MATHEMATICS OF COMPUTATION
VOLUME 50, NUMBER 181
JANUARY 1988, PAGES 229-234

Asymptotic Expansion of $\int_0^{\pi/2} J_\nu^2(\lambda\cos\theta)\,d\theta$

By R. Wong*

Abstract. An asymptotic expansion is obtained, as $\lambda \to +\infty$, for the integral

$$I(\lambda) = \int_0^{\pi/2} J_\nu^2(\lambda\cos\theta)\,d\theta,$$

where $J_\nu(t)$ is the Bessel function of the first kind and $\nu > -\frac{1}{2}$. This integral arises in studies of crystallography and diffraction theory. We show in particular that $I(\lambda) \sim \ln\lambda/\lambda\pi$.

Recently, G. Weiss at the National Institute of Health posed to me the problem of finding an asymptotic expansion for the integral

$$I(\lambda) = \int_0^{\pi/2} J_\nu^2(\lambda\cos\theta)\,d\theta, \qquad \nu > -\frac{1}{2}, \tag{1}$$

as $\lambda \to +\infty$, where $J_\nu(t)$ is the Bessel function of the first kind. According to him, this problem arises in his study on crystallography. At about the same time, Stoyanov and Farrell [5] encountered a similar integral, namely

$$I = \int_0^{\pi/2} J_0^2(\lambda\sin\theta)\,d\theta, \tag{2}$$

in their work on diffraction theory, and showed that as $\lambda \to +\infty$,

$$I = \frac{1}{\lambda\pi}(\ln\lambda + 4\ln 2 + \gamma) + \frac{1}{2\sqrt{\pi}\lambda^{3/2}}\sin\left(2\lambda - \frac{\pi}{4}\right) + O(\lambda^{-5/2}), \tag{3}$$

where γ is the Euler constant.

The purpose of this note is to provide a solution to the problem of Weiss. Our approach is entirely different from that of Stoyanov and Farrell. We shall show that the integral in (1) has the asymptotic expansion

$$\begin{aligned} I(\lambda) = {} & \frac{1}{\pi\lambda}\left[\ln\lambda + 2\ln 2 - \psi\left(\frac{1}{2}+\nu\right)\right] + \frac{1}{2\sqrt{\pi}\lambda^{3/2}}\sin\left(2\lambda - \nu\pi - \frac{\pi}{4}\right) \\ & + \frac{1}{32\sqrt{\pi}\lambda^{5/2}}(16\nu^2 - 9)\cos\left(2\lambda - \nu\pi - \frac{\pi}{4}\right) \\ & + \frac{1}{16\pi\lambda^3}\left\{[4\nu^2 - 1]\left[\ln\lambda + 2\ln 2 - \psi\left(\frac{1}{2}+\nu\right)\right] - (4\nu^2 - 3)\right\} \\ & + O(\lambda^{-7/2}), \end{aligned} \tag{4}$$

Received November 24, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 41A60.
Key words and phrases. Asymptotic expansion, Bessel functions, Mellin transforms.
*This research was partially supported by the Natural Sciences and Engineering Research Council of Canada under Grant A7359.

where $\psi(z)$ is the logarithmic derivative of the gamma function. The result in (3) can be deduced from (4) by setting $\nu = 0$.

To derive the expansion in (4), we first observe that the integral $I(\lambda)$ in (1) clearly can be written as

$$I(\lambda) = \int_0^1 \frac{J_\nu^2(\lambda t)}{\sqrt{1-t^2}}\, dt. \tag{5}$$

The integral I in (2) also can be written in this form with $\nu = 0$. Furthermore, if we set

$$h(t) = J_\nu^2(t) \tag{6}$$

and

$$f(t) = \begin{cases} \dfrac{1}{\sqrt{1-t^2}}, & 0 < t < 1, \\ 0, & t \geq 1, \end{cases} \tag{7}$$

then $I(\lambda)$ can be expressed in the form of a convolution integral

$$I(\lambda) = \int_0^\infty f(t) h(\lambda t)\, dt. \tag{8}$$

This immediately suggests the use of the Mellin transform technique described in [1, Chapter 4]. (For a quick summary, see also [7, Section 6].) From integral table [3] it is easily found that

$$M[h; z] = \frac{2^{z-1}\Gamma(1-z)\Gamma(\nu + z/2)}{\Gamma^2(1-z/2)\Gamma(1+\nu-z/2)}, \qquad -2\nu < \operatorname{Re} z < 1, \tag{9}$$

and

$$M[f; 1-z] = \pi 2^{z-1} \frac{\Gamma(1-z)}{\Gamma^2(1-z/2)}, \qquad \operatorname{Re} z < 1. \tag{10}$$

Thus, by the Parseval identity [7, Eq. (6.2)],

$$I(\lambda) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \lambda^{-z} M[h; z] M[f; 1-z]\, dz, \tag{11}$$

where $-2\nu < c < 1$. The usual procedure now is to move the vertical line of integration to the right. The terms in the asymptotic expansion are then picked up as residues. This is, however, not permitted in the present case, since here we have

$$M[h; c+iy] = O(|y|^{c-3/2}) \tag{12}$$

and

$$M[f; 1-c-iy] = O(|y|^{-1/2}). \tag{13}$$

Therefore, the integrand in (11) decays only for $c < 2$, and the integral in (11) is absolutely convergent only for $c < 1$.

The reason for the failure of this simple procedure is partly due to the fact that the kernel function $h(t)$ in (8) consists of an algebraic as well as an oscillatory part. To see this, we recall the identities

$$J_\nu(t) = \frac{1}{2}\{H_\nu^{(1)}(t) + H_\nu^{(2)}(t)\} \tag{14}$$

and

$$H_\nu^{(1)}(t)H_\nu^{(2)}(t) = J_\nu^2(t) + Y_\nu^2(t), \tag{15}$$

where $H_\nu^{(1)}(t)$ and $H_\nu^{(2)}(t)$ are the Hankel functions and $Y_\nu(t)$ is the Bessel function of the second kind. From these it follows that

$$h(t) = h_1(t) + h_2(t), \tag{16}$$

where

$$h_1(t) = \frac{1}{4}\{[H_\nu^{(1)}(t)]^2 + [H_\nu^{(2)}(t)]^2\} = \frac{1}{2}[J_\nu^2(t) - Y_\nu^2(t)] \tag{17}$$

and

$$h_2(t) = \frac{1}{2}H_\nu^{(1)}(t)H_\nu^{(2)}(t) = \frac{1}{2}[J_\nu^2(t) + Y_\nu^2(t)]. \tag{18}$$

The function $h_1(t)$ is oscillatory, as we can see from the well-known behavior of $H_\nu^{(1)}(t)$ and $H_\nu^{(2)}(t)$, whereas the function $h_2(t)$ is algebraic, as evidenced by the asymptotic expansion of $J_\nu^2(t) + Y_\nu^2(t)$. More specifically, we have

$$h_2(t) \sim \frac{1}{\pi t}\sum_{s=0}^{\infty} 1\cdot 3\cdot 5\cdots(2s-1)\frac{A_s(\nu)}{t^{2s}}, \qquad t\to+\infty, \tag{19}$$

and

$$\begin{aligned} h_1(t) \sim{}& \frac{1}{2\pi t}e^{i(2t-\nu\pi-\pi/2)}\sum_{s=0}^{\infty} i^s\frac{C_s(\nu)}{t^s} \\ &+ \frac{1}{2\pi t}e^{-i(2t-\nu\pi-\pi/2)}\sum_{s=0}^{\infty}(-i)^s\frac{C_s(\nu)}{t^s}, \qquad t\to+\infty, \end{aligned} \tag{20}$$

where

$$C_s(\nu) = \sum_{l=0}^{s} A_l(\nu)A_{s-l}(\nu), \tag{21}$$

$A_0(\nu) = 1$ and

$$A_s(\nu) = \frac{(4\nu^2-1^2)(4\nu^2-3^2)\cdots\{4\nu^2-(2s-1)^2\}}{s!8^s}; \tag{22}$$

cf. [4, pp. 238 and 342]. Also, from integral tables (e.g., [3, p. 203, Eq. 32(1), and p. 209, Eq. 45(1)]) we have

$$M[h_2; z] = \frac{2^{z-1}}{\pi^2}\cos(\nu\pi)\frac{\Gamma(z/2)\Gamma(z/2+\nu)\Gamma(z/2-\nu)}{\Gamma(1-z/2)\Gamma(z)}\frac{\pi}{\sin\pi z}, \tag{23}$$

for $2|\nu| < \operatorname{Re} z < 1$, and

$$M[h_1; z] = -\frac{2^{z-1}}{\pi}\frac{\Gamma(1+z/2)\Gamma(z/2)\Gamma(z/2+\nu)\Gamma(z/2-\nu)}{\Gamma(1+z)\Gamma((z+1)/2-\nu)\Gamma((1-z)/2+\nu)}, \tag{24}$$

for $2|\nu| < \operatorname{Re} z < 2$.

Inserting (16) in (8), we obtain

$$I(\lambda) = I_1(\lambda) + I_2(\lambda), \tag{25}$$

where

$$I_i(\lambda) = \int_0^\infty f(t) h_i(\lambda t)\, dt, \qquad i = 1, 2. \tag{26}$$

To the integral $I_2(\lambda)$ we shall apply the result (3.2) in [8]; see, also, Theorem A in [2]. Since

$$f(t) = 1 + \frac{1}{2}t^2 + \frac{3}{2^2 \cdot 2!}t^4 + \frac{3 \cdot 5}{2^3 \cdot 3!}t^6 + \frac{3 \cdot 5 \cdot 7}{2^4 \cdot 4!}t^8 + \cdots \tag{27}$$

for $0 \le t < 1$, with

$$a_1 = a_3 = \cdots = 0, \quad a_0 = 1, \quad a_2 = \frac{1}{2}, \quad a_4 = \frac{3}{2^2 \cdot 2!}, \ldots,$$
$$b_1 = b_3 = \cdots = 0, \quad b_0 = \frac{1}{\pi}, \quad b_{2s} = \frac{1 \cdot 3 \cdot 5 \cdots (2s-1)}{\pi} A_s(\nu),$$

we have

$$I_2(\lambda) \sim \ln \lambda \sum_{s=0}^{\infty} a_s b_s \lambda^{-s-1} + \sum_{s=0}^{\infty} c_s \lambda^{-s-1}, \tag{28}$$

where

$$c_s = a_s b_s^* + a_s^* b_s$$

and

$$a_s^* = \lim_{z \to s+1} \left\{ M[f; 1-z] + \frac{a_s}{z-s-1} \right\},$$
$$b_s^* = \lim_{z \to s+1} \left\{ M[h_2; z] + \frac{b_s}{z-s-1} \right\}.$$

Simple calculation gives

$$a_0^* = \ln 2, \qquad a_2^* = \frac{1}{2}\ln 2 - \frac{1}{4},$$
$$b_0^* = \frac{1}{\pi}\ln 2 - \frac{1}{\pi}\psi\left(\frac{1}{2} + \nu\right) + \frac{1}{2}\tan \nu\pi,$$
$$b_2^* = \frac{1}{8\pi}\left\{ [4\nu^2 - 1]\left[\ln 2 - \psi\left(\frac{1}{2} + \nu\right) + \frac{\pi}{2}\tan \nu\pi\right] - \frac{1}{2}(4\nu^2 - 5) \right\}.$$

Hence

$$c_0 = \frac{2}{\pi}\ln 2 - \frac{1}{\pi}\psi\left(\frac{1}{2} + \nu\right) + \frac{1}{2}\tan \nu\pi,$$
$$c_2 = \frac{1}{16\pi}\left\{ [4\nu^2 - 1]\left[2\ln 2 - \psi\left(\frac{1}{2} + \nu\right) + \frac{\pi}{2}\tan \nu\pi\right] - (4\nu^2 - 3) \right\}.$$

Since both a_s and b_s vanish for odd s, it follows from (28) that

$$\begin{aligned} I_2(\lambda) = {} & \frac{1}{\pi\lambda}\left[\ln \lambda + 2\ln 2 - \psi\left(\frac{1}{2} + \nu\right) + \frac{\pi}{2}\tan \nu\pi\right] \\ & + \frac{1}{16\pi\lambda^3}\left\{ [4\nu^2 - 1]\left[\ln \lambda + 2\ln 2 - \psi\left(\frac{1}{2} + \nu\right) + \frac{\pi}{2}\tan \nu\pi\right] \right. \\ & \left. \qquad - (4\nu^2 - 3) \right\} \\ & + O(\ln \lambda / \lambda^5). \end{aligned} \tag{29}$$

Higher terms in this expansion can also be obtained.

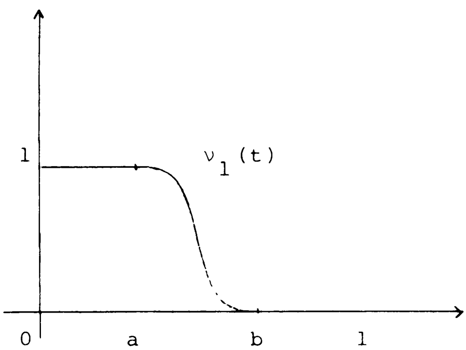
The asymptotic expansion of $I_1(\lambda)$ needs some special attention. This is due to the fact that the kernel $h_1(t)$ is oscillatory and the function $f(t)$ is discontinuous at $t = 1$. By using neutralizers [1, p. 87], we may write

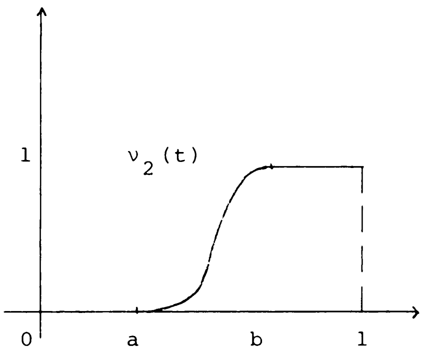
$$I_1(\lambda) = \int_0^\infty f(t)\nu_1(t)h_1(\lambda t)\,dt + \int_0^\infty f(t)\nu_2(t)h_1(\lambda t)\,dt = I_{11}(\lambda) + I_{21}(\lambda), \tag{30}$$

where $\nu_1(t)$ and $\nu_2(t)$ are C^∞-functions satisfying

$$\nu_1(t) = 1 \quad \text{for } 0 \le t \le a, \qquad \nu_1(t) = 0 \quad \text{for } t \ge b,\ 0 < a < b < 1,$$
$$\nu_2(t) = 0 \quad \text{for } 0 \le t \le a, \qquad \nu_2(t) = 1 \quad \text{for } t \ge b,$$
$$\nu_1^{(s)}(a^+) = \nu_2^{(s)}(b^-) = 0, \qquad s = 1, 2, \ldots,$$
$$\nu_1(t) + \nu_2(t) = 1 \quad \text{for } a \le t \le b.$$

The graphs of $\nu_1(t)$ and $\nu_2(t)$ are depicted in the figures below.





For small values of t, the Maclaurin series of $f(t)\nu_1(t)$ is the same as that of $f(t)$. Thus

$$f(t)\nu_1(t) = 1 + \frac{1}{2}t^2 + \frac{3}{8}t^4 + \cdots.$$

The asymptotic expansion of $I_{11}(\lambda)$ now follows immediately from Eq. (4.5) in [6]. The result is

$$I_{11}(\lambda) \sim M[h_1; 1]\lambda^{-1} + \frac{1}{2}M[h_1; 3]\lambda^{-3} + \cdots. \tag{31}$$

The values of the Mellin transform $M[h_1; z]$ at $z = 1, 2, \ldots$ can be obtained from (24). For instance,

$$M[h_1; 1] = -\frac{1}{2}\tan\nu\pi, \qquad M[h_1; 3] = -\frac{4\nu^2 - 1}{16}\tan\nu\pi.$$

Since $\nu_2(t)$ and $f(t)$ vanish in $(0, a)$ and $(1, \infty)$, respectively, the integral $I_{21}(\lambda)$ in (30) can be expressed as

$$I_{21}(\lambda) = \int_0^{1-a} f(1-t)\nu_2(1-t)h_1[\lambda(1-t)]\,dt. \tag{32}$$

Inserting (20) in (32), we obtain

$$
\begin{aligned}
I_{21}(\lambda) = {} & \frac{1}{2\pi} e^{i(2\lambda - \nu\pi - \pi/2)} \sum_{s=0}^{N-1} i^s C_s(\nu) F_{s+1}^{(-)}(\lambda) \lambda^{-s-1} \\
& + \frac{1}{2\pi} e^{-i(2\lambda - \nu\pi - \pi/2)} \sum_{s=0}^{N-1} (-i)^s C_s(\nu) F_{s+1}^{(+)}(\lambda) \lambda^{-s-1} + O(\lambda^{-N-1})
\end{aligned}
\tag{33}
$$

for any $N \geq 1$, where

$$
F_{s+1}^{(\pm)}(\lambda) = \int_0^{1-a} f(1-t)\nu_2(1-t) \frac{e^{\pm i2\lambda t}}{(1-t)^{s+1}}\, dt. \tag{34}
$$

Since $\nu_2(1-t)$ vanishes to infinite order at $t = 1-a$, only the lower limit of integration, $t = 0$, contributes to the asymptotic expansions of the integrals $F_{s+1}^{(\pm)}(\lambda)$, $s = 0, 1, 2, \ldots$. For small t, we have

$$
\begin{aligned}
f(1-t)\nu_2(1-t)\frac{1}{1-t} &= \frac{1}{\sqrt{2}} t^{-1/2} + \frac{5}{4\sqrt{2}} t^{1/2} + \frac{43}{32\sqrt{2}} t^{3/2} + \cdots, \\
f(1-t)\nu_2(1-t)\frac{1}{(1-t)^2} &= \frac{1}{\sqrt{2}} t^{-1/2} + \frac{9}{4\sqrt{2}} t^{1/2} + \frac{115}{32\sqrt{2}} t^{3/2} + \cdots.
\end{aligned}
$$

The asymptotic expansions of $F_1^{(\pm)}(\lambda)$ and $F_2^{(\pm)}(\lambda)$ can now be easily written down by using some well-known procedures; see, e.g., [4, Chapter 3]. Recalling $C_0(\nu) = 1$ and $C_1(\nu) = \nu^2 - \frac{1}{4}$, we have

$$
\begin{aligned}
I_{21}(\lambda) = {} & \frac{1}{2\sqrt{\pi}} \sin\left(2\lambda - \nu\pi - \frac{\pi}{4}\right) \lambda^{-3/2} - \frac{5}{32\sqrt{\pi}} \cos\left(2\lambda - \nu\pi - \frac{\pi}{4}\right) \lambda^{-5/2} \\
& + \frac{1}{2\sqrt{\pi}} \left(\nu^2 - \frac{1}{4}\right) \cos\left(2\lambda - \nu\pi - \frac{\pi}{4}\right) \lambda^{-5/2} + O(\lambda^{-7/2})
\end{aligned}
\tag{35}
$$

as $\lambda \to +\infty$. A combination of (29), (30), (31) and (35) gives the desired result in (4).

Acknowledgment. I would like to thank the referee for pointing out an error in an earlier version of this paper.

Department of Applied Mathematics
University of Manitoba
Winnipeg, Manitoba, Canada R3T 2N2

1. N. BLEISTEIN & R. A. HANDELSMAN, *Asymptotic Expansions of Integrals*, Holt, Rinehart and Winston, New York, 1975.

2. B. C. CARLSON & J. L. GUSTAFSON, "Asymptotic expansion of the first elliptic integral," *SIAM J. Math. Anal.*, v. 16, 1985, pp. 1072–1092.

3. O. I. MARICHEV, *Handbook of Integral Transforms of Higher Transcendental Functions, Theory and Algorithmic Tables*, Ellis Horwood Ltd., West Sussex, England, 1983.

4. F. W. J. OLVER, *Asymptotics and Special Functions*, Academic Press, New York, 1974.

5. B. J. STOYANOV & R. A. FARRELL, "On the asymptotic evaluation of $\int_0^{\pi/2} J_0^2(\lambda \sin x)\, dx$," *Math. Comp.*, v. 49, 1987, pp. 275–279.

6. R. WONG, "Explicit error terms for asymptotic expansions of Mellin convolutions," *J. Math. Anal. Appl.*, v. 72, 1979, pp. 740–756.

7. R. WONG, "Error bounds for asymptotic expansions of integrals," *SIAM Rev.*, v. 22, 1980, pp. 401–435.

8. R. WONG, *Applications of Some Recent Results in Asymptotic Expansions*, Proc. 12th Winnipeg Conf. on Numerical Methods of Computing, *Congress. Numer.*, v. 37, 1983, pp. 145–182.