x_k} + i\xi_k\phi\right) \overline{\left(\frac{\partial \phi}{\partial x_\ell} + i\xi_\ell\phi\right)} dx = \lambda(\xi) \int_Y \phi\overline{\varphi}\rho(x) dx.$$

Since the operator associated with (4.56) is uniformly elliptic and self-adjoint, defined in a bounded domain, it is known (see [3] and [4]) that the above spectral problem admits a discrete sequence of eigenvalues with the following properties:

$$\begin{cases} 0 \leq \lambda_1(\xi) \leq \cdots \leq \lambda_m(\xi) \leq \cdots \to \infty, \\ \lambda_m(\xi) \text{ is a Lipschitz function of } \xi \in Y', \forall m \geq 1. \end{cases} \tag{4.57}$$

Besides, the corresponding eigenfunctions denoted by $\psi_m(\cdot;\xi) = e^{i\xi\cdot x}\phi_m(\cdot;\xi)$, where the functions $\{\phi_m(x;\xi)\}$ form orthonormal basis in the space of periodic functions in $L^2_{loc}(\mathbb{R}^N; \rho(x)dx)$, i.e,

$$\int_Y \phi_m \overline{\phi}_n \rho(x) dx = \delta_{mn} \qquad \text{(Kronecker's delta)}.$$

The eigenfunctions $\psi_m(\cdot, \xi)$ and $\phi_m(\cdot, \xi)$ are (ξ, Y)-periodic and Y-periodic, respectively. Moreover, as a consequence of the min-max principle (see [4]) we have

$$\lambda_2(\xi) \geq \frac{\lambda_2^{(N)}}{\rho_1} > 0, \quad \forall \xi \in Y', \tag{4.58}$$

where $\lambda_2^{(N)}$ is the second eigenvalue of A in the cell Y with Neumann boundary condition on ∂Y for $\rho \equiv 1$ and ρ_1 is defined in (1.3).

Now, with the orthonormal basis of Bloch waves $\{e^{ix\cdot\xi}\phi_m(x;\xi) : m \geq 1,\, \xi \in Y'\}$, we have a similar Bloch wave decomposition as in Proposition 1:

Proposition 4 *Let $g \in L^2(\boldsymbol{R}^N)$. The m^{th} Bloch coefficient of g is defined as follows:*

$$\widehat{g}_m(\xi) = \int_{R^N} g(x)e^{-ix\cdot\xi}\bar{\phi}_m(x;\xi)\rho(x)dx \quad \forall m \geq 1,\ \xi \in Y'.$$

Then the following inverse formula holds:

$$g(x) = \int_{Y'} \sum_{m=1}^{\infty} \widehat{g}_m(\xi)e^{ix\cdot\xi}\phi_m(x;\xi)d\xi.$$

Further, we have Parseval's identity:

$$\|g\|^2_{L^2(\rho)} = \int_{R^N} |g(x)|^2\rho(x)dx = \int_{Y'} \sum_{m=1}^{\infty} |\widehat{g}_m(\xi)|^2 d\xi.$$

Finally, for all g in the domain of A, we have

$$Ag(x) = \rho(x)\int_{Y'} \sum_{m=1}^{\infty} \lambda_m(\xi)\widehat{g}_m(\xi)e^{ix\cdot\xi}\phi_m(x;\xi)d\xi.$$

Using Proposition 4, the equation (1.1) can be written as follows:

$$\int_{Y'} \sum_{m=1}^{\infty} \left(\partial_t^2\widehat{u}_m(\xi,t) + \lambda_m(\xi)\widehat{u}_m(\xi,t) + a_0\partial_t\widehat{u}_m(\xi,t)\right) e^{ix\cdot\xi}\phi_m(x;\xi)\rho(x)d\xi = 0.$$

Since $\{e^{ix\cdot\xi}\phi_m(x;\xi) : m \geq 1,\, \xi \in Y'\}$ form an orthonormal basis, this is equivalent to the family of the differential equations

$$\partial_t^2\widehat{u}_m(\xi,t) + \lambda_m(\xi)\widehat{u}_m(\xi,t) + a_0\partial_t\widehat{u}_m(\xi,t) = 0, \quad \forall m \geq 1,\, \xi \in Y'.$$

Once these differential equations are solved, (1.1) is solved as in (2.16) and Lemma 1 holds. The developments of Section 3 apply with minor changes and Theorem 1 holds.

In order to understand the type of changes that the variable density $\rho(\cdot)$ causes in the fundamental solution, we are going to study the Taylor expansion of the first Bloch eigenvalue and eigenvector. For a more complete analysis the reader is referred to [2]. We observe that

$$\lambda_1(0) = 0 \quad \text{and} \quad \phi(x;0) = (2\pi)^{-\frac{N}{2}}\bar{\rho}^{-\frac{1}{2}},$$

where $\bar{\rho}$ is defined in (1.8). We consider the equation

$$A(\xi)\phi_1(\cdot;\xi) = \lambda_1(\xi)\rho(\cdot)\phi_1(\cdot;\xi), \tag{4.59}$$

where

$$A(\xi) \;=\; -\left(\frac{\partial}{\partial x_k} + i\xi_k\right)\left[a_{k\ell}(x)\left(\frac{\partial}{\partial x_\ell} + i\xi_\ell\right)\right].$$

If we differentiate the equation (4.59) with respect to ξ_k with $k = 1, \ldots, N$ and if we take scalar product with $\phi_1(x;\xi)$ in $\xi = 0$, we get

$$\partial_k \lambda_1(0) = 0.$$

Futhermore, we observe that

$$A\partial_k\phi_1(\cdot;0) = i(2\pi)^{-\frac{N}{2}}\bar{\rho}^{-\frac{1}{2}}\frac{\partial a_{k\ell}}{\partial x_\ell},$$

then

$$\partial_k\phi_1(x;0) = i(2\pi)^{-\frac{N}{2}}\bar{\rho}^{-\frac{1}{2}}\chi^k(x),$$

where χ^k is the classical test function in homogenization theory, solution of the cell problem

$$\begin{cases} A\chi^k = \dfrac{\partial a_{k\ell}}{\partial y_\ell} & \text{in} \quad Y, \\ \chi^k \in H^1_\#(Y), & \frac{1}{|Y|}\int\limits_Y \chi^k dy = 0. \end{cases} \tag{4.60}$$

This is the same test function as in the case $\rho \equiv 1$. If we differentiate again the eigenvalue equation, we have that

$$\partial^2_{k\ell}\lambda_1(0) = \frac{1}{\bar{\rho}}\frac{1}{(2\pi)^N}\int\limits_Y (2a_{k\ell} + a_{km}\frac{\partial\chi^\ell}{\partial x_m} + a_{m\ell}\frac{\partial\chi^k}{\partial x_m})dx = \frac{2q_{k\ell}}{\bar{\rho}},$$

with $q_{k\ell}$ the homogenized coefficients as in previous section (see in [4]).

Since $\alpha_1^1(\xi)$ is defined in (2.17) and thanks to the analysis above for the eigenvalue λ_1, we obtain

$$\begin{aligned} \alpha_1^1(0) = \partial_k\alpha_1^1(0) = 0 \qquad & k = 1,\cdots,N, \\ \partial^2_{k\ell}\alpha_1^1(0) = \frac{2q_{k\ell}}{\bar{\rho}a_0} \qquad & k,\ell = 1,\cdots,N. \end{aligned}$$

Then, for all $\xi \in B_\delta$ we have

$$e^{-\alpha_1^1(\xi)t} \sim e^{-\frac{q_{k\ell}}{\bar{\rho}a_0}\xi_k\xi_\ell t}.$$

5. Analysis of the periodic functions and constants entering in the asymptotic expansion

To finish this work we describe the periodic functions $c_\alpha(\cdot)$ and constants $c_{\beta,n}$, where $\alpha,\beta \in (\mathbb{N}\cup\{0\})^N$ and $n \geq 1$, that appear in the statement of Theorem 1.

Computation of $c_\alpha(\cdot)$. According to (3.38), (3.41) and (3.55),

$$c_\alpha(x) = \sum_{\beta\leq\alpha} \frac{(2\pi)^N}{(\alpha-\beta)!\beta!} \partial^{\alpha-\beta}\phi_1(x;0)\partial^\beta \beta_1^1(0), \tag{5.61}$$

where, recalling the definition of $\beta_1^1(\xi)$ given in (2.19), we have

$$\partial^\beta \beta_1^1(0) = \partial^\beta \widehat{\varphi}_1^0(0) + \sum_{0\neq\gamma\leq\beta} \frac{a_0}{2} \partial^{\beta-\gamma}\widehat{\varphi}_1^0(0)\partial^\gamma \left((a_0^2 - 4\lambda_1(\xi))^{-\frac{1}{2}}\right)(0) + \sum_{\gamma\leq\beta} \partial^{\beta-\gamma}\widehat{\varphi}_1^1(0)\partial^\gamma \left((a_0^2 - 4\lambda_1(\xi))^{-\frac{1}{2}}\right)(0)$$

and for $j = 0,1$ (the first Bloch coefficients of the initial data)

$$\partial^\gamma \widehat{\varphi}_1^j(0) = \int_{R^N} \varphi^j(x) \sum_{\alpha\leq\gamma} [(-i)^{|\gamma-\alpha|} x^{\gamma-\alpha} \partial^\alpha \phi_1(x;0)] dx.$$

We observe that the higher order derivatives of λ_1 and ϕ_1 in $\xi = 0$ may be computed as in the previous section.

First, note that

$$c_0(x) = (2\pi)^N \phi_1(x;0)\beta_1^1(0) = (2\pi)^N \phi_1(x,0) \left(\widehat{\varphi}_1^0(0) + \frac{\widehat{\varphi}_1^1(0)}{a_0}\right),$$

and since $\phi_1(x,0) = (2\pi)^{-\frac{N}{2}} \bar{\rho}^{-\frac{1}{2}}$, it follows that c_0 is constant. Futhermore, according to Proposition 4, we have for $i = 0,1$

$$\widehat{\varphi}_1^i(0) = (2\pi)^{-\frac{N}{2}} \bar{\rho}^{-\frac{1}{2}} \int_{R^N} \varphi^i(x)\rho(x) dx = (2\pi)^{-\frac{N}{2}} \bar{\rho}^{-\frac{1}{2}}\, m_\rho(\varphi^i).$$

Thus,

$$c_0 := c_0(x) = \frac{1}{\bar{\rho}} m_\rho(\varphi^0 + \frac{1}{a_0}\varphi^1),$$

and, since (1.4) is satisfied, c_0 is defined as in (1.10).

For $|\alpha| = 1$, we consider $\alpha = e_k$, any of the canonical vectors. The corresponding periodic function is

$$c_k(x) = \frac{-i}{\bar{\rho}} \left(\chi^k(x)\, m_\rho(\varphi^0 + \frac{\varphi^1}{a_0}) + m_\rho((\chi^k + x_k)(\varphi^0 + \frac{\varphi^1}{a_0})\right),$$

with χ^k the periodic test function, solution of (4.60). Observe that two different terms appear in c_k. First the total mass of the solution multiplied by the periodic function $\chi^k(\cdot)$. Second, a constant term in which the periodic function $\chi^k(\cdot)$ enters as well, but this time as a weight when computing the corresponding moment of the initial data. In both cases we see how the periodicity of the medium affects the value of c_k that varies substantially with respect to the case of constant coefficients. This fact was already pointed out in [7] when studying the heat equation with periodic coeffcients.

The remaining values of the functions $c_\alpha(\cdot)$ with multiindexes α with $|\alpha| \geq 2$ may be computed by taking successive derivatives in (4.59).

Computation of $c_{\beta,n}$. We recall that the constants $c_{\beta,n}$ were defined in (3.52) and satisfy

$$\sum_{m=1}^{\infty} \sum_{|\beta|=4n+2m} \xi^\beta c_{\beta,n} = (-1)^n \left(\sum_{m=0}^{\infty} \sum_{|\beta|=4+2m} \frac{1}{\beta!} \xi^\beta \partial^\beta \alpha_1^1(0) \right)^n,$$

where $\alpha_1^1(\xi)$ is given in (2.17). Moreover, we have for $|\beta|$ even and $|\beta| \geq 4$ that

$$\partial^\beta \alpha_1^1(0) = \sum_{m=1}^{|\beta|} \sum_{\substack{\beta', \beta_1, \ldots, \beta_N \in \mathbb{N}^N \\ \beta' + \beta_1 + \cdots + \beta_N = \beta}}^{|\beta'|=m} \frac{2^{m-1} f(m)}{a_0^{2m-1}} \partial^{\beta_1} \left[(\partial_1 \lambda_1)^{\beta_1'} \right](0) \cdots \partial^{\beta_m} \left[(\partial_N \lambda_1)^{\beta_N'} \right](0),$$

where, for $m \in \mathbb{N}$,

$$f(m) = \begin{cases} 1 & m = 1, \\ \frac{(2m-3)!}{2^{m-2}(m-2)!} & m \geq 2. \end{cases}$$

Observe that the constants $\partial^\beta \alpha_1^1(0)$ depend on the derivatives of λ_1 at $\xi = 0$, computed in Section 4. Thus, we may write the constants $c_{\beta,n}$ as

$$c_{\beta,n} = \sum_{\substack{s_1, \ldots, s_n \in \mathbb{N} \\ s_1 + \cdots + s_n = \frac{|\beta| - 4n}{2}}} \sum_{\substack{\beta_1, \ldots, \beta_n \in \mathbb{N}^N \\ \beta_1 + \cdots + \beta_n = \beta}}^{|\beta_i| = 4 + s_i} (-1)^n \frac{1}{\beta_1!} \cdots \frac{1}{\beta_n!} \partial^{\beta_1} \alpha_1^1(0) \cdots \partial^{\beta_n} \alpha_1^1(0).$$

Acknowledgment

This work was done while the second author was a visiting scholar at *Universidad Complutense de Madrid* in post-doctoral program supported by grant BEX1432/99-0 of the CAPES (MCT-Brasil). The first and third authors were supported by grant PB96-0663 of the DGES (Spain) and the TMR project of the EU "*Homogenization and Multiple Scales*".

References

1. C. M. Bender and S. A. Orszag, **Advanced mathematical methods for scientists and engineers**, (McGraw-Hill, 1978).

2. C. Conca, R. Orive, and M. Vanninathan, *Bloch approximation in homogenization and applications*, *preprint* (2000).
3. C. Conca, J. Planchard, and M. Vanninathan, **Fluids and Periodic Structures**, no. 38 in Research in Applied Mathematics, (J. Wiley/Masson, 1995).
4. C. Conca and M. Vanninathan, *Homogenization of periodic structures via Bloch decomposition*, *SIAM J. Appl. Math.* **57** (1997) 1639–1659.
5. J. Duoandikoetxea and E. Zuazua, *Moments, masses de Dirac et décomposition de fonctions*, *C. R. Acad. Sci. Paris Sér. I Math.* **315** (1992) 693–698.
6. G. Duro and E. Zuazua, *Large time behavior for convection-diffusion equations in* $\boldsymbol{R}^N$ *with asymptotically constant diffusion Comm. Partial Differential Equations* **24** (1999) 1283–1340.
7. J. H. Ortega and E. Zuazua, *Large time behavior in* $\boldsymbol{R}^N$ *for linear parabolic equations with periodic coefficients*, *Asymptotic Anal.* **22** (2000) 51–85.