ASYMPTOTIC EXPANSION FOR THE FUNCTIONAL OF MARKOVIAN EVOLUTION IN R^d IN THE CIRCUIT OF DIFFUSION APPROXIMATION

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We study the asymptotic expansion for solution of singularly perturbed equation for functional of Markovian evolution in R^d. The view of regular and singular parts of solution is found.

1. Introduction

The problems of asymptotic expansion for solutions of PDE and PDE systems were studied by many authors. A lot of references could be found in [5]. As a rule, border problems are studied with the small parameter being denoted at the higher derivative by t.

For example, in [8, page 155] the system of first-order equations is studied with the small parameter denoted by t and x that corresponds to the telegraph equation.

In this paper we study asymptotic expansion for solution of singularly perturbed equation for functional of Markovian evolution in R^d.

Let $x \in R^d$ and $\xi(s)$ is an ergodic Markovian process in the set $E = \{1,\dots,N\}$ with the intensity matrix $Q = \{q_{ij},\ i,j = \overline{1,N}\}$.

The probability of being in the ith state longer than t is $P\{\theta_i > t\} = e^{-q_i t}$, where $q_i = \sum_{j\neq i} q_{ij}$.

Let $a(i) = (a_1(i),\dots,a_d(i))$ be a vector-function on E. We regard a vector-function as a corresponding vector-column.

Put matrix $A = \{a_k(i),\ k = \overline{1,d},\ i = \overline{1,N}\}$.

We study evolution

$$x^\varepsilon(t) = x + \varepsilon^{-1}\int_0^t a\left(\xi\frac{s}{\varepsilon^2}\right)ds = x + \varepsilon\int_0^{t/\varepsilon^2} a(\xi(s))\,ds. \tag{1.1}$$

It is well known [6], that the functionals of evolution, determined by a test-function $\overline{f}(x) \in C^\infty(R^d)$ (here $\overline{f}(x)$ is integrable on R^d and has equal components $\overline{f}(x) = (f(x),\dots,f(x))$) such that $u_i^\varepsilon(x,t) = E_i\overline{f}(x^\varepsilon(t))$, $i = \overline{1,N}$ (here i is a start state of $\xi(s)$) satisfy the system of Kolmogorov backward differential equations:

$$\frac{\partial}{\partial t}u^\varepsilon(x,t) = \varepsilon^{-2}Qu^\varepsilon(x,t) + \varepsilon^{-1}A\nabla u^\varepsilon(x,t), \tag{1.2}$$

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where $u^\varepsilon(x,t) = (u_1^\varepsilon(x,t),\dots,u_N^\varepsilon(x,t))$, $A\nabla = \operatorname{diag}[(a(i),\nabla),\ i=\overline{1,N}]$, $\nabla = (\partial/\partial x_1,\dots,\partial/\partial x_d)$.

As an example we will describe a well-known model, where an equation of type (1.2) appears.

Example 1.1. In [6, 7] functionals of the view

$$u_i(x,t) = E_i\overline{f}\left(x+v\int_0^t \overline{\tau}_{\xi(s)}ds\right), \quad i=\overline{0,n} \tag{1.3}$$

were studied. Here $\xi(u)$ is the Poisson process with parameter λ, $\xi(0)=0$, v is the velocity of particle's motion, $\overline{\tau}_i$, $i=\overline{0,n}$ are vectors that determine the directions of motion. The systems of Kolmogorov backward differential equations were obtained for the functionals $u_i(x,t)$, $i=\overline{0,n}$ in case of cyclic and uniform change of motion directions.

In a matrix form we have

$$\frac{\partial}{\partial t}u^\varepsilon(x,t) = [\lambda Q + vA\nabla]u^\varepsilon(x,t), \tag{1.4}$$

where $u^\varepsilon(x,t) = (u_0^\varepsilon(x,t),\dots,u_n^\varepsilon(x,t))$, $A\nabla = \operatorname{diag}[(\overline{\tau}_i,\nabla),\ i=\overline{0,n}]$, $Q=[q_{ij},\ i,j=\overline{0,n}]$. Here $q_{ii}=-1$, $q_{ii+1}=1$, $q_{ij}=0$, $j\neq i$, $j\neq i+1$ in case of cyclic change of directions, and $q_{ii}=-1$, $q_{ij}=1/n$, $i\neq j$ in case of uniform change.

If we put in (1.4) $v=\varepsilon^{-1}$, $\lambda=\varepsilon^{-2}$, where ε is a small parameter, we will have a singularly perturbed equation of type (1.2):

$$\frac{\partial}{\partial t}u^\varepsilon(x,t) = [\varepsilon^{-2}Q + \varepsilon^{-1}A\nabla]u^\varepsilon(x,t). \tag{1.5}$$

Initial condition $u^\varepsilon(x,0) = \overline{f}(x) := (f(x),\dots,f(x))$.

Equations of type (1.2) were also studied in [2, 4]. It was partially shown in [2] that for the distribution of absorption time of Markov chain with continuous time that depends on small parameter ε, the following equation was obtained $\varepsilon(d/dx)u^\varepsilon(x) = (Q-\varepsilon G)u^\varepsilon(x)$, $Q=P-I$. Asymptotic expansion of its solution was found there.

In this paper, we study system (1.2) with the second-order singularity. This problem has interesting probabilistic sense: hyperbolic equation of high degree, corresponding to system (1.4) (see [7]) becomes parabolic equation of Wiener process in hydrodynamic limit, when $\varepsilon\to 0$. The fact that solution of (1.4) in hydrodynamic limit tends to the functional of Wiener process is well known and studied, for example, in [3].

To find asymptotic expansion of the solution of (1.2), we use the method proposed in [8]. The solution consists of two parts, regular terms and singular terms, which are determined by different equations. Asymptotic expansion allows not only the determination of the terms of asymptotic, but also allows us to see the velocity of convergence in hydrodynamic limit.

Besides, when studying this problem, we improved the algorithm of asymptotic expansion. Partially, the initial conditions for the regular terms of asymptotic are determined without the use of singular terms, that is, the regular part of the solution may be found by a separate recursive algorithm; scalar part of the regular term is found and without the use of singular terms. These and other improvements in the algorithm are pointed later.

2. Asymptotic expansion of the solution

Let $P(t) = e^{Qt} = \{p_{ij}(t);\ i,j = \overline{1,N}\}$. Put $\pi_j = \lim_{t\to\infty} p_{ij}(t)$ and $-R_0 = \{\int_0^\infty (p_{ij}(t) - \pi_j)dt;\ i,j = \overline{1,N}\} = \{r_{ij};\ i,j = \overline{1,N}\}$. Let Π be a projecting matrix on the null-space N_Q of the matrix Q. For any vector g we have $\Pi g = \hat{g}\mathbf{1}$, where $\hat{g} = \sum_{i=1}^N g_i\pi_i$, $\mathbf{1} = (1,\ldots,1)$. Then for the matrix Q the following correlation is true: $\Pi Q\Pi = 0$ (see [3, Chapter 3]).

Let the matrix A satisfy balance condition

$$\Pi A\Pi = 0. \tag{2.1}$$

We put

$$\begin{aligned} R_0A\nabla &= \{r_{ij}(a(j),\nabla),\ i,j = \overline{1,N}\} = \left\{\sum_{k=1}^d r_{ij}a_k(j)\frac{\partial}{\partial x_k},\ i,j = \overline{1,N}\right\}, \\ A\nabla R_0 &= \{(a(i),\nabla)r_{ij},\ i,j = \overline{1,N}\} = \left\{\sum_{k=1}^d a_k(i)r_{ij}\frac{\partial}{\partial x_k},\ i,j = \overline{1,N}\right\}, \\ A\nabla R_0A\nabla &= \{(a(i),\nabla)r_{ij}(a(j),\nabla),\ i,j = \overline{1,N}\} \\ &= \left\{\sum_{k=1}^d\sum_{l=1}^d a_k(i)r_{ij}a_l(j)\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_l},\ i,j = \overline{1,N}\right\}, \end{aligned} \tag{2.2}$$

$$\exp_0(Qt) := e^{Qt} - \Pi,$$

$$\hat{a}_{kl} = \sum_{i,j=1}^N \pi_i a_k(i)r_{ij}a_l(j)\pi_j.$$

Here, following [6], we need the condition

$$\hat{a}_{kl} > 0. \tag{2.3}$$

Theorem 2.1. *The solution of (1.2) with initial condition $u^\varepsilon(x,0) = \overline{f}(x)$, where $\overline{f}(x) \in C^\infty(R^d)$ and integrable on R^d has asymptotic expansion*

$$u^\varepsilon(x,t) = u^{(0)}(x,t) + \sum_{n=1}^\infty \varepsilon^n\left(u^{(n)}(x,t) + v^{(n)}(x,t/\varepsilon^2)\right). \tag{2.4}$$

Regular terms of the expansion are $u^{(0)}(x,t)$ which represent the solution of equation

$$\frac{\partial}{\partial t}u^{(0)}(x,t) = \sum_{k,l=1}^d \hat{a}_{kl}\frac{\partial^2 u^{(0)}(x,t)}{\partial x_k\partial x_l} \tag{2.5}$$

with initial condition $u^{(0)}(x,0)=\overline{f}(x)$,

$$u^{(1)}(x,t)=R_0A\nabla u^{(0)}(x,t)=\left[\sum_{k=1}^{d}\sum_{j=1}^{N}r_{ij}a_k(j)\frac{\partial u_j^{(0)}(x,t)}{\partial x_k},\ i=\overline{1,N}\right]. \tag{2.6}$$

For $k\geq 2$,

$$\begin{aligned}u^{(k)}(x,t)&=R_0\left[\frac{\partial}{\partial t}u^{(k-2)}(x,t)-A\nabla u^{(k-1)}(x,t)\right]+c^{(k)}(t)\\&:=R_0\Phi\left[u^{(k-2)}(x,t),u^{(k-1)}(x,t)\right]+c^{(k)}(t),\end{aligned} \tag{2.7}$$

where

$$c^{(k)}(t)\in N_Q,\quad c^{(k)}(t)=c^{(k)}(0)+\int_0^t\hat{L}_kc^{(0)}(s)ds. \tag{2.8}$$

Here

$$\begin{aligned}c^{(0)}(t)=u^{(0)}(x,t),\qquad &L_0=\left\{\sum_{k=1}^{d}\sum_{l=1}^{d}a_k(i)r_{ij}a_l(j)\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_l},\ i,j=\overline{1,N}\right\},\\ \hat{L}_k=\Pi L_k\Pi,\quad &L_k=(-1)^{k+1}\left(A\nabla R_0\right)^k£_0,\quad k\geq 1,\\ £_0=&\left\{\frac{\partial}{\partial t}-\sum_{k=1}^{d}\sum_{l=1}^{d}a_k(i)r_{ij}a_l(j)\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_l},\ i,j=\overline{1,N}\right\}.\end{aligned} \tag{2.9}$$

The singular terms of the expansion have the view

$$v^{(1)}(x,t)=\exp_0(Qt)A\nabla\overline{f}(x). \tag{2.10}$$

For $k>1$,

$$\begin{aligned}v^{(k)}(x,t)=\exp_0(Qt)v^{(k)}(x,0)+\int_0^t\exp_0\left(Q(t-s)\right)A\nabla v^{(k-1)}(x,s)ds\\ -\Pi\int_t^\infty A\nabla v^{(k-1)}(x,s)ds.\end{aligned} \tag{2.11}$$

Initial conditions are

$$\begin{aligned}c^{(0)}(0)&=\overline{f}(x),\\ u^{(1)}(x,0)=R_0A\nabla\overline{f}(x),\qquad &v^{(1)}(x,0)=-\frac{1}{2}A\nabla\Pi\overline{f}(x).\end{aligned} \tag{2.12}$$

For $k>1$,

$$\begin{aligned}v^{(k)}(x,0)&=\Phi\left[u^{(k-2)}(x,0),u^{(k-1)}(x,0)\right],\\ c^{(k)}(0)&=-A\nabla\widetilde{v}^{(k-1)}(x,0),\end{aligned} \tag{2.13}$$

where $\widetilde{v}^{(1)}(x,0) = -R_0 A\nabla \overline{f}(x)$,

$$\begin{aligned}\widetilde{v}^{(k)}(x,0) &= R_0\Phi\Big[u^{(k-2)}(x,0), u^{(k-1)}(x,0)\Big] + R_0 A\nabla\widetilde{v}^{(k-1)}(x,0)\\ &\quad + \Pi A\nabla\left(\widetilde{v}^{(k-1)}(x,\lambda)\right)'_{\lambda}\Big|_{\lambda=0},\\ \left(\widetilde{v}^{(k)}(x,\lambda)\right)'_{\lambda}\Big|_{\lambda=0} &= R_0^2\Phi\Big[u^{(k-2)}(x,0), u^{(k-1)}(x,0)\Big] + R_0^2 Q_1\widetilde{v}^{(k-1)}(x,0)\\ &\quad + R_0 A\nabla\left(\widetilde{v}^{(k-1)}(x,\lambda)\right)'_{\lambda}\Big|_{\lambda=0}.\end{aligned} \tag{2.14}$$

Remark 2.2. The initial conditions for the regular terms of asymptotic are determined without the use of singular terms, that is, the regular part of the solution may be found by a separate recursive algorithm (cf. [2]).

Remark 2.3. In case of evolution described in Example 1.1 (2.5) has the view

$$\frac{\partial}{\partial t}u^{(0)}(x,t) = \frac{1}{(n+1)^2}\triangle u^{(0)}(x,t) \tag{2.15}$$

with initial condition $u^{(0)}(x,0) = \overline{f}(x)$.

Solution of this problem in the class of integrable and infinitely differentiable functions of exponential growth is

$$u^{(0)}(x,t) = (2\pi t)^{-n/2}\frac{1}{(n+1)^2}\int_{R^n} e^{-(n+1)^2(\langle (x-y),(x-y)\rangle/2t)}\overline{f}(y)dy. \tag{2.16}$$

Proof of Theorem 2.1. We substitute the solution $u^{\varepsilon}(x,t)$ in the view (2.4) to (1.2) and equal the terms at ε degrees. We will have the system for the regular terms of asymptotic:

$$\begin{aligned} Qu^{(0)} &= 0,\\ Qu^{(1)} + A\nabla u^{0} &= 0,\\ Qu^{(k)} &= \frac{\partial}{\partial t}u^{(k-2)} - A\nabla u^{(k-1)}, \quad k \geq 2,\end{aligned} \tag{2.17}$$

and for the singular terms,

$$\begin{aligned}\frac{\partial}{\partial t}v^{(1)} &= Qv^{(1)},\\ \frac{\partial}{\partial t}v^{(k)} - Qv^{(k)} &= A\nabla v^{(k-1)}, \quad k > 1.\end{aligned} \tag{2.18}$$

From (2.17) we have $u^{(0)} \in N_Q$, $u^{(1)} = R_0 A\nabla u^{(0)} + c^{(1)}(t)$. For $u^{(2)}$ we obtain $Qu^{(2)} = (\partial/\partial t)u^{(0)} - A\nabla u^{(1)} = (\partial/\partial t)u^{(0)} - A\nabla R_0 A\nabla u^{(0)} = (\partial/\partial t)u^{(0)} - L_0 u^{(0)}$.

The solvability condition for $u^{(2)}$ has the view

$$\Pi Q\Pi u^{(2)} = 0 = \frac{\partial}{\partial t}u^{(0)} - \Pi L_0 \Pi u^{(0)}. \tag{2.19}$$

So, we have (2.5) for $u^{(0)}(x,t)$.

We note that in [2] solvability condition is written for the equation that contains the terms $u^{(0)}(x,t)$ and $u^{(1)}(x,t)$. In this paper we have to express $u^{(1)}(x,t)$ through $u^{(0)}(x,t)$ and only then we can write down solvability condition for the equation that contains the terms $u^{(0)}(x,t)$ and $u^{(2)}(x,t)$.

For $u^{(1)}$ we have

$$u^{(1)} = R_0 A \nabla u^{(0)} + c^{(1)}(t). \tag{2.20}$$

Using the last equation from (2.17) we obtain

$$\begin{aligned} u^{(k)}(x,t) &= R_0\left[\frac{\partial}{\partial t}u^{(k-2)}(x,t) - A\nabla u^{(k-1)}(x,t)\right] + c^{(k)}(t) \\ &:= R_0\Phi\left[u^{(k-2)}(x,t), u^{(k-1)}(x,t)\right] + c^{(k)}(t), \end{aligned} \tag{2.21}$$

where $c^{(k)}(t) \in N_Q$.

To find $c^{(k)}(t)$ we will use the fact that $u^{(0)} \in N_Q$. We put $c^{(0)}(t) = u^{(0)}(x,t)$. From the equation $Qu^{(2)} = (\partial/\partial t)c^{(0)}(t) - L_0 c^{(0)}(t) = £_0 c^{(0)}(t)$, we have

$$u^{(2)} = R_0 £_0 c^{(0)}(t). \tag{2.22}$$

For $u^{(3)}$,

$$Qu^{(3)} = \frac{\partial}{\partial t}c^{(1)}(t) - A\nabla u^{(2)} = \left(c^{(1)}(t)\right)' - A\nabla R_0 £_0 c^{(0)}(t) = £_1 c^{(0)}(t). \tag{2.23}$$

From the solvability condition $\Pi Q \Pi u^{(3)} = 0 = (\partial/\partial t)c^{(1)}(t) - \Pi A\nabla R_0 £_0 \Pi c^{(0)}(t) = (c^{(1)}(t))' - \widehat{L}_1 c^{(0)}(t)$ we find

$$c^{(1)}(t) = c^{(1)}(0) + \int_0^t \widehat{L}_1 c^{(0)}(s)ds, \tag{2.24}$$

and $u^{(3)} = R_0 £_1 c^{(0)}(t)$, where $£_1 = (-L_1)c^{(0)}(t)$, as soon as $R_0\widehat{L}_1 = 0$.

By induction

$$c^{(k)}(t) = c^{(k)}(0) + \int_0^t \widehat{L}_k c^{(0)}(s)ds, \tag{2.25}$$

where $\widehat{L}_k = \Pi L_k \Pi$, $L_k = (-1)^{k+1}(A\nabla R_0)^k £_0, £_0 = (\partial/\partial t) - L_0$, $k \geq 2$.

In contrast to [2], where the equations for $c^{(k)}(t)$ were found, in this paper we may find $c^{(k)}(t)$ explicitly through $c^{(0)}(t)$.

For the singular terms we have from (2.18),

$$v^{(1)}(x,t) = \exp_0(Qt)v^{(1)}(x,0). \tag{2.26}$$

Here we should note that the ordinary solution $v^{(1)}(x,t) = \exp(Qt)v^{(1)}(x,0)$ is corrected by the term $-\Pi v^{(1)}(x,0)$ in order to obtain the following limit: $\lim_{t\to\infty} v^{(1)}(x,t) = 0$. This limit is true for all singular terms due to uniform ergodicity of switching Markovian process.

For the homogenous part of the second equation of the system, we have the following solution:

$$v^{(k)}(x,t) = \exp_0(Qt)v^{(k)}(x,0). \tag{2.27}$$

But as soon as the equation is not homogenous, the corresponding solution should be

$$v^{(k)}(x,t) = \exp_0(Qt)v^{(k)}(x,0) + \int_0^t \exp_0\left(Q(t-s)\right)A\nabla v^{(k-1)}(x,s)ds. \tag{2.28}$$

But here we should again correct the solution in order to obtain the limit $\lim_{t\to\infty} v^{(k)}(x,t) = 0$, by the term $-\Pi\int_t^\infty A\nabla v^{(k-1)}(x,s)ds$.

And so the solution is

$$\begin{aligned} v^{(k)}(x,t) = \exp_0(Qt)v^{(k)}(x,0) &+ \int_0^t \exp_0\left(Q(t-s)\right)A\nabla v^{(k-1)}(x,s)ds \\ &- \Pi\int_t^\infty A\nabla v^{(k-1)}(x,s)ds. \end{aligned} \tag{2.29}$$

We should finally find the initial conditions for the regular and singular terms.

We put $c^{(0)}(t) = u^{(0)}(x,t)$, so $c^{(0)}(0) = u^{(0)}(x,0) = \overline{f}(x)$.

From the initial condition for the solution $u^\varepsilon(x,0) = u^{(0)}(x,0) = (f(x),\dots,f(x))$, we have $u^{(k)}(x,0) + v^{(k)}(x,0) = 0$, $k \geq 1$. We rewrite this equation for the null-space N_Q of matrix Q:

$$\Pi u^{(k)}(x,0) + \Pi v^{(k)}(x,0) = 0, \quad k \geq 1, \tag{2.30}$$

and the space of values R_Q:

$$(I-\Pi)u^{(k)}(x,0) + (I-\Pi)v^{(k)}(x,0) = 0, \quad k \geq 1. \tag{2.31}$$

As we proved for $k > 1$,

$$\begin{aligned} u^{(k)}(x,0) &= R_0\Phi\left[u^{(k-2)}(x,0),u^{(k-1)}(x,0)\right] + c^{(k)}(0) \\ &= (I-\Pi)\Phi\left[u^{(k-2)}(x,0),u^{(k-1)}(x,0)\right] + \Pi c^{(k)}(0), \\ v^{(k)}(x,0) &= (I-\Pi)v^{(k)}(x,0) - \Pi\int_0^\infty A\nabla v^{(k-1)}(x,s)ds. \end{aligned} \tag{2.32}$$

Functions $v^{(k-1)}(x,s)$, $u^{(k-2)}(x,0)$, $u^{(k-1)}(x,0)$ are known from the previous steps of induction. So, we have found $\Pi v^{(k)}(x,0)$ in (2.30) and $(I-\Pi)u^{(k)}(x,0)$ in (2.31).

Now we may use the correlations (2.30), (2.31) to find the unknown initial conditions

$$\begin{aligned} c^{(k)}(0) &= -\int_0^\infty A\nabla v^{(k-1)}(x,s)ds, \\ v^{(k)}(x,0) &= \Phi\left[u^{(k-2)}(x,0),u^{(k-1)}(x,0)\right]. \end{aligned} \tag{2.33}$$

In [2], an analogical correlation was found for $c^{(k)}(0)$. To find $c^{(k)}(0)$ explicitly and without the use of singular terms, we will find Laplace transform for the singular term. The following lemma is true.

Lemma 2.4. *Laplace transform for the singular term of asymptotic expansion*

$$\tilde{v}^{(k)}(x,\lambda) = \int_0^\infty e^{-\lambda s} v^{(k)}(x,s)\,ds \tag{2.34}$$

has the view

$$\begin{aligned}
\tilde{v}^{(1)}(x,\lambda) &= \left(\lambda - \Pi + (R_0+\Pi)^{-1}\right)^{-1}\left[-R_0 A\nabla \overline{f}(x)\right],\\
\tilde{v}^{(k)}(x,\lambda) &= \left(\lambda - \Pi + (R_0+\Pi)^{-1}\right)^{-1}\Phi\left[u^{(k-2)}(x,0),u^{(k-1)}(x,0)\right]\\
&\quad + \left(\lambda - \Pi + (R_0+\Pi)^{-1}\right)^{-1} A\nabla\tilde{v}^{(k-1)}(x,\lambda)\\
&\quad + \frac{1}{\lambda}\Pi A\nabla\left[\tilde{v}^{(k-1)}(x,\lambda) - \tilde{v}^{(k-1)}(x,0)\right],
\end{aligned} \tag{2.35}$$

where

$$\begin{aligned}
\tilde{v}^{(1)}(x,0) &= -R_0 A\nabla\overline{f}(x),\\
\left(\tilde{v}^{(1)}(x,\lambda)\right)'_\lambda\Big|_{\lambda=0} &= -R_0^2 A\nabla\Pi\overline{f}(x),\\
\tilde{v}^{(k)}(x,0) &= R_0\Phi\left[u^{(k-2)}(x,0),u^{(k-1)}(x,0)\right] + R_0 A\nabla\tilde{v}^{(k-1)}(x,0)\\
&\quad + \Pi A\nabla\left(\tilde{v}^{(k-1)}(x,\lambda)\right)'_\lambda\Big|_{\lambda=0},\\
\left(\tilde{v}^{(k)}(x,\lambda)\right)'_\lambda\Big|_{\lambda=0} &= R_0^2\Phi\left[u^{(k-2)}(x,0),u^{(k-1)}(x,0)\right] + R_0^2 Q_1\tilde{v}^{(k-1)}(x,0)\\
&\quad + R_0 A\nabla\left(\tilde{v}^{(k-1)}(x,\lambda)\right)'_\lambda\Big|_{\lambda=0}.
\end{aligned} \tag{2.36}$$

Proof.

$$\begin{aligned}
\tilde{v}^{(1)}(x,\lambda) &= \int_0^\infty e^{-\lambda s} v^{(1)}(x,s)\,ds = \int_0^\infty e^{-\lambda s}\left[e^{Qs} - \Pi\right]ds\, v^{(1)}(x,0)\\
&= \left(\lambda - \Pi + (R_0+\Pi)^{-1}\right)^{-1}\left[-A\nabla\overline{f}(x)\right],
\end{aligned} \tag{2.37}$$

where the correlation for the resolvent was found in [3]. Moreover,

$$\begin{aligned}
\tilde{v}^{(1)}(x,0) &= -R_0 A\nabla\overline{f}(x),\\
\left(\tilde{v}^{(1)}(x,\lambda)\right)'_\lambda\Big|_{\lambda=0} &= \lim_{\lambda\to 0}\frac{R(\lambda) - R_0}{\lambda}\left[-A\nabla\overline{f}(x)\right] = -R_0^2 A\nabla\overline{f}(x).
\end{aligned} \tag{2.38}$$

For the next terms we have

$$\begin{aligned}\widetilde{v}^{(k)}(x,\lambda) &= \left(\lambda - \Pi + (R_0+\Pi)^{-1}\right)^{-1}\Phi\left[u^{(k-2)}(x,0),u^{(k-1)}(x,0)\right]\\ &+\left(\lambda - \Pi + (R_0+\Pi)^{-1}\right)^{-1}A\nabla\widetilde{v}^{(k-1)}(x,\lambda)\\ &+\frac{1}{\lambda}\Pi A\nabla\left[\widetilde{v}^{(k-1)}(x,\lambda)-\widetilde{v}^{(k-1)}(x,0)\right].\end{aligned}\tag{2.39}$$

Here the last term was found using the following correlation:

$$\begin{aligned}\int_0^\infty e^{-\lambda s}\int_s^\infty A\nabla v^{(k-1)}(x,\tau)d\tau\, ds &= \int_0^\infty\int_0^\tau e^{-\lambda s}A\nabla v^{(k-1)}(x,\tau)ds\, d\tau\\ &=\int_0^\infty\left(-\frac{1}{\lambda}\right)\left(e^{-\lambda\tau}-1\right)A\nabla v^{(k-1)}(x,\tau)d\tau\\ &=\frac{1}{\lambda}A\nabla\left[\widetilde{v}^{(k-1)}(x,\lambda)-\widetilde{v}^{(k-1)}(x,0)\right].\end{aligned}\tag{2.40}$$

So,

$$\begin{aligned}\widetilde{v}^{(k)}(x,0) &= R_0\Phi\left[u^{(k-2)}(x,0),u^{(k-1)}(x,0)\right]+R_0A\nabla\widetilde{v}^{(k-1)}(x,0)\\ &+\Pi A\nabla\left(\widetilde{v}^{(k-1)}(x,\lambda)\right)'_\lambda\Big|_{\lambda=0},\\ \left(\widetilde{v}^{(k)}(x,\lambda)\right)'_\lambda\Big|_{\lambda=0} &= R_0^2\Phi\left[u^{(k-2)}(x,0),u^{(k-1)}(x,0)\right]\\ &+R_0^2Q_1\widetilde{v}^{(k-1)}(x,0)+R_0A\nabla\left(\widetilde{v}^{(k-1)}(x,\lambda)\right)'_\lambda\Big|_{\lambda=0}\\ &-\lim_{\lambda\to 0}\left\{\frac{1}{\lambda^2}\Pi A\nabla\left[\widetilde{v}^{(k-1)}(x,\lambda)-\widetilde{v}^{(k-1)}(x,0)\right]\right.\\ &\left.-\frac{1}{\lambda}\Pi A\nabla\left(\widetilde{v}^{(k-1)}(x,\lambda)\right)'_\lambda\right\},\end{aligned}\tag{2.41}$$

where the last limit tends to 0.

Lemma is proved. □

So, the obvious view of the initial condition for the $c^{(k)}(t)$ is

$$c^{(k)}(0) = -A\nabla\widetilde{v}^{(k-1)}(x,0).\tag{2.42}$$

Theorem is proved. □

3. Estimate of the remainder

Let function $f(x,i)$ in the definition of the functional $u^\varepsilon(x,t)$ belong to Banach space twice continuously differentiable by x functions $C^2(R^d\times E)$.

We write (1.2) in the view

$$\tilde{u}_2^\varepsilon(x,t) = u^\varepsilon(x,t) - u_2^\varepsilon(x,t),\tag{3.1}$$

where $u_2^\varepsilon(x,t) = u^{(0)}(x,t) + \varepsilon(u^{(1)}(x,t) + v^{(1)}(x,t)) + \varepsilon^2(u^{(2)}(x,t) + v^{(2)}(x,t))$, and the explicit view of the functions $u^{(i)}(x,t)$, $v^{(j)}(x,t)$, $i = \overline{0,2}$, $j = 1,2$ is given in Theorem 2.1.

By [3, Theorem 3.2.1], in Banach space $C^2(R^d \times E)$ for the generator of Markovian evolution $L^\varepsilon = \varepsilon^{-2}Q + \varepsilon^{-1}A\nabla$, there exists bounded inverse operator $(L^\varepsilon)^{-1} = \varepsilon^2[Q + \varepsilon A\nabla]^{-1}$.

We substitute the function (3.1) into (1.2):

$$\frac{d}{dt}\tilde{u}^\varepsilon - L^\varepsilon \tilde{u}^\varepsilon = \frac{d}{dt}u_2^\varepsilon - L^\varepsilon u_2^\varepsilon := \varepsilon w^\varepsilon. \tag{3.2}$$

Here $\varepsilon w^\varepsilon = \varepsilon[(d/dt)((u^{(1)} + v^{(1)}) + \varepsilon(u^{(2)} + v^{(2)})) - (\varepsilon^{-1}Q(u^{(1)} + v^{(1)}) + Q(u^{(2)} + v^{(2)}) + A\nabla(u^{(1)} + v^{(1)}) + \varepsilon A\nabla(u^{(2)} + v^{(2)}))]$.

The initial condition has the order ε, so we may write it in the view

$$\tilde{u}^\varepsilon(0) = \varepsilon\tilde{u}^\varepsilon(0). \tag{3.3}$$

Let $L_t^\varepsilon f(x,i) = E[f(x^\varepsilon(t),\xi^\varepsilon(t/\varepsilon^2)) \mid x^\varepsilon(0) = x, \xi^\varepsilon(0) = i]$ be the semigroup corresponding to the operator L^ε.

Theorem 3.1. *The following estimate is true for the remainder (3.1) of the solution of (1.2):*

$$||\tilde{u}^\varepsilon(t)|| \le \varepsilon||\tilde{u}^\varepsilon(0)||\exp\{\varepsilon L||w^\varepsilon||\}, \tag{3.4}$$

where $L = 2\|(L^\varepsilon)^{-1}\|$.

Proof. The solution of (3.2) is

$$\tilde{u}_2^\varepsilon(t) = \varepsilon\left[L_t^\varepsilon\tilde{u}^\varepsilon(0) + \int_0^t L_{t-s}^\varepsilon w^\varepsilon(s)ds\right]. \tag{3.5}$$

For the semigroup we have $L_t^\varepsilon = I + L^\varepsilon\int_0^t L_s^\varepsilon ds$, so $\int_0^t L_s^\varepsilon ds = (L^\varepsilon)^{-1}(L_t^\varepsilon - I)$.

Using Gronwell-Bellman inequality [1], we obtain

$$||\tilde{u}^\varepsilon(t)|| \le \varepsilon L_t^\varepsilon||\tilde{u}^\varepsilon(0)||\exp\left\{\varepsilon\int_0^t L_s^\varepsilon w^\varepsilon(t-s)ds\right\} \le \varepsilon L_t^\varepsilon||\tilde{u}^\varepsilon(0)||\exp\{\varepsilon L||w^\varepsilon||\}, \tag{3.6}$$

where $L = 2\|(L^\varepsilon)^{-1}\|$.

Theorem is proved. □

Remark 3.2. For the remainder of asymptotic expansion (2.4) of the view,

$$\tilde{u}_{N+1}^\varepsilon(x,t) := u^\varepsilon(x,t) - u_{N+1}^\varepsilon(x,t), \tag{3.7}$$

where $u_{N+1}^\varepsilon(x,t) = u^{(0)}(x,t) + \sum_{n=1}^{N+1}\varepsilon^n(u^{(n)}(x,t) + v^{(n)}(x,t))$ we have analogical estimate

$$||\tilde{u}_{N+1}^\varepsilon(t)|| \le \varepsilon^N||\tilde{u}^\varepsilon(0)||\exp\{\varepsilon^N L||w_N^\varepsilon||\}, \tag{3.8}$$

where $(d/dt)u_{N+1}^\varepsilon - L^\varepsilon u_{N+1}^\varepsilon := \varepsilon^N w_N^\varepsilon$.

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