Asymptotic expansion of β matrix models in the multi-cut regime

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Abstract

We push further our study of the all-order asymptotic expansion in β matrix models with a confining, offcritical potential, in the regime where the support of the equilibrium measure is a reunion of segments. We first address the case where the filling fractions of those segments are fixed, and show the existence of a $1/N$ expansion to all orders. Then, we study the asymptotic of the sum over filling fractions, in order to obtain the full asymptotic expansion for the initial problem in the multi-cut regime. We describe the application of our results to study the all-order small dispersion asymptotics of solutions of the Toda chain related to the one hermitian matrix model ($\beta = 2$) as well as orthogonal polynomials outside the bulk.

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1 Introduction

This paper deals with the all-order asymptotic expansion for the partition function and multilinear statistics of β matrix models. These laws represent a generalization of the joint distribution of the N eigenvalues of the Gaussian Unitary Ensemble [Meh04]. The convergence of the empirical measure of the eigenvalues is well-known (see e.g. [dMPS95]), and we are interested in the all-order finite size corrections to the moments of this empirical measure. This problem has received a lot of attention in the regime when the eigenvalues condensate on a single segment, usually called the one-cut regime. In this case, a central limit theorem for linear statistics has been proved by Johansson [Joh98], while a full $1/N$ expansion was derived first for $\beta = 2$ [APS01, EM03], then for any $\beta > 0$ in [BG11]. On the other hand, the multi-cut regime remained poorly understood at a rigorous level until recently, except for $\beta = 2$ which is related to integrable systems, and can be treated with the powerful asymptotic analysis techniques for Riemann-Hilbert problems, see e.g. [DKM$^+$99b]. Nevertheless, a heuristic derivation of the asymptotic expansion for the multi-cut regime was proposed to leading order by Bonnet, David and Eynard [BDE00], and extended to all orders in [Eyn09], in terms of Theta functions and their derivatives. It features oscillatory behavior, whose origin lies in the tunneling of eigenvalues between the different connected components of the support. These heuristics, initially written for $\beta = 2$, trivially extend to $\beta > 0$, see e.g. [Bor11].

Lately, M. Shcherbina has established this asymptotic expansion up to terms of order 1 [Shc11, Shc12]. This allows for instance the observation that linear statistics do not always satisfy a central limit theorem (this fact was already noticed for $\beta = 2$ in [Pas06]). In this paper, we go beyond the $O(1)$ and put the heuristics of [Eyn09] to all orders on a firm mathematical ground. As a consequence for $\beta = 2$, we can establish the full asymptotic expansion outside of the bulk for the orthogonal polynomials with real-analytic potentials, and the all-order asymptotic expansion of certain solutions of the Toda lattice in the continuum limit. The same method would allow to justify rigorously the asymptotics of skew-orthogonal polynomials ($\beta = 1$ and 4) outside of the bulk, derived heuristically in [Eyn01].

1.1 Definitions

We consider the probability measure $\mu_{N,\beta}^{V}$ on B^N given by:

$$\mathrm{d}\mu_{N,\beta}^{V;\mathsf{B}}(\lambda) = \frac{1}{Z_{N,\beta}^{V;\mathsf{B}}} \prod_{i=1}^{N} \mathrm{d}\lambda_i\, \mathbf{1}_{\mathsf{B}}(\lambda_i)\, e^{-\frac{\beta N}{2}V(\lambda_i)} \prod_{1 \leqslant i < j \leqslant N} |\lambda_i - \lambda_j|^{\beta}. \tag{1.1}$$

B is a reunion of closed intervals of $\mathbb{R} \cup \{\pm\infty\}$, β is a positive number, and $Z_{N,\beta}^{V;\mathsf{B}}$ is the partition function so that (1.1) has total mass 1. This model is usually called the β ensemble [Meh04, DE02, For10]. We introduce the unnormalized empirical measure M_N of the eigenvalues:

$$M_N = \sum_{i=1}^{N} \delta_{\lambda_i}, \tag{1.2}$$

and we consider several types of statistics for M_N. We sometimes denote $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$.

Correlators

We introduce the Stieltjes transform of the n-th order moments of the empirical measure, called *disconnected correlators*:

$$\widetilde{W}_n(x_1, \ldots, x_n) = \mu_{N,\beta}^{V;\mathsf{B}}\Big[\Big(\int \frac{\mathrm{d}M_N(\xi_1)}{x_1 - \xi_1} \cdots \int \frac{\mathrm{d}M_N(\xi_n)}{x_n - \xi_n}\Big)\Big]. \tag{1.3}$$

They are holomorphic functions of $x_i \in \mathbb{C}\backslash\mathsf{B}$. For reasons related to concentration of measures, it is more convenient to consider the *correlators* to study large N asymptotics:

$$\begin{aligned} W_n(x_1,\ldots,x_n) &= \partial_{t_1}\cdots\partial_{t_n}\Big(\ln Z_{N,\beta}^{V-\frac{2}{\beta N}\sum_{i=1}^{n}\frac{t_i}{x_i-\bullet};\mathsf{B}}\Big)\Big|_{t_i=0} \\ &= \mu_{N,\beta}^{V;\mathsf{B}}\Big[\prod_{i=1}^{n}\mathrm{Tr}\,\frac{1}{x_j-\Lambda}\Big]_c. \end{aligned} \tag{1.4}$$

By construction, the coefficients of their expansion as a Laurent series in the variable $x_i \to \infty$ give the n-th order cumulants of M_N. If I is a set, we introduce the notation $x_I = (x_i)_{i\in I}$ for a set of variables indexed by I. The two type of correlators are related by:

$$\widetilde{W}_n(x_1,\ldots,x_n) = \sum_{s=1}^{n}\sum_{J_1\dot\cup\cdots\dot\cup J_s=I}\prod_{i=1}^{s}W_{|J_i|}(x_{J_i}). \tag{1.5}$$

If φ_n is an analytic function in n variables in a neighborhood of B^n, the n-linear statistics can be deduced as contour integrals of the disconnected correlators:

$$\mu_{N,\beta}^{V;\mathsf{B}}\Big[\sum_{i_1,\ldots,i_n=1}^{N}\varphi_n(\lambda_{i_1},\ldots,\lambda_{i_n})\Big] = \oint_{\mathsf{B}}\frac{\mathrm{d}\xi_1}{2\mathrm{i}\pi}\cdots\oint_{\mathsf{B}}\frac{\mathrm{d}\xi_n}{2\mathrm{i}\pi}\,\varphi_n(\xi_1,\ldots,\xi_n)\,\widetilde{W}_n(\xi_1,\ldots,\xi_n). \tag{1.6}$$

We remark that the knowledge of the correlators for a smooth family of potentials $(V_t)_t$ determines the partition function up to an integration constant, since:

$$\partial_t \ln Z_{N,\beta}^{V_t;\mathsf{B}} = -\frac{\beta N}{2}\,\mu_{N,\beta}^{V_t;\mathsf{B}}\Big[\sum_{i=1}^{N}\partial_t V_t(\lambda_i)\Big] = -\frac{\beta N}{2}\oint_{\mathsf{B}}\frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\,\partial_t V_t(\xi)\,W_1(\xi) \tag{1.7}$$

Kernels

Let $\mathbf{c}$ be a n-uple of non zero complex numbers. We introduce the n-kernels:

$$\begin{aligned} K_{n,\mathbf{c}}(x_1,\ldots,x_n) &= \mu_{N,\beta}^{V;\mathsf{B}}\left[\prod_{j=1}^{n}\det{}^{c_j}(x_j-\Lambda)\right] \\ &= \frac{Z_{N,\beta}^{V-\frac{2}{\beta N}\sum_{j=1}^{n}c_j\ln(x_j-\bullet);\mathsf{B}}}{Z_{N,\beta}^{V;\mathsf{B}}}. \end{aligned} \tag{1.8}$$

When c_j are integers, the kernels are holomorphic functions of $x_j \in \mathbb{C}\backslash\mathsf{B}$. When c_j are not integers, the kernels are multivalued holomorphic functions of x_j in $\mathbb{C}\backslash\mathsf{B}$, with monodromies around the connected components of B and around ∞.

In particular, for $\beta = 2$, $K_{1,1}(x)$ is the monic N-th orthogonal polynomial associated to the weight $\mathbf{1}_{\mathsf{B}}(x)\,e^{-N\,V(x)}\mathrm{d}x$ on the real line, and $K_{2,(1,-1)}(x,y)$ is the N-th Christoffel-Darboux kernel associated to those orthogonal polynomials, see Section 2.

1.2 Equilibrium measure and multi-cut regime

By standard results of potential theory, see [Joh98] or the textbooks [Dei99, Theorem 6] or [AGZ10, Theorem 2.6.1 and Corollary 2.6.3], we have:

Theorem 1.1 *Assume that $V : \mathsf{B} \to \mathbb{R}$ is a continuous function, and if $\tau\infty \in \mathsf{B}$, assume that:*

$$\liminf_{x\to\tau\infty}\frac{V(x)}{2\ln|x|} > 1. \tag{1.9}$$

If V depends on N, assume also that $V \to V^{\{0\}}$ in the space of continuous function over B for the sup norm. Then, the normalized empirical measure $L_N = N^{-1}\,M_N$ converges almost surely and in expectation towards the unique probability measure $\mu_{\mathrm{eq}} := \mu_{\mathrm{eq}}^{V;\mathsf{B}}$ on B which minimizes:

$$E[\mu] = \int \mathrm{d}\mu(\xi) V^{\{0\}}(\xi) - \iint \mathrm{d}\mu(\xi)\mathrm{d}\mu(\eta)\ln|\xi - \eta|. \tag{1.10}$$

μ_{eq} has compact support, denoted S. It is characterized by the existence of a constant C such that:

$$\forall x \in \mathsf{B}, \qquad 2\int_{\mathsf{B}} \mathrm{d}\mu_{\mathrm{eq}}(\xi)\ln|x - \xi| - V^{\{0\}}(x) \leqslant C, \tag{1.11}$$

with equality realized μ_{eq} almost surely. Moreover, if $V^{\{0\}}$ is real-analytic in a neighborhood of B, the support consists of a finite disjoint union of segments:

$$\mathsf{S} = \bigcup_{h=0}^{g} \mathsf{S}_h, \qquad \mathsf{S}_h = [\alpha_h^-, \alpha_h^+], \tag{1.12}$$

μ_{eq} has a density of the form:

$$\frac{\mathrm{d}\mu_{\mathrm{eq}}}{\mathrm{d}x} = \frac{S(x)}{\pi} \prod_{h=0}^{g} (\alpha_h^+ - x)^{\rho_h^+/2}(x - \alpha_h^-)^{\rho_h^-/2}, \tag{1.13}$$

where $\rho_h^\bullet$ is $+1$ (resp. -1) if the corresponding edge is soft (resp. hard), and S is analytic in a neighborhood of S.

The goal of this article is to establish an all-order expansion of the partition function, the correlators and the kernels, in all such situations.

1.3 Assumptions

We will refer throughout the text to the following set of assumptions.

Hypothesis 1.1

- *(Regularity) $V : \mathsf{B} \to \mathbb{R}$ is continuous, and if V depends on N, it has a limit $V^{\{0\}}$ in the space of continuous functions over $[b_-, b_+]$ for the sup norm.*
- *(Confinement) If $\tau\infty \in \mathsf{B}$, $\liminf_{x\to\tau\infty} \frac{V(x)}{2\ln|x|} > 1$.*
- *($g+1$-cut regime) The support of $\mu_{\mathrm{eq}}^{V;\mathsf{B}}$ is of the form $\mathsf{S} = \bigcup_{h=0}^{g} \mathsf{S}_h$ where $\mathsf{S}_h = [\alpha_h^-, \alpha_h^+]$ with $\alpha_h^- < \alpha_h^+$.*
- *(Control of large deviations) The effective potential $U^{V;\mathsf{B}}(x) = V(x) - 2\int \ln|x - \xi| d\mu_{\mathrm{eq}}^{V;\mathsf{B}}(\xi)$ achieves its minimum value on S only.*
- *(Offcriticality) In the equilibrium measure (1.13), $S(x) > 0$ in S.*

At some point, we shall need to add a stronger assumption concerning off-criticality:

Hypothesis 1.2 *The same as Hypothesis 1.3, and:*

- *(Strong off-criticality) For any soft edge $\alpha_h^\bullet$, we have $S'(\alpha_h^\bullet) \neq 0$.*

We will also require regularity of the potential:

Hypothesis 1.3

- *(Analyticity)* V *extends as a holomorphic function in some open neighborhood* U *of* S.
- *($1/N$ expansion of the potential) There exists a sequence* $(V^{\{k\}})_{k\geqslant 0}$ *of holomorphic functions in* U *and constants* $(v^{\{k\}})_{k\geqslant 0}$ *such that, for any* $K\geqslant 0$,

$$\sup_{\xi\in\mathsf{U}}\Big|V(\xi)-\sum_{k=0}^{K}N^{-k}\,V^{\{k\}}(\xi)\Big|\leqslant v^{\{K\}}\,N^{-(K+1)}. \tag{1.14}$$

In Section 6, we shall weaken Hypothesis 1.3 by allowing complex perturbations of order $1/N$ and harmonic functions instead of analytic functions:

Hypothesis 1.4 $V\,:\,\mathsf{B}\rightarrow\mathbb{C}$ *can be decomposed as* $V=\mathcal{V}_1+\overline{\mathcal{V}_2}$ *where:*

- *For* $j=1,2$, $\mathcal{V}_j$ *extends to a holomorphic function in some neighborhood* U *of* B. *There exists a sequence of holomorphic functions* $(\mathcal{V}_j^{\{k\}})_{k\geqslant 0}$ *and constants* $(v_j^{\{k\}})_{k\geqslant 0}$ *so that, for any* $K\geqslant 0$:

$$\sup_{\xi\in\mathsf{U}}\Big|\mathcal{V}_j(\xi)-\sum_{k=0}^{K}N^{-k}\,\mathcal{V}_j^{\{k\}}(\xi)\Big|\leqslant v_j^{\{K\}}\,N^{-(K+1)}. \tag{1.15}$$

- $V^{\{0\}}=\mathcal{V}_1^{\{0\}}+\overline{\mathcal{V}_2^{\{0\}}}$ *is real-valued on* B.

The topology for which we study the large N expansion of the correlators is described in § 5.1, and amounts to controlling the (moments of order m)$\times C^m$ uniformly in m for some constant $C>0$. We now describe our strategy and announce our results.

1.4 Main result with fixed filling fractions

Before coming to the multi-cut regime, we analyze a different model where the number of λ's in a small enlargment of S_h is fixed. Let $\mathsf{A}=\bigcup_{h=0}^{g}\mathsf{A}_h$ where $\mathsf{A}_h=[a_h^-,a_h^+]$ are pairwise disjoint segments such that $a_h^-<\alpha_h^-<\alpha_h^+<a_h^+$. We introduce the set:

$$\mathcal{E}_g=\Big\{\boldsymbol{\epsilon}\in]0,1[^g,\quad\sum_{h=1}^{g}\epsilon_h<1\Big\}. \tag{1.16}$$

If $\boldsymbol{\epsilon}\in\mathcal{E}_g$, we denote $\epsilon_0=1-\sum_{h=1}^{g}\epsilon_h$, we let $\boldsymbol{N}=(\lfloor N\epsilon_0\rfloor,\lfloor N\epsilon_1\rfloor,\ldots,\lfloor N\epsilon_g\rfloor)$, and consider the probability measure on $\prod_{h=0}^{g}\mathsf{A}_h^{N_h}$:

$$\begin{aligned}\mathrm{d}\mu_{N,\boldsymbol{\epsilon},\beta}^{V;\mathsf{A}}(\boldsymbol{\lambda}) &= \frac{1}{Z_{N,\boldsymbol{\epsilon},\beta}^{V;A}}\prod_{h=0}^{g}\Big[\prod_{i=1}^{N_h}\mathrm{d}\lambda_{h,i}\,\mathbf{1}_{\mathsf{A}_h}(\lambda_{h,i})\,e^{-\frac{\beta N}{2}\,V(\lambda_{h,i})}\prod_{1\leqslant i<j\leqslant N}|\lambda_{h,i}-\lambda_{h,j}|^{\beta}\Big] \\ &\quad\times\prod_{0\leqslant h<h'\leqslant g}\ \prod_{\substack{1\leqslant i\leqslant N_h\\ 1\leqslant j\leqslant N_{h'}}}|\lambda_{h,i}-\lambda_{h',j}|^{\beta}.\end{aligned} \tag{1.17}$$

The empirical measure $M_{N,\boldsymbol{\epsilon}}$ and the correlators $W_{n,\boldsymbol{\epsilon}}(x_1,\ldots,x_n)$ for this model are defined as in § 1.1. We call ϵ_h the *filling fraction* of A_h. It follows from the definitions that:

$$\oint_{\mathsf{A}_h}\frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\,W_{n,\boldsymbol{\epsilon}}(\xi,x_2,\ldots,x_n)=\delta_{n,1}\,N\epsilon_h. \tag{1.18}$$

We will refer to (1.1) as the initial model, and to (1.17) as the model with fixed filling fractions. Standard results from potential theory imply:

Theorem 1.2 *Assume V regular and confining on A. Then, the normalized empirical measure $N^{-1}\,M_{N,\boldsymbol{\epsilon}}$ converges almost surely and in expectation towards the unique probability measure $\mu_{\mathrm{eq},\boldsymbol{\epsilon}}$ on A which minimizes:*

$$E[\mu] = \iint \left(\frac{1}{2}(V^{\{0\}}(\xi) + V^{\{0\}}(\eta)) - \ln|\xi - \eta|\right)\mathrm{d}\mu(\xi)\mathrm{d}\mu(\eta)\,. \tag{1.19}$$

among probability measures with partial masses $\mu[\mathsf{A}_h] = \epsilon_h$. They are characterized by the existence of constants $C_{\boldsymbol{\epsilon},h}$ such that:

$$\forall x \in \mathsf{A}_h, \qquad 2\int_{\mathsf{B}} \mathrm{d}\mu_{\mathrm{eq},\boldsymbol{\epsilon}}(\xi)\ln|x-\xi| - V^{\{0\}}(x) \leqslant C_{\boldsymbol{\epsilon},h}, \tag{1.20}$$

with equality realized $\mu_{\mathrm{eq},\boldsymbol{\epsilon}}$ almost surely. $\mu_{\mathrm{eq},\boldsymbol{\epsilon}}$ can be decomposed as a sum of positive measures $\mu_{\mathrm{eq},\boldsymbol{\epsilon},h}$ having compact support in A_h, denoted $\mathsf{S}_{\boldsymbol{\epsilon},h}$. Moreover, if $V^{\{0\}}$ is real-analytic in a neighborhood of A, $\mathsf{S}_{\boldsymbol{\epsilon},h}$ consists of a finite reunion of segments.

μ_{eq} appearing in Theorem 1.1 coincides with $\mu_{\mathrm{eq},\boldsymbol{\epsilon}_\star}$ for the optimal value $\boldsymbol{\epsilon}_\star = (\mu_{\mathrm{eq}}[\mathsf{A}_h])_{1\leqslant h\leqslant g}$, and in this case $\mathsf{S}_{\boldsymbol{\epsilon}_\star,h}$ is actually the segment $[\alpha_h^-, \alpha_h^+]$. The key point is that, for $\boldsymbol{\epsilon}$ close enough to $\boldsymbol{\epsilon}_\star$, the support $\mathsf{S}_{\boldsymbol{\epsilon},h}$ remains connected, and the model with fixed filling fraction enjoys a $1/N$ expansion.

Theorem 1.3 *If V satisfies Hypotheses 1.1 and 1.4 on A, there exists $t > 0$ such that, uniformly for $\boldsymbol{\epsilon} \in \mathcal{E}_g$ such that $|\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_\star| < t$, we have an expansion for the correlators:*

$$W_{n,\boldsymbol{\epsilon}}(x_1,\ldots,x_n) = \sum_{k\geqslant n-2} N^{-k}\,W_{n,\boldsymbol{\epsilon}}^{\{k\}}(x_1,\ldots,x_n) + O(N^{-\infty}). \tag{1.21}$$

Up to a fixed $O(N^{-K})$ and for a fixed n, (1.21) holds uniformly for $x_1,\ldots,x_n$ in compact regions of $\widehat{\mathbb{C}}\backslash\mathsf{A}$. Besides, if the strong off-criticality of Hypothesis 1.2 is satisfied, $W_{n,\boldsymbol{\epsilon}}^{\{k\}}$ are smooth functions of $\boldsymbol{\epsilon}$ close enough to $\boldsymbol{\epsilon}_\star$.

We prove this theorem, independently of the nature soft/hard of the edges, in Section 5 with real-analytic potential (i.e. Hypothesis 1.3 instead of 1.4). The result is extended to harmonic potentials (i.e. Hypothesis 1.4) in Section 6.2. Actually, we provide in Proposition 5.5 an explicit control of the errors in terms of the distance of $x_1,\ldots,x_k$ to A, and its proof makes clear that the expansion of the correlators is not expected to be uniform for $x_1,\ldots,x_n$ chosen in a compact of $\widehat{\mathbb{C}}\backslash\mathsf{A}$ independently of n and K.

We then compute in Section 7.1 the expansion of the partition function thanks to the expansion of $W_{1,\boldsymbol{\epsilon}}$, by a two-step interpolation preserving Hypotheses 1.2-1.4 between our potential V and a reference situation where the partition function is exactly computable for finite N, in terms of Selberg integrals.

Theorem 1.4 *If V satisfies Hypotheses 1.2 and 1.4 on A, there exists $t > 0$ such that, uniformly for $\boldsymbol{\epsilon} \in \mathcal{E}_g$ such that $|\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_\star| < t$, we have:*

$$\frac{N!}{\prod_{h=0}^{g}(N\epsilon_h)!}\,Z_{N,\boldsymbol{\epsilon},\beta}^{V;\mathsf{A}} = N^{(\beta/2)N+e}\exp\Big(\sum_{k\geqslant -2} N^{-k}\,F_{\boldsymbol{\epsilon},\beta}^{\{k\}} + O(N^{-\infty})\Big), \tag{1.22}$$

with $e = \sum_{h=0}^{g} e_{\rho_h^-,\rho_h^+}$, where:

$$e_{++} = \frac{3+\beta/2+2/\beta}{12}, \qquad e_{+-} = e_{-+} = \frac{\beta/2+2/\beta}{6}, \qquad e_{--} = \frac{-1+2/\beta+\beta/2}{4}, \tag{1.23}$$

and we recall $\rho_h^\bullet = 1$ for a soft edge and $\rho_h^\bullet = -1$ for a hard edge. Besides, $F_{\boldsymbol{\epsilon},\beta}^{\{k\}}$ is a smooth function of $\boldsymbol{\epsilon}$ close enough to $\boldsymbol{\epsilon}_\star$, and at the value $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_\star$, the derivative of $F_{\boldsymbol{\epsilon},\beta}^{\{-2\}}$ vanishes and its Hessian is negative definite.

Up to a given $O(N^{-K})$, all expansions are uniform with respect to parameters of the potential and of $\boldsymbol{\epsilon}$ chosen in a compact set so that the assumptions hold. The power of N in prefactor is universal in the sense that it only depends on the nature of the edges, and its value can be extracted from the large N expansion of Selberg type integrals. Theorems 1.3-1.4 are the generalizations to the fixed filling fraction model of our earlier results about existence of the $1/N$ expansion in the one-cut regime [BG11] (see also [Joh98, APS01, EM03, GMS07, KS10] for previous results concerning the one-cut regime in $\beta = 2$ or general β ensembles).

1.5 Main results in the multi-cut regime

Let us come back to the initial model (1.1), and take $\mathsf{A} = \bigcup_{h=0}^{g} \mathsf{A}_h \subseteq \mathsf{B}$ a small enlargement of the support S as in the previous paragraph. It is well-known that the partition function $Z_{N,\beta}^{V;\mathsf{B}}$ can be replaced by $Z_{N,\beta}^{V;\mathsf{A}}$ up to exponentially small corrections when N is large (see [] for results in this direction, and we give a proof for completeness in § 3.1 below). The latter can be decomposed as a sum over all possible ways of sharing the λ's between the segments A_h, namely:

$$Z_{N,\beta}^{V,\mathsf{A}} = \sum_{0 \leqslant N_1,\ldots N_g \leqslant N} \frac{N!}{\prod_{h=0}^{g} N_h!}\, Z_{N,\boldsymbol{N}/N,\beta}^{V;\mathsf{A}}, \tag{1.24}$$

where we have denoted $N_0 = N - \sum_{h=1}^{g} N_h$ the number of λ's put in the segment A_0. So, we can use our results for the model with fixed filling fractions to analyze the asymptotic behavior of each term in the sum, and then find the asymptotic expansion of the sum taking into account the interference of all contributions.

In order to state the result, we need to introduce the Siegel Theta function with characteristics $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathbb{C}^g$. If $\boldsymbol{\tau}$ be a $g \times g$ matrix of complex numbers such that $\mathrm{Im}\,\boldsymbol{\tau} > 0$, it is the entire function of $\boldsymbol{v} \in \mathbb{C}^g$ defined by the converging series:

$$\vartheta\begin{bmatrix}\boldsymbol{\mu}\\ \boldsymbol{\nu}\end{bmatrix}(\boldsymbol{v}|\boldsymbol{\tau}) = \sum_{\boldsymbol{m} \in \mathbb{Z}^g} \exp\Big(\mathrm{i}\pi(\boldsymbol{m}+\boldsymbol{\mu})\cdot\boldsymbol{\tau}\cdot(\boldsymbol{m}+\boldsymbol{\mu}) + 2\mathrm{i}\pi(\boldsymbol{v}+\boldsymbol{\nu})\cdot(\boldsymbol{m}+\boldsymbol{\mu})\Big). \tag{1.25}$$

Among its essential properties, we mention:

- for any characteristics $\boldsymbol{\mu}, \boldsymbol{\nu}$, it satisfies the diffusion-like equation $4\mathrm{i}\pi\partial_{\tau_{h,h'}}\vartheta = \partial_{v_h}\partial_{v_{h'}}\vartheta$.

- it is a quasiperiodic function on the lattice $\mathbb{Z}^g \oplus \boldsymbol{\tau}(\mathbb{Z}^g)$: for any $\boldsymbol{m}_0, \boldsymbol{n}_0 \in \mathbb{Z}^g$,

$$\vartheta\begin{bmatrix}\boldsymbol{\mu}\\ \boldsymbol{\nu}\end{bmatrix}(\boldsymbol{v}+\boldsymbol{m}_0+\boldsymbol{\tau}\cdot\boldsymbol{n}_0|\boldsymbol{\tau}) = \exp\big(2\mathrm{i}\pi\boldsymbol{m}_0\cdot\boldsymbol{\mu} - 2\mathrm{i}\pi\boldsymbol{n}_0\cdot(\boldsymbol{v}+\boldsymbol{\nu}) - \mathrm{i}\pi\boldsymbol{n}_0\cdot\boldsymbol{\tau}\cdot\boldsymbol{n}_0\big)\,\vartheta\begin{bmatrix}\boldsymbol{\mu}\\ \boldsymbol{\nu}\end{bmatrix}(\boldsymbol{v}|\boldsymbol{\tau}). \tag{1.26}$$

- it has a nice transformation law under $\boldsymbol{\tau} \to (\boldsymbol{A}\boldsymbol{\tau}+\boldsymbol{B})(\boldsymbol{C}\boldsymbol{\tau}+\boldsymbol{D})^{-1}$ where $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ are the $g \times g$ blocks of a $2g \times 2g$ symplectic matrix [Mum84].

- when $\boldsymbol{\tau}$ is the matrix of periods of a genus g Riemann surface, it satisfies the Fay identity [Fay70].

We define the operator $\nabla_{\boldsymbol{v}}$ acting on the variable $\boldsymbol{v}$ of this function. For instance, the diffusion equation takes the form $4\mathrm{i}\pi\partial_{\boldsymbol{\tau}}\vartheta = \nabla_{\boldsymbol{v}}^{\otimes 2}\vartheta$.

Theorem 1.5 *Assume Hypotheses 1.2 and 1.4. Let $\boldsymbol{\epsilon}_\star = (\mu_{\mathrm{eq}}[\mathsf{S}_h])_{1 \leqslant h \leqslant g}$. Given the coefficients of the expansion in the fixed filling fraction model from Theorem 1.4, we denote $(F_{\star,\beta}^{\{k\}})^{(\ell)}$ their tensor of*

ℓ-th order derivatives with respect to $\boldsymbol{\epsilon}$, evaluated at $\boldsymbol{\epsilon}_\star$. Then, the partition function has an asymptotic expansion of the form:

$$Z_{N,\beta}^{V;\mathsf{A}} = Z_{N,\boldsymbol{\epsilon}_\star,\beta}^{V;\mathsf{A}} \left\{ \Big(\sum_{k \geqslant 0} N^{-k}\, T_{\star,\beta}^{\{k\}}\big[\frac{\nabla_{\boldsymbol{v}}}{2\mathrm{i}\pi}\big]\Big) \vartheta\begin{bmatrix} -N\boldsymbol{\epsilon}_\star \\ \mathbf{0} \end{bmatrix}(\boldsymbol{v}_{\star,\beta}|\boldsymbol{\tau}_{\star,\beta}) + O(N^{-\infty}) \right\}. \tag{1.27}$$

In this expression, if $\boldsymbol{X}$ is a vector with g components, $T_{\boldsymbol{\epsilon},\beta}^{\{0\}}[\boldsymbol{X}] = 1$, and for $k \geqslant 1$:

$$T_{\boldsymbol{\epsilon},\beta}^{\{k\}}[\boldsymbol{X}] = \sum_{r=1}^{k} \frac{1}{r!} \sum_{\substack{\ell_1,\ldots,\ell_r \geqslant 1 \\ m_1,\ldots,m_r \geqslant -2 \\ \sum_{i=1}^r \ell_i + m_i = k}} \Big(\bigotimes_{i=1}^{r} \frac{(F_{\boldsymbol{\epsilon},\beta}^{\{m_i\}})^{(\ell_i)}}{\ell_i!} \Big) \cdot \boldsymbol{X}^{\otimes(\sum_{i=1}^r \ell_i)}, \tag{1.28}$$

where $\cdot$ denotes the contraction of tensors. We have also introduced:

$$\boldsymbol{v}_{\star,\beta} = \frac{(F_{\star,\beta}^{\{-1\}})'}{2\mathrm{i}\pi}, \qquad \boldsymbol{\tau}_{\star,\beta} = \frac{(F_{\star,\beta}^{\{-2\}})''}{2\mathrm{i}\pi}. \tag{1.29}$$

Being more explicit but less compact, we may rewrite:

$$T_{\star,\beta}^{\{k\}}\big[\frac{\nabla_{\boldsymbol{v}}}{2\mathrm{i}\pi}\big] \vartheta\begin{bmatrix} -N\boldsymbol{\epsilon}_\star \\ \mathbf{0} \end{bmatrix}(\boldsymbol{v}_{\star,\beta}|\boldsymbol{\tau}) = \sum_{r=1}^{k} \frac{1}{r!} \sum_{\substack{\ell_1,\ldots,\ell_r \geqslant 1 \\ m_1,\ldots,m_r \geqslant -2 \\ \sum_{i=1}^r \ell_i + m_i = k}} \Big(\bigotimes_{i=1}^{r} \frac{(F_{\boldsymbol{\epsilon},\beta}^{\{m_i\}})^{(\ell_i)}}{\ell_i!} \Big) \times \Big(\sum_{\boldsymbol{m} \in \mathbb{Z}^g} (\boldsymbol{m} - N\boldsymbol{\epsilon}_\star)^{\otimes(\sum_{i=1}^r \ell_i)}\, e^{\mathrm{i}\pi(\boldsymbol{m}-N\boldsymbol{\epsilon}_\star)\cdot\boldsymbol{\tau}_{\star,\beta}\cdot(\boldsymbol{m}-N\boldsymbol{\epsilon}_\star) + 2\mathrm{i}\pi \boldsymbol{v}_{\star,\beta}\cdot(\boldsymbol{m}-N\boldsymbol{\epsilon}_\star)} \Big). \tag{1.30}$$

For $\beta = 2$, this result has been derived heuristically to leading order in [BDE00], and to all orders in [Eyn09], and the arguments there can be extended straightforwardly to all values of β, see e.g. [Bor11]. Our work justifies their heuristic argument. We exploit the Schwinger-Dyson equations for the β ensemble with fixed filling fractions taking advantage of a rough control on the large N behavior of the correlators. The result of Theorem 1.5 has been derived up to $o(1)$ by Shcherbina [Shc12] for real-analytic potentials, with different techniques, based on the representation of $\prod_{1 \leqslant h < h' \leqslant g} |\lambda_{h,i} - \lambda_{h',j}|^\beta$, which is the exponential of a quadratic statistic, as expectation value of a linear statistics coupled to a Brownian motion. The rough a priori controls on the correlators do not allow at present the description of the $o(1)$ by such methods. The results in [Shc12] were also written in a different form: $F^{\{0\}}$ was identified with a combination of Fredholm determinants (see also the physics paper [WZ06]), whereas this representation does not come naturally in our approach). Also, the step of the analysis of Section 8 consisting in replacing the sum over nonnegative integers such that $N_0 + \ldots + N_g = N$ in (1.24), by a sum over $\boldsymbol{N} \in \mathbb{Z}^g$, thus reconstructing the theta function, was not performed in [Shc12]

Let us make a few remarks. The $2\mathrm{i}\pi$ appears because we used the standard definition of the Siegel theta function, and should not hide the fact that all terms in (1.30) are real-valued. Here, the matrix:

$$\boldsymbol{\tau}_{\star,\beta} = \frac{\mathrm{Hessian}(F_{\star,\beta}^{\{-2\}})}{2\mathrm{i}\pi} \tag{1.31}$$

involved in the theta function has purely imaginary entries, and $\mathrm{Im}\,\boldsymbol{\tau}_{\star,\beta}$ is definite positive according to Theorem 1.4, hence the theta function in the right-hand side makes sense. Notice also that for it is $\mathbb{Z}^g$-periodic in its characteristics $\boldsymbol{\mu}$, hence we can replace $-N\boldsymbol{\epsilon}_\star$ by $-N\boldsymbol{\epsilon}_\star + \lfloor N\boldsymbol{\epsilon}_\star \rfloor$, and this is responsible for modulations of frequency $O(1/N)$ in the asymptotic expansion, and thus breakdown of the $1/N$ expansion. Still, "subsequential" asymptotic expansions in $1/N$ may occur. For instance, in a symmetric two cuts ($g = 1$) model, we have $\epsilon_\star = 1/2$ and thus the right-hand side is an asymptotic expansion in powers of $1/N$ depending on the parity of N.

Let us give the two first orders of (1.30):

$$T^{\{1\}}_{\star,\beta}[\boldsymbol{X}] = \frac{1}{6}\,(F^{\{-2\}}_{\star,\beta})''' \cdot \boldsymbol{X}^{\otimes 3} + \frac{1}{2}\,(F^{\{-1\}}_{\star,\beta})'' \cdot \boldsymbol{X}^{\otimes 2} + (F^{\{0\}}_{\star,\beta})' \cdot \boldsymbol{X}, \tag{1.32}$$

and:

$$\begin{aligned} T^{\{2\}}_{\star,\beta}[\boldsymbol{X}] \;=\; & \frac{1}{72}\big[(F^{\{-2\}}_{\star,\beta})'''\big]^{\otimes 2} \cdot \boldsymbol{X}^{\otimes 6} + \frac{1}{12}\big[(F^{\{-2\}}_{\star,\beta})''' \otimes (F^{\{-1\}}_{\star,\beta})''\big] \cdot \boldsymbol{X}^{\otimes 5} \\ & + \Big(\frac{1}{6}\big[(F^{\{-2\}}_{\star,\beta})''' \otimes (F^{\{0\}}_{\star,\beta})'\big] + \frac{1}{8}\big[(F^{\{-1\}}_{\star,\beta})''\big]^{\otimes 2} + \frac{1}{24}\,(F^{\{-2\}}_{\star,\beta})^{(4)}\Big) \cdot \boldsymbol{X}^{\otimes 4} \\ & + \Big(\frac{1}{2}\big[(F^{\{-1\}}_{\star,\beta})'' \otimes (F^{\{0\}}_{\star,\beta})'\big] + \frac{1}{6}\,(F^{\{-1\}}_{\star,\beta})'''\Big) \cdot \boldsymbol{X}^{\otimes 3} \\ & + \Big(\frac{1}{2}\big[(F^{\{0\}}_{\star,\beta})'\big]^{\otimes 2} + \frac{1}{2}\,(F^{\{0\}}_{\star,\beta})''\Big) \cdot \boldsymbol{X}^{\otimes 2} + (F^{\{1\}}_{\star,\beta})' \cdot \boldsymbol{X}. \end{aligned} \tag{1.33}$$

If the potential V is independent of β, we observe that $\boldsymbol{\epsilon}_\star$ does not depend on β, and it is well-known [CE06] that the coefficients in the expansion (1.22) have a simple dependence in β:

$$F^{\{k\}}_{\boldsymbol{\epsilon},\beta} = \sum_{G=0}^{\lfloor k/2\rfloor+1} \Big(\frac{\beta}{2}\Big)^{1-G} \Big(1-\frac{2}{\beta}\Big)^{k+2-2G} \mathcal{F}^{[G,k+2-2G]}_{\boldsymbol{\epsilon}}. \tag{1.34}$$

In particular, those coefficients vanish for odd k. The first few ones are:

$$F^{\{-2\}}_{\boldsymbol{\epsilon},\beta} = \frac{\beta}{2}\,\mathcal{F}^{[0,0]}_{\boldsymbol{\epsilon}}, \qquad F^{\{-1\}}_{\boldsymbol{\epsilon},\beta} = \Big(\frac{\beta}{2}-1\Big)\,\mathcal{F}^{[0,1]}_{\boldsymbol{\epsilon}}, \qquad F^{\{0\}}_{\boldsymbol{\epsilon},\beta} = \mathcal{F}^{[1,0]}_{\boldsymbol{\epsilon}} + \Big(\frac{\beta}{2}+\frac{2}{\beta}-2\Big)\,\mathcal{F}^{[0,2]}_{\boldsymbol{\epsilon}}, \tag{1.35}$$

and have been first identified in [WZ06]. In particular, for the argument of the theta function:

$$\boldsymbol{v}_{\star,\beta} = \Big(\frac{\beta}{2}-1\Big)\,\frac{(\mathcal{F}^{[0,1]}_\star)'}{2\mathrm{i}\pi}, \qquad \boldsymbol{\tau}_{\star,\beta} = \frac{\beta}{2}\,\frac{(\mathcal{F}^{[0,0]}_\star)''}{2\mathrm{i}\pi}. \tag{1.36}$$

Similarly for the correlators in the fixed filling fraction model, the dependence in β takes the form:

$$W^{\{k\}}_{n,\boldsymbol{\epsilon}}(x_1,\ldots,x_n) = \sum_{G=0}^{\lfloor (k-n+2)/2\rfloor} \Big(\frac{\beta}{2}\Big)^{1-G-n} \Big(1-\frac{2}{\beta}\Big)^{k+2-2G-n} \mathcal{W}^{[G;k+2-2G-n]}_{n,\boldsymbol{\epsilon}}(x_1,\ldots,x_n). \tag{1.37}$$

All coefficients $F^{[G,K]}_{\boldsymbol{\epsilon}}$ and functions $W^{[G,K]}_{n,\boldsymbol{\epsilon}}(x_1,\ldots,x_n)$ can be computed with the β deformation of the topological recursion formulated by Chekhov and Eynard [CE06], applied to the spectral curve determined by the equilibrium measure $\mu_{\mathrm{eq},\boldsymbol{\epsilon}}$ and $W^{\{0\}}_{2,\boldsymbol{\epsilon}}$, which encodes the covariance of linear statistics at leading order in the model with fixed filling fractions. We stress now a point of this theory relevant in the present case. When V is a polynomial and $\boldsymbol{\epsilon}$ is close enough to $\boldsymbol{\epsilon}_\star$, the density of the equilibrium measure can be analytically continued to a hyperelliptic curve of genus g, denoted $\mathcal{C}_{\boldsymbol{\epsilon}}$ and called spectral curve. Its equation is:

$$y^2 = \prod_{h=0}^{g} (x-\alpha^-_{\boldsymbol{\epsilon},h})^{\rho^+_h}(x-\alpha^+_{\boldsymbol{\epsilon},h})^{\rho^-_h}. \tag{1.38}$$

Let $\mathcal{A}_h$ be the cycle in $\mathcal{C}_{\boldsymbol{\epsilon}}$ surrounding $\mathsf{A}_{\boldsymbol{\epsilon},h} = [\alpha^-_{\boldsymbol{\epsilon},h},\alpha^+_{\boldsymbol{\epsilon},h}]$. The family $\boldsymbol{\mathcal{A}} = (\mathcal{A}_h)_{1\leqslant h\leqslant g}$ can be completed by a family of cycles $\boldsymbol{\mathcal{B}}$ so that $(\boldsymbol{\mathcal{A}},\boldsymbol{\mathcal{B}})$ is a symplectic basis of homology of $\mathcal{C}_{\boldsymbol{\epsilon}}$. The correlators $W^{[G,K]}_{n,\boldsymbol{\epsilon}}$ are meromorphic functions on $\mathcal{C}^n_{\boldsymbol{\epsilon}}$, computed recursively by a residue formula on $\mathcal{C}_{\boldsymbol{\epsilon}}$. In particular, the analytic continuation of

$$\omega^0_2(x_1,x_2) = \mathcal{W}^{[0,0]}_{2,\boldsymbol{\epsilon}}(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2 + \frac{2}{\beta}\,\frac{\mathrm{d}x_1\,\mathrm{d}x_2}{(x_1-x_2)^2} \tag{1.39}$$

is the unique 2-form on $\mathcal{C}_{\boldsymbol{\epsilon}}$, which has vanishing $\boldsymbol{\mathcal{A}}$ periods, and has for only singularity a double pole with leading coefficient $\frac{2}{\beta}$ and without residue at coinciding points. Then, it is a property of the topological recursion that the derivatives of $F_{\boldsymbol{\epsilon}}^{[G,K]}$ can be computed as $\mathcal{B}$-cycle integrals of the correlators: for any $(G,K) \neq (0,0),(0,1)$,

$$(\mathcal{F}_{\boldsymbol{\epsilon}}^{[G,K]})^{(\ell)} = \oint_{\boldsymbol{\mathcal{B}}} \mathrm{d}x_1 \cdots \oint_{\boldsymbol{\mathcal{B}}} \mathrm{d}x_\ell\, \mathcal{W}_{\ell,\boldsymbol{\epsilon}}^{[G,K]}(x_1,\ldots,x_\ell), \tag{1.40}$$

and for any (n,G,K):

$$(\mathcal{W}_n^{[G,K]})^{(\ell)}(x_1,\ldots,x_n) = \oint_{\boldsymbol{\mathcal{B}}} \mathrm{d}x_{n+1} \cdots \oint_{\boldsymbol{\mathcal{B}}} \mathrm{d}x_{n+\ell}\, \mathcal{W}_{n+\ell,\boldsymbol{\epsilon}}^{[G,K]}(x_1,\ldots,x_{n+\ell}). \tag{1.41}$$

In particular:

$$(W_{1,\boldsymbol{\epsilon}}^{[0,0]})'(x)\mathrm{d}x = 2\mathrm{i}\pi\, \boldsymbol{\varpi}(x) \tag{1.42}$$

where $\boldsymbol{\varpi}$ is the basis of holomorphic 1-forms on $\mathcal{C}_{\boldsymbol{\epsilon}}$ dual to $\boldsymbol{\mathcal{A}}$, i.e. characterized by $\oint_{\mathcal{A}_h} \varpi_{h'} = \delta_{h,h'}$. This formula at $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_\star$ can be used to compute the functions $T_\beta^{\{k\}}[\boldsymbol{X}]$ appearing in (1.28). The derivation wrt $\boldsymbol{\epsilon}$ is not a natural operation in the initial model when N is finite, since $N\epsilon_h$ are forced to be integers in (1.17). We rather show that the coefficients of expansion themselves are smooth functions of $\boldsymbol{\epsilon}$, and thus $\partial_{\boldsymbol{\epsilon}}$ makes sense.

For $\beta = 2$, we remark from (1.34) that the coefficients $F_{\star,\beta=2}^{\{2k+1\}}$ all vanishes, so that we retrieve the celebrated $1/N^2$ expansion in the one-cut regime or in the fixed filling fraction model. This is in general not true anymore in the multi-cut regime. For instance, we have a term of order $1/N$ involving:

$$T_{\star,\beta=2}^{\{1\}}[\boldsymbol{X}] = \frac{1}{6}\,(F_{\star,\beta}^{\{-2\}})''' \cdot \boldsymbol{X}^{\otimes 3} + (F_{\star,\beta}^{\{0\}})' \cdot \boldsymbol{X}. \tag{1.43}$$

In a two-cut regime ($g = 1$), a sufficient condition for all terms of order $N^{-(2k+1)}$ to vanish is that $\epsilon_\star = 1/2$ and $Z_{N,\epsilon}^{V;\mathsf{A}} = Z_{N,1-\epsilon}^{V;\mathsf{A}}$, i.e. the potential has two symmetric wells. In this case, we have an expansion in powers of $1/N^2$ for the partition function, whose coefficients depends on the parity of N. In general, we also observe that $\boldsymbol{v}_{\star,\beta=2} = \boldsymbol{0}$, i.e. Thetanullwerten appear in the expansion.

1.6 Asymptotic expansion of kernels and correlators

Once the result on large N expansion of the partition function is obtained, we can easily infer the asymptotic expansion of the correlators and the kernels by perturbing the potential by terms of order $1/N$, maybe complex-valued, as allowed by Hypothesis 1.4.

1.6.1 Leading behavior of the correlators

Although we could write down the expansion for the correlators as a corollary of Theorem 1.5, we bound ourselves to point out their leading behavior. Whereas W_n behaves as $O(N^{2-n})$ in the one-cut regime or in the model with fixed filling fractions, W_n for $n \geqslant 3$ does not decay when N is large in a $(g+1)$-cut regime with $g \geqslant 1$. More precisely:

Theorem 1.6 *Assume Hypothesis 1.2, 1.4 and number of cuts $(g+1) \geqslant 2$. We have, for uniform convergence when $x_1,\ldots,x_n$ belongs to any compact of $(\widehat{\mathbb{C}}\backslash\mathsf{A})^n$:*

$$W_2(x_1,x_2) \underset{N\to\infty}{\sim} W_{2,\star}^{\{0\}}(x_1,x_2) + \Big(\frac{\boldsymbol{\varpi}(x_1)}{\mathrm{d}x_1} \otimes \frac{\boldsymbol{\varpi}(x_2)}{\mathrm{d}x_2}\Big) \cdot \nabla_{\boldsymbol{v}}^{\otimes 2} \ln\vartheta\begin{bmatrix} -N\boldsymbol{\epsilon}_\star \\ \boldsymbol{0} \end{bmatrix}\big(\boldsymbol{v}_{\star,\beta}\big|\boldsymbol{\tau}_{\star,\beta}\big), \tag{1.44}$$

and for any $n \geqslant 3$:

$$W_n(x_1,\ldots,x_n) \underset{N\to\infty}{\sim} \Big(\bigotimes_{i=1}^{n} \frac{\boldsymbol{\varpi}(x_i)}{\mathrm{d}x_i}\Big)\cdot \nabla_{\boldsymbol{v}}^{\otimes n} \ln\vartheta\begin{bmatrix} -N\boldsymbol{\epsilon}_{\star} \\ \mathbf{0}\end{bmatrix}\big(\boldsymbol{v}_{\star,\beta}\big|\boldsymbol{\tau}_{\star,\beta}\big). \tag{1.45}$$

Integrating this result over $\boldsymbol{\mathcal{A}}$-cycles provide the leading order behavior of n-th order moments of the filling fractions $\boldsymbol{N}$. We will also describe in Section 8.2 the fluctuations of the filling fractions: we find that they converge to a discrete Gaussian random variable.

1.6.2 Kernels

We explain in § 6.4 that the following result concerning the kernel – defined in (1.8) – is a consequence of Theorem 1.3:

Corollary 1.7 *Assume Hypothesis 1.2 and 1.4. There exists $t > 0$ such that, for any $\boldsymbol{\epsilon} \in \mathcal{E}_g$ such that $|\boldsymbol{\epsilon}-\boldsymbol{\epsilon}_{\star}| < t$, the n-kernels in the model with fixed filling fractions have an asymptotic expansion of the form:*

$$K_{n,\boldsymbol{c},\boldsymbol{\epsilon}}(x_1,\ldots,x_n) = \exp\Big[\sum_{k\geqslant -1} N^{-k}\Big(\sum_{n=1}^{k+2}\frac{1}{k!}\mathcal{L}_{\boldsymbol{x},\boldsymbol{c}}^{\otimes n}[W_{n,\boldsymbol{\epsilon}}^{\{k\}}]\Big) + O(N^{-\infty})\Big], \tag{1.46}$$

where $\mathcal{L}_{\boldsymbol{x},\boldsymbol{c}}$ is the linear form :

$$\mathcal{L}_{\boldsymbol{x},\boldsymbol{c}} = \sum_{j=1}^{n} c_j \int_{\infty}^{x_j}. \tag{1.47}$$

Up to a given $O(N^{-K})$, this expansion is uniform for $x_1,\ldots,x_n$ in any compact of $\widehat{\mathbb{C}}\backslash\mathsf{A}$.

Hereafter, if γ is a smooth path in $\widehat{\mathbb{C}}\backslash\mathsf{S}_{\boldsymbol{\epsilon}}$, we set $\mathcal{L}_{\gamma} = \int_{\gamma}$, and $\mathcal{L}_{\gamma}^{\otimes n}$ is given by:

$$\mathcal{L}_{\gamma}^{\otimes n}[W_n^{\{k\}}] = \int_{\gamma}\mathrm{d}x_1\cdots\int_{\gamma}\mathrm{d}x_n\, W_n^{\{k\}}(x_1,\ldots,x_n).$$

A priori, the integrals in the right-hand side of (1.46) depend on the homology class in $\mathbb{C}\backslash\mathsf{A}$ of paths $\infty \to x_i$. A basis of homology cycles in $\mathbb{C}\backslash\mathsf{A}$ is given by $\overline{\boldsymbol{\mathcal{A}}} = (\mathcal{A}_h)_{0\leqslant h\leqslant g}$. We also denote for convenience $\overline{\boldsymbol{\epsilon}} = (\epsilon_h)_{0\leqslant h\leqslant g}$. We deduce from (1.18) that:

$$\oint_{\overline{\boldsymbol{\mathcal{A}}}}\frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\, W_{n,\boldsymbol{\epsilon}}^{\{k\}}(\xi,x_2,\ldots,x_n) = \delta_{n,1}\delta_{k,-1}\,\overline{\boldsymbol{\epsilon}}. \tag{1.48}$$

Therefore, the only multivaluedness of the right-hand side comes from the first term $N\int\mathrm{d}\xi\, W_{1,\boldsymbol{\epsilon}}^{\{-1\}}(\xi)$, and given (1.48) and observing that $N\epsilon_h$ are integers, we see that it exactly reproduces the monodromies of the kernels depending on c_j.

We now come to the multi-cut regime of the initial model. If $\boldsymbol{X}$ is a vector with g components, and $\mathcal{L}$ is a linear form on the space of holomorphic functions on $\widehat{\mathbb{C}}\backslash\mathsf{S}_{\boldsymbol{\epsilon}}$, let us define:

$$\tilde{T}_{\boldsymbol{\epsilon},\beta}^{\{k\}}[\mathcal{L},\boldsymbol{X}] = \sum_{r=1}^{k}\frac{1}{r!}\sum_{\substack{\ell_1,\ldots,\ell_r\geqslant 1\\ m_1,\ldots,m_r\geqslant -2\\ n_1,\ldots,n_r\geqslant 0\\ \sum_{i=1}^r \ell_i+m_i+n_i=k}}\Big(\bigotimes_{i=1}^{r}\frac{\mathcal{L}^{\otimes n_i}\big[(W_{n_i,\boldsymbol{\epsilon}}^{\{m_i\}})^{(\ell_i)}\big]}{n_i!\,\ell_i!}\Big)\cdot\boldsymbol{X}^{\otimes(\sum_{i=1}^r \ell_i)}, \tag{1.49}$$

where we took as convention $W_{n=0,\boldsymbol{\epsilon}}^{\{k\}} = F_{\boldsymbol{\epsilon}}^{\{k\}}$. Then, as a consequence of Theorem 1.5:

Corollary 1.8 *Assume Hypothesis 1.2 and 1.4. With the notations of Corollary 1.7, the n-kernels have an asymptotic expansion*[1]*:*

$$K_{n,\mathbf{c}}(\mathbf{x}) = K_{n,\mathbf{c},\star}(\mathbf{x})(1+O(N^{-\infty})) \times \frac{\Big(\sum_{k\geqslant 0} N^{-k}\,\tilde{T}^{\{k\}}_{\star,\beta}\big[\mathcal{L}_{\boldsymbol{x},\mathbf{c}},\,\frac{\nabla_{\boldsymbol{v}}}{2\mathrm{i}\pi}\big]\Big)\vartheta\begin{bmatrix}-N\boldsymbol{\epsilon}_\star \\ \mathbf{0}\end{bmatrix}\big(\boldsymbol{v}_{\star,\beta}+\mathcal{L}_{\boldsymbol{x},\mathbf{c}}[\boldsymbol{\varpi}]\big|\boldsymbol{\tau}_{\star,\beta}\big)}{\Big(\sum_{k\geqslant 0} N^{-k}\,T^{\{k\}}_{\star,\beta}\big[\frac{\nabla_{\boldsymbol{v}}}{2\mathrm{i}\pi}\big]\Big)\vartheta\begin{bmatrix}-N\boldsymbol{\epsilon}_\star \\ \mathbf{0}\end{bmatrix}\big(\boldsymbol{v}_{\star,\beta}\big|\boldsymbol{\tau}_{\star,\beta}\big)}, \qquad (1.50)$$

where $\mathcal{L}_{\boldsymbol{x},\mathbf{c}} = \sum_{j=1}^{n} c_j \int_{\infty}^{x_j}$ *and* $\boldsymbol{\varpi}$ *is the basis of holomorphic* 1*-forms.*

A diagrammatic representation for the terms of such expansion was proposed in [BE12, Appendix A].

2 Application to orthogonal polynomials and integrable systems

Since orthogonal polynomials are related to the 1-hermitian matrix model (i.e. $\beta = 2$), our results can be used to establish the all-order asymptotics of orthogonal polynomials outside the bulk (Theorem 2.2 below). We will illustrate it for orthogonal polynomials with respect to an analytic weight defined on the whole real line, but it could be applied equally well to orthogonal polynomials with respect to an analytic weight on a finite union of segments of the real axis.

The leading order asymptotic of orthogonal polynomials is well-known since the work of Deift et al. [DKM+97, DKM+99b, DKM+99a], using the asymptotic analysis of Riemann-Hilbert problem which was pioneered in [DZ95]. In principle, it is possible to push the Riemann-Hilbert analysis beyond leading order, but this approach being very cumbersome, it has not been performed yet to our knowledge. Notwithstanding, the all-order expansion has a nice structure, and was heuristically derived by Eynard [Eyn06] based on the general works [BDE00, Eyn09]. In this article, we provide a proof of those heuristics.

Unlike the Riemann-Hilbert technique which becomes cumbersome to study the asymptotics of skew-orthogonal polynomials (i.e. $\beta = 1$ and 4) and thus has not been performed up to now, our method could be applied without difficulty to those values of β, and would allow to justify the heuristics of Eynard [Eyn01] formulated for the leading order, and describe all subleading orders. In other words, it provides a purely probabilistic approach to address asymptotic problems in integrable systems. It also suggest that the appearance of theta functions is not intrinsically related to integrability. In particular, we see in Theorem 2.2 that for $\beta = 2$, the theta function appearing in the leading order is associated to the matrix of periods of the hyperelliptic curve $\mathcal{C}_{\boldsymbol{\epsilon}_\star}$ defined by the equilibrium measure. Actually the theta function is just the basic block to construct analytic functions on this curve, and this is the reason why it pops up in the Riemann-Hilbert analysis. However, for $\beta \neq 2$, the theta function comes is associated to $(\beta/2)$ times the matrix of periods of $\mathcal{C}_{\boldsymbol{\epsilon}_\star}$, which might be or not the matrix of period of a curve, and anyway is not that of $\mathcal{C}_{\boldsymbol{\epsilon}_\star}$. So, the monodromy problem solved by this theta function is not directly related to the equilibrium measure, which makes for instance for $\beta = 1$ or 4 its construction via Riemann-Hilbert techniques a priori more involved.

Contrarily to Riemann-Hilbert techniques however, we are not yet in position within our method to consider the asymptotic in the bulk, at the edges, or the double-scaling limit for varying weights close to a critical point, or the case of complex-values weights which has been studied in [BM09]. We hope those technical restrictions to be removable in a near future.

[1] We warn the reader that $'$ denotes a derivative with respect to filling fractions, not with respect to variables of the correlators.

2.1 Setting

We first review the standard relations between orthogonal polynomials on the real line, random matrices and integrable systems, see e.g. [CG12, Section 5]. In this section, $\beta = 2$ and we omit to precise it in the notations. Let $V_{\mathbf{t}}(\lambda) = V(\lambda) + \sum_{k=1}^{d} t_k \lambda^k$. Let $(P_{n,N}(x))_{n \geqslant 0}$ be the monic orthogonal polynomials associated to the weight $\mathrm{d}w(x) = \mathrm{d}x\, e^{-NV_{\mathbf{t}}(x)}$ on $\mathsf{B} = \mathbb{R}$. We choose V and restrict in consequence t_k so that the weight decreases quickly at $\pm\infty$. If we denote $h_{n,N}$ the $L^2(\mathrm{d}w)$ norm of $P_{n,N}$, the polynomials $\hat{P}_{n,N} = P_{n,N}/\sqrt{h_{n,N}}$ are orthonormal. They satisfy a three-term recurrence relation:

$$x\hat{P}_{n,N}(x) = \sqrt{h_{n,N}}\hat{P}_{n+1}(x) + \beta_{n,N}\hat{P}_n(x) + \sqrt{h_{n-1,N}}\hat{P}_{n-1}(x). \tag{2.1}$$

The recurrence coefficients are solutions of a Toda chain: if we set

$$u_{n,N} = \ln h_{n,N}, \qquad v_{n,N} = -\beta_{n,N}, \tag{2.2}$$

we have:

$$\partial_{t_1} u_{n,N} = v_{n,N} - v_{n-1,N}, \qquad \partial_{t_1} v_{n,N} = e^{u_{n+1,N}} - e^{u_{n,N}}, \tag{2.3}$$

and the coefficients t_k generate higher Toda flows. The recurrence coefficients also satisfy the string equations:

$$\sqrt{h_{n,N}}[V'(\mathbf{Q}_N)]_{n,n-1} = \frac{n}{N}, \qquad [V'(\mathbf{Q}_N)]_{n,n} = 0, \tag{2.4}$$

where $\mathbf{Q}_N$ is the semi-infinite matrix:

$$\mathbf{Q}_N = \begin{pmatrix} \sqrt{h_{1,N}} & \beta_{1,N} & & & \\ \beta_{1,N} & \sqrt{h_{2,N}} & \beta_{2,N} & & \\ & \beta_{2,N} & \sqrt{h_{3,N}} & \beta_{3,N} & \\ & \ddots & \ddots & \ddots & \\ & & & & \end{pmatrix}. \tag{2.5}$$

Eqns 2.4 determines in terms of V the initial condition for the system (2.3). The partition function $\mathcal{T}(\mathbf{t}) = Z_N^{V_{\mathbf{t}};\mathbb{R}}$ is the Tau function associated to the solution $(u_{n,N}(\mathbf{t}), v_{n,N}(\mathbf{t}))_{n\geqslant 1}$ of (2.3). The partition function itself can be computed as [Meh04, PS11]:

$$Z_N^{V;\mathbb{R}} = N! \prod_{j=1}^{N-1} h_{j,N}. \tag{2.6}$$

We insist on the dependence on N and V by writing $h_{j,N} = h_j(NV)$. Therefore, the norms can be retrieved as:

$$h_n(NV) = \frac{\prod_{j=1}^{n} h_j(NV)}{\prod_{j=1}^{n-1} h_j(NV)} = \frac{1}{n+1}\,\frac{Z_{n+1}^{NV/(n+1);\mathbb{R}}}{Z_n^{NV/n;\mathbb{R}}} = \frac{1}{n+1}\,\frac{Z_{n+1}^{\frac{V}{s(1+1/n)};\mathbb{R}}}{Z_n^{V/s;\mathbb{R}}}, \qquad s = \frac{n}{N}. \tag{2.7}$$

The regime where $n, N \to \infty$ but $s = n/N$ remains fixed and positive correspond to the small dispersion regime in the Toda chain, where $1/n$ plays the role of the dispersion parameter.

2.2 Small dispersion asymptotics of $h_{n,N}$

When $V_{\mathbf{t}_0}/s_0$ satisfies Hypotheses 1.2 and 1.3 for a given set of times $(s_0, \mathbf{t}_0)$, $V_{\mathbf{t}}/s$ satisfies the same assumptions at least for $(s, \mathbf{t})$ in some neighborhood $\mathcal{U}$ of $(s_0, \mathbf{t}_0)$, and Theorem 1.5 determines the asymptotic expansion of $\mathcal{T}_N(\mathbf{t}) = Z_N^{V_{\mathbf{t}},\mathbb{R}}$ up to $O(N^{-\infty})$. Besides, we can apply Theorem 1.5 to study the ratio in the right-hand side of (2.7) when $n \to \infty$.

Theorem 2.1 *In the regime $n, N \to \infty$, $s = n/N > 0$ fixed, and Hypotheses 1.2 and 1.3 are satisfied with soft edges, we have the asymptotic expansion:*

$$
\begin{aligned}
u_{n,N} \quad = \quad & n\big(2\mathcal{F}_{\star}^{[0]} - \mathcal{L}_{V_{\mathbf{t}}/s}[\mathcal{W}_{1,\star}^{[0]}]\big) + \mathcal{F}_{\star}^{[0]} - \mathcal{L}_{V_{\mathbf{t}}/s}[\mathcal{W}_{1,\star}^{[0]}] + \frac{1}{2}\mathcal{L}_{V_{\mathbf{t}}/s}^{\otimes 2}[\mathcal{W}_{2,\star}^{[0]}] \\
& + \ln\left(\frac{\vartheta\begin{bmatrix}-(n+1)\,\boldsymbol{\epsilon}_{\star} \\ \mathbf{0}\end{bmatrix}\big(\mathcal{L}_{V_{\mathbf{t}}/s}[\boldsymbol{\varpi}]\big|\boldsymbol{\tau}_{\star}\big)}{\vartheta\begin{bmatrix}-n\,\boldsymbol{\epsilon}_{\star} \\ \mathbf{0}\end{bmatrix}\big(\mathbf{0}\big|\boldsymbol{\tau}_{\star}\big)}\right) \\
& - \ln\Big(1 + \frac{1}{n}\Big) + \sum_{\substack{G \geqslant 0,\ m \geqslant 0 \\ 2-2G-m<0}} (n+1)^{2-2G-m}\,\mathcal{L}_{V_{\mathbf{t}}/s}^{\otimes m}[\mathcal{W}_{m,\star}^{[G]}] \\
& + \ln\left(1 + \frac{\Big(\sum_{k \geqslant 1}(n+1)^{-k}\,\tilde{T}_{\star}^{\{k\}}\big[\mathcal{L}_{V_{\mathbf{t}}/s}\,;\,\frac{\nabla}{2\mathrm{i}\pi}\big]\Big)\vartheta\begin{bmatrix}-(n+1)\,\boldsymbol{\epsilon}_{\star} \\ \mathbf{0}\end{bmatrix}\big(\mathcal{L}_{V_{\mathbf{t}}/s}[\boldsymbol{\varpi}]\big|\boldsymbol{\tau}_{\star}\big)}{\vartheta\begin{bmatrix}-(n+1)\,\boldsymbol{\epsilon}_{\star} \\ \mathbf{0}\end{bmatrix}\big(\mathcal{L}_{V_{\mathbf{t}}/s}[\boldsymbol{\varpi}]\big|\boldsymbol{\tau}_{\star}\big)}\right) \\
& - \ln\left(1 + \frac{\Big(\sum_{k \geqslant 1} n^{-k}\,T_{\star}^{\{k\}}\big[\frac{\nabla}{2\mathrm{i}\pi}\big]\Big)\vartheta\begin{bmatrix}-n\,\boldsymbol{\epsilon}_{\star} \\ \mathbf{0}\end{bmatrix}\big(\mathbf{0}\big|\boldsymbol{\tau}_{\star}\big)}{\vartheta\begin{bmatrix}-n\,\boldsymbol{\epsilon}_{\star} \\ \mathbf{0}\end{bmatrix}\big(\mathbf{0}\big|\boldsymbol{\tau}_{\star}\big)}\right)
\end{aligned}
\tag{2.8}
$$

Here, $\boldsymbol{\epsilon}_{\star}$ are the filling fractions of $\mu_{\mathrm{eq}}^{V_{\mathbf{t}}/s}$ and $\mathcal{L}_{V_{\mathbf{t}}/s}$ is the linear form defined by:

$$
\mathcal{L}_{V_{\mathbf{t}}/s}[f] = \oint_{\mathsf{S}} \frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\,\frac{V_{\mathbf{t}}(\xi)}{s}\,f(\xi) \tag{2.9}
$$

We have not performed the expansion of $1/(n+1)$ in powers of $1/n$ to make the structure more transparent. We recall that all the quantities $\mathcal{W}_{m,\star}^{[G]}$ can be computed from the equilibrium measure associated to the potential $V_{\mathbf{t}}$, so making those asymptotic explicit just requires to solve the scalar Riemann-Hilbert problem for $\mu_{\mathrm{eq}}^{sV_{\mathbf{t}}}$. Notice that the number $g+1$ of cuts a priori depends on $(s_0, \mathbf{t}_0)$, and we do not address the issue of transitions between regimes with different number of cuts (because we cannot relax at present our off-criticality assumption), which are expected to be universal [Dub08].

We also collect here in one place and for $\beta = 2$ other notations appearing throughout the text:

$$
\mathcal{W}_{0,\star}^{[G]} = \mathcal{F}_{\star}^{[G]} = F_{\boldsymbol{\epsilon}_{\star}}^{\{2G-2\}}, \qquad \mathcal{W}_{n,\star}^{[G]} = W_{n,\boldsymbol{\epsilon}_{\star}}^{\{2G-2+n\}}, \qquad \boldsymbol{\tau}_{\star} = \frac{(\mathcal{F}_{\star}^{[0]})''}{2\mathrm{i}\pi}, \tag{2.10}
$$

and

$$
T_{\star}^{\{k\}}[\boldsymbol{X}] \quad = \quad \sum_{r=1}^{k}\frac{1}{r!}\sum_{\substack{\ell_1,\ldots,\ell_r \geqslant 1 \\ G_1,\ldots,G_r \geqslant 0 \\ \ell_i + 2G_i - 2 > 0 \\ \sum_{i=1}^{r}(\ell_i + 2G_i - 2) = k}} \Big(\bigotimes_{i=1}^{r}\frac{(\mathcal{F}_{\star}^{[G_i]})^{(\ell_i)}}{\ell_i!}\Big)\cdot\boldsymbol{X}^{\otimes(\sum_{i=1}^{r}\ell_i)}, \tag{2.11}
$$

$$
\tilde{T}_{\star}^{\{k\}}[\mathcal{L}\,;\,\boldsymbol{X}] \quad = \quad \sum_{r=1}^{k}\frac{1}{r!}\sum_{\substack{\ell_1,\ldots,\ell_r \geqslant 1 \\ G_1,\ldots,G_r \geqslant 0 \\ n_1,\ldots,n_r \geqslant 0 \\ \ell_i + 2G_i - 2 + n_i > 0 \\ \sum_{i=1}^{r}(\ell_i + 2G_i - 2 + n_i) = k}} \Big(\bigotimes_{i=1}^{r}\frac{\mathcal{L}^{\otimes n_i}[(\mathcal{W}_{n_i,\star}^{[G_i]})^{(\ell_i)}]}{n_i!\,\ell_i!}\Big)\cdot\boldsymbol{X}^{\otimes(\sum_{i=1}^{r}\ell_i)}. \tag{2.12}
$$

2.3 Asymptotic expansion of orthogonal polynomials away from the bulk

The orthogonal polynomials can be computed thanks to Heine formula [Sze39]:

$$P_n(x) = \mu_n^{V_{\mathbf{t}}/s;\mathbb{R}}\Big[\prod_{i=1}^{n}(x-\lambda_i)\Big] = K_{1,1}(x). \tag{2.13}$$

Hence, as a corollary of Theorem 1.8:

Theorem 2.2 *In the regime $n, N \to \infty$, $s = n/N > 0$ fixed, and Hypotheses 1.2 and 1.3 are satisfied, for $x \in \mathbb{C}\setminus\mathsf{S}$, we have the asymptotic expansion:*

$$\begin{aligned} P_n(x) &= \exp\Big(\sum_{m\geqslant 1}\sum_{G\geqslant 0} n^{2-2G-m}\frac{\mathcal{L}_x^{\otimes m}[\mathcal{W}_{m,\star}^{[G]}]}{m!}\Big)\big(1+O(n^{-\infty})\big) \\ &\quad\times \frac{\Big(\sum_{k\geqslant 0} n^{-k}\,\tilde{T}^{\{k\}}\big[\mathcal{L}_x\,;\,\frac{\nabla_{\boldsymbol{v}}}{2\mathrm{i}\pi}\big]\Big)\vartheta\begin{bmatrix}-n\,\boldsymbol{\epsilon}_\star \\ \mathbf{0}\end{bmatrix}\big(\mathcal{L}_x[\boldsymbol{\varpi}]\big|\boldsymbol{\tau}_\star\big)}{\Big(\sum_{k\geqslant 0} n^{-k}\,T^{\{k\}}\big[\frac{\nabla_{\boldsymbol{v}}}{2\mathrm{i}\pi}\big]\Big)\vartheta\begin{bmatrix}-n\,\boldsymbol{\epsilon}_\star \\ \mathbf{0}\end{bmatrix}\big(\mathbf{0}\big|\boldsymbol{\tau}_\star\big)}, \end{aligned} \tag{2.14}$$

where $\mathcal{L}_x = \int_\infty^x$. Up to a given $O(N^{-K})$, this expansion is uniform for x in any compact of $\mathbb{C}\setminus\mathsf{S}$.

We remark that $\mathcal{L}_x[\boldsymbol{\varpi}]$ is the Abel map evaluated between the points x and ∞.

As such, the results presented in this article do not allow the study of the asymptotic expansion of orthogonal polynomials in the bulk, i.e. for $x \in \mathsf{S}$. Indeed, this requires to perturb the potential $V(\lambda)$ by a term $-\frac{1}{n}\ln(\lambda - x)$ having a singularity at $x \in \mathsf{S}$, a case going beyond our Hypothesis 1.4. Similarly, we cannot address at present the regime of transitions between a g cut regime and and g'-cut regime with $g \neq g'$, because offcriticality was a key assumption in our derivation. Although it is the most interesting in regard of universality, the question of deriving uniform asymptotics, even at the leading order, valid for the crossover around a critical point is still open from the point of view of our methods.

3 Large deviations and concentration of measure

3.1 Restriction to a vicinity of the support

Our first step is to show that the interval of integration in (1.1) can be restricted to a vicinity of the support of the equilibrium measure, up to exponentially small corrections when N is large. The proofs are very similar to the one-cut case [BG11], and we remind briefly their idea in § 3.2. Let V be a regular and confining potential, and $\mu_{\mathrm{eq}}^{V;\mathsf{B}}$ the equilibrium measure determined by Theorem 1.1 or Theorem 1.2. We denote S its (compact) support. We define the effective potential by:

$$U^{V;\mathsf{B}}(x) = V(x) - 2\int_{\mathsf{B}} \mathrm{d}\mu_{\mathrm{eq}}^{V;\mathsf{B}}(\xi)\,\ln|x-\xi|, \qquad \tilde{U}^{V;\mathsf{B}}(x) = U^{V;\mathsf{B}}(x) - \inf_{\xi\in\mathsf{B}} U^{V;\mathsf{B}}(\xi) \tag{3.1}$$

when $x \in \mathsf{B}$, and $+\infty$ otherwise.

Lemma 3.1 *If V is regular and confining, we have large deviation estimates: for any $\mathsf{F} \subseteq \overline{\mathsf{B}\setminus\mathsf{S}}$ closed and $\mathsf{O} \subseteq \mathsf{B}\setminus\mathsf{S}$ open,*

$$\limsup_{N\to\infty}\frac{1}{N}\ln\mu_{N,\beta}^{V;\mathsf{B}}\big[\exists i\quad \lambda_i \in \mathsf{F}\big] \;\leqslant\; -\frac{\beta}{2}\inf_{x\in\mathsf{F}}\tilde{U}^{V;\mathsf{B}}(x), \tag{3.2}$$

$$\liminf_{N\to\infty}\frac{1}{N}\ln\mu_{N,\beta}^{V;\mathsf{B}}\big[\exists i\quad \lambda_i \in \mathsf{O}\big] \;\geqslant\; -\frac{\beta}{2}\inf_{x\in\mathsf{O}}\tilde{U}^{V;\mathsf{B}}(x). \tag{3.3}$$

We say that V satisfies a control of large deviations on B if $\tilde{U}^{V,\mathsf{B}}$ is positive on $\mathsf{B}\backslash\mathsf{S}$. Note that $\tilde{U}^{V,\mathsf{B}}$ vanishes at the boundary of S. According to Lemma 3.1, such a property implies that large deviations outside S are exponentially small when N is large.

Corollary 3.2 *Let V be regular, confining, satisfying a control of large deviations on B, and assume $\partial\mathsf{B} \cap \mathsf{S} = \varnothing$. Let $\mathsf{A} \subseteq \mathsf{B}$ be a finite union of segments such that $\mathsf{S} \subset \mathring{\mathsf{A}}$. There exists $\eta(\mathsf{A}) > 0$ so that:*

$$Z_{N,\beta}^{V;\mathsf{B}} = Z_{N,\beta}^{V;\mathsf{A}}(1 + O(e^{-N\eta(\mathsf{A})})), \tag{3.4}$$

and for any $n \geqslant 1$, there exists a universal constant $\gamma_n > 0$ so that, for any $x_1, \ldots, x_n \in (\mathbb{C}\backslash\mathsf{B})^n$:

$$\big|W_n^{V;\mathsf{B}}(x_1, \ldots, x_n) - W_n^{V;\mathsf{A}}(x_1, \ldots, x_n)\big| \leqslant \frac{\gamma_n\, e^{-N\eta(\mathsf{A})}}{\prod_{i=1}^n d(x_i, \mathsf{B})}. \tag{3.5}$$

It is useful to have a local version of this result:

Corollary 3.3 *Let V be regular, confining, satisfying a control of large deviations on B, and assume $\partial\mathsf{B} \cap \mathsf{S} = \varnothing$. Let $\mathsf{A} \subseteq \mathsf{B}$ be a finite union of segments such that $\mathsf{S} \subseteq \mathring{\mathsf{A}}$. If a_0 is the left (resp. right) edge of a connected component of A, let us define $\mathsf{A}_a = \mathsf{A} \cup [a, a_0]$. For any $\varepsilon > 0$ small enough, there exists $\eta_\varepsilon > 0$ so that, for N large enough and any $a \in]a_0 - \varepsilon, a_0 + \varepsilon[$, we have:*

$$\Big|\partial_a \ln Z_N^{V;\mathsf{A}_a}\Big| \leqslant e^{-N\eta_\varepsilon}, \tag{3.6}$$

and, for N large enough and any $n \geqslant 1$ and $x_1, \ldots, x_n \in (\mathbb{C}\backslash\mathsf{A}_a)$:

$$\Big|\partial_{a'_-} W_n^{V;\mathsf{A}_a}(x_1, \ldots, x_n)\Big| \leqslant \frac{\gamma_n\, e^{-N\eta_\varepsilon}}{\prod_{i=1}^n d(x_i, \mathsf{A}_a)}. \tag{3.7}$$

From now on, even though we want initially to study the model on B^N, we are going first to study the model on A^N, where A is small (but fixed) enlargement of S as allowed above, in particular we choose A bounded.

Proposition 3.4 *For any fixed $\boldsymbol{\epsilon} \in \overline{\mathcal{E}}_g$, the same results holds for the partition function and the correlators in the fixed filling fraction model.*

3.2 Sketch of the proof of Lemma 3.1

We only sketch the proof, since it is similar to [BG11].

Recall that $L_N = N^{-1}\sum_{i=1}^N \delta_{\lambda_i}$ denotes the normalized empirical measure, either in the initial model, or in the fixed filling fraction model. We represent:

$$\mu_{N,\beta}^{V;\mathsf{B}}\big[\exists i \quad \lambda_i \in \mathsf{F}\big] = N\,\frac{\Upsilon_{N,\beta}^{V;\mathsf{B}}(\mathsf{F})}{\Upsilon_{N,\beta}^{V;\mathsf{B}}(\mathsf{B})} \tag{3.8}$$

where, for any measurable set X:

$$\Upsilon_{N,\beta}^{V;\mathsf{B}}(\mathsf{X}) = \mu_{N-1,\beta}^{\frac{NV}{N-1};\mathsf{B}}\Big[\int_{\mathsf{X}} \mathrm{d}\xi\, \exp\Big\{-\frac{N\beta}{2}\,V(\xi) + (N-1)\beta\int_{\mathsf{B}} \mathrm{d}L_{N-1}(\lambda)\ln|\xi - \lambda|\Big)\Big\}\Big] \tag{3.9}$$

We first prove a lower bound for $\Upsilon_{N,\beta}^{V;\mathsf{B}}(\mathsf{X})$ assuming X contains at least an open interval, of size larger than some $\varepsilon > 0$. Let $\kappa_1(V)$ be the Lipschitz constant for V on B, and:

$$\mathsf{X}^\varepsilon = \{x \in \mathsf{B}, \quad \inf_{\xi \in \mathsf{B}\backslash\mathsf{X}} |x - \xi| > \varepsilon/2\} \tag{3.10}$$

Using twice Jensen's inequality, we get

$$\begin{aligned}\Upsilon^{V;\mathsf{B}}_{N,\beta}(\mathsf{X}) & \geqslant \sup_{x\in\mathsf{X}^{\varepsilon}} \mu^{\frac{NV}{N-1};\mathsf{B}}_{N-1,\beta}\bigg[\int_{x-\varepsilon/4}^{x+\varepsilon/4}\mathrm{d}\xi\,\exp\Big\{-\frac{N\beta}{2}\,V(\xi)+(N-1)\beta\int_{\mathsf{B}}\mathrm{d}L_{N-1}(\eta)\ln|\xi-\eta|\Big)\Big\}\bigg] \\ & \geqslant \sup_{x\in\mathsf{X}^{\varepsilon}} e^{-\frac{N\beta}{2}\big(V(x)+\frac{\kappa_1(V)}{2}\varepsilon\big)}\,\mu^{\frac{NV}{N-1};\mathsf{B}}_{N-1,\beta}\bigg[\int_{x-\varepsilon/4}^{x+\varepsilon/4}\mathrm{d}\xi\,\exp\Big\{(N-1)\beta\int_{\mathsf{B}}\mathrm{d}L_{N-1}(\lambda)\ln|\xi-\lambda|\Big\}\bigg] \\ & \geqslant \frac{\varepsilon}{2}\sup_{x\in\mathsf{X}^{\varepsilon}} e^{-\frac{N\beta}{2}\big(V(x)+\frac{\kappa_1(V)}{2}\varepsilon\big)}\,\exp\Big\{(N-1)\beta\,\mu^{\frac{NV}{N-1};\mathsf{B}}_{N-1,\beta}\Big[\int_{\mathsf{B}}\mathrm{d}L_{N-1}(\lambda)\,H_{x,\varepsilon}(\lambda)\Big]\Big\} \end{aligned} \tag{3.11}$$

where we have set:

$$H_{x,\varepsilon}(\lambda)=\int_{x-\varepsilon/4}^{x+\varepsilon/4}\frac{\mathrm{d}\xi}{\varepsilon/2}\,\ln|\xi-\lambda| \tag{3.12}$$

For any fixed $\varepsilon>0$, $H_{x,\varepsilon}$ is bounded continuous on any compact, so we have by Theorem 1.1 in the initial model (or Theorem 1.2 in the fixed filling fraction model):

$$\Upsilon^{V;\mathsf{B}}_{N,\beta}(\mathsf{X})\geqslant\frac{\varepsilon}{2}\sup_{x\in\mathsf{X}^{\varepsilon}} e^{-\frac{N\beta}{2}\big(V(x)+\frac{\kappa_1(V)}{2}\varepsilon\big)}\exp\Big\{(N-1)\beta\int_{\mathsf{B}}\mathrm{d}\mu^{V;\mathsf{B}}_{\mathrm{eq}}(\lambda)\,H_{x,\varepsilon}(\lambda)+NR(\varepsilon,N)\Big\} \tag{3.13}$$

with $\lim_{N\to\infty}R(\varepsilon,N)=0$. Letting $N\to\infty$, we deduce:

$$\liminf_{N\to\infty}\frac{1}{N}\,\ln\Upsilon^{V;\mathsf{B}}_{N,\beta}(\mathsf{X})\geqslant-\frac{\beta}{4}\,\kappa_1(V)\,\varepsilon-\frac{\beta}{2}\inf_{x\in\mathsf{X}^{\varepsilon}}\Big(V(x)-2\int\mathrm{d}\mu^{V;\mathsf{B}}_{\mathrm{eq}}(\lambda)\,H_{x,\varepsilon}(\lambda)\Big) \tag{3.14}$$

Interchanging the integration over ξ and x, observing that $\xi\to\int\mathrm{d}\mu^{V;\mathsf{B}}_{\mathrm{eq}}(\lambda)\,\ln|\xi-\lambda|$ is smooth and then letting $\varepsilon\to0$ we conclude

$$\liminf_{N\to\infty}\frac{1}{N}\,\ln\Upsilon^{V;\mathsf{B}}_{N,\beta}(\mathsf{X})\geqslant-\frac{\beta}{2}\,\inf_{x\in\mathsf{X}}U^{V;\mathsf{B}}(x) \tag{3.15}$$

where we have recognized the effective potential of (3.1). To prove the upper bound, we observe that for any $M>0$,

$$\Upsilon^{V;\mathsf{B}}_{N,\beta}(\mathsf{X})\leqslant\mu^{\frac{NV}{N-1};\mathsf{B}}_{N-1,\beta}\bigg[\int_{\mathsf{X}}\mathrm{d}\xi\,\exp\Big\{-\frac{N\beta}{2}\,V(\xi)-(N-1)\beta\int_{\mathsf{B}}\mathrm{d}L_{N-1}(\lambda)\ln\max\big(|\xi-\lambda|,M^{-1}\big)\Big\}\bigg] \tag{3.16}$$

As $\lambda\to\ln\min\big(|\xi-\lambda|,M^{-1}\big)$ is bounded continuous on compacts, we can use Theorem 1.1 in the initial model (or Theorem 1.2 in the fixed filling fraction model) to deduce that for any $\varepsilon>0$

$$\Upsilon^{V;\mathsf{B}}_{N,\beta}(\mathsf{X})\leqslant\int_{\mathsf{X}}\mathrm{d}\xi\,\exp\Big\{-\frac{N\beta}{2}\,V(\xi)-(N-1)\beta\int_{\mathsf{B}}\mathrm{d}\mu^{V;\mathsf{B}}_{\mathrm{eq}}(\lambda)\ln\max\big(|\xi-\lambda|,M^{-1}\big)+NM\varepsilon\Big\}+e^{N^2\tilde{R}(\varepsilon,N)} \tag{3.17}$$

with

$$\limsup_{N\to\infty}\tilde{R}(\varepsilon,N)=\limsup_{N\to\infty}\frac{1}{N^2}\ln\mu^{\frac{NV}{N-1};\mathsf{B}}_{N-1,\beta}\big(d(L_{N-1},\mu^{V;\mathsf{B}}_{\mathrm{eq}})>\varepsilon\big)<0. \tag{3.18}$$

Moreover, $\xi\to V(\xi)-\int\mathrm{d}\mu^{V;\mathsf{B}}_{\mathrm{eq}}(\lambda)\ln\max\big(|\xi-\lambda|,M^{-1}\big)$ is bounded continuous so that a standard Laplace method yields

$$\liminf_{N\to\infty}\frac{1}{N}\,\ln\Upsilon^{V;\mathsf{B}}_{N,\beta}(\mathsf{X})\leqslant-\inf_{\xi\in\mathsf{X}}\Big\{\frac{\beta}{2}\Big(V(\xi)-\int_{\mathsf{B}}\mathrm{d}\mu^{V;\mathsf{B}}_{\mathrm{eq}}(\lambda)\ln|\xi-\lambda|\vee M^{-1}\Big)\Big\}. \tag{3.19}$$

Finally, we conclude by monotone convergence theorem which implies that $\int\mathrm{d}\mu^{V;\mathsf{B}}_{\mathrm{eq}}(\lambda)\ln\max\big(|\xi-\lambda|,M^{-1}\big)$ increases as M goes to infinity towards $\int\mathrm{d}\mu^{V;\mathsf{B}}_{\mathrm{eq}}(\lambda)\ln|\xi-\lambda|$.

3.3 Concentration of measure and consequences

We will need rough a priori bounds on the correlators, which can be derived by purely probabilistic methods. This type of result first appeared in the work of [dMPS95, Joh98] and more recently [KS10, MMS12]. Given their importance, we find useful to prove independently the bound we need by elementary means.

Hereafter, we will say that a function $f : \mathbb{R} \to \mathbb{C}$ is b-Lipschitz if

$$\kappa_b(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^b} < \infty. \tag{3.20}$$

Our final goal is to control $(L_N - \mu_{\mathrm{eq}})[\varphi]$ for a class of functions φ which is large enough, in particular contains analytic functions on a neighborhood of the interval of integration A. This problem can be settled by controlling the "distance" between L_N and μ_{eq} for an appropriate notion of distance. We introduce the pseudo-distance between probability measures:

$$\mathfrak{D}(\mu, \nu) = -\iint \mathrm{d}[\mu - \nu](x)\mathrm{d}[\mu - \nu](y) \, \ln|x - y| \tag{3.21}$$

which can be represented in terms of Fourier transform of the measures by:

$$\mathfrak{D}(\mu, \nu) = \int_0^{\infty} \frac{\mathrm{d}s}{|s|} \left|(\widehat{\mu} - \widehat{\nu})(s)\right|^2 \tag{3.22}$$

Since L_N has atoms, its pseudo-distance to another measure is in general infinite. There are several methods to circumvent this issue, and one of them, that we borrow from [MMS12], is to define a regularized measure $\widetilde{L}_N^{\mathrm{u}}$ (see the beginning of § 3.4.1 below) from L_N. Then, the result of concentration, takes the form:

Lemma 3.5 *Let V be regular, $\mathcal{C}^3$, confining, satisfying a control of large deviations on A. There exists $C > 0$ so that, for t small enough and N large enough:*

$$\mu_{N,\beta}^{V;\mathsf{A}}\big[\mathfrak{D}[\widetilde{L}_N^{\mathrm{u}}, \mu_{\mathrm{eq}}^{V;\mathsf{A}}] \geqslant t\big] \leqslant e^{CN\ln N - N^2 t^2}. \tag{3.23}$$

We prove it in § 3.4.1 below. The assumption V of class $\mathcal{C}^3$ ensures that the effective potential (3.1) defined from the equilibrium measure is a $\frac{1}{2}$-Lipschitz function (and even Lipschitz if all edges are soft) on the compact set A, as one can observe on (A.4) given in Appendix A.

This lemma allows a priori control of expectation values of test functions:

Corollary 3.6 *Let V be regular, $\mathcal{C}^3$, confining, satisfying a control of large deviations on A. Let $b > 0$, and assume $\varphi : \mathbb{R} \to \mathbb{C}$ is a b-Lipschitz function with constant $\kappa_b(\varphi)$, and such that:*

$$|\varphi|_{1/2} := \Big(\int_{\mathbb{R}} \mathrm{d}s\, |s|\, |\widehat{\varphi}(s)|^2\Big)^{1/2} < \infty. \tag{3.24}$$

Then, there exists $C_3 > 0$ such that, for t small enough and N large enough:

$$\mu_{N,\beta}^{V;\mathsf{A}}\Big[\Big|\int_{\mathsf{A}} \mathrm{d}[L_N - \mu_{\mathrm{eq}}^{V;\mathsf{A}}](x)\, \varphi(x)\Big| \geqslant \frac{2\kappa_b(\varphi)}{(b+1)N^{2b}} + t\, |\varphi|_{1/2}\Big] \leqslant e^{C_3 N\ln N - N^2 t^2}. \tag{3.25}$$

As a special case, we can obtain a rough a priori control on the correlators:

Corollary 3.7 *Let V be regular, $\mathcal{C}^3$, confining and satisfying a control of large deviations on A. Let $D' > 0$, and:*

$$w_N = \sqrt{\frac{\ln N}{N}}, \qquad f(\delta) = \frac{\sqrt{|\ln\delta|}}{\delta}, \qquad d(x, \mathsf{A}) = \inf_{\xi \in \mathsf{A}} |x - \xi| \geqslant \frac{D'}{\sqrt{N\ln N}} \tag{3.26}$$

There exists a constant $\gamma_1(\mathsf{A},D')>0$ so that, for N large enough:

$$\big|W_1(x)-NW_1^{\{-1\}}(x)\big|\leqslant\gamma_1(\mathsf{A},D')\,w_N\,f\big(d(x,\mathsf{A})\big). \tag{3.27}$$

Similarly, for any $n\geqslant 2$, there exists constants $\gamma_n(\mathsf{A},D')>0$ so that, for N large enough:

$$\big|W_n(x_1,\ldots,x_n)\big|\leqslant\gamma_n(\mathsf{A},D')\,w_N^n\prod_{i=1}^{n}f\big(d(x_i,\mathsf{A})\big). \tag{3.28}$$

In the $(g+1)$-cut regime with $g\geqslant 1$, we denote $(\mathsf{S}_h)_{0\leqslant h\leqslant g}$ the connected components of the support of $\mu_{\mathrm{eq}}^{V;\mathsf{B}}$, and we take $\mathsf{A}=\bigcup_{h=0}^{g}\mathsf{A}_h$, where $\mathsf{A}_h=[a_h^-,a_h^+]\subseteq\mathsf{B}$ are pairwise disjoint bounded segments such that $\mathsf{S}_h\subseteq\mathring{\mathsf{A}}_h$. For any configuration $\lambda\in\mathsf{A}^N$, we denote N_h the number of λ_i's in A_h, and $\boldsymbol{N}=(N_h)_{1\leqslant h\leqslant g}$. The following result gives an estimate for large deviations of $\boldsymbol{N}$ away from $N\boldsymbol{\epsilon}_\star$ in the large N limit.

Corollary 3.8 *Let A be as above, and V be $\mathcal{C}^3$, confining, satisfying a control of large deviations on A, and leading to a $(g+1)$-cut regime. There exists a positive constant C such that, for N large enough and uniformly in t:*

$$\mu_{N,\beta}^{V;\mathsf{A}}\big[|\boldsymbol{N}-N\boldsymbol{\epsilon}_\star|>t\sqrt{N\ln N}\big]\leqslant e^{N\ln N(C-t^2)}. \tag{3.29}$$

As an outcome of this article, we will be more precise in Section 8.2 about large deviations of filling fractions when the potential satisfies the stronger Hypotheses 1.1-1.4.

3.4 Large deviation of L_N: distance (Lemma 3.5)

3.4.1 Regularization of L_N

We start by following an idea introduced by Maïda and Maurel-Segala [MMS12, Proposition 3.2]. Let $\sigma_N,\eta_N\to 0$ be two sequences of positive numbers. To any configuration of points $\lambda_1\leqslant\ldots\leqslant\lambda_N$ in A, we associate another configuration $\widetilde{\lambda}_1,\ldots,\widetilde{\lambda}_N$ by the formula:

$$\widetilde{\lambda}_1=\lambda_1,\qquad\widetilde{\lambda}_{i+1}=\widetilde{\lambda}_i+\max(\lambda_{i+1}-\lambda_i,\sigma_N), \tag{3.30}$$

It has the properties:

$$\forall i\neq j,\qquad|\widetilde{\lambda}_i-\widetilde{\lambda}_j|\geqslant\sigma_N,\qquad|\lambda_i-\lambda_j|\leqslant|\widetilde{\lambda}_i-\widetilde{\lambda}_j|,\qquad|\widetilde{\lambda}_i-\lambda_i|\leqslant(i-1)\sigma_N. \tag{3.31}$$

Let us denote $\widetilde{L}_N=N^{-1}\sum_{i=1}^{N}\delta_{\widetilde{\lambda}_i}$ the new counting measure. Then, we define $\widetilde{L}_N^{\mathrm{u}}$ be the convolution of $\widetilde{L}_N$ with the uniform measure on $[0,\eta_N\sigma_N]$.

We are going to compare the logarithmic energy of L_N to that of $\widetilde{L}_N^{\mathrm{u}}$, which has the advantage of having no atom. We may write by (3.31):

$$\Sigma_\Delta(L_N)=\iint_{x\neq y}\mathrm{d}L_N(x)\mathrm{d}L_N(y)\ln|x-y|\leqslant\iint_{x\neq y}\mathrm{d}\widetilde{L}_N(x)\mathrm{d}\widetilde{L}_N(y)\,\ln|x-y|=\Sigma_\Delta(\widetilde{L}_N) \tag{3.32}$$

and, if we denote $\mathcal{U},\mathcal{U}'$ are two independent random variables uniformly distributed on $[0,1]$, we find:

$$\begin{aligned}\Sigma_\Delta(\widetilde{L}_N)-\Sigma_\Delta(\widetilde{L}_N^{\mathrm{u}}) &= \int_{x\neq y}\mathrm{d}\widetilde{L}_N(x)\mathrm{d}\widetilde{L}_N(y)\,\mathbb{E}\Big[\ln\Big(1+\eta_N\sigma_N\frac{\mathcal{U}-\mathcal{U}'}{(x-y)}\Big)\Big]\\ &\leqslant \int_{x\neq y}\mathrm{d}\widetilde{L}_N(x)\mathrm{d}\widetilde{L}_N(y)\,\frac{\eta_N\sigma_N}{|x-y|}\leqslant\eta_N,\end{aligned}$$

thanks to the minimal distance between $\widetilde{\lambda}_i$'s enforced in (3.30). Eventually, we compute with $\Sigma(\mu) = \iint \ln|x-y|\mathrm{d}\mu(x)\mathrm{d}\mu(y)$,

$$\begin{aligned}\Sigma_{\Delta}(\widetilde{L}_N^{\mathrm{u}}) - \Sigma(\widetilde{L}_N^{\mathrm{u}}) &= -\iint_{x=y} \mathrm{d}\widetilde{L}_N^{\mathrm{u}}(x)\mathrm{d}\widetilde{L}_N^{\mathrm{u}}(y)\,\ln|x-y| \\ &= -\frac{1}{N}\,\mathbb{E}\big[\ln|\eta_N\sigma_N(\mathcal{U}-\mathcal{U}')|\big] = \frac{1}{N}\Big(\frac{3}{2} - 3\ln(\eta_N\sigma_N)\Big). \end{aligned} \tag{3.33}$$

Besides, if $b > 0$ and $\varphi : \mathsf{A} \to \mathbb{C}$ is a b-Lipschitz function with constant $\kappa_b(\varphi)$, we have by (3.31):

$$\Big|\int_{\mathsf{A}} \mathrm{d}[L_N - \widetilde{L}_N^{\mathrm{u}}](x)\,\varphi(x)\Big| \leqslant \frac{\kappa_b(\varphi)}{N}\sum_{i=1}^{N}(i-1)^b[\sigma_N(1+2\eta_N)]^b \leqslant \frac{2\kappa_b(\varphi)}{(1+b)}\,(N\sigma_N)^b \tag{3.34}$$

3.4.2 Large deviations of L_N^{u}

We would like to estimate the probability of large deviations of $\widetilde{L}_N^{\mathrm{u}}$ from the equilibrium measure $\mu_{\mathrm{eq}} = \mu_{\mathrm{eq}}^{V;\mathsf{A}}$. We need first a lower bound on $Z_{N,\beta}^{V;\mathsf{A}}$ similar to that of [AG97] obtained by localizing the ordered eigenvalues at a distance N^{-3} of the quantiles λ_i^{cl} of the equilibrium measure $\mu_{\mathrm{eq}}^{V;\mathsf{A}}$, which are defined as:

$$\lambda_i^{\mathrm{cl}} = \inf\Big\{x \in \mathsf{A},\qquad \mu_{\mathrm{eq}}^{V;\mathsf{A}}\big([-\infty,x]\big) \geqslant \frac{i}{N}\Big\}. \tag{3.35}$$

Since V is $\mathcal{C}^2$, $\mathrm{d}\mu_{\mathrm{eq}}^{V;\mathsf{A}}$ is continuous on the interior of its support, and diverge only at hard edges, where it blows at most like the inverse of a squareroot. Therefore, there exists a constant $C > 0$ such that, for N large enough:

$$|\lambda_i^{\mathrm{cl}} - \lambda_{i-1}^{\mathrm{cl}}| \geqslant \frac{C}{N^2}. \tag{3.36}$$

Then, since V is a fortiori $\mathcal{C}^1$ on A compact,

$$\begin{aligned} Z_{N,\beta}^{V;\mathsf{A}} &\geqslant N!\int_{|\delta_i|\leqslant N^{-3}} \prod_{1\leqslant i<j\leqslant N}|\lambda_i^{\mathrm{cl}} - \lambda_j^{\mathrm{cl}} + \delta_i - \delta_j|^{\beta}\prod_{i=1}^{N} e^{-\frac{\beta N}{2}V(\lambda_i^{\mathrm{cl}}+\delta_i)}\mathrm{d}\delta_i \\ &\geqslant N!\,N^{-3N}e^{-C_1 N}\prod_{1\leqslant i<j\leqslant N}|\lambda_i^{\mathrm{cl}} - \lambda_j^{\mathrm{cl}}|^{\beta}\prod_{i=1}^{N} e^{-\frac{N\beta}{2}\sum_{i=1}^{N}V(\lambda_i^{\mathrm{cl}})}, \end{aligned} \tag{3.37}$$

for some constant $C_1 > 0$. Therefore, since:

$$\begin{aligned} \iint_{x\leqslant y}\ln|x-y|\mathrm{d}\mu_{\mathrm{eq}}^{V;\mathsf{A}}(x)\mathrm{d}\mu_{\mathrm{eq}}^{V;\mathsf{A}}(y) &\leqslant \sum_{i<j}\int_{\lambda_i^{\mathrm{cl}}}^{\lambda_{i+1}^{\mathrm{cl}}}\int_{\lambda_j^{\mathrm{cl}}}^{\lambda_{j+1}^{\mathrm{cl}}}\ln|x-y|\mathrm{d}\mu_{\mathrm{eq}}^{V;\mathsf{A}}(x)\mathrm{d}\mu_{\mathrm{eq}}^{V;\mathsf{A}}(y) \\ &\leqslant \frac{1}{N^2}\sum_{i<j-1}\ln|\lambda_i^{\mathrm{cl}} - \lambda_j^{\mathrm{cl}}| \\ &\leqslant \frac{1}{N^2}\sum_{i<j}\ln|\lambda_i^{\mathrm{cl}} - \lambda_j^{\mathrm{cl}}| + \frac{1}{N}\ln\Big(\frac{N^2}{C}\Big), \end{aligned} \tag{3.38}$$

we find:

$$Z_{N,\beta}^{V;\mathsf{A}} \geqslant \exp\Big\{\frac{\beta}{2}\Big(-C_2\,N\ln N - N^2\,E[\mu_{\mathrm{eq}}^{V;\mathsf{A}}]\Big)\Big\}. \tag{3.39}$$

for some positive constant C_2 and with the energy introduced in (1.10).

Now, let us denote $\mathcal{S}_N(t)$ the event $\{\mathfrak{D}[\widetilde{L}_N^{\mathrm{u}}, \mu_{\mathrm{eq}}^{V;\mathsf{A}}] \geqslant t\}$. We have:

$$\mu_{N,\beta}^{V;\mathsf{A}}[\mathcal{S}_N(t)] = \frac{1}{Z_{N,\beta}^{V;\mathsf{A}}}\int_{\mathcal{S}_N(t)} e^{\frac{\beta N^2}{2}\left(\iint_{x\neq y}\mathrm{d}L_N(x)\mathrm{d}L_N(y)\ln|x-y| - \int \mathrm{d}L_N(x)\,V(x)\right)}\prod_{i=1}^{N}\mathrm{d}\lambda_i, \tag{3.40}$$

and using the comparisons (3.32)-(3.34), we find, with the notations of Theorem 1.2:

$$\begin{aligned}\mu_{N,\beta}^{V;\mathsf{A}}[\mathcal{S}_N(t)] &\leqslant \frac{e^{\frac{\beta}{2}R_N}}{Z_{N,\beta}^{V;\mathsf{A}}}\int_{\mathcal{S}_N(t)} e^{\frac{\beta N^2}{2}\left(\iint \mathrm{d}\widetilde{L}_N^{\mathrm{u}}(x)\mathrm{d}\widetilde{L}_N^{\mathrm{u}}(y)\ln|x-y| - \int \mathrm{d}\widetilde{L}_N^{\mathrm{u}}(x)\,V(x)\right)}\prod_{i=1}^N \mathrm{d}\lambda_i \\ &\leqslant \frac{e^{\frac{\beta}{2}\left(R_N - N^2\,E[\mu_{\mathrm{eq}}^{V;A}]\right)}}{Z_{N,\beta}^{V;\mathsf{A}}}\int_{\mathcal{S}_N(t)} e^{-\frac{\beta N^2}{2}\left(\mathfrak{D}[\widetilde{L}_N^{\mathrm{u}},\mu_{\mathrm{eq}}] + \int \mathrm{d}\widetilde{L}_N^{\mathrm{u}}(x)\,U^{V;\mathsf{A}}(x)\right)}\prod_{i=1}^N \mathrm{d}\lambda_i, \end{aligned} \tag{3.41}$$

where:

$$R_N = N^3\sigma_N\,\kappa_1(V) + N^2\eta_N + \frac{3N}{2} - 3N\ln(\eta_N\sigma_N), \tag{3.42}$$

and the effective potential $U^{V;\mathsf{A}}$ was defined in (3.1). Since $U^{V;\mathsf{A}}$ is at least $\frac{1}{2}$-Lipschitz on A (and even 1-Lipschitz if all edges are soft),we find:

$$\mu_{N,\beta}^{V;\mathsf{A}}[\mathcal{S}_N(t)] \leqslant \frac{e^{\frac{\beta}{2}\left(R_N + \kappa_{1/2}(U^{V;\mathsf{A}})\,N^{5/2}\sigma_N^{1/2} - N^2\,E[\mu_{\mathrm{eq}}^{V;\mathsf{A}}]\right)}}{Z_{N,\beta}^{V;\mathsf{A}}}\int_{\mathcal{S}_N(t)} e^{-\frac{\beta N^2}{2}\,\mathfrak{D}[\widetilde{L}_N^{\mathrm{u}},\mu_{\mathrm{eq}}]}\prod_{i=1}^N e^{-\frac{\beta N}{2}\,U^{V;\mathsf{A}}(\lambda_i)}\,\mathrm{d}\lambda_i. \tag{3.43}$$

We now use the lower bound (3.39) for the partition function, and the definition of the event $\mathcal{S}_N(t)$, in order to obtain:

$$\mu_{N,\beta}^{V;\mathsf{A}}[\mathcal{S}_N(t)] \leqslant e^{\frac{\beta}{2}(R_N + \kappa_{1/2}(U)\,N^{5/2}\sigma_N^{1/2} + C_2\,N\ln N - N^2t^2)}\Big(\int_{\mathsf{A}}\mathrm{d}\lambda\,e^{-\frac{\beta N}{2}\,U^{V;\mathsf{A}}(\lambda)}\Big)^N \leqslant e^{\frac{\beta}{2}(\tilde{R}_N + C_2\,N\ln N - N^2t^2)}, \tag{3.44}$$

with:

$$\tilde{R}_N = R_N + \kappa_{1/2}(U)\,N^{5/2}\sigma_N^{1/2} + \frac{2N}{\beta}\,\ln \ell(\mathsf{A}). \tag{3.45}$$

Indeed, since $U^{V;\mathsf{A}}$ is nonnegative on A, we observed that the integral in bracket is bounded by the total length $\ell(\mathsf{A})$ of the range of integration, which is here finite. We now choose:

$$\sigma_N = \frac{1}{N^3},\qquad \eta_N = \frac{1}{\sqrt{N}}, \tag{3.46}$$

which guarantee that $\tilde{R}_N \in O(N\ln N)$. Thus, there exists a positive constant C_3 such that, for N large enough:

$$\mu_{N,\beta}^{V;\mathsf{A}}[\mathcal{S}_N(t)] \leqslant e^{\frac{\beta}{2}(C_3\,N\ln N - N^2t^2)}, \tag{3.47}$$

which concludes the proof of Proposition 3.5. We may rephrase this result by saying that the probability of $\mathcal{S}_N(t)$ becomes small for t larger than $w_N = \sqrt{2C_3\ln N/N}$.

□

3.5 Large deviations for test functions

3.5.1 Proof of Corollary 3.6

Since φ is b-Lipschitz, we can use the comparison (3.34) with $\sigma_N = N^{-3}$ chosen in (3.46):

$$\Big|\int_{\mathsf{A}}\mathrm{d}[L_N - \widetilde{L}_N^{\mathrm{u}}](x)\,\varphi(x)\Big| \leqslant \frac{2\kappa_b(\varphi)}{(b+1)N^{2b}} \tag{3.48}$$

Then, we compute:

$$\begin{aligned}\Big|\int_{\mathsf{A}}\mathrm{d}[\widetilde{L}_N^{\mathrm{u}} - \mu_{\mathrm{eq}}](x)\,\varphi(x)\Big| &= \Big|\int_{\mathbb{R}}\mathrm{d}s\big(\widehat{\widetilde{L}}{}_N^{\mathrm{u}} - \widehat{\mu_{\mathrm{eq}}}\big)(s)\,\widehat{\varphi}(-s)\Big| \\ &\leqslant |\varphi|_{1/2}\Big(\int_{\mathbb{R}}\frac{\mathrm{d}s}{|s|}\,\big|(\widehat{\widetilde{L}}{}_N^{\mathrm{u}} - \widehat{\mu_{\mathrm{eq}}})(s)\big|^2\Big)^{1/2}, \end{aligned} \tag{3.49}$$

and we recognize in the last factor the definition (3.22) of the pseudo-distance:

$$\Big|\int_{\mathsf{A}} \mathrm{d}[\widetilde{L}_N^{\mathrm{u}} - \mu_{\mathrm{eq}}](x)\,\varphi(x)\Big| \leqslant \sqrt{2}\,|\varphi|_{1/2}\,\mathfrak{D}[\widetilde{L}_N^{\mathrm{u}}, \mu_{\mathrm{eq}}]. \qquad (3.50)$$

Corollary 3.6 then follows from this inequality combined with Lemma 3.5.

3.5.2 Bounds on correlators and filling fractions (Proof of Corollary 3.7 and 3.8)

If μ is a probability measure, let $\mathcal{W}_\mu$ denote its Stieltjes transform. We have:

$$[\mathcal{W}_{L_N} - \mathcal{W}_{\mu_{\mathrm{eq}}}](x) = \int_{\mathsf{A}} \mathrm{d}[L_N - \mu_{\mathrm{eq}}](\xi)\,\psi_x(\xi), \qquad \psi_x(\xi) = \psi_x^R(\xi) + \mathrm{i}\psi_x^I(\xi) = \frac{\mathbf{1}_{\mathsf{A}}(\xi)}{x-\xi} \qquad (3.51)$$

Since ψ_x is 1-Lipschitz with constant $\kappa_1(\psi_x) = d^{-2}(x,\mathsf{A})$, we have for N large enough:

$$\big|[\mathcal{W}_{L_N} - \mathcal{W}_{\widetilde{L}_N^{\mathrm{u}}}](x)\big| \leqslant \frac{1}{N^2\,d^2(x,\mathsf{A})} \qquad (3.52)$$

We now focus on estimating $\mathcal{W}_{\widetilde{L}_N^{\mathrm{u}}} - \mathcal{W}_{\mu_{\mathrm{eq}}}$. Since the support of $\widetilde{L}_N^{\mathrm{u}}$ is included in

$$\mathsf{A}_{1/N^2} = \{x \in \mathbb{R}, \qquad d(x,\mathsf{A}) \leqslant 1/N^2\}, \qquad (3.53)$$

we have the freedom to replace $\psi_x^\bullet$ by any function $\phi_x^\bullet$ which coincides with $\psi_x^\bullet$ on A_{1/N^2}. We also find:

$$\big|[\mathcal{W}_{\widetilde{L}_N^{\mathrm{u}}} - \mathcal{W}_{\mu_{\mathrm{eq}}}](x)\big| \leqslant \sqrt{2}\,\big(|\phi_x^R|_{1/2} + |\phi_x^I|_{1/2}\big)\,\mathfrak{D}[\widetilde{L}_N^{\mathrm{u}}, \mu_{\mathrm{eq}}] \qquad (3.54)$$

Let $\phi : \mathbb{R} \to \mathbb{R}$ be $\mathcal{C}^1$ function which decays as $\phi(x) \in O(1/x^2)$ when $|x| \to \infty$. We observe:

$$\begin{aligned} |\phi|_{1/2}^2 &= \int_{\mathbb{R}} |s|\,|\widehat{\phi}(s)|^2 \mathrm{d}s = \int_{\mathbb{R}} \frac{1}{|s|}\,|\widehat{\phi'}(s)|^2 ds \\ &= -2\int_{\mathbb{R}^2} \ln|\xi_1 - \xi_2|\,\phi'(\xi_1)\phi'(\xi_2)\mathrm{d}\xi_1\mathrm{d}\xi_2 = -2\int_{\mathbb{R}^2} \ln\frac{|\xi_1-\xi_2|}{M}\,\phi'(\xi_1)\phi'(\xi_2)\mathrm{d}\xi_1\mathrm{d}\xi_2 \\ &\leqslant 2\int_{\mathbb{R}^2} \Big|\ln\Big(\frac{\xi_1-\xi_2}{M}\Big)\Big|\,|\phi'(\xi_1)|\,|\phi'(\xi_2)|\mathrm{d}\xi_1\mathrm{d}\xi_2 \end{aligned} \qquad (3.55)$$

where $M > 0$ can be chosen arbitrarily. Let $a_{x,h} \in \mathsf{A}_h$ the point such that $d(x,\mathsf{A}_h) = |x - a_{x,h}|$. We claim that, for $d(x,\mathsf{A})$ small enough, we can always choose ϕ_x^R and ϕ_x^I such that:

$$\big|(\phi_x^\bullet)'(\xi)\big| \leqslant \sum_{h=0}^{g} \frac{1}{(\xi - a_{x,h})^2 + d(x,\mathsf{A}_h)^2} \qquad (3.56)$$

This family (indexed by x) of functions is uniformly bounded by $(g+1)/\xi^2$ at ∞, which is integrable at ∞. Therefore, we can choose M in (3.55) independent of x so that:

$$|\phi_x^\bullet|_{1/2}^2 \leqslant 1 - \int_{\mathbb{R}^2} 2\ln\Big(\frac{\xi_1-\xi_2}{M}\Big)\,\big|(\phi_x^\bullet)'(\xi_1)\big|\,\big|(\phi_x^\bullet)'(\xi_2)\big|\mathrm{d}\xi_1\mathrm{d}\xi_2 \qquad (3.57)$$

If we plug the bound (3.56) in the right-hand side, the integral can be explicitly computed and we find a finite constant $D > 0$ which depends only on A, such that:

$$|\phi_x^\bullet|_{1/2}^2 \leqslant \frac{D\ln d(x,\mathsf{A})}{d^2(x,\mathsf{A})} \qquad (3.58)$$

when x approaches A. Combining (3.52)-(3.54)-(3.58) with Lemma 3.5, we find:

$$\begin{aligned} \Big|\frac{1}{N}W_1(x) - W_1^{\{-1\}}(x)\Big| &= \Big|\mu_{N,\beta}^{V;\mathsf{A}}\big[\mathcal{W}_{L_N}(x) - \mathcal{W}_{\mu_{\mathrm{eq}}}(x)\big]\Big| \\ &\leqslant \frac{1}{Nd^2(x,\mathsf{A})} + 2D\sqrt{\frac{\ln N}{N}}\,\frac{\sqrt{|\ln d(x,\mathsf{A})|}}{d(x,\mathsf{A})} \end{aligned} \qquad (3.59)$$

If we restrict ourselves to $x \in \mathbb{C}\backslash\mathsf{A}$ such that:

$$d(x, \mathsf{A}) \geqslant \frac{D'}{\sqrt{N \ln N}} \tag{3.60}$$

for some constant D', then:

$$\Big|\frac{1}{N}\, W_1(x) - W_1^{\{-1\}}(x)\Big| \leqslant (2D + D')\,\sqrt{\frac{\ln N}{N}}\,\frac{\sqrt{|\ln d(x,\mathsf{A})|}}{d(x,\mathsf{A})}. \tag{3.61}$$

Now let us consider the higher correlators. For any $n \geqslant 2$, $W_n^{V;\mathsf{A}}$ is the expectation value of some homogeneous polynomial of degree 1 in the quantities $(\mathcal{W}_{L_N} - W_{\mu_{\mathrm{eq}}})(x_i)$ and $\mu_{N,\beta}^{V;\mathsf{A}}\big[(\mathcal{W}_{L_N} - \mathcal{W}_{\mu_{\mathrm{eq}}})(x_i)\big]$. Accordingly, Lemma 3.5 yields:

$$|W_n(x_1, \ldots, x_n)| \leqslant \gamma_n \left(\frac{\ln N}{N}\right)^{n/2} \prod_{i=1}^{n} \frac{\sqrt{|\ln d(x_i,\mathsf{A})|}}{d(x_i,\mathsf{A})}. \tag{3.62}$$

for some constant $\gamma_n > 0$, which depends only on A. This concludes the proof of Corollary 3.7.

Similarly, to have a hand on filling fraction, we write:

$$N_h - N\epsilon_{\star,h} = N \int \mathrm{d}[L_N - \mu_{\mathrm{eq}}](\xi)\, \mathbf{1}_{\mathsf{A}_h}(\xi). \tag{3.63}$$

After replacing the function $\mathbf{1}_{\mathsf{A}_h}$ by a smooth function which assumes the value 1 on A_h, vanishes on $\mathsf{A}_{h'}$ for $h' \neq h$, and has compact support, we can apply Corollary 3.6 to deduce Corollary 3.8. □

4 Schwinger-Dyson equations for β ensembles

Let $\mathsf{A} = \bigcup_{h=0}^{g} \mathsf{A}_h$ be a finite union of pairwise disjoint bounded segments, and V be a $\mathcal{C}^1$ function of A. Schwinger-Dyson equation for the initial model $\mu_{N,\beta}^{V;\mathsf{A}}$ can be derived by integration by parts. Since the result is well-known (and has been reproved in [BG11]), we shall give them without proof. They can be written in several equivalent forms, and here we recast them in a way which is convenient for our analysis. We actually assume that V extends to a holomorphic function in a neighborhood of A, so that they can written in terms of contour integrals of correlators, and an extension to V harmonic will be mentioned in § 6.2. We introduce (arbitrarily for the moment) a partition $\partial\mathsf{A} = (\partial\mathsf{A})_+ \dot\cup (\partial\mathsf{A})_-$ of the set of edges of the range of integration, and

$$L(x) = \prod_{a \in (\partial\mathsf{A})_-} (x - a), \qquad L_1(x, \xi) = \frac{L(x) - L(\xi)}{x - \xi}, \qquad L_2(x; \xi_1, \xi_2) = \frac{L_1(x,\xi_1) - L_1(x,\xi_2)}{\xi_1 - \xi_2}. \tag{4.1}$$

Theorem 4.1 *Schwinger-Dyson equation at level* 1. *For any* $x \in \mathbb{C}\backslash\mathsf{A}$, *we have:*

$$\begin{aligned} & W_2(x,x) + \big(W_1(x)\big)^2 + \Big(1 - \frac{2}{\beta}\Big)\partial_x W_1(x) \\ & -N \oint_A \frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\, \frac{L(\xi)}{L(x)}\, \frac{V'(\xi)\, W_1(\xi)}{x - \xi} - \frac{2}{\beta} \sum_{a \in (\partial\mathsf{A})_+} \frac{L(a)}{x - a}\, \partial_a \ln Z_{N,\beta}^{V;\mathsf{A}} \\ & + \Big(1 - \frac{2}{\beta}\Big) \oint_{\mathsf{A}} \frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\, \frac{L_2(x;\xi,\xi)}{L(x)}\, W_1(\xi) \\ & - \oint\!\!\oint_{\mathsf{A}^2} \frac{\mathrm{d}\xi_1 \mathrm{d}\xi_2}{(2\mathrm{i}\pi)^2}\, \frac{L_2(x;\xi_1,\xi_2)}{L(x)} \big(W_2(\xi_1,\xi_2) + W_1(\xi_1)W_1(\xi_2)\big) \quad = \quad 0. \end{aligned} \tag{4.2}$$

And similarly, for higher correlators:

Theorem 4.2 *Schwinger-Dyson equation at level $n \geqslant 2$. For any $x, x_2, \ldots, x_n \in \mathbb{C}\backslash \mathsf{A}$, if we denote $I = [\![2, n]\!]$, we have:*

$$\begin{aligned} & W_{n+1}(x, x, x_I) + \sum_{J \subseteq I} W_{|J|+1}(x, x_J) W_{n-|J|}(x, x_{I \backslash J}) + \Big(1 - \frac{2}{\beta}\Big) \partial_x W_n(x, x_I) \\ & -N \oint_{\mathsf{A}} \frac{\mathrm{d}\xi}{2\mathrm{i}\pi} \frac{L(\xi)}{L(x)} \frac{V'(\xi)\, W_n(\xi, x_I)}{x - \xi} - \frac{2}{\beta} \sum_{a \in (\partial\mathsf{A})_+} \frac{L(a)}{x - a}\, \partial_a W_{n-1}(x_I) \\ & + \frac{2}{\beta} \sum_{i \in I} \oint_{\mathsf{A}} \frac{\mathrm{d}\xi}{2\mathrm{i}\pi} \frac{L(\xi)}{L(x)} \frac{W_{n-1}(\xi, x_{I\backslash\{i\}})}{(x - \xi)(x_i - \xi)^2} + \Big(1 - \frac{2}{\beta}\Big) \oint_{\mathsf{A}} \frac{\mathrm{d}\xi}{2\mathrm{i}\pi} \frac{L_2(x; \xi, \xi)}{L(x)}\, W_n(\xi, x_I) \\ & - \oiint_{\mathsf{A}^2} \frac{\mathrm{d}\xi_1 \mathrm{d}\xi_2}{(2\mathrm{i}\pi)^2} \frac{L_2(x; \xi_1, \xi_2)}{L(x)} \Big(W_{n+1}(\xi_1, \xi_2, x_I) + \sum_{J \subseteq I} W_{|J|+1}(\xi_1, x_J) W_{n-|J|}(\xi_2, x_{I\backslash J})\Big) \quad = \quad 0. \end{aligned} \tag{4.3}$$

The last line in (4.2) or (4.3) is a rational fraction in x, with poles at $a \in (\partial\mathsf{A})_+$, whose coefficients are linear combination of moments of λ_i.

As a matter of fact, if $\boldsymbol{N} \in [0, N]^g$ so that $|\boldsymbol{N}| = \sum_{h=1}^{g} N_h \leqslant N$, the correlators in the model with fixed filling fractions $\mu^{V;\mathsf{B}}_{N,\boldsymbol{\epsilon}=\boldsymbol{N}/N,\beta}$ satisfy the *same* equations. Indeed, in the process of integration by parts, one does not make use of the information about the location of the λ_i's. By linearity, in a model $\boldsymbol{N}$ where is random, the partition function $Z = \mathbb{E}[Z^{V;\mathsf{A}}_{N,\boldsymbol{N}/N,\beta}]$ and the correlators $W_n(x_1, \ldots, x_n) = \mathbb{E}[W_{\boldsymbol{N}/N,n}(x_1, \ldots, x_n)]$ satisfy the same equations. The initial model $\mu^{V;\mathsf{A}}_{N,\beta}$ and the model with fixed filling fractions are just special cases of the model with random filling fractions.

When $g \geqslant 1$, we denote $\boldsymbol{\mathcal{A}} = (\mathcal{A}_h)_{1 \leqslant h \leqslant g}$ a family of contours surrounding A_h in $\widehat{\mathbb{C}}\backslash\mathsf{A}$, and introduce the vector-valued linear operator:

$$\mathcal{L}_{\boldsymbol{\mathcal{A}}}[f] = \Big(\oint_{\mathcal{A}_1} \frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\, f(\xi), \ldots, \oint_{\mathcal{A}_g} \frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\, f(\xi)\Big) \tag{4.4}$$

on the space of holomorphic functions in $\widehat{\mathbb{C}}\backslash\mathsf{A}$. For any $m \in [\![1, n]\!]$, We denote:

$$\boldsymbol{W}_{n|m}(x_1, \ldots, x_{n-m}) = \mathcal{L}_{\boldsymbol{\mathcal{A}}}^{\otimes m}[W_n(x_1, \ldots, x_{n-m}, \bullet)], \tag{4.5}$$

which means that we integrate the remaining m variables on $\boldsymbol{\mathcal{A}}$-cycles. By definition, if we denote $(\boldsymbol{e}_h)_{1 \leqslant h \leqslant g}$ the canonical basis of $\mathbb{C}^g$,

$$\boldsymbol{W}_{n|m}(x_1, \ldots, x_{n-m}) = \sum_{1 \leqslant h_1, \ldots, h_m \leqslant g} \mu^{V;\mathsf{A}}_{N,\bullet,\beta}\Big[\prod_{i=n-m+1}^{n} N_{h_i} \prod_{j=1}^{n-m} \mathrm{Tr}\, \frac{1}{x_j - \Lambda}\Big]_c \bigotimes_{i=1}^{m} \boldsymbol{e}_{h_i}. \tag{4.6}$$

In particular, $\boldsymbol{W}_{n|n}$ is the tensor of n-th order cumulants of the numbers N_h of λ's in the segment A_h. We take as convention $\boldsymbol{W}_{n|0}(x_1, \ldots, x_n) = W_n(x_1, \ldots, x_n)$. Here, $\mu^{V;\mathsf{A}}_{N,\bullet,\beta}$ denotes the probability measure in a model with N λ's and random filling fractions. If $\boldsymbol{\epsilon} \in \overline{\mathcal{E}_g}$ and if we specialize to the model with fixed filling fraction $\boldsymbol{\epsilon}$, we have:

$$\boldsymbol{W}_{n|m}(x_1, \ldots, x_{n-m}) = \delta_{n,1}\delta_{m,1}\, N\boldsymbol{\epsilon}. \tag{4.7}$$

In general, we deduce from Theorem 4.2:

Corollary 4.3 *For any $n \geqslant 1$ and $m \in [\![0, n-1]\!]$, for any $x, x_2, \ldots, x_{n-m} \in \mathbb{C}\backslash\mathsf{A}$, if we denote*

$I = [\![2, n-m]\!]$, *we have:*

$$\boldsymbol{W}_{n+1|m}(x, x, x_I) + \sum_{\substack{J \subseteq I \\ 0 \leqslant m' \leqslant m}} \binom{m}{m'} \boldsymbol{W}_{|J|+1+m'|m'}(x, x_J) \otimes \boldsymbol{W}_{n-|J|-m'|m-m'}(x, x_{I \setminus J}) \tag{4.8}$$
$$+ \Big(1 - \frac{2}{\beta}\Big) \partial_x \boldsymbol{W}_{n|m}(x, x_I) - N \oint_{\mathsf{A}} \frac{\mathrm{d}\xi}{2\mathrm{i}\pi} \, \frac{L(\xi)}{L(x)} \, \frac{V'(\xi)\, \boldsymbol{W}_{n|m}(\xi, x_I)}{x - \xi}$$
$$- \frac{2}{\beta} \sum_{a \in (\partial \mathsf{A})_+} \frac{L(a)}{x - a} \, \partial_a \boldsymbol{W}_{n-1|m}(x_I) + \frac{2}{\beta} \sum_{i \in I} \oint_{\mathsf{A}} \frac{\mathrm{d}\xi}{2\mathrm{i}\pi} \, \frac{L(\xi)}{L(x)} \, \frac{\boldsymbol{W}_{n-1|m}(\xi, x_{I \setminus \{i\}})}{(x - \xi)(x_i - \xi)^2}$$
$$+ \Big(1 - \frac{2}{\beta}\Big) \oint_{\mathsf{A}} \frac{\mathrm{d}\xi}{2\mathrm{i}\pi} \, \frac{L_2(x; \xi, \xi)}{L(x)} \, \boldsymbol{W}_{n|m}(\xi, x_I) - \oiint_{\mathsf{A}^2} \frac{\mathrm{d}\xi_1 \mathrm{d}\xi_2}{(2\mathrm{i}\pi)^2} \, \frac{L_2(x; \xi_1, \xi_2)}{L(x)} \Big(\boldsymbol{W}_{n+1|m}(\xi_1, \xi_2, x_I)$$
$$+ \sum_{\substack{J \subseteq I \\ 0 \leqslant m' \leqslant m}} \binom{m}{m'} \boldsymbol{W}_{|J|+1+m'|m'}(\xi_1, x_J) \otimes \boldsymbol{W}_{n-|J|-m'|m-m'}(\xi_2, x_{I \setminus J}) \Big) = 0.$$

Proof. Straightforward from (4.3), once we notice that the integrals over a closed cycle of a total derivative or a holomorphic integrand in neighborhoods of A in $\mathbb{C}$ give a zero contribution. □

5 Fixed filling fractions: $1/N$ expansion of correlators

5.1 Norms on analytic functions and assumptions

In this Section, we analyze the Schwinger-Dyson equations in the model with random filling fractions (which contains the model with fixed filling fractions as special case) and the following assumptions:

Hypothesis 5.1

- A *is a disjoint finite union of bounded segments* $\mathsf{A}_h = [a_h^-, a_h^+]$.
- *(Real-analyticity)* $V : \mathsf{A} \to \mathbb{R}$ *extends as a holomorphic function in a neighborhood* $\mathsf{U} \subseteq \mathbb{C}$ *of* A.
- *($1/N$ expansion for the potential) There exists a sequence* $(V^{\{k\}})_{k \geqslant 0}$ *of holomorphic functions in* U *and constants* $(v^{\{k\}})_{k \geqslant 0}$*, so that, for any* $K \geqslant 0$*:*

$$\sup_{\xi \in \mathsf{U}} \Big| V(\xi) - \sum_{k=0}^{K} N^{-k} \, V^{\{k\}}(\xi) \Big| \leqslant v^{\{K\}} \, N^{-(K+1)}. \tag{5.1}$$

- *($g+1$-cut regime)* $W_1^{\{-1\}}(x) = \lim_{N \to \infty} N^{-1} \, W_1(x)$ *exists, is uniform for* x *in any compact of* $\mathbb{C} \backslash \mathsf{A}$*, and extends to a holomorphic function on* $\mathbb{C} \backslash \mathsf{S}$*, where* S *is a disjoint finite union of segments* $\mathsf{S}_h = [\alpha_h^-, \alpha_h^+] \subseteq \mathsf{A}_h$.
- *(Offcriticality)* $y(x) = \frac{(V^{\{0\}})'(x)}{2} - W_1^{\{-1\}}(x)$ *takes the form:*

$$y(x) = S(x) \prod_{h=0}^{g} \sqrt{(x - \alpha_h^+)^{\rho_h^+} (x - \alpha_h^-)^{\rho_h^-}}, \tag{5.2}$$

where S *does not vanish on* A*,* $\alpha_h^\bullet$ *are all pairwise distinct, and* $\rho_h^\bullet = 1$ *if* $\alpha_h^\bullet \in \partial \mathsf{A}$*, and* $\rho_h^\bullet = -1$ *else.*

- *($1/N$ expansion for correlations of filling fractions) For any $n \geqslant 1$, there exist a sequence $(\boldsymbol{W}^{\{k\}}_{n|n})_{k\geqslant n-2}$ of elements of $(\mathbb{C}^g)^{\otimes n}$, and positive constants $(w^{\{k\}}_{n|n})_{k\geqslant n-2}$, so that, for any $K \geqslant -1$, we have:*

$$\Big|\boldsymbol{W}_{n|n} - \sum_{k=n-2}^{K} N^{-k}\boldsymbol{W}^{\{k\}}_{n|n}\Big| \leqslant N^{-(K+1)}\,w^{\{K\}}_{n|n} \tag{5.3}$$

We say that Hypothesis 5.1 holds up to $o(N^{-K})$ if we only have a $1/N$ expansion of the potential at least up to $o(N^{-K})$, and an asymptotic expansion for correlations of filling fractions up to $o(N^{-(K-1)})$.

Remark 5.2 *In the model with fixed filling fractions, this last point is automatically satisfied since $\boldsymbol{W}_{n|n}$ is given by (4.7). Then, $W^{\{-1\}}_1(x)$ is the Stieltjes transform of the equilibrium measure determined by Theorem 1.2, and Hypothesis 5.1 then constrains the choice of V and $\boldsymbol{\epsilon}$.*

We fix once for all a neighborhood U of A so that $S^{-1}(0)\cap\mathsf{U} = \varnothing$, and contours $\boldsymbol{\mathcal{A}} = (\mathcal{A}_h)_{1\leqslant h\leqslant g}$ surrounding A_h in U.

Definition 5.3 *If $\delta > 0$, we introduce the norm $\|\cdot\|_{\delta}$ on the space $\mathcal{H}^{(n)}_{m_1,\ldots,m_n}(\mathsf{A})$ of holomorphic functions on $(\mathbb{C}\backslash\mathsf{A})^n$ which behave like $O(1/x_i^{m_i})$ when $x_i \to \infty$:*

$$\|f\|_{\delta} = \sup_{d(x,\mathsf{A})\geqslant\delta} |f(x_1,\ldots,x_n)| = \sup_{d(x_i,\mathsf{A})=\delta} |f(x_1,\ldots,x_n)|, \tag{5.4}$$

the last equality following from the maximum principle. If $n \geqslant 2$, we denote $\mathcal{H}^{(n)}_m = \mathcal{H}^{(n)}_{m,\ldots,m}$.

From Cauchy residue formula, we have a naive bound on the derivatives of a function $f \in \mathcal{H}^{(1)}_1$ in terms of f itself:

$$\|\partial_x^m f(x)\|_{\delta} \leqslant \frac{2^{m+1}C}{\delta^{m+1}}\,\|f\|_{\delta/2}. \tag{5.5}$$

Definition 5.4 *We fix once for all a sequence δ_N of positive numbers, so that:*

$$\lim_{N\to\infty} \frac{\ln^{1/3}\delta_N}{\delta_N}\Big(\frac{\ln N}{N}\Big)^{1/3} = 0 \tag{5.6}$$

If $f \in \mathcal{H}^{(n)}_m(\mathsf{A})$ is a sequence of functions indexed by N, we will denote $f \in O(R_N(\delta))$ when, for any $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ independent of δ and N, so that:

$$\|f\|_{\delta} \leqslant C(\varepsilon)\delta^{-\varepsilon}\,R_N(\delta) \tag{5.7}$$

for N large enough and δ small enough but larger than δ_N.

This choice will be justified at the end of § 5.3.1, and we can simplify the condition by taking δ_N of order $N^{-1/3+\varepsilon}$ for some $\varepsilon > 0$ arbitrarily small. We notice it also guarantees the assumptions $d(x,\mathsf{A})$ not much smaller than $(N\ln N)^{-1/2}$ as it appears in Corollary 3.7.

Definition 5.5 *If $\boldsymbol{X}$ is an element of $(\mathbb{C}^g)^{\otimes n}$, we define its norm as:*

$$|\boldsymbol{X}| = \sum_{1\leqslant h_1,\ldots,h_n\leqslant g} |X_{h_1,\ldots,h_n}|. \tag{5.8}$$

To perform the asymptotic analysis to all order, we need a rough a priori estimate on the correlators. We have established (actually under weaker assumptions on the potential) in § 3.3 that:

$$\Big(W_1 - \frac{1}{N}\,W_1^{\{-1\}}\Big) \in O\big(\sqrt{N\ln N}\,\delta^{-\theta}\big) \tag{5.9}$$

and for any $n \geqslant 2$ and $m \in [\![0,n]\!]$:

$$\boldsymbol{W}_{n|m} \in O\big((N\ln N)^{n/2}\,\delta^{-(n-m)\theta}\big) \tag{5.10}$$

with exponent $\theta = 1$.

Our goal in this section is to establish under those assumptions Proposition 5.5 below about the $1/N$ expansion for the correlators. We are going to recast the Schwinger-Dyson equations in a form which makes the asymptotic analysis easier. We already notice that it is convenient to choose

$$(\partial\mathsf{A})_{\pm} = \{a_h^{\bullet} \in (\partial\mathsf{A}),\quad \rho_h^{\bullet} = \pm 1\}, \tag{5.11}$$

as bipartition of $\partial\mathsf{A}$ to write down the Schwinger-Dyson equation, since the terms involving $\partial_a \ln Z$ and $\partial_a W_{n-1}$ for $a \in (\partial\mathsf{A})_+$ will be exponentially small according to Corollary 3.3. If $a = a_h^{\bullet}$, we denote $\alpha(a) = \alpha_h^{\bullet}$.

5.2 Some relevant linear operators

5.2.1 The operator $\mathcal{K}$

We introduce the following linear operator defined on the space $\mathcal{H}_2^{(1)}(\mathsf{A})$:

$$\mathcal{K}f(x) = 2W_1^{\{-1\}}(x)f(x) - \frac{1}{L(x)}\oint_{\mathsf{A}}\frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\Big[\frac{L(\xi)\,(V^{\{0\}})'(\xi)}{x-\xi} + P^{\{-1\}}(x;\xi)\Big]f(\xi), \tag{5.12}$$

where:

$$P^{\{-1\}}(x;\xi) = \oint_{\mathsf{A}}\frac{\mathrm{d}\eta}{2\mathrm{i}\pi}\,2L_2(x;\xi,\eta)\,W_1^{\{-1\}}(\eta) \tag{5.13}$$

We remind that $L(x) = \prod_{a\in(\partial\mathsf{A})_-}(x-\alpha(a))$ and L_2 was defined in (4.1). Notice that $W_1^{\{-1\}}(x) \sim 1/x$ when $x \to \infty$, and $P^{\{-1\}}(x,\xi)$ is a polynomial in two variables, of maximal total degree $|(\partial\mathsf{A})_-| - 2$. Hence:

$$\mathcal{K}\,:\,\mathcal{H}_2^{(1)}(\mathsf{A}) \to \mathcal{H}_1^{(1)}(\mathsf{A}). \tag{5.14}$$

Notice also that:

$$y(x) = \frac{\big(V^{\{0\}}\big)'(x)}{2} - W_1^{\{-1\}}(x) = S(x)\sqrt{\frac{\tilde{L}(x)}{L(x)}}, \tag{5.15}$$

where $\tilde{L}(x) = \prod_{a\in(\partial\mathsf{A})_+}(x-\alpha(a))$, and by offcriticality assumption the zeroes of S are away from A. Let us define $\sigma(x) = \sqrt{\prod_{a\in(\partial\mathsf{A})}(x-\alpha(a))} = \sqrt{\tilde{L}(x)L(x)}$, so that $\frac{\sigma(x)}{y(x)} = \frac{L(x)}{S(x)}$. We may rewrite:

$$\mathcal{K}f(x) = -2y(x)f(x) + \frac{\mathcal{Q}f(x)}{L(x)}, \tag{5.16}$$

where:

$$\mathcal{Q}f(x) = -\oint_{\mathsf{A}}\frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\Big[\frac{L(\xi)\,(V^{\{0\}})'(\xi) - L(x)\,(V^{\{0\}})'(x)}{x-\xi} + P^{\{-1\}}(x;\xi)\Big]f(\xi). \tag{5.17}$$

For any $f \in \mathcal{H}_2^{(1)}(\mathsf{A})$, $\mathcal{Q}f$ is analytic in $\mathbb{C}\backslash\mathsf{A}$, with singularities only where $(V^{\{0\}})'(\xi)$ has singularities, in particular it is holomorphic in the neighborhood of A. It is clear that $\mathrm{Im}\,\mathcal{K} \subseteq \mathcal{H}_1^{(1)}(\mathsf{A})$. Let $\varphi \in \mathrm{Im}\,\mathcal{K}$, and $f \in \mathcal{H}_2^{(1)}(\mathsf{A})$ such that $\varphi = \mathcal{K}f$. We can write:

$$\sigma(x)\,f(x) = \operatorname*{Res}_{\xi\to x} \frac{\mathrm{d}\xi}{\xi - x}\,\sigma(\xi)\,f(\xi) = \psi(x) - \oint_{\mathsf{A}} \frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\,\frac{\sigma(\xi)\,f(\xi)}{\xi - x}, \tag{5.18}$$

where:

$$\psi(x) = -\operatorname*{Res}_{\xi\to\infty} \frac{\mathrm{d}\xi}{\xi - x}\,\sigma(\xi)\,f(\xi). \tag{5.19}$$

Since $f(x) \in O(1/x^2)$, $\psi(x)$ is a polynomial in x of degree at most $g-1$. We then pursue the computation:

$$\begin{aligned}
\sigma(x)f(x) &= \psi(x) - \oint_A \frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\,\frac{1}{\xi - x}\,\frac{\sigma(\xi)}{2y(\xi)}\Big(-\varphi(\xi) + \frac{\mathcal{Q}f(\xi)}{L(\xi)}\Big) \\
&= \psi(x) + \oint_A \frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\,\frac{1}{\xi - x}\,\frac{1}{2S(\xi)}\Big[L(\xi)\,\varphi(\xi) + (\mathcal{Q}f)(\xi)\Big] \\
&= \psi(x) + \oint_A \frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\,\frac{1}{\xi - x}\,\frac{L(\xi)}{2S(\xi)}\,\varphi(\xi),
\end{aligned} \tag{5.20}$$

using the fact that S has no zeroes on A. Let us denote $\mathcal{G} : \mathrm{Im}\,\mathcal{K} \to \mathcal{H}_2^{(1)}(\mathsf{A})$ the linear operator defined by:

$$[\mathcal{G}\varphi](x) = \frac{1}{\sigma(x)} \oint_{\mathsf{A}} \frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\,\frac{1}{\xi - x}\,\frac{L(\xi)}{2S(\xi)}\,\varphi(\xi). \tag{5.21}$$

One also obtains:

$$f(x) = \frac{\psi(x)}{\sigma(x)} + (\mathcal{G}\circ\mathcal{K})[f](x). \tag{5.22}$$

5.2.2 The extended operator $\widehat{\mathcal{K}}$ and its inverse

It was first observed in [Ake96] that $\frac{\psi(x)\mathrm{d}x}{\sigma(x)}$ defines a holomorphic 1-form on the Riemann surface $\Sigma\,:\,\sigma^2 = \prod_{a\in(\partial\mathsf{A})}(x - \alpha(a))$. The space $H^1(\Sigma)$ of holomorphic 1-forms on Σ has dimension g if all $\alpha(a)$ are pairwise distinct (which is the case by offcriticality) and the number of cuts is $(g+1)$ and. So, if $g \geqslant 1$, $\mathcal{K}$ is not invertible. But we can define an extended operator:

$$\begin{aligned}
\widehat{\mathcal{K}}\,:\,\mathcal{H}_2^{(1)}(\mathsf{A}) &\longrightarrow \mathrm{Im}\,\mathcal{K}\times\mathbb{C}^r \\
f &\longmapsto (\mathcal{K}f, \mathcal{L}_{\boldsymbol{\mathcal{A}}}[f\mathrm{d}x]).
\end{aligned} \tag{5.23}$$

Since $\Big(\frac{x^{j-1}\mathrm{d}x}{\sigma(x)}\Big)_{0\leqslant j\leqslant g-1}$ are independent, they form a basis of $H^1(\Sigma)$. On the other hand, the family of linear forms:

$$\mathcal{L}_{\boldsymbol{\mathcal{A}}} = \Big(\oint_{\mathcal{A}_1}, \ldots, \oint_{\mathcal{A}_g}\Big) \tag{5.24}$$

are independent, hence they determine a unique basis $\varpi_h(x) = \frac{\psi_h(x)\mathrm{d}x}{\sigma(x)} \in H^1(\Sigma)$ so that:

$$\forall h, h' \in [\![1, g]\!], \qquad \oint_{\mathcal{A}_h} \varpi_{h'}(x) = \delta_{h,h'}. \tag{5.25}$$

$\mathcal{L}_{\boldsymbol{\mathcal{A}}}$ thus induces a linear isomorphism of the space of $H^1(\Sigma)$. Its inverse can be written:

$$\mathcal{L}_{\boldsymbol{\mathcal{A}}}^{-1}[\boldsymbol{w}] = \sum_{h=1}^{g} w_h\,\varpi_h(x) \tag{5.26}$$

We deduce that $\widehat{\mathcal{K}}$ is an isomorphism, its inverse being given by:

$$\widehat{\mathcal{K}}^{-1}[\varphi, \boldsymbol{w}](x) = \frac{\mathcal{L}_{\boldsymbol{\mathcal{A}}}^{-1}\big[\boldsymbol{w} - \mathcal{L}_{\boldsymbol{\mathcal{A}}}[(\mathcal{G}\varphi)\mathrm{d}x]\big](x)}{\mathrm{d}x} + \mathcal{G}\varphi(x), \tag{5.27}$$

where $\mathcal{G}$ is defined in (5.21). We will use the notation $\widehat{\mathcal{K}}_{\boldsymbol{w}}^{-1}\varphi = \widehat{\mathcal{K}}^{-1}[\varphi, \boldsymbol{w}]$. The continuity of this inverse operator is the key ingredient of our method:

Lemma 5.1 $\mathrm{Im}\,\mathcal{K}$ *is closed in* $\mathcal{H}_2^{(1)}(\mathsf{A})$*, and there exists s constant* $k_\Gamma(\mathsf{A}), k'_\Gamma(\mathsf{A}) > 0$ *such that:*

$$\forall (\varphi, \boldsymbol{w}) \in \mathrm{Im}\,\mathcal{K} \times \mathbb{C}^g, \qquad \|\widehat{\mathcal{K}}_{\boldsymbol{w}}^{-1}\varphi\|_\Gamma \leqslant k_\Gamma(\mathsf{A})\,\|\varphi\|_\Gamma + k'_\Gamma(\mathsf{A})|\boldsymbol{w}| \tag{5.28}$$

□

Remark 1. If one is interested in controlling the large N expansion of the correlators explicitly in terms of the distance of x_i's to A, it is useful to give an explicit bound on the norm of $\widehat{\mathcal{K}}_{\boldsymbol{w}}^{-1}$. For this purpose, let $\delta_0 > 0$ be small enough but fixed once for all, and we move the contour in (5.21) to a contour close staying at distance larger than δ_0 from A. If we choose now a point x so that $d(x, \mathsf{A}) < \eta$, we can write:

$$\mathcal{G}\varphi(x) = \frac{\varphi(x)}{2S(x)\sigma(x)} - \frac{\varphi(x)}{\sigma(x)} \oint_{d(\xi,\mathsf{A})=\delta_0} \frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\,\frac{L(\xi)}{2S(\xi)}\,\frac{1}{x-\xi} + \frac{1}{\sigma(x)} \oint_{d(\xi,\mathsf{A})=\delta_0} \frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\,\frac{L(\xi)}{2S(\xi)}\,\frac{\varphi(\xi)}{x-\xi} \tag{5.29}$$

Hence, there exist constants $C, C' > 0$ depending only on the position of the pairwise disjoint segments A_h such that, for any $\delta > 0$ smaller than $\delta_0/2$:

$$\|\mathcal{G}\varphi\|_\delta \leqslant (C D_c(\delta) + C')\,\delta^{-1/2}\,\|\varphi\|_\delta + \delta^{-1/2}\,\|\varphi\|_{\delta_0} \tag{5.30}$$

We set:

$$D_c(\delta) = \sup_{d(\xi,\mathsf{A})=\delta} \Big|\frac{L(\xi)}{S(\xi)}\Big| \tag{5.31}$$

For δ small enough but fixed, $D_c(\delta)$ blows up when the parameters of the model are tuned to achieve a critical point, i.e. it measures the distance to criticality. Besides, we have for the operator $\mathcal{L}_{\boldsymbol{\mathcal{A}}}^{-1}$ written in (5.26):

$$\|\mathcal{L}_{\boldsymbol{\mathcal{A}}}^{-1}[\boldsymbol{w}]\|_\delta \leqslant \frac{\max_{1\leqslant h\leqslant g}\|\psi_h\|_\infty^{\mathsf{U}}}{\inf_{d(\xi,\mathsf{A})=\delta}|\sigma(x)|}\,|\boldsymbol{w}|, \tag{5.32}$$

and the denominator behaves like $\delta^{-1/2}$ when $\delta \to 0$. We then deduce from (5.27) the existence of a constant $C'' > 0$ so that:

$$\|\widehat{\mathcal{K}}_{\boldsymbol{w}}^{-1}\varphi\|_\delta \leqslant (C D_c(\delta) + C')\delta^{-\kappa}\|\varphi\|_\delta + \delta^{-\kappa}\,|\boldsymbol{w}|, \tag{5.33}$$

with exponent $\kappa = 1/2$.

Remark 2. From the expression (5.27) for the inverse, we observe that, if φ is holomorphic in $\mathbb{C}\backslash\mathsf{S}$, so is $\widehat{\mathcal{K}}_{\boldsymbol{w}}^{-1}\varphi$ for any $\boldsymbol{w} \in \mathbb{C}^g$, in other words $\widehat{\mathcal{K}}_{\boldsymbol{w}}^{-1}(\mathrm{Im}\,\mathcal{K} \cap \mathcal{H}_1^{(1)}(\mathsf{S})) \subseteq \mathcal{H}_2^{(1)}(\mathsf{S})$.

5.2.3 Other linear operators

Some other linear operators appear naturally in the Schwinger-Dyson equation. We collect them below. Let us first define:

$$\Delta_{-1}W_1(x) = N^{-1}\,W_1(x) - W_1^{\{-1\}}(x), \tag{5.34}$$

$$\Delta_{-1}P(x;\xi) = \oint_{\mathsf{A}} \frac{\mathrm{d}\eta}{2\mathrm{i}\pi}\,2L_2(x;\xi,\eta)\Delta_{-1}W_1(\eta), \tag{5.35}$$

$$\Delta_0 V(x) = V(x) - V^{\{0\}}(x). \tag{5.36}$$

Let also h_1, h_2 be two holomorphic functions in U. We define:

$$\begin{aligned}
\mathcal{L}_1 \,:\, \mathcal{H}_1^{(1)}(\mathsf{A}) \to \mathcal{H}_2^{(1)}(\mathsf{A}) \qquad & \mathcal{L}_1 f(x) = \oint_{\mathsf{A}} \frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\,\frac{L_2(x;\xi,\xi)}{L(x)}\,f(\xi)\,, \\
\mathcal{L}_2 \,:\, \mathcal{H}_1^{(2)}(\mathsf{A}) \to \mathcal{H}_1^{(1)}(\mathsf{A}) \qquad & \mathcal{L}_2 f(x) = \oint_{\mathsf{A}} \frac{\mathrm{d}\xi_1\mathrm{d}\xi_2}{(2\mathrm{i}\pi)^2}\,\frac{L_2(x;\xi_1,\xi_2)}{L(x)}\,f(\xi_1,\xi_2)\,, \\
\mathcal{M}_{x'} \,:\, \mathcal{H}_1^{(1)}(\mathsf{A}) \to \mathcal{H}_1^{(2)}(\mathsf{A}) \qquad & \mathcal{M}_{x'} f(x) = \oint_{\mathsf{A}} \frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\,\frac{L(\xi)}{L(x)}\,\frac{f(\xi)}{(x-\xi)(x'-\xi)^2}\,, \\
\mathcal{N}_{h_1,h_2} \,:\, \mathcal{H}_1^{(1)}(\mathsf{A}) \to \mathcal{H}_1^{(1)}(\mathsf{A}) \qquad & \mathcal{N}_{h_1,h_2} f(x) = \frac{1}{L(x)}\oint_{\mathsf{A}} \frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\Big(\frac{L(\xi)h_1(\xi)}{x-\xi} + h_2(\xi)\Big)f(\xi)\,, \\
\Delta\mathcal{K} \,:\, \mathcal{H}_1^{(1)}(\mathsf{A}) \to \mathcal{H}_1^{(1)}(\mathsf{A}) \qquad & (\Delta\mathcal{K})f(x) = -\mathcal{N}_{(\Delta_0 V)',\Delta_{-1}P(x;\bullet)}[f](x) + 2\Delta_{-1}W_1(x)\,f(x) \\
& \qquad + \frac{1}{N}\Big(1-\frac{2}{\beta}\Big)(\partial_x + \mathcal{L}_1)f(x).
\end{aligned} \tag{5.37}$$

All those operators are continuous for appropriate norms, since we have the bounds, for δ_0 small enough but fixed, and δ small enough:

$$\begin{aligned}
\|\mathcal{L}_1 f(x)\|_{\delta} & \leqslant \frac{C\,\|L''\|_{\mathsf{U}}^{\infty}}{D_L(\delta)}\,\|f\|_{\delta_0}\,, \\
\|\mathcal{L}_2 f(x)\|_{\delta} & \leqslant \frac{C^2\,\|L''\|_{\mathsf{U}}^{\infty}}{D_L(\delta)}\,\|f\|_{\delta_0}\,, \\
\|\mathcal{M}_{x'} f\|_{\delta} & \leqslant \frac{C\|L\|_{\mathsf{U}}^{\infty}}{D_L(\delta)\,\delta^3}\,\|f\|_{\delta/2}\,, \\
\|\mathcal{N}_{h_1,h_2}\|_{\delta} & \leqslant \|h_1\|_{\mathsf{U}}^{\infty}\,\|f\|_{\delta} + C\,\frac{\|Lh_1\|_{\mathsf{U}}^{\infty} + \|h_2\|_{\mathsf{U}}^{\infty}}{\delta_0\,D_L(\delta)}\,\|f\|_{\delta_0}\,, \\
\|(\Delta\mathcal{K})f\|_{\delta} & \leqslant \big(\|(\Delta_0 V)'\|_{\mathsf{U}}^{\infty} + 2\,\|\Delta_{-1}W_1\|_{\delta}\big)\|f\|_{\delta} + \Big|1-\frac{2}{\beta}\Big|\,\frac{2C}{N\delta^2}\,\|f\|_{\delta/2} \\
& \quad + C\,\frac{\|L\,(\Delta_0 V)'\|_{\mathsf{U}}^{\infty} + \|\Delta_{-1}P\|_{\mathsf{U}^2}^{\infty}}{D_L(\delta)\,\delta_0}\,\|f\|_{\delta_0}
\end{aligned} \tag{5.38}$$

for any f in the domain of definition of the corresponding operator, and:

$$C = \ell(\mathsf{A})/\pi + (g+1), \qquad D_L(\delta) = \inf_{d(x,\mathsf{A})\geqslant\delta} |L(x)|. \tag{5.39}$$

If all edges are soft, $D_L(\delta) \equiv 1$, whereas if there exist at least one hard edge, $D_L(\delta)$ scales like δ when $\delta \to 0$.

5.3 Recursive expansion of the correlators

5.3.1 Rewriting Schwinger-Dyson equations

For $n \geqslant 2$ and $m \in [\![0, n-1]\!]$, we can organize the Schwinger-Dyson equation of Corollary 4.2 as follows:

$$\Big[\mathcal{K} + \Delta\mathcal{K} + \frac{1}{N}\Big(1-\frac{2}{\beta}\Big)\partial_x\Big]\boldsymbol{W}_{n|m}(x, x_I) = \boldsymbol{A}_{n+1|m} + \boldsymbol{B}_{n|m} + \boldsymbol{C}_{n-1|m} + \boldsymbol{D}_{n-1|m}, \tag{5.40}$$

where:

$$\begin{aligned}
\boldsymbol{A}_{n+1|m}(x;x_I) &= N^{-1}(\mathcal{L}_2-\mathrm{id})\boldsymbol{W}_{n+1|m}(x,x,x_I),\\
\boldsymbol{B}_{n|m}(x;x_I) &= N^{-1}(\mathcal{L}_2-\mathrm{id})\Big\{\sum_{\substack{J\subseteq I,\ 0\leqslant m'\leqslant m\\ (J,m')\neq(\varnothing,0),(I,m)}}\binom{m}{m'}\boldsymbol{W}_{|J|+1+m'|m'}(x,x_J)\otimes\boldsymbol{W}_{n-|J|-m'|m-m'}(x,x_{I\setminus J})\Big\},\\
\boldsymbol{C}_{n-1|m}(x;x_I) &= -\frac{2}{\beta N}\sum_{i\in I}\mathcal{M}_{x_i}\boldsymbol{W}_{n-1|m}(x,x_{I\setminus\{i\}}),\\
\boldsymbol{D}_{n-1|m}(x;x_I) &= \frac{2}{\beta N}\sum_{a\in(\partial\mathsf{A})_+}\frac{L(a)}{x-a}\,\partial_a\boldsymbol{W}_{n-1|m}(x_I).
\end{aligned}\tag{5.41}$$

This equation can be rewritten, with the notation $\widehat{\mathcal{K}}_{\boldsymbol{w}}^{-1}\varphi=\widehat{\mathcal{K}}^{-1}[\varphi,\boldsymbol{w}]$ and by definition (4.5) of $\boldsymbol{W}_{n|m}$,

$$\begin{aligned}
\boldsymbol{W}_{n|m}(x,x_I) &= \widehat{\mathcal{K}}^{-1}_{\boldsymbol{W}_{n|m+1}(x_I)}\Big[\boldsymbol{A}_{n+1|m}(x,x_I)+\boldsymbol{B}_{n|m}(x,x_I)+\boldsymbol{C}_{n-1|m}(x,x_I)+\boldsymbol{D}_{n-1|m}(x,x_I)\\
&\quad -(\Delta\mathcal{K})[\boldsymbol{W}_{n|m}(x,x_I)]-\frac{1}{N}\Big(1-\frac{2}{\beta}\Big)(\partial_x+\mathcal{L}_1)\boldsymbol{W}_{n|m}(x,x_I)\Big],
\end{aligned}\tag{5.42}$$

where it is understood that the operators all act on the first variable (or the two first variables for $\mathcal{L}_2$).

For $n=1$ and $m=0$, we have almost the same equation; with the notation of (5.34), and in view of (4.2),

$$\begin{aligned}
\Delta_{-1}W_1(x) &= \widehat{\mathcal{K}}^{-1}_{\Delta_{-1}\boldsymbol{W}_{1|1}}\Big[\frac{A_2(x)+D_0}{N}-\frac{1}{N}\Big(1-\frac{2}{\beta}\Big)(\partial_x+\mathcal{L}_1)W_1(x)\\
&\quad -\mathcal{N}_{(\Delta_0 V)',0}W_1(x)-\big(\Delta_{-1}W_1(x)\big)^2\Big],
\end{aligned}\tag{5.43}$$

where:

$$\Delta_{-1}\boldsymbol{W}_{1|1}=\mathcal{L}_{\boldsymbol{\mathcal{A}}}[\Delta_{-1}W_1]=N^{-1}\,\mathcal{L}_{\boldsymbol{\mathcal{A}}}[W_1]-\mathcal{L}_{\boldsymbol{\mathcal{A}}}[W_1^{\{-1\}}]\tag{5.44}$$

is the first correction to the expected filling fractions.

We would like $\Delta\mathcal{K}$ to be negligible compared to $\mathcal{K}$ in (5.40), that is

$$\|\widehat{\mathcal{K}}_{\boldsymbol{w}}(\Delta\mathcal{K}f)\|_\delta \ll \|f\|_\delta\tag{5.45}$$

This can be controlled thanks to (5.33) and (5.38). From our assumptions, we know that:

$$\|\Delta_{-1}P\|^\infty_{\mathsf{U}^2}\in o(1),\qquad \|\Delta_0 V\|^\infty_{\mathsf{U}}\in O(1/N).\tag{5.46}$$

Taking into account those estimates in (5.38), we observe that it will be possible independently of the nature of the edges provided δ is restricted to be larger than δ_N such that $\lim_{N\to\infty}\delta_N^{-1/2}\,\|\Delta_{-1}W_1\|_{\delta_N}=0$, and $\lim_{N\to\infty}\frac{1-2/\beta}{N\delta_N^{5/2}}=0$. Given the a priori bound in Corollary 3.7 on $\Delta_{-1}W_1$, this can be realized independently of β if:

$$\lim_{N\to\infty}\sqrt{\frac{\ln N}{N}}\,\frac{\sqrt{\ln\delta_N}}{\delta_N}D_c(\delta_N)\,\delta_N^{-\kappa}=0\tag{5.47}$$

Since we consider here a fixed, off-critical potential, $D_c(\delta)$ remains bounded, and since $\kappa=1/2$, this condition is equivalent to:

$$\lim_{N\to\infty}\Big(\frac{\ln N}{N}\Big)^{1/3}\frac{\ln^{1/3}\delta_N}{\delta_N}=0\tag{5.48}$$

It justifies the introduction of the sequence δ_N in Definition 5.4. Then, by similar arguments, $\|\boldsymbol{D}_{n|m}\|_\delta$ will always be exponentially small when N is large provided $\delta\geqslant\delta_N$, and thus negligible in front of the other terms.

5.3.2 Initialization and order of magnitude of W_n

We remind $\theta = 1$ and $\kappa = 1/2$ here. The goal of this section is to prove the following bounds.

Proposition 5.2 *There exists a function* $W_1^{\{0\}} \in \mathcal{H}_2^{(1)}(\mathsf{S})$ *independent of* N *so that:*

$$W_1 = NW_1^{\{-1\}} + W_1^{\{0\}} + \Delta_0 W_1, \qquad \Delta_0 W_1 \in O\Big(\sqrt{\frac{\ln^3 N}{N}}\,\Big(\frac{D_c(\delta)}{D_L(\delta)}\Big)^2\,\delta^{-(2\theta+\kappa)}\Big) \tag{5.49}$$

It is given by:

$$W_1^{\{0\}}(x) = \widehat{\mathcal{K}}^{-1}_{\boldsymbol{W}_{1|1}^{\{0\}}}\Big\{\Big[-\Big(1-\frac{2}{\beta}\Big)(\partial_x + \mathcal{L}_1) - \mathcal{N}_{(V^{\{1\}})',0}\Big]W_1^{\{-1\}}\Big\}(x) \tag{5.50}$$

Proposition 5.3 *For any* $n \geqslant 2$*, there exists a function* $W_n^{\{n-2\}} \in \mathcal{H}_2^{(n)}(\mathsf{S})$ *so that:*

$$\boldsymbol{W}_{n|m} = N^{2-n}(\boldsymbol{W}_{n|m}^{\{n-2\}} + \Delta_{n-2}\boldsymbol{W}_{n|m}) \tag{5.51}$$

where:

$$\Delta_{n-2}W_n \in O\Big(N^{-1/2}(\ln N)^{2n-3/2}\,\Big(\frac{D_c(\delta)}{D_L(\delta)}\Big)^{3n-3}\,\delta^{-\theta(n-m)-(\kappa+\theta)(3n-3)}\Big) \tag{5.52}$$

Prior to those results, we are going to prove the following bound:

Lemma 5.4 *Denote* $r_n^* = 3n-4$*. For any integers* $n \geqslant 2$ *and* $r \geqslant 0$ *such that* $r \leqslant r_n^*$*, we have:*

$$\boldsymbol{W}_{n|m} \in O\Big(N^{\frac{n-r}{2}}(\ln N)^{\frac{n+r}{2}}\,\Big(\frac{D_c(\delta)}{D_L(\delta)}\Big)^r\,\delta^{-\theta(n-m)-(\kappa+\theta)r}\Big) \tag{5.53}$$

Proof. The a priori control of correlators (3.28) provides the result for $r = 0$. Let s be an integer, and assume the result is true for any $r \in [\![0, s]\!]$. Let n be such that $s+1 \leqslant r_n^* = 3n-4$. We consider (5.42) which gives $\boldsymbol{W}_{n|m}$ in terms of $\boldsymbol{W}_{n+1|m}$ and $\boldsymbol{W}_{n'|m'}$ for $n' < n$ or $n' = n$ but $m' > m$, and we exploit the control (5.33) on the inverse of $\widehat{\mathcal{K}}$. We obtain the following bound on the A-term:

$$\widehat{\mathcal{K}}^{-1}_{\boldsymbol{W}_{n|m+1}}(\boldsymbol{A}_{n+1|m}) \in O\Big(N^{\frac{n-(s+1)}{2}}(\ln N)^{\frac{n+s+1}{2}}\,\Big(\frac{D_c(\delta)}{D_L(\delta)}\Big)^{s+1}\,\delta^{-(n-m)\theta-(s+1)(\kappa+\theta)} + \|\boldsymbol{W}_{n|m+1}\|_{\delta}\,\delta^{-1/2}\Big), \tag{5.54}$$

and we now argue that it will be the worse estimate among all other terms, given that $\delta \geqslant \delta_N$. The B-term involves linear combinations of $\boldsymbol{W}_{j+1|m'} \otimes \boldsymbol{W}_{n-j|m-m'}$. Notice that:

$$s - r_{j+1}^* \leqslant r_n^* - r_{j+1}^* = r_{n-j}^* \tag{5.55}$$

Therefore, we can use the recursion hypothesis with $r = r_{j+m'+1}^* = 3(j+m')-1$ to bound $\boldsymbol{W}_{j+m'+1|m'}$, and with $r = s - r_{j+m'+1}^*$ to bound $\boldsymbol{W}_{n-j-m'|m-m'}$, and we find:

$$\widehat{\mathcal{K}}^{-1}_{\boldsymbol{W}_{n|m+1}}(\boldsymbol{B}_{n|m}) \in O\Big(N^{\frac{n-(s+1)}{2}}(\ln N)^{\frac{n+(s+1)}{2}}\,\Big(\frac{D_c(\delta)}{D_L(\delta)}\Big)^{s+1}\,\delta^{-(n-m)\theta-(s+1)(\kappa+\theta)} + \|\boldsymbol{W}_{n|m+1}\|_{\delta}\,\delta^{-\kappa}\Big) \tag{5.56}$$

which is of the same order as (5.54). The C-term involves $\boldsymbol{W}_{n-1|m}$. If $s \leqslant r_{n-1}^*$, we can use the recursion hypothesis at $r = s$ to bound it as

$$\widehat{\mathcal{K}}^{-1}_{\boldsymbol{W}_{n|m+1}}(\boldsymbol{C}_{n-1|m}) \in O\Big(\frac{\delta^{2\theta-3}}{N^2\ln N}\,N^{\frac{n-(s+1)}{2}}(\ln N)^{\frac{n+1+s}{2}}\,\Big(\frac{D_c(\delta)}{D_L(\delta)}\Big)^{s+1}\,\delta^{-(n-m)\theta-(s+1)(\kappa+\theta)}\Big) \tag{5.57}$$

The prefactor makes this term negligible comparing to (5.54). If $s > r^*_{n-1}$, we can only use the recursion hypothesis for $r = r^*_{n-1}$, and find the bound:

$$\begin{aligned}\widehat{\mathcal{K}}^{-1}_{\boldsymbol{W}_{n|m+1}}(\boldsymbol{C}_{n-1|m}) \quad &\in \quad O\Big(N^{2-n}\,(\ln N)^{2n-4}\Big(\frac{D_c(\delta)}{D_L(\delta)}\Big)^{3n-6}\,\delta^{-(n-m)\theta-(3n-4)(\kappa+\theta)+2(\theta-\kappa)} \\ &\qquad + \|\boldsymbol{W}_{n|m+1}\|_{\delta}\,\delta^{-\kappa}\Big), \qquad (5.58)\end{aligned}$$

and thus still negligible compared to the A-term when $N \to \infty$ and $\delta \to 0$. We can conclude by recursion from $m = n$ to $m = 0$, taking into account the assumed expansion (5.3) for $\boldsymbol{W}_{n|n}$. □

Proof of Proposition 5.2. Lemma 5.4 for $r = 1$ gives the bound:

$$W_2 \in O\Big(\sqrt{N\ln^3 N}\,\frac{D_c(\delta)}{D_L(\delta)}\,\delta^{-(3\theta+\kappa)}\Big) \qquad (5.59)$$

Then, in (5.43), we find that $A_2 \in O\big(\frac{W_2}{ND_L(\delta)}\big)$, and we already know that $|\Delta_{-1}W_1|$ goes to zero for uniform convergence in any compact subset of $\mathbb{C}\backslash\mathsf{A}$, so that $\|\widehat{\mathcal{K}}^{-1}_{\boldsymbol{W}_{n|m+1}}(\Delta_{-1}W_1)^2 l_\delta$ is neglectable with respect to $\Delta_{-1}W_1$ at list for δ not going to zero with N. By continuity of $\widehat{\mathcal{K}}^{-1}$, we deduce that $N\Delta_{-1}W_1(x)$ has a limit $W_1^{\{0\}}$ for convergence in any compact subset of $\mathbb{C}\backslash\mathsf{A}$, given by (5.50). Reminding Remark 2 page 28, this limit belongs to $\mathcal{H}_2^{(1)}(\mathsf{S})$, and given the behavior of $W_1^{\{-1\}}$ at the edges, we have a naive bound:

$$(\partial_x + \mathcal{L}_1)W_1^{\{0\}} \in O\Big(\frac{\delta^{-3/2}}{D_L(\delta)}\Big), \qquad (5.60)$$

and we have already argued at the end of § 5.3.1 that $\|N^{-1}(\partial_x + \mathcal{L}_1)(\Delta_{-1}W_1)\|_\delta$ was negligible compared to $\|\Delta_{-1}W_1\|_{\delta/2}$ provided $\delta \geqslant \delta_N$. Moreover, Therefore, the worse estimate on the error $\Delta_0 W_1$ is given by the term A_2. Taking into account the effect of $\widehat{\mathcal{K}}^{-1}$ given in (5.33), we find:

$$\Delta_0 W_1 \in O\Big(\frac{(\ln N)^{3/2}}{\sqrt{N}}\,\Big(\frac{D_c(\delta)}{D_L(\delta)}\Big)^2\,\delta^{-(3\theta+2\kappa)}\Big) \qquad (5.61)$$

which is the desired result. □

Proof of Proposition 5.3 We already know the result for $n = 1$. Let $n \geqslant 2$, $m \in [\![0, n-1]\!]$, and assume the result holds for all $n' \in [\![1, n-1]\!]$ and $m' \in [\![m, n]\!]$. We want to use (5.40) once more to compute $\boldsymbol{W}_{n|m}$. Applying Lemma 5.4 to $r = 3n-4$ for $n \geqslant 2$, we find:

$$\boldsymbol{A}_{n+1|m} \in O\Big(N^{2-n}\,\frac{(\ln N)^{2n-3/2}}{\sqrt{N}}\,\big(D_L(\delta)\big)^{-(3n-4)}\,\delta^{-(n+1-m)\theta-(3n-4)(\theta+\kappa)}\Big), \qquad (5.62)$$

whereas the recursion hypothesis implies that $\boldsymbol{B}_{n|m}$ and $\boldsymbol{C}_{n-1|m}$ are of order $O(N^{2-n})$. Hence, $\boldsymbol{W}_{n|m} = N^{2-n}(\boldsymbol{W}^{\{n-2\}}_{n|m} + \Delta_{n-2}\boldsymbol{W}_{n|m})$ with:

$$\begin{aligned}\boldsymbol{W}^{\{n-2\}}_{n|m}(x, x_I) \quad &= \quad \widehat{\mathcal{K}}^{-1}_{\boldsymbol{W}^{\{n-2\}}_{n|m+1}(x_I)}\Big[-\frac{2}{\beta}\sum_{i\in I}\mathcal{M}_{x_i}\boldsymbol{W}^{\{n-3\}}_{n-1|m}(x, x_I) \qquad (5.63) \\ &+(\mathcal{L}_2 - \mathrm{id})\Big\{\sum_{\substack{0\leqslant m'\leqslant m \\ J\subseteq I}}\binom{m}{m'}\boldsymbol{W}^{\{|J|-1+m'\}}_{|J|+1+m'|m'}(x, x_J)\otimes\boldsymbol{W}^{\{n-|J|-2-m'\}}_{n-|J|-m'|m-m'}(x, x_{I\backslash J})\Big\}\Big]\end{aligned}$$

The error term $\Delta_{n-2}\boldsymbol{W}_{n|m}$ receives contribution either from errors $\Delta_{n'-2}\boldsymbol{W}_{n'|m'}$ appearing in $\boldsymbol{B}_{n|m}$ and $\boldsymbol{C}_{n-1|m}$, which we already know how to bound. And the restriction that $\lim_{N\to\infty} N\delta_N^2 = +\infty$

guarantees that they are negligible in front of $\boldsymbol{A}_{n+1|m}$ computed in (5.62). Hence, we deduce by substracting $\boldsymbol{W}_{n|m}^{\{n-2\}}$ and inverting $\mathcal{K}$ that:

$$\Delta_{n-2}\boldsymbol{W}_{n|m} \in O\Big(\sqrt{\frac{(\ln N)^{4n-3}}{N}}\,\Big(\frac{D_c(\delta)}{D_L(\delta)}\Big)^{3n-3}\,\delta^{-(n-m)\theta-(3n-3)(\theta+\kappa)}\Big) \tag{5.64}$$

which is the desired result at step (n,m). We conclude by recursion. □

5.4 Recursive expansion of the correlators

Proposition 5.5 *For any $k_0 \geqslant 0$, we have for any $n \geqslant 1$:*

$$\boldsymbol{W}_{n|m}(x_1,\ldots,x_n) = \sum_{k=n-2}^{k_0} N^{-k}\,\boldsymbol{W}_{n|m}^{\{k\}}(x_1,\ldots,x_n) + N^{-k_0}(\Delta_{k_0}\boldsymbol{W}_{n|m})(x_1,\ldots,x_n), \tag{5.65}$$

where:

(i) for any $n \geqslant 1$ and any $k \in [\![0,\ldots,k_0]\!]$, $\boldsymbol{W}_{n|m}^{\{k\}}$ has a limit when $N \to \infty$ in $\mathcal{H}_2^{(n-m)}(\mathsf{S})$ for pointwise convergence in any compact of $(\mathbb{C}\backslash\mathsf{A})^{n-m}$, and:

$$\boldsymbol{W}_{n|m}^{\{k\}} \in O\Big((\ln N)^{\frac{n+k}{2}}\,\Big(\frac{D_c(\delta)}{D_L(\delta)}\Big)^{n+2k}\,\delta^{-(n-m)\theta-(n+2k)(\kappa+\theta)}\Big). \tag{5.66}$$

(ii) for any $n \geqslant 1$, $\Delta_{k_0}\boldsymbol{W}_{n|m} \in \mathcal{H}_2^{(n-m)}(\mathsf{A})$ and:

$$\Delta_{k_0}\boldsymbol{W}_{n|m} \in O\Big(\frac{(\ln N)^{n+k_0+1/2}}{\sqrt{N}}\,\Big(\frac{D_c(\delta)}{D_L(\delta)}\Big)^{n+2k_0+1}\,\delta^{-(n-m)\theta-(n+2k_0+1)(\kappa+\theta)}\Big). \tag{5.67}$$

Proof. The case $k_0 = 0$ follows from § 5.3.2, and we prove the general case by recursion on k_0, which can be seen as the continuation of the proof of Proposition 5.3. Assume the result holds for some $k_0 \geqslant 0$. Let us decompose:

$$V = \sum_{k=0}^{k_0+2} N^{-k}\,V^{\{k\}} + N^{-(k_0+2)}\Delta_{k_0+2}V. \tag{5.68}$$

We already know that the loop equations are satisfied up to order N^{1-k_0}. We can decompose the remainder as:

$$N^{-(k_0-1)}\Big\{\mathcal{K} + \Delta\mathcal{K} + \frac{1}{N}\Big(1-\frac{2}{\beta}\Big)\partial_x\Big\}\Delta_{k_0}\boldsymbol{W}_{n|m}(x,x_I) = N^{-k_0}\big(\boldsymbol{E}_{n|m}^{\{k_0\}}(x;x_I) + \boldsymbol{R}_{n|m}^{\{k_0\}}(x;x_I)\big). \tag{5.69}$$

It is understood that all linear operators appearing here (and defined in § 5.2) act on the variable(s) x. We have set:

$$\begin{aligned}\boldsymbol{E}_{n|m}^{\{k_0\}}(x;x_I) \quad = \quad & (\mathcal{L}_2 - \mathrm{id})\big[\boldsymbol{W}_{n+1|m}^{\{k_0-1\}}(x,x,x_I)\big] \\ & + \sum_{\substack{0\leqslant k\leqslant k_0 \\ 0\leqslant m'\leqslant m}}\sum_{J\subseteq I}\binom{m}{m'}(\mathcal{L}_2-\mathrm{id})\big[\boldsymbol{W}_{|J|+1+m'|m'}^{\{k\}}(x,x_J)\otimes\boldsymbol{W}_{n-|J|-m'|m-m'}^{\{k_0-k\}}(x,x_{I\backslash J})\big] \\ & -\Big(1-\frac{2}{\beta}\Big)(\partial_x+\mathcal{L}_1)\big[\boldsymbol{W}_{n|m}^{\{k_0\}}(x,x_I)\big] + \sum_{k=n_0-2}^{k_0}\mathcal{N}_{(V^{\{k_0+1-k\}})',0}\big[\boldsymbol{W}_{n|m}^{\{k\}}(x,x_I)\big] \\ & -\frac{2}{\beta}\sum_{i\in I}\mathcal{M}_{x_i}\big[\boldsymbol{W}_{n-1|m}^{\{k_0\}}(x,x_{I\backslash\{i\}})\big],\end{aligned} \tag{5.70}$$

and:

$$\begin{aligned}\boldsymbol{R}^{\{k_0\}}_{n|m}(x;x_I) &= (\mathcal{L}_2-\mathrm{id})\big[\Delta_{k_0}\boldsymbol{W}_{n+1|m}(x,x,x_I)\big] \\ &+ \sum_{\substack{0\leqslant m'\leqslant m \\ J\subseteq I}} (\mathcal{L}_2-\mathrm{id})\big[\big(\Delta_{k_0}\boldsymbol{W}_{|J|+1+m'|m'}(x,x_J)\big)\otimes \boldsymbol{W}_{n-|J|-m'|m-m'}(x,x_{I\setminus J})\big] \\ &+ \sum_{\substack{0\leqslant k\leqslant k_0 \\ 0\leqslant m'\leqslant m}}\sum_{J\subseteq I}(\mathcal{L}_2-\mathrm{id})\big[\boldsymbol{W}^{\{k_0-k\}}_{|J|+1+m'|m'}(x,x_J)\otimes\big(\Delta_k\boldsymbol{W}_{n-|J|-m'|m-m'}(x,x_{I\setminus J})\big)\big] \\ &+ \sum_{l=-1}^{k}\mathcal{N}_{(\Delta_{l+1}V)',0}\big[\boldsymbol{W}^{\{k_0-l\}}_{n|m}(x,x_I)\big]-\frac{2}{\beta}\sum_{i\in I}\mathcal{M}_{x_i}\big[\Delta_{k_0}\boldsymbol{W}_{n-1|m}(x,x_{I\setminus\{i\}})\big] \\ &-\frac{2}{\beta}\sum_{a\in(\partial\mathsf{A})_+}\frac{L(a)}{x-a}\,\partial_a\boldsymbol{W}_{n-1|m}(x_I).\end{aligned}$$

Let us denote:

$$w^{\{k\}}_{n|m}(N,\delta)=\frac{(\ln N)^{n+k+1/2}}{\sqrt{N}}\Big(\frac{D_c(\delta)}{D_L(\delta)}\Big)^{n+2k+1}\delta^{-(n-m)\theta-(n+2k+1)(\kappa+\theta)}. \tag{5.71}$$

Thanks to the recursion hypothesis and the bound on the norm of the inverse of $\widehat{\mathcal{K}}$ from (5.33), we find:

$$\widehat{\mathcal{K}}_{\mathbf{0}}^{-1}\boldsymbol{R}_{n|m}\in O\big(w^{\{k+1\}}_{n}(N,\delta)\big). \tag{5.72}$$

This bound arise from the three first lines. Indeed, the two last terms are negligible compared to $w_n^{\{k+1\}}$ as noticed in the proof of Lemma 5.4, and the term involving $\Delta_{l+1}V$ is of order $(\ln N)^p/N$ for some p, with a dependence in δ which is less divergent than that appearing of $w_n^{\{k+1\}}$. Therefore, we deduce by recursion on m from $m=n$ to $m=0$ that $N\Delta_{k_0}\boldsymbol{W}_{n|m}$ converges to:

$$\boldsymbol{W}^{\{k_0+1\}}_{n|m}(x,x_I)=\widehat{\mathcal{K}}^{-1}_{\boldsymbol{W}^{\{k_0+1\}}_{n|m+1}(x_I)}\big[\boldsymbol{E}^{\{k_0\}}_{n}(x,x_I)\big], \tag{5.73}$$

pointwise and uniformly on any compact of $(\mathbb{C}\backslash\mathsf{A})^{n-m}$. Besides, the estimate (5.72) yields the bound (5.67) for the error, while the recursion hypothesis combined with (5.73) leads to (5.66). □

This proves the first part of Theorem 1.3 for real-analytic potentials (i.e. the stronger Hypothesis 1.3 instead of 1.4). For given n and k, the bound on the error $\Delta_k W_n$ depend only on a finite number of constants $v^{\{k'\}}, w^{\{k'\}}_{n'}$ appearing in Hypotheses 5.1.

5.5 Central limit theorem

With Proposition 5.2 at our disposal, we can already establish a central limit theorem for linear statistics of analytic functions in the fixed filling fraction model. It will be refined for non-analytic but smooth enough functions in § 6.1.

Proposition 5.6 *Assume the result of Proposition 5.2. Let $\varphi:\mathsf{A}\to\mathbb{R}$ extending to a holomorphic function in a neighborhood of S. Then:*

$$\mu^{V;\mathsf{A}}_{N,\bullet,\beta}\Big[\exp\Big(\sum_{i=1}^{N}\varphi(\lambda_i)\Big)\Big]=\exp\Big(NL[\varphi]+M[\varphi]+\frac{1}{2}\,Q[\varphi,\varphi]+o(1)\Big), \tag{5.74}$$

where:

$$L[\varphi]=\oint_{\mathsf{A}}\frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\,\varphi(\xi)\,W_1^{\{-1\}}(\xi),\qquad M[h]=\oint_{\mathsf{A}}\frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\,\varphi(\xi)\,W_1^{\{0\}}(\xi) \tag{5.75}$$

$W_1^{\{0\}}$ has been introduced in (5.49), and Q is a quadratic form given in (5.78) or (5.79) below.

Proof. Let us define $V_t = V - \frac{2t}{\beta N}\,\varphi$. Since the equilibrium measure is the same for V_t and V, we still have the result of Proposition 5.2 for the model with potential V_t for any $t \in [0,1]$, with uniform errors. We can thus write:

$$\begin{aligned}\ln \mu_{N,\bullet,\beta}^{V;\mathsf{A}}\Big[\exp\Big(\sum_{i=1}^{N} t\,\varphi(\lambda_i)\Big)\Big] &= \int_0^1 \mathrm{d}t \oint_{\mathsf{A}} \frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\, W_1^{V_t}(\xi)\,\varphi(\xi) \\ &= \int_0^1 \mathrm{d}t \oint_{\mathsf{A}} \frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\, \varphi(\xi)\big[N\,W_1^{V_t;\{-1\}} + W_1^{V_t;\{0\}}(\xi)\big] + o(1) \qquad (5.76)\end{aligned}$$

As already pointed out, $W_1^{V_t;\{0\}} = W_1^{V;\{0\}}$, and from (5.49):

$$W_1^{V_t;\{0\}} = W_1^{V;\{0\}} - \frac{2t}{\beta}\big(\widehat{\mathcal{K}}^{-1}_{\boldsymbol{W}_{1|1}^{\{0\}}} \circ \mathcal{N}_{\varphi',0}\big)[W_1^{V;\{-1\}}] \qquad (5.77)$$

Hence (5.74), with:

$$Q[\varphi,\varphi] = -\frac{2}{\beta}\oint_{\mathsf{A}} \frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\,\varphi(\xi)\big(\widehat{\mathcal{K}}^{-1}_{\boldsymbol{W}_{1|1}^{\{0\}}} \circ \mathcal{N}_{\varphi',0}\big)[W_1^{V;\{-1\}}](\xi) \qquad (5.78)$$

If we restrict to the model with fixed filling fraction, it can be simplified to:

$$Q[\varphi,\varphi] = \oiint_{\mathsf{A}} \frac{\mathrm{d}\xi_1\,\mathrm{d}\xi_2}{(2\mathrm{i}\pi)^2}\,\varphi(\xi_1)\varphi(\xi_2)\,W_2^{V;\{0\}}(\xi_1,\xi_2) \qquad (5.79)$$

where $W_2^{V;\{0\}}$ has been introduced in (5.51) and we recall $\boldsymbol{W}_{2|1}^{V} = 0$ for the model with fixed filling fractions. From the proof of Proposition 5.2, we observe that the $o(1)$ in (5.74) is uniform in h such that $\sup_{\xi\in\Gamma_E}|\varphi(\xi)|$ is bounded by a fixed constant. □

In other words, the random variable $\Phi = \sum_{i=1}^N \varphi(\lambda_i) - L[\varphi]$ converges almost surely to a Gaussian variable with mean $M[\varphi]$ and variance $Q[\varphi,\varphi]$. This is a generalization of the central limit theorem already known in the one-cut regime [Joh98, BG11]. A similar result was recently obtained in [Shc12]. In the next Section, we are going to extend it to holomorphic h which could be complex-valued on A (Proposition 6.3). In general, to establish the central limit theorem, one could be tempted to use the definition of the correlators:

$$\partial_t^n \ln \mu_{N,\bullet,\beta}^{V;\mathsf{A}}\Big[\exp\Big(\sum_{i=1}^{N} t\,\varphi(\lambda_i)\Big)\Big]\Big|_{t=0} = \oint_{\mathsf{A}^n} \prod_{i=1}^{n} \frac{\mathrm{d}\xi_i}{2\mathrm{i}\pi}\,\varphi(\xi_i)\,W_n^V(\xi_1,\ldots,\xi_n), \qquad (5.80)$$

then represent $G_N(t) = \ln \mu_{N,\bullet,\beta}^{V;\mathsf{A}}[e^{t\Phi}]$ by its Taylor expansion up to $t = 1$, and use the result of Proposition 5.2 that $W_n^V \in O(N^{2-n})$ to conclude. However, for any fixed N, $G_N(t)$ is analytic in the domain of the complex plane where $\mu_{N,\bullet,\beta}^{V;\mathsf{A}}[e^{t\Phi}]$ does not vanish, and it is not obvious that for N large enough (although it will turn out to be true) that this does not happen for some $t_0 \in \mathbb{C}$ with $|t_0| < 1$, i.e. that the Taylor series converges in the appropriate domain.

6 Fixed filling fraction: refined results

In this section, we show how the asymptotic expansion of multilinear statistics for non-analytic test functions can be deduced from our results, thanks to their explicit dependence on the distance of the variables x (appearing in the correlators) to A. We also show how to extend our results to the case of harmonic potentials, and potentials containing a complex-valued term of order $O(1/N)$. The latter is performed by using fine properties of analytic functions (the two-constants theorem) as was recently proposed in [Shc12].

6.1 Multilinear statistics for non-analytic test functions

Our methods establish a control on n-point correlators $W_n(x_1,\ldots,x_n)$, depending on how $x_1,\ldots,x_n$ approach the range of integration A. We now argue that it gives a control n-linear statistics for test functions with regularity lower than analytic. If s is a finite-dimensional vector, we denote $|s|_1 = \sum_i |s_i|$.

Lemma 6.1 *Let f_n be a holomorphic function defined in a neighborhood of A^n in $(\mathbb{C}\backslash\mathsf{A})^n$. Assume there exists $C, r,$ and $\eta \in (0,1)$ small enough, such that*

$$\forall \delta \geqslant \eta, \qquad \sup_{d(\xi_i,\mathsf{A})\geqslant\delta} |f_n(\xi_1,\ldots,\xi_n)| \leqslant \frac{C}{\delta^r} \tag{6.1}$$

Then, there exists a constant C' so that, for any s satisfying $|s|_1 \in [r, r/\eta]$, we have:

$$\Big|\oint_{\mathsf{A}^n} \prod_{j=1}^{n} \frac{\mathrm{d}\xi_j}{2\mathrm{i}\pi}\, e^{\mathrm{i}s_j\xi_j}\, f_n(\xi_1,\ldots,\xi_n)\Big| \leqslant C'\,|s|_1^r \tag{6.2}$$

Proof. For δ small enough but larger than η, let $\mathcal{C}(\delta)$ be the contour surrounding A such that $d(\xi,\mathsf{A}) = \delta$ for any $\xi \in \mathcal{C}(\delta)$. When A has $(g+1)$ connected components, its length is $2(\ell(\mathsf{A}) + (g+1)\pi\delta)$. For any $s \in \mathbb{R}$, we find:

$$\begin{aligned}\Big|\oint_{\mathsf{A}^n} \prod_{j=1}^{n} \frac{\mathrm{d}\xi_j}{2\mathrm{i}\pi}\, e^{-\mathrm{i}s_j\xi_j}\, f_n(\xi_1,\ldots,\xi_n)\Big| & = \Big|\oint_{\mathcal{C}^n(\delta)} \frac{\mathrm{d}\xi_j}{2\mathrm{i}\pi}\, e^{-\mathrm{i}s_j\xi_j}\, f_n(\xi_1,\ldots,\xi_n)\Big| \\ & \leqslant C\Big(\frac{\ell(\mathsf{A})}{\pi} + (g+1)\delta\Big)^n\, e^{|s|_1\delta - r\ln\delta}\end{aligned} \tag{6.3}$$

We now optimize this inequality keeping $|s|_1$ large in mind, by choosing $\delta = r/|s|_1$, which leads to the desired result. ☐

Corollary 6.2 *Let $\varphi : \mathbb{R}^n \to \mathbb{C}$ be a continuous function with compact support, so that its Fourier transform satisfies:*

$$\widehat{\varphi}(s) \in o(|s|^{-r}), \qquad |s| \to \infty \tag{6.4}$$

Then, for any integer k_0 such that $r \geqslant 1 + \kappa + 2\theta + \frac{2k_0}{n}(\kappa+\theta)$, we have an expansion of the form:

$$\mu_{N,\beta}^{V;\mathsf{A}}\Big[\prod_{j=1}^{n}\Big(\sum_{i_j=1}^{N}\varphi(\lambda_{i_j})\Big)\Big]_c = \sum_{k=n-2}^{k_0} N^{-k}\,\mathcal{M}_n^{\{k\}}[\varphi] + o\big(N^{-(k_0+1/2)}(\ln N)^{n+k_0+1/2}\big) \tag{6.5}$$

Proof. Let $\eta > 0$, and define a function φ_η by its Fourier transform $\widehat{\varphi}_\eta(s) = e^{-\eta|s|}\,\widehat{\varphi}(s)$. It is analytic in the strip $\{\xi \in \mathbb{C}, \quad |\mathrm{Im}\,\xi| < \eta\}$, and we may write:

$$\begin{aligned}\mu_{N,\beta}^{V;\mathsf{A}}\Big[\prod_{j=1}^{n}\Big(\sum_{i_j=1}^{N}\varphi_\eta(\lambda_{i_j})\Big)\Big]_c & = \oint_{\mathsf{A}^n}\Big(\prod_{j=1}^{n}\frac{\mathrm{d}\xi_j}{2\mathrm{i}\pi}\,\varphi_\eta(\xi_j)\Big)W_n(\xi_1,\ldots,\xi_n) \\ & = \int_{\mathbb{R}^n}\Big(\prod_{j=1}^{n}\mathrm{d}s_j\, e^{-\eta|s_j|}\,\widehat{\varphi}(s_j)\Big)\oint_{\mathsf{A}^n}\Big(\prod_{j=1}^{n}\frac{\mathrm{d}\xi_j}{2\mathrm{i}\pi}\, e^{-\mathrm{i}s_j\xi_j}\Big)W_n(\xi_1,\ldots,\xi_n)\end{aligned} \tag{6.6}$$

We may insert the large N expansion of the correlators established in Proposition 5.5:

$$W_n(\xi_1,\ldots,\xi_n) = \sum_{k=n-2}^{k_0} N^{-k}\, W_n^{\{k\}}(\xi_1,\ldots,\xi_n) + N^{-k_0}\,\Delta_{k_0}W_n(\xi_1,\ldots,\xi_n) \tag{6.7}$$

where:

$$\|\Delta_{k_0} W_n\|_\delta \in O\Big(\frac{(\ln N)^{n+k_0+1/2}}{\sqrt{N}}\,\Big(\frac{D_c(\delta)}{D_L(\delta)}\Big)^{n+2k_0+1}\,\delta^{-n\theta-(n+2k_0)(\kappa+\theta)}\Big)\Big) \tag{6.8}$$

We may pass to the limit $\eta \to 0$ in (6.6) when the integrand in the right-hand side is integrable near $|s|_1 = \infty$. It constrains the allowed behavior for $\widehat{\varphi}(s)$ at $|s| \to \infty$. The worse behavior at $|s|_1 = \infty$ comes from the error term $\Delta_{k_0} W_n$. Lemma 6.1 implies that, for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that:

$$\Big|\oint_{\mathsf{A}^n}\Big(\prod_{j=1}^{n}\frac{\mathrm{d}\xi_j}{2\mathrm{i}\pi}\,e^{-\mathrm{i}s_j\xi_j}\Big)W_n(\xi_1,\ldots,\xi_n)\Big| \leqslant C_\varepsilon\,\frac{(\ln N)^{n+k_0+1/2}}{\sqrt{N}}\,\Big(\frac{D_c(|s|_1^{-1})}{D_L(|s|_1^{-1})}\Big)^{n+2k_0+1}\,|s|_1^{n\theta+(n+2k_0)(\kappa+\theta)+\varepsilon} \tag{6.9}$$

Assume now that $\widehat{\varphi}(s) \in o(|s|^{-r})$. Then, integrability at $|s|_1$ in (6.6) requires:

$$n\theta + (n+2k_0)(\kappa+\theta) - n(r+\varepsilon) < -n \tag{6.10}$$

In other words, performing an expansion up to $o(N^{-k_0})$ if the regularity exponent r satisfies:

$$r \geqslant 1 + \kappa + 2\theta + \frac{2k_0}{n}(\kappa+\theta). \tag{6.11}$$

□

6.2 Extension to harmonic potentials

The main use of the assumption that V is analytic came from the representation (1.6) of n-linear statistics described by a holomorphic function, in terms of contour integrals of the n-point correlator. If φ is holomorphic in a neighborhood of A, its complex conjugate $\overline{\varphi}$ is antiholomorphic, and we can also represent:

$$\mu_{N,\beta}^{V;\mathsf{A}}\Big[\sum_{i=1}^{N}\overline{\varphi(\lambda_i)}\Big] = \overline{\oint_{\mathsf{A}}\frac{\mathrm{d}x}{2\mathrm{i}\pi}\,\varphi(x)\,W_1(x)} \tag{6.12}$$

In this paragraph, we explain how to use a weaker set of assumptions than Hypothesis 1.3 , where "analyticity" and "$1/N$ expansion of the potential" are weakened as follows.

Hypothesis 6.1

- *(Harmonicity) $V : \mathsf{A} \to \mathbb{R}$ can be decomposed $V = \mathcal{V}_1 + \overline{\mathcal{V}_2}$, where $\mathcal{V}_1, \mathcal{V}_2$ extends to holomorphic functions in a neighborhood U of A.*
- *($1/N$ expansion of the potential) For $j = 1, 2$, there exists a sequence of holomorphic functions $(\mathcal{V}_j^{\{k\}})_{k\geqslant 0}$ and constants $(v_j^{\{k\}})_k$ so that, for any $K \geqslant 0$:*

$$\sup_{\xi\in\mathsf{U}}\Big|\mathcal{V}_j(\xi) - \sum_{k=0}^{K}N^{-k}\,\mathcal{V}_j^{\{k\}}(\xi)\Big| \leqslant v_j^{\{K\}}\,N^{-(K+1)} \tag{6.13}$$

In other words, we only assume $\mathcal{V}$ to be harmonic. "Analyticity" corresponds to the special case $\mathcal{V}_2 \equiv 0$. The main difference lies in the representation (6.12) of expectation values of antiholomorphic statistics, which come into play at various stages, but do not affect the reasoning. Below chronologically Section 5, we enumerate below the small changes to take into account.

In § 4, in the Schwinger-Dyson equations (Theorem 4.2 and 4.2), we encounter a term:

$$\mu_{N,\beta}^{V;\mathsf{A}}\Big[\sum_{i=1}^{N}\frac{L(\lambda_i)}{L(x)}\,\frac{V'(\lambda_i)}{x-\lambda_i}\,\prod_{j=2}^{n}\Big(\sum_{i_j=1}^{N}\frac{1}{x_j-\lambda_{i_j}}\Big)\Big]_c. \tag{6.14}$$

It is now equal to:

$$\frac{1}{L(x)}\oint_{\mathsf{A}}\frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\,L(\xi)\,\frac{\mathcal{V}_1'(\xi)}{x-\xi}\,W_n(\xi,x_I)-\frac{1}{L(x)}\overline{\oint_{\mathsf{A}}\frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\,L(\xi)\,\frac{\mathcal{V}_2'(\xi)}{\overline{x}-\xi}\,W_n(\xi,x_I)}. \tag{6.15}$$

Remark that (6.14) or (6.15) still defines a holomorphic function of x in $\mathbb{C}\backslash A$. The second line in Corollary 4.3 has to be modified similarly. In § 5.2, we can define the operator $\mathcal{K}$ by (5.16) with $\mathcal{Q}(x)$ now given by:

$$\begin{aligned}\mathcal{Q}f(x) \quad = \quad & -\oint_{\mathsf{A}}\frac{\mathrm{d}\xi}{2\mathrm{i}\pi}P^{\{-1\}}(\xi)(x;\xi)\,f(\xi)\\ & +\oint_{\mathsf{A}}\frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\,\frac{L(\xi)(\mathcal{V}_1^{\{0\}})'(\xi)-L(x)(\mathcal{V}_1^{\{0\}})'(x)}{\xi-x}\,f(\xi)\\ & \overline{+\oint_{\mathsf{A}}\frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\,\frac{L(\xi)(\mathcal{V}_2^{\{0\}})'(\xi)-L(x)(\mathcal{V}_2^{\{0\}})'(x)}{\xi-\overline{x}}\,f(\xi)}.\end{aligned} \tag{6.16}$$

It is still a holomorphic function of x in a neighborhood of A, thus it disappears in the computation leading to formula 5.22 for the inverse of $\mathcal{K}$, which still holds. In § 5.2.3, the expression (5.37) for the operator $\Delta\mathcal{K}$ used in (5.40) should be replaced by:

$$\begin{aligned}(\Delta\mathcal{K})f(x) \quad = \quad & 2\Delta_{-1}W_1(x)\,f(x)+\frac{1}{N}\Big(1-\frac{2}{\beta}\Big)\mathcal{L}_1 f(x)\\ & -\mathcal{N}_{(\Delta_0\mathcal{V}_1)',\Delta_{-1}P(x;\bullet)}[f](x)-\overline{\mathcal{N}_{(\Delta_0\mathcal{V}_2)',0}[f](\overline{x})},\end{aligned} \tag{6.17}$$

and the bound (5.38) still holds, where v_0 is replaced by $v_{1,0}+v_{2,0}$ introduced in (6.13). In § 5.3.1-5.4, all occurences of $\mathcal{N}_{V',0}[f](x)$ should be replaced by $\mathcal{N}_{(\mathcal{V}_1)',0}[f](x)+\overline{\mathcal{N}_{(\mathcal{V}_2)',0}[f](\overline{x})}$ (and similarly for $\mathcal{N}_{(\Delta_k V)',0}$ or $\mathcal{N}_{(V^{\{k\}})',0}$). The key remark is that the terms where $\overline{\mathcal{V}_2}$ appear involve complex conjugates of contour integrals of the type $g(\xi)\,W_n^{\{k\}}(\xi,x_I)$ or $g(\xi)\,\Delta_k W_n(\xi,x_I)$ where g is some holomorphic function in a neighborhood of A. Their norm can be controlled in terms of the norms of $W_n^{\{k\}}$ or $\Delta_k W_n$ on contours Γ, as were the terms involving $\mathcal{V}_1$, so the recursive control of errors in the $1/N$ expansion of correlators for the fixed filling fraction model is still valid, leading to the first part of Theorem 1.3, and to the central limit theorem (Proposition 5.6) for harmonic potentials in a neighborhood of A, which are still real-valued on A.

6.3 Complex perturbations of the potential

Proposition 6.3 *The central limit theorem (5.74) holds for $\varphi:\mathsf{A}\to\mathbb{C}$, which can be decomposed as $\varphi=\varphi+\overline{\varphi_2}$, where φ_1,φ_2 are holomorphic functions in a neighborhood of A.*

Proof. We present the proof for $\varphi=t\,f$, where $t\in\mathbb{C}$ and $f:\mathsf{A}\to\mathbb{R}$ extends to a holomorphic function in a neighborhood of A. Indeed, the case of $f:\mathsf{A}\to\mathbb{R}$ which can be decomposed as $f=f_1+\overline{f_2}$ with f_1,f_2 extending to holomorphic functions in a neighborhood of A, can be treated similarly with the modifications pointed out in § 6.2. Then, if $\varphi:\mathsf{A}\to\mathbb{C}$ can be decomposed as $\varphi=\varphi_1+\overline{\varphi_2}$ with φ_1,φ_2 holomorphic, we may decompose further $\varphi_j=\varphi_j^R+\mathrm{i}\varphi_j^I$, then write $\tilde{V}=V-\frac{2}{\beta N}(\varphi_1^R+\varphi_2^R)$ and $f=(\varphi_1^I-\varphi_2^I)$, and:

$$\mu_{N,\boldsymbol{\epsilon},\beta}^{V;\mathsf{A}}\Big[\exp\Big(\sum_{i=1}^N h(\lambda_i)\Big)\Big]=\mu_{N,\boldsymbol{\epsilon},\beta}^{V;\mathsf{A}}\Big[\exp\Big(\sum_{i=1}^N(\varphi_1^R+\varphi_2^R)\Big)\Big]\,\mu_{N,\boldsymbol{\epsilon},\beta}^{\tilde{V};\mathsf{A}}\Big[\exp\Big(\sum_{i=1}^N \mathrm{i}f(\lambda_i)\Big)\Big]. \tag{6.18}$$

The first factor can be treated with the initial central limit theorem (Proposition 5.6), while an equivalent of the second factor for large N will be deduced from the following proof applied to the potential $\tilde{V}$.

This proof is inspired from that of [Shc12, Lemma 1]. From Theorem 1.3 applied to V up to $o(1)$, we introduce $W_{n,\boldsymbol{\epsilon}}^{\{k\}}$ for $(n,k) = (1,-1),(2,0),(1,0)$ (see (5.51)-(5.49)). If $t \in \mathbb{R}$, the central limit theorem (Proposition 5.6) applied to $\varphi = t\,f$ implies:

$$\mu_{N,\bullet,\beta}^{V;\mathsf{A}}\Big[\Big(\sum_{i=1}^{N} t\,f(\lambda_i)\Big)\Big] = G_N(t)(1+R_N(t)), \qquad G_N(t) = \exp\Big(Nt\,L[f] + t\,M[f] + \frac{t^2}{2}\,Q[f,f]\Big), \tag{6.19}$$

where $\sup_{t\in[-T_0,T_0]} |R_N(t)| \leqslant C(T_0)\,\eta_N$ and $\lim_{N\to\infty}\eta_N = 0$. Let $T_0 > 0$, and introduce the function:

$$\tilde{R}_N(t) = \frac{1}{C(T_0)\eta_N}\,R_N(t). \tag{6.20}$$

For any fixed N, it is an entire function of t, and by construction

$$\sup_{t\in[-T_0,T_0]} |\tilde{R}_N(t)| \leqslant 1. \tag{6.21}$$

Besides, for any $t \in \mathbb{C}$, we have

$$\Big|\mu_{N,\bullet;\beta}^{V;\mathsf{A}}\Big[\exp\Big(\sum_{i=1}^{N} t\,f(\lambda_i)\Big)\Big]\Big| \leqslant \mu_{N,\bullet,\beta}^{V;\mathsf{A}}\Big[\exp\Big(\sum_{i=1}^{N}(\mathrm{Re}\,t)\,f(\lambda_i)\Big)\Big] \tag{6.22}$$

Therefore, we deduce that

$$\begin{aligned}\sup_{|t|\leqslant T_0} |\tilde{R}_N(t)| & \leqslant \frac{1}{C(T_0)\eta_N}\sup_{|t|\leqslant T_0}\frac{G_N(\mathrm{Re}\,t)}{|G_N(t)|} \\ & \leqslant \frac{1}{C(T_0)\eta_N}\sup_{|t|\leqslant T_0}\exp\Big(\frac{(\mathrm{Im}\,t)^2}{2}\,Q[f,f]\Big) \\ & \leqslant \frac{1}{C'(T_0)\eta_N}\end{aligned} \tag{6.23}$$

for some constant $C'(T_0)$. By the two-constants lemma [NN22], (6.21)-(6.23) imply

$$\forall T \in]0,T_0[, \qquad \sup_{|t|\leqslant T} |\tilde{R}_N(t)| \leqslant (C'(T_0)\eta_N)^{-2\phi(T,T_0)/\pi}, \qquad \phi(T,T_0) = \arctan\Big(\frac{2T/T_0}{1-(T/T_0)^2}\Big). \tag{6.24}$$

In particular, for any compact K of the complex plane, we can find an open disk of radius T_0 which contains K, and thus show (6.19) with $R_N(t) \in o(1)$ uniformly in K. □

We observe from the proof that Proposition 6.3 cannot be easily extended to $T_N\,|t| \in O(1)$ with $T_N \to +\infty$. Indeed, the ratio $G_N(T_N(\mathrm{Re}\,t))/|G_N(T_N t)|$ in (6.23) will not be bounded when $N \to \infty$, hence applying the two-constants lemma as above does not show $R_N(t) \to 0$.

Corollary 6.4 *In the model with fixed filling fractions $\boldsymbol{\epsilon}$, assume the potential V_0 satisfies Hypotheses 5.1. Then, if $\varphi : \mathsf{A} \to \mathbb{C}$ can be decomposed as $\varphi = \varphi_1 + \overline{\varphi_2}$ with φ_1, φ_2 extending to holomorphic functions in a neighborhood of A, then the model with fixed filling fractions $\boldsymbol{\epsilon}$ and potential $V = V_0 + \varphi/N$ satisfies Hypotheses 5.1. Therefore, the result of Proposition 5.5 also holds: the correlators have a $1/N$ expansion.*

Proof. Hypothesis 5.1 contrains only the leading order of the potential, i.e. it holds for $(V_0, \boldsymbol{\epsilon})$ iff it holds for $(V = V_0 + h/N, \boldsymbol{\epsilon})$. Proposition 6.3 implies a fortiori the existence of constants $C_+, C_-, C > 0$ such that:

$$C_-\,C^N \leqslant |Z_{N,\boldsymbol{\epsilon},\beta}^{V;\mathsf{A}}| \leqslant C_+\,C^N \tag{6.25}$$

Using this inequality as an input, we can repeat the proof given in Section 3 to check to obtain Corollary 3.7 (i.e. the a priori control reminded in (5.9)-(5.10)) for the potential V. Then, in the recursive analysis of the Schwinger-Dyson equation of Section 5 for the model with fixed filling fractions, the fact that the potential is complex-valued does not matter, so we have proved the $1/N$ expansion of the correlators. □

This proves Theorem 1.3 in full generality.

6.4 $1/N$ expansion of n-kernels

We can apply Corollary 6.4 to study potentials of the form:

$$V_{\mathbf{c},\mathbf{x}}(\xi) = V - \frac{2}{\beta N}\sum_{j} c_j \ln(x_j - \xi) \tag{6.26}$$

where $x_j \in \mathbb{C}\backslash\mathsf{A}$, and thus derive the asymptotic expansion of the kernels in the complex plane, i.e. Corollary 1.7 and 1.8. Indeed, let us introduce the random variable $H_{\mathbf{c}}(\mathbf{x}) = \sum_{j=1}^{n} c_j \sum_{i=1}^{N} \ln(x_j - \lambda_i)$. We now know from Proposition 6.3 that $\ln \mu_{N,\boldsymbol{\epsilon},\beta}^{V;\mathsf{A}}\big[e^{tH_{\mathbf{c}}(\mathbf{x})}\big]$ is an entire function. Therefore, its Taylor series is convergent for any $t \in \mathbb{C}$, and we have:

$$\begin{aligned} K_{n,\mathbf{c}}(\mathbf{x}) &= \exp\Big(\ln \mu_{N,\boldsymbol{\epsilon},\beta}^{V;\mathsf{A}}\big[e^{tH_{\mathbf{c}}(\mathbf{x})}\big]\Big) \\ &= \exp\Big(\sum_{r\geqslant 1}\frac{1}{r!}\oint_{\mathsf{A}^r}\prod_{i=1}^{n}\frac{\mathrm{d}\xi_i}{2\mathrm{i}\pi}\Big(\sum_{j=1}^{n} c_j \ln(x_j-\xi_i)\Big)W_r(\xi_1,\ldots,\xi_r)\Big) \end{aligned} \tag{6.27}$$

which can also be rewritten:

$$K_{n,\mathbf{c}}(\mathbf{x}) = \exp\Big(\sum_{r\geqslant 1}\frac{1}{r!}\mathcal{L}_{\mathbf{c},\mathbf{x}}^{\otimes r}[W_r]\Big) \tag{6.28}$$

where we introduced:

$$\mathcal{L}_{\mathbf{c},\mathbf{x}} f(x) = \sum_{j=1}^{n} c_j \int_{\infty}^{x_j} \tag{6.29}$$

As a consequence of Proposition 5.5, $W_n \in O(N^{2-n})$ and has a $1/N$ expansion. Therefore, only a finite number of terms contribute to each order in the n-kernels, and we find:

Proposition 6.5 *Assume Hypothesis 1.3. Then, for any $K \geqslant -1$, we have the asymptotic expansion:*

$$K_{n,\mathbf{c}}(\mathbf{x}) = \exp\Big\{\sum_{k=-1}^{K} N^{-k}\Big(\sum_{r=1}^{k+2}\frac{1}{r!}\mathcal{L}_{\boldsymbol{x},\mathbf{c}}[W_r^{\{k\}}]\Big)\Big\}\,, \tag{6.30}$$

where $\delta = \inf_j d(x_j,\mathsf{A})$ is assumed larger than δ_N introduced in Definition 5.4. For a fixed K, it is uniform for $\boldsymbol{x}$ in any compact of $(\mathbb{C}\backslash\mathsf{A})^n$. □

If $\varphi : \mathsf{A} \to \mathbb{C}$ is a function such that $\widehat{\varphi}(s) \in o(|s|^{-r})$ when $|s| \to \infty$, one could study by similar methods the asymptotic expansion of the exponential statistics $\mu_{N,\beta}^{V;\mathsf{A}}[e^{H[\varphi]}]$ where $H[\varphi] = \sum_{i=1}^{N}\varphi(\lambda_i)$, that we would establish thanks to Corollary 6.2 up to $o(N^{-K(r)})$, where

$$K(r) = \Big\lfloor \frac{r-(1+\kappa+2\theta)}{2(\kappa+\theta)} \Big\rfloor \tag{6.31}$$

Note that $K(r) \geqslant 0$ implies $r \geqslant 1+\kappa+2\theta = 7/2$. In particular, we can deduce:

Proposition 6.6 *The central limit theorem 5.6 holds for test functions φ such that $|\widehat{\varphi}(s)| \in o(|s|^{-(1+\kappa+2\theta)})$ when $|s| \to \infty$.*

7 Fixed filling fractions: $1/N$ expansion of the partition function

In this Section, we restrict ourselves to the fixed filling fraction model, i.e. we study $\mu_{N,\boldsymbol{\epsilon},\beta}^{V;\mathsf{A}}$ for some $\boldsymbol{\epsilon} \in \mathcal{E}_g$. Following § 6.2, it is again not difficult to consider potentials of the form $V = \mathcal{V}_1 + \overline{\mathcal{V}_2}$, with $\mathcal{V}_1, \mathcal{V}_2$ is holomorphic, so we will write down proofs only for holomorphic V.

7.1 Interpolation principle

Recall that, if $(V_t)_t$ is a smooth family of potentials so that $\partial_t V_t$ is holomorphic in a neighborhood of A, we have:

$$\partial_t \ln Z_{N,\boldsymbol{\epsilon},\beta}^{V_t;\mathsf{A}} = -\frac{\beta N}{2} \oint_{\mathsf{A}} \frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\, \partial_t V_t(\xi)\, W_1^{V_t}(\xi). \tag{7.1}$$

We are going to interpolate in two steps between the initial potential V, and a potential for which the partition function can be computed exactly by means of a Selberg β integral.

7.1.1 Reference potentials

We first describe a set of reference potentials. Let $\gamma = [\gamma^-, \gamma^+]$ be a segment not reduced to a point, and $\rho^{\pm}$ two elements of $\{\pm 1\}$. We introduce a probability measure supported on γ:

$$\mathrm{d}\sigma_{\gamma,\rho}(x) = \frac{c_{\gamma,\rho}}{\pi} \sqrt{(x - \gamma^-)^{\rho^-} (\gamma^+ - x)^{\rho^+}}\, \mathrm{d}x, \tag{7.2}$$

where the constant $c_{\Gamma,\rho}$ ensures that the total mass is 1. It is well-known that $\sigma_{\gamma,\rho} = \mu_{\mathrm{eq}}^{V_{\gamma,\rho};\tilde{\gamma}}$ for the following data:

- if $(\rho^-, \rho^+) = (1, 1)$, $\sigma_{\gamma,\rho}$ is a semi-circle law, and it is the equilibrium measure for the Gaussian potential
$$V_{\gamma,\rho} = \frac{8}{(\gamma^+ - \gamma^-)^2} \Big(x - \frac{\gamma^- + \gamma^+}{2}\Big)^2, \qquad c_{\gamma,\rho} = \frac{8}{(\gamma^+ - \gamma^-)^2} \tag{7.3}$$
on $\tilde{\gamma}$, any interval of $\gamma_m = \mathbb{R}$ which is a neighborhood of γ.

- if $(\rho^-, \rho^+) = (-1, 1)$, $\sigma_{\gamma,\rho}$ is a Marčenko-Pastur law, and it is the equilibrium measure for a linear potential:
$$V_{\gamma,\rho} = \frac{4(x - \gamma^-)}{\gamma^+ - \gamma^-}, \qquad c_{\gamma,\rho} = \frac{2}{\gamma^+ - \gamma^-} \tag{7.4}$$
on $\tilde{\gamma}$, any interval of $\gamma_m = [\gamma^-, +\infty[$ which is a neighborhood of γ.

- if $(\rho^-, \rho^+) = (1, -1)$, we have similarly a linear potential:
$$V_{\gamma,\rho} = \frac{4(\gamma^+ - x)}{\gamma^+ - \gamma^-}, \qquad c_{\gamma,\rho} = \frac{2}{\gamma^+ - \gamma^-} \tag{7.5}$$
on $\tilde{\gamma}$, any intevral of $\gamma_m =]-\infty, \gamma^+]$ which is a neighborhood of γ.

- if $(\rho^-, \rho^+) = (-1, -1)$, $\sigma_{\gamma,\rho}$ is an arcsine law, and it is the equilibrium measure for a constant potential:
$$V_{\gamma,\rho} = 0, \qquad c_{\gamma,\rho} = 1 \tag{7.6}$$
on $\tilde{\gamma} = \gamma = \gamma_m$.

When we choose $\tilde{\gamma} = \gamma_m$, the partition function $Z_{N,\beta}^{V_{\gamma,\rho};\gamma_m}$ of the initial model with such potentials are special cases of Selberg β integrals, and therefore can be computed exactly. We have in general:

$$Z_{N,\beta}^{V_{\gamma,\rho};\gamma_m} = \exp\Big(- \big[(\beta/2)N^2 + (1-\beta/2)N\big] \ln\Big(\frac{\gamma^+ - \gamma^-}{4}\Big)\Big) \mathcal{Z}_{N,\beta}^{\rho}, \tag{7.7}$$

where $\mathcal{Z}_{N,\beta}^{\rho}$ do not depend on γ and their expression is given in Appendix B.2.

7.1.2 Step 1: interpolation with a reference potential

Given $\mathsf{A} = \bigcup_{h=0}^{g} \mathsf{A}_h = \bigcup_{h=0}^{g} [a_h^-, a_h^+]$, we consider the support of the equilibrium measure $\mu_{\mathrm{eq},\boldsymbol{\epsilon}}^{V;A}$, which is of the form $\mathsf{S}_{\boldsymbol{\epsilon}} = \bigcup_{h=0}^{g} \mathsf{S}_{h,\boldsymbol{\epsilon}} = \bigcup_{h=0}^{g} [\alpha_{h,\boldsymbol{\epsilon}}^-, \alpha_{h,\boldsymbol{\epsilon}}^+]$, with signs $\rho_h = (\rho_h^-, \rho_h^+)$ indicating the soft of hard nature of the edges. Let $(U_h)_{0 \leqslant h \leqslant g}$ be a family of pairwise distinct neighborhoods of A_h. We denote:

$$V_{\mathrm{ref},\boldsymbol{\epsilon}}(x) = \sum_{h=0}^{g} \mathbf{1}_{U_h}(x) \Big(\epsilon_h \, V_{\mathsf{S}_{h,\boldsymbol{\epsilon}},\rho_h}(x) + \sum_{h' \neq h} 2\epsilon_{h'} \int_{\mathsf{S}_{h',\boldsymbol{\epsilon}}} \ln|x-\xi| \mathrm{d}\sigma_{\mathsf{S}_{h',\boldsymbol{\epsilon}},\rho_{h'}}(\xi)\Big). \tag{7.8}$$

By construction, $V_{\mathrm{ref},\boldsymbol{\epsilon}}$ is holomorphic in the neighborhood $\mathsf{U} = \bigcup_{h=0}^{g} \mathsf{U}_h$ of A, and:

$$\sigma_{\mathrm{ref},\boldsymbol{\epsilon}}(x) = \sum_{h=0}^{g} \epsilon_h \, \sigma_{\mathsf{S}_{h,\boldsymbol{\epsilon}},\rho_h} \tag{7.9}$$

is the equilibrium measure for the potential $V_{\mathrm{ref},\boldsymbol{\epsilon}}$ on A. Indeed, it satisfies the characterization (1.20). Notice that σ_{ref} has same support as $\mu_{\mathrm{eq},\boldsymbol{\epsilon}}$, edges of the same nature, and same filling fractions. Besides, V_{ref} satisfy the assumptions of 1.2. Then, if we consider the convex combination of potentials $V_s = (1-s)V + sV_{\mathrm{ref}}$, it follows from the characterization (1.20) that the equilibrium measure associated to V_s on A with filling fraction $\boldsymbol{\epsilon}$ is precisely:

$$\mu_{\mathrm{eq},\boldsymbol{\epsilon}}^{V_s;\mathsf{A}} = (1-s)\mu_{\mathrm{eq},\boldsymbol{\epsilon}}^{V;\mathsf{A}} + s\,\sigma_{\mathrm{ref},\boldsymbol{\epsilon}}. \tag{7.10}$$

Besides, since both $\mu_{\mathrm{eq},\boldsymbol{\epsilon}}^{V;\mathsf{A}}$ and $\sigma_{\mathrm{ref},\boldsymbol{\epsilon}}$ satisfy (5.2) with edges of the same nature, we conclude that if V satisfies 1.2, so does the family $(V_s)_{s\in[0,1]}$. Therefore, we can use Theorem 5.5 to deduce from (7.1) the asymptotic expansion:

$$\frac{Z_{N,\boldsymbol{\epsilon},\beta}^{V;\mathsf{A}}}{Z_{N,\boldsymbol{\epsilon},\beta}^{V_{\mathrm{ref}};\mathsf{A}}} = \exp\left\{ \frac{\beta}{2} \sum_{k \geqslant -2} N^{-k} \int_0^1 \mathrm{d}s \oint_{\mathsf{S}} \frac{\mathrm{d}\xi}{2\mathrm{i}\pi} (V_{\mathrm{ref}}(\xi) - V(\xi))\, W_{1,\boldsymbol{\epsilon}}^{\{k+1\};V_s}(\xi) \right\}. \tag{7.11}$$

7.1.3 Step 2: localizing the supports

We now have to analyze the partition function for the reference potential V_{ref} defined by (7.8). When $g = 0$ (the one-cut regime), V_{ref} coincides with one of the reference potentials, and we know that up to exponentially small correction, $Z_{N,\beta}^{V_{\mathrm{ref}};A}$ will be given by the Selberg integrals described case by case in (B.6)-(B.8), so there is nothing more to do.

Assume now $g \geqslant 1$. Let us define shortening flows on the support:

- if $(\rho_h^-, \rho_h^+) = (1,1)$ or $(-1,-1)$, we set $\mathsf{S}_{h,\boldsymbol{\epsilon}}^t = \Big[\frac{\alpha_{h,\boldsymbol{\epsilon}}^- + \alpha_{h,\boldsymbol{\epsilon}}^+}{2} - \frac{\alpha_{h,\boldsymbol{\epsilon}}^+ - \alpha_{h,\boldsymbol{\epsilon}}^-}{2} t, \frac{\alpha_{h,\boldsymbol{\epsilon}}^- + \alpha_{h,\boldsymbol{\epsilon}}^+}{2} + \frac{\alpha_{h,\boldsymbol{\epsilon}}^- + \alpha_{h,\boldsymbol{\epsilon}}^+}{2} t\Big]$,
- if $(\rho_h^-, \rho_h^+) = (-1,1)$, we set $\mathsf{S}_{h,\boldsymbol{\epsilon}}^t = \big[\alpha_{h,\boldsymbol{\epsilon}}^-, \alpha_{h,\boldsymbol{\epsilon}}^+ + t(\alpha_{h,\boldsymbol{\epsilon}}^+ - \alpha_{h,\boldsymbol{\epsilon}}^-)\big]$,
- if $(\rho_h^-, \rho_h^+) = (1,-1)$, we set $\mathsf{S}_{h,\boldsymbol{\epsilon}}^t = \big[\alpha_{h,\boldsymbol{\epsilon}}^+ - t(\alpha_{h,\boldsymbol{\epsilon}}^+ - \alpha_{h,\boldsymbol{\epsilon}}^-), \alpha_{h,\boldsymbol{\epsilon}}^+\big]$.

We consider the family of potentials on A:

$$V^t_{\mathrm{ref},\boldsymbol{\epsilon}}(x) = \sum_{h=0}^{g} \mathbf{1}_{\mathsf{U}_h}(x)\Big(\epsilon_h\, V_{\mathsf{S}^t_{\boldsymbol{\epsilon},h},\rho_h}(x) - \sum_{h'\neq h} 2\epsilon_{h'}\int_{\mathsf{S}^t_{h',\boldsymbol{\epsilon}}} \ln|x-\xi|\mathrm{d}\sigma_{\mathsf{S}^t_{\boldsymbol{\epsilon},h'},\rho_{h'}}(\xi)\Big), \tag{7.12}$$

for which the equilibrium measure is obviously $\sum_{h=0}^{g}\epsilon_h\,\sigma_{\mathsf{S}^t_{h,\boldsymbol{\epsilon}},\rho_h}$ and has support $\mathsf{S}^t_{\boldsymbol{\epsilon}} = \bigcup_{h=0}^{g}\mathsf{S}^t_{h,\boldsymbol{\epsilon}}$. Accordingly, $V^t_{\mathrm{ref},\boldsymbol{\epsilon}}$ satisfies Hypothesis 5.1, uniformly for t in any compact of $]0,1]$. Besides, the partition function $Z^{V^t_{\mathrm{ref},\boldsymbol{\epsilon}};\mathsf{A}}_{N,\boldsymbol{\epsilon},\beta}$ can be computed exactly in the limit $t\to 0$. If we introduce:

$$\alpha^0_{\boldsymbol{\epsilon},h} = \begin{cases} (\alpha^-_{\boldsymbol{\epsilon},h}+\alpha^+_{\boldsymbol{\epsilon},h})/2 & \text{if}\quad (\rho_h^-,\rho_h^+) = (1,1)\text{ or }(-1,-1) \\ \alpha^-_{\boldsymbol{\epsilon},h} & \text{if}\quad (\rho_h^-,\rho_h^+) = (-1,1) \\ \alpha^+_{\boldsymbol{\epsilon},h} & \text{if}\quad (\rho_h^-,\rho_h^+) = (1,-1)\end{cases}, \tag{7.13}$$

we find that:

$$\lim_{t\to 0}\frac{Z^{V^t_{\mathrm{ref},\boldsymbol{\epsilon}};A}_{N,\boldsymbol{\epsilon},\beta}}{\prod_{h=0}^{g} Z^{V_{\mathsf{S}^t_h,\rho_h};\mathsf{T}^t_h}_{N\epsilon_h,\beta}} = \prod_{0\leqslant h<h'\leqslant g}\big|\alpha^0_{\boldsymbol{\epsilon},h}-\alpha^0_{\boldsymbol{\epsilon},h'}\big|^{N^2\epsilon_h\epsilon_{h'}}, \tag{7.14}$$

where T^t_h is the maximum allowed interval associated to S^t_h which depends on the nature of the edges (i.e. on ρ_h) as described in § 7.1.1. We remark that the dependence in t factors and:

$$Z^{V_{\mathsf{S}^t_h,\rho_h};\mathsf{T}^t_h}_{N\epsilon_h,\beta} = \exp\Big\{-\big[(\beta/2)N^2\epsilon_h^2 + (1-\beta/2)N\epsilon_h\big]\ln\Big(\frac{\alpha^+_{\boldsymbol{\epsilon},h}-\alpha^-_{\boldsymbol{\epsilon},h}}{4t}\Big)\Big\}\,\mathscr{Z}^{\rho_h}_{N\epsilon_h,\beta}, \tag{7.15}$$

where $\mathscr{Z}^{\rho_h}_{N\epsilon_h}$ is an analytic function of $N\epsilon_h$. The asymptotic expansion when $N\to\infty$ of those factors associated to reference potentials is described in Appendix B.2. We just mention that it is of the form:

$$\mathscr{Z}^{\gamma,\rho}_{N,\beta} = N^{(\beta/2)N+\gamma'_\rho}\exp\Big(\sum_{k\geqslant -2} N^2\,\mathcal{F}^{\rho}_{\beta}\Big), \tag{7.16}$$

for $\gamma = [\gamma^-,\gamma^+]$. Therefore, we obtain:

$$\begin{aligned} Z^{V_{\mathrm{ref}};\mathsf{A}}_{N,\boldsymbol{\epsilon},\beta} \;=\; & \prod_{0\leqslant h<h'\leqslant g}|\alpha^0_{\boldsymbol{\epsilon},h}-\alpha^0_{\boldsymbol{\epsilon},h'}|^{\beta}\prod_{h=0}^{g}\mathscr{Z}^{\rho_h}_{N\epsilon_h,\beta}\exp\Big\{-\Big[(\beta/2)N^2\Big(\sum_{h=0}^{g}\epsilon_h^2\Big)+(1-\beta/2)N\Big]\ln\Big(\frac{\alpha^+_{\boldsymbol{\epsilon},h}-\alpha^-_{\boldsymbol{\epsilon},h}}{4}\Big)\Big\} \\ & \times\exp\Big\{\sum_{k\geqslant -2} N^{-k}\int_0^1\mathrm{d}s\,\Big[(\beta/2)\Big(\sum_{h=0}^{g}\epsilon_h^2\Big)\frac{\delta_{k,-2}}{s} + (1-\beta/2)\frac{\delta_{k,-1}}{s} \\ & \qquad + \oint_{\mathsf{S}_{\boldsymbol{\epsilon}}}\frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\,(\partial_s V^s_{\mathrm{ref},\boldsymbol{\epsilon}})(\xi)\,W^{\{k+1\};V^s_{\mathrm{ref},\boldsymbol{\epsilon}}}_{1,\boldsymbol{\epsilon}}(\xi)\Big]\Big\}. \end{aligned} \tag{7.17}$$

By construction, the integrand in the right-hand side is finite when $s\to 0$. The expression does not make it obvious for the terms $k=-2$ and $k=-1$, but it can be checked explicitly since the eigenvalues in different $\mathsf{S}^s_{h,\boldsymbol{\epsilon}}$ decouple in the limit $s\to 0$, in the sense that:

$$W_1^{V^s_{\mathrm{ref},\boldsymbol{\epsilon}}}(x) \underset{s\to 0}{=} \sum_{h=0}^{g} W_1^{V_{\mathsf{S}^s_h,\rho_h}}(x) + o(1), \tag{7.18}$$

and the expressions for the non decaying contributions to $W_1^{V^{\gamma,\rho}_{\mathrm{ref}}}$ when N is large are given in Appendix B.1.

7.2 Expansion of the partition function

We establish in Lemma B.1 that the partition functions for the reference potentials $\mathcal{Z}^{\rho}_{N,\beta}$ do have an asymptotic expansion of the form:

$$\mathcal{Z}^{\rho}_{N,\beta} = N^{(\beta/2)N+e_{\rho}} \exp\Big(\sum_{k\geqslant -2} N^{-k}\,\mathcal{F}^{\rho}_{\beta} + O(N^{-\infty})\Big), \tag{7.19}$$

where:

$$\gamma'_{++} = \frac{3+\beta/2+2/\beta}{12}, \qquad e_{+-} = e_{-+} = \frac{\beta/2+2/\beta}{6}, \qquad e_{--} = \frac{-1+\beta/2+2/\beta}{4}. \tag{7.20}$$

Therefore, we have proved a part of Theorem 1.3:

Proposition 7.1 *If $(V,\boldsymbol{\epsilon})$ satisfy Hypothesis 1.1 and 1.3 on A (instead of B), we have:*

$$Z^{V;\mathsf{A}}_{N,\boldsymbol{\epsilon},\beta} = N^{(\beta/2)N+e} \exp\Big(\sum_{k\geqslant -2} N^{-k}\,F^{\{k\}}_{\boldsymbol{\epsilon},\beta} + O(N^{-\infty})\Big). \tag{7.21}$$

with a universal constant $e = \sum_{h=0}^{g} e_{\rho_h}$ depending on β and the nature of the edges.

Let $\boldsymbol{\epsilon}_{\star}$ the equilibrium filling fractions in the initial model $\mu^{V;\mathsf{A}}_{N,\beta}$. In order to finish the proof of Theorem 1.3, it remains to show that the stronger Hypotheses 1.2-1.3 for $\mu^{V;\mathsf{A}}_{N,\beta}$ imply Hypothesis 5.1 for the model $\mu^{V;\mathsf{A}}_{N,N\boldsymbol{\epsilon},\beta}$ for the model with fixed filling fractions $\boldsymbol{\epsilon} \in \mathcal{E}_g$ close enough to $\boldsymbol{\epsilon}_{\star}$, that all coefficients of the expansion are smooth functions of $\boldsymbol{\epsilon}$, and that the Hessian of $F^{\{-2\}}_{\boldsymbol{\epsilon}}$ with respect to filling fractions is negative definite. This last part is justified in Proposition A.3 proved in Appendix A.1, whereas to prove the first part, we rely on the basic result also proved in Lemma A.1 and Corollary A.2 in Appendix A.1:

Lemma 7.2 *If V satisfies Hypotheses 1.2-1.4, then $(V,\boldsymbol{\epsilon})$ satisfies Hypotheses 5.1 for $\boldsymbol{\epsilon} \in \mathcal{E}_g$ close enough to $\boldsymbol{\epsilon}_{\star}$. Besides, the soft edges $\alpha^{\bullet}_h$ and $W^{\{-1\}}_{1,\boldsymbol{\epsilon}}(x)$ are $\mathcal{C}^{\infty}$ functions of $\boldsymbol{\epsilon}$, while the hard edges remain unchanged, at least for $\boldsymbol{\epsilon}$ close enough to $\boldsymbol{\epsilon}_{\star}$.*

We observe that, once $W^{\{-1\}}_{1,\boldsymbol{\epsilon}}$ and the edges of the support $\alpha^{\bullet}_{\boldsymbol{\epsilon},h}$ are known, the $W^{\{k\}}_{n,\boldsymbol{\epsilon}}$ for any $n \geqslant 1$ and $k \geqslant 0$ are determined recursively by (5.51)-(5.49) and (5.70)-(5.73), where the linear operator $\mathcal{K}^{-1}$ is given explicitly in (5.21)-(5.27), and thus depend also analytically on $\boldsymbol{\epsilon}$ close enough to $\boldsymbol{\epsilon}_{\star}$. Similarly, $F^{\{k\}}_{\boldsymbol{\epsilon}}$ for $k \geqslant 0$ are obtained from (7.11)-(7.17), which shows their analyticity for $\boldsymbol{\epsilon}$ close enough to $\boldsymbol{\epsilon}_{\star}$.

Corollary 7.3 *If V satisfies Hypotheses 1.2-1.4, then $W^{\{k\}}_{n,\boldsymbol{\epsilon}}$ and $F^{\{k\}}_{\boldsymbol{\epsilon}}$ are $\mathcal{C}^{\infty}$ functions of $\boldsymbol{\epsilon} \in \mathcal{E}_g$ close enough to $\boldsymbol{\epsilon}_{\star}$.* □

This concludes the proofs of Theorem 1.3 and Corollary 1.7 announced in Section 1.4.

8 Asymptotic expansion in the initial model in the multi-cut regime

8.1 The partition function

We come back to the initial model $\mu^{V;\mathsf{A}}_{N,\beta}$, and we assume Hypotheses 1.2-1.4 with number of cuts $(g+1) \geqslant 2$. We remind the notation $\boldsymbol{N} = (N_h)_{1\leqslant h\leqslant g}$ for the number of eigenvalues in A_h, and the number of eigenvalue in A_0 is $N_0 = N - \sum_{h=1}^{g} N_h$. The N_h are here random variables, which take the value $N\boldsymbol{\epsilon}$ with probability $Z^{V;\mathsf{A}}_{N,\boldsymbol{\epsilon},\beta}/Z^{V;\mathsf{A}}_{N,\beta}$. We denote $\boldsymbol{\epsilon}_{\star}$ the vector of equilibrium filling fractions, and $\boldsymbol{N}_{\star} = N\boldsymbol{\epsilon}_{\star}$. Let us summarize four essential points:

- We have established in Theorem 1.4 an expansion for the partition function with fixed filling fractions:
$$\frac{N!}{\prod_{h=0}^{g}(N\epsilon_h)!}\,Z^{V;\mathsf{A}}_{N,\boldsymbol{\epsilon},\beta} = N^{(\beta/2)N+e}\exp\Big(\sum_{k\geqslant -2} N^{-k}\,F^{\{k\}}_{\boldsymbol{\epsilon},\beta}\Big), \tag{8.1}$$
where e are independent of the filling fractions.

- By concentration of measures, we have established in Corollary 3.8 the existence of constants $C, C' > 0$ such that, for N large enough,
$$\mu^{V;\mathsf{A}}_{N,\beta}\big[|\boldsymbol{N}-\boldsymbol{N}_{\star}| > \ln N\big] \leqslant e^{CN\ln N - C'\,N\ln^2 N}. \tag{8.2}$$

- Thanks to the strong offcriticality assumption, we have after Lemma 7.2 that $F^{\{k\}}_{\boldsymbol{\epsilon},\beta}$ is smooth when $\boldsymbol{\epsilon}$ is in the vicinity of $\boldsymbol{\epsilon}_{\star}$. From there we deduce that, for any $K, k \geqslant -2$, there exists a constant $C_{k,K} > 0$ such that:
$$\Big|N^{-k}\,F^{\{k\}}_{\boldsymbol{N}/N,\beta} - \sum_{j=0}^{K-k} N^{-(k+j)}\,\frac{(F^{\{k\}}_{\star,\beta})^{(j)}}{j!}\cdot(\boldsymbol{N}-\boldsymbol{N}_{\star})^{\otimes j}\Big| \leqslant C_{k,K}\,N^{-(K+1)}|\boldsymbol{N}-\boldsymbol{N}_{\star}|^{K-k+1}. \tag{8.3}$$

- We establish in Proposition A.3 in Appendix A.1 that the Hessian $(F^{\{-2\}}_{\star,\beta})''$ is negative definite.

We now proceed with the proof of Theorem 1.5.

8.1.1 Taylor expansion around the equilibrium filling fraction

By the estimate (8.2), we can write:
$$\frac{Z^{V;\mathsf{A}}_{N,\beta}}{Z^{V;\mathsf{A}}_{N,\epsilon_{\star},\beta}} = \sum_{\substack{0\leqslant N_1,\cdots,N_g\leqslant N\\ |\boldsymbol{N}|\leqslant N}} \frac{N!}{\prod_{h=0}^{g} N_h!}\,\frac{Z^{V;\mathsf{A}}_{N,\boldsymbol{N}/N,\beta}}{Z^{V;\mathsf{A}}_{N,\epsilon_{\star},\beta}} = \Big(\sum_{\substack{0\leqslant N_1,\cdots,N_g\leqslant N\\ |\boldsymbol{N}-\boldsymbol{N}_{\star}|\leqslant \ln N}} \frac{N!}{\prod_{h=0}^{g} N_h!}\,Z^{V;\mathsf{A}}_{N,\boldsymbol{N}/N,\beta}\Big)(1+r_N), \tag{8.4}$$
with:
$$r_N \leqslant (g+1)^N\,e^{-\frac{C'}{2}\,N\ln^2 N}. \tag{8.5}$$
And, we have, for any $K \geqslant -2$:
$$\begin{aligned} & \sum_{\substack{0\leqslant N_1,\ldots,N_r\leqslant N\\ |\boldsymbol{N}_{\bullet}-\boldsymbol{N}_{\star}|\leqslant \ln N}} \frac{N!}{\prod_{h=0}^{g} N_h!}\,Z^{V;\mathsf{A}}_{N,\boldsymbol{N}/N,\beta} \\ = & \sum_{\substack{0\leqslant N_1,\ldots,N_r\leqslant N\\ |\boldsymbol{N}-\boldsymbol{N}_{\star}|\leqslant \ln N}} \exp\Big(\sum_{k=-2}^{K}\sum_{j=0}^{K-k} N^{-(k+j)}\,\frac{(F^{\{k\}}_{\star})^{(j)}}{j!}\cdot(\boldsymbol{N}-\boldsymbol{N}_{\star})^{\otimes j} + N^{-(K+1)}R_K\Big). \end{aligned} \tag{8.6}$$
And, since $N^{-(K+1)R_K} \leqslant 1$ for N large enough:
$$\begin{aligned} |e^{N^{-(K+1)}R_K} - 1| & \leqslant 2|N^{-(K+1)}\,R_K| \\ & \leqslant N^{-(K+1)}\sum_{k=-2}^{K} 2C_{k,K}\,(\ln N)^{K-k+1} \leqslant C'_K\,N^{-(K+1)}(\ln N)^{K+3}. \end{aligned} \tag{8.7}$$
where we finally used (8.3). Notice that, since $\boldsymbol{\epsilon}_{\star}$ is the equilibrium filling fraction, we have $(F^{\{-2\}})'_{\star} = 0$, and therefore, for any $K \geqslant 0$:
$$\begin{aligned} & \exp\Big(\sum_{k=-2}^{K}\sum_{j=1}^{K-k} N^{-(k+j)}\,\frac{(F^{\{k\}}_{\star,\beta})^{(j)}}{j!}\cdot(\boldsymbol{N}-\boldsymbol{N}_{\star})^{\otimes j}\Big) \\ = & \; e^{\mathrm{i}\pi\boldsymbol{\tau}_{\star,\beta}\cdot(\boldsymbol{N}-\boldsymbol{N}_{\star})^{\otimes 2} + 2\mathrm{i}\pi\boldsymbol{v}_{\star,\beta}\cdot(\boldsymbol{N}-\boldsymbol{N}_{\star})}\Big(1 + \sum_{k=1}^{K} N^{-k}\,T^{\{k\}}_{\star,\beta}[\boldsymbol{N}-\boldsymbol{N}_{\star}] + O(N^{-(K+1)}(\ln N)^{K+3})\Big), \end{aligned} \tag{8.8}$$

where we have introduced:

$$\boldsymbol{v}_{\star,\beta} = \frac{(F^{\{-1\}}_{\star,\beta})'}{2\mathrm{i}\pi}, \qquad \boldsymbol{\tau}_{\star} = \frac{(F^{\{-2\}}_{\star,\beta})''}{2\mathrm{i}\pi}, \tag{8.9}$$

and for any vector $\boldsymbol{X}$ with g components:

$$T^{\{k\}}_{\boldsymbol{\epsilon},\beta}[\boldsymbol{X}] = \sum_{r=1}^{k} \frac{1}{r!} \sum_{\substack{\ell_1,\ldots,\ell_r \geqslant 1 \\ m_1,\ldots,m_r \geqslant -2 \\ \sum_{i=1}^{r} \ell_i + m_i = k}} \Big(\bigotimes_{i=1}^{r} \frac{(F^{\{m_i\}}_{\boldsymbol{\epsilon},\beta})^{(\ell_i)}}{\ell_i!} \Big) \cdot \boldsymbol{X}^{\otimes(\sum i=1^r \ell_i)}. \tag{8.10}$$

Since the number of lattice points $\boldsymbol{N}$ satisfying $|\boldsymbol{N} - \boldsymbol{N}_\star| \leqslant \ln N$ is a $O(\ln N)$, we can write:

$$\begin{aligned} \frac{Z^{V;\mathsf{A}}_{N,\beta}}{Z^{V;\mathsf{A}}_{N,\star,\beta}} &= \Big\{ \sum_{\substack{N_1,\ldots,N_g \in \mathbb{Z}^g \\ |\boldsymbol{N}-\boldsymbol{N}_\star| \leqslant \eta_N}} e^{\mathrm{i}\pi \boldsymbol{\tau}_{\star,\beta}\cdot(\boldsymbol{N}-\boldsymbol{N}_\star)^{\otimes 2} + 2\mathrm{i}\pi \boldsymbol{v}_{\star,\beta}\cdot(\boldsymbol{N}-\boldsymbol{N}_\star)} \Big(1 + \sum_{k=1}^{K} N^{-k}\, T^{\{k\}}_{\star,\beta}[\boldsymbol{N}-\boldsymbol{N}_\star]\Big)\Big\} \\ & \quad + O(N^{-(K+1)}(\ln N)^{K+4}). \end{aligned} \tag{8.11}$$

where we have set $\eta_N = \ln N$.

8.1.2 Waiving the constraint on the sum over filling fractions

Now, we would like to extend the sum over the whole lattice $\mathbb{Z}^g$. Let us denote $\lambda_{\star,\beta} = \min \mathrm{Sp}\,(-F^{\{-2\}}_{\star,\beta})'' > 0$. For any $\alpha > 0$ small enough, there exists a constant $C'' > 0$ so that:

$$\begin{aligned} & \Big| \sum_{\substack{\boldsymbol{N}\in\mathbb{Z}^r \\ |\boldsymbol{N}-\boldsymbol{N}_\star| \geqslant \eta_N}} e^{\mathrm{i}\pi \boldsymbol{\tau}_{\star,\beta}\cdot(\boldsymbol{N}-\boldsymbol{N}_\star)^{\otimes 2} + 2\mathrm{i}\pi \boldsymbol{v}_{\star,\beta}\cdot(\boldsymbol{N}-\boldsymbol{N}_\star)} (\boldsymbol{N}-\boldsymbol{N}_\star)^{\otimes j} \Big| \\ \leqslant\ & C'' \sum_{\substack{\boldsymbol{N}\in\mathbb{Z}^g \\ |\boldsymbol{N}-\boldsymbol{N}_\star| \geqslant \eta_N}} e^{-\lambda_{\star,\beta}(1-\alpha)g|\boldsymbol{N}-\boldsymbol{N}_\star|^2}\, |\boldsymbol{N}-\boldsymbol{N}_\star|^j \\ \leqslant\ & C'' \sum_{n \geqslant \eta_N} \mathrm{Vol}_g(n)\,(n+1)^j\, e^{-\lambda_{\star,\beta}(1-\alpha)g\,n^2}, \end{aligned} \tag{8.12}$$

where $\mathrm{Vol}_g(n) = (2n+1)^g - (2n-1)^g \leqslant g\,2^g\,n^{g-1}$ is the number of points in $\mathbb{Z}^g$ so that $n \leqslant |\boldsymbol{N}-\boldsymbol{N}_\star| < n+1$. Therefore:

$$\begin{aligned} & \Big| \sum_{\substack{\boldsymbol{N}\in\mathbb{Z}^r \\ |\boldsymbol{N}-\boldsymbol{N}_\star| \geqslant \eta_N}} e^{\mathrm{i}\pi \boldsymbol{\tau}_{\star,\beta}\cdot(\boldsymbol{N}-\boldsymbol{N}_\star)^{\otimes 2} + 2\mathrm{i}\pi \boldsymbol{v}_{\star,\beta}\cdot(\boldsymbol{N}-\boldsymbol{N}_\star)} (\boldsymbol{N}-\boldsymbol{N}_\star)^{\otimes j} \Big| \\ \leqslant\ & C_2(1+\alpha)g\,2^g \Big(\sum_{n \geqslant \eta_N} (n+1)^{g-1+j}\, e^{-\lambda_\star(1-\alpha)gn\eta_N} \Big) \\ \leqslant\ & C_{3,j}\, e^{-\lambda_\star(1-\alpha)g\eta_N^2}, \end{aligned} \tag{8.13}$$

where $C_{3,j}$ is a constant depending on j. In other words, by unrestricting the sum in (8.11), we only make an error of order $O(e^{-C_4(\ln N)^2})$, which is $O(N^{-\infty})$. Then, we remark that:

$$\sum_{\boldsymbol{N}\in\mathbb{Z}^g} e^{\mathrm{i}\pi(\boldsymbol{N}-\boldsymbol{N}_\star)\cdot\boldsymbol{\tau}_{\star,\beta}\cdot(\boldsymbol{N}-\boldsymbol{N}_\star) + 2\mathrm{i}\pi \boldsymbol{v}_{\star,\beta}\cdot(\boldsymbol{N}-\boldsymbol{N}_\star)}\,(\boldsymbol{N}-\boldsymbol{N}_\star)^{\otimes j} = \Big(\frac{\nabla_{\boldsymbol{v}}}{2\mathrm{i}\pi}\Big)^{\otimes j} \vartheta\begin{bmatrix} -\boldsymbol{N}_\star \\ \boldsymbol{0} \end{bmatrix}(\boldsymbol{v}_{\star,\beta}|\boldsymbol{\tau}_{\star,\beta}). \tag{8.14}$$

We have thus proved:

$$\frac{Z^{V;\mathsf{A}}_{N,\beta}}{Z^{V;\mathsf{A}}_{N,\star,\beta}} = \Big\{ \sum_{k=0}^{K} N^{-k}\, T^{\{k\}}_{\star,\beta}\Big[\frac{\nabla_{\boldsymbol{v}}}{2\mathrm{i}\pi}\Big] \Big\} \vartheta\begin{bmatrix} -\boldsymbol{N}_\star \\ \boldsymbol{0} \end{bmatrix}(\boldsymbol{v}_{\star,\beta}|\boldsymbol{\tau}_{\star,\beta}) + O(N^{-(K+1)}(\ln N)^{K+4}). \tag{8.15}$$

The term appearing as a prefactor of N^{-k} is bounded when $N \to \infty$. So, by pushing the expansion one step further, the error $O(N^{-(K+1)}(\ln N)^{K+4})$ can be replaced by $O(N^{-(K+1)})$. This concludes the proof of Theorem 1.5.

8.2 Deviations of filling fractions from their mean value

We now describe the fluctuations of the number of eigenvalues in each segment. Let $\boldsymbol{P} = (P_0, \ldots, P_g)$ be a vector of integers such that $\boldsymbol{P} - N\epsilon_{\star,h} \in o(N^{1/3})$ when $N \to \infty$. The joint probability for $h \in [\![0, g]\!]$ to find P_h eigenvalues in the segment A_h is:

$$\mu_{N,\beta}^{V;\mathsf{A}}[\boldsymbol{N} = \boldsymbol{P}] = \frac{N!}{\prod_{h=0}^{g} P_h!} \frac{Z_{N,\boldsymbol{P}/N,\beta}^{V;\mathsf{A}}}{Z_{N,\beta}^{V;\mathsf{A}}} \tag{8.16}$$

We remind that the coefficients of the large N expansion of the numerator are smooth functions of $\boldsymbol{P}/N$. Therefore, we can perform a Taylor expansion in $\boldsymbol{P}/N$ close to $\boldsymbol{\epsilon}_\star$, and we find that provided $\boldsymbol{P} - N\boldsymbol{\epsilon}_\star \in o(N^{1/3})$, only the quadratic term of the Taylor expansion remains when N is large:

$$\begin{aligned} \mu_{N,\beta}^{V;\mathsf{A}}[\boldsymbol{N} = \boldsymbol{P}] \quad = \quad & \Big(\vartheta\begin{bmatrix} -\boldsymbol{N}_\star \\ \boldsymbol{0} \end{bmatrix}(\boldsymbol{v}_{\star,\beta}|\boldsymbol{\tau}_{\star,\beta})\Big)^{-1} \exp\Big[\Big(\sum_{h=0}^{g} (\beta/2)(P_h - N\epsilon_{\star,h})\Big)\ln N\Big] \\ & \times \exp\Big(\frac{1}{2}\,(F_{\star,\beta}^{\{-2\}})'' \cdot (\boldsymbol{P} - N\boldsymbol{\epsilon}_\star)^{\otimes 2} + (F_{\star,\beta}^{\{-1\}})' \cdot (\boldsymbol{P} - N\boldsymbol{\epsilon}_\star) + o(1)\Big) \end{aligned} \tag{8.17}$$

In other words, the random vector $\Delta\boldsymbol{N} = (\Delta N_1, \ldots, \Delta N_g)$ defined by:

$$\Delta N_h = N_h - N\epsilon_{\star,h} + \sum_{h'=1}^{g} [(F_{\star,\beta}^{\{-2\}})'']_{h,h'}^{-1}\,(F_{\star,\beta}^{\{-1\}})'_{h'} \tag{8.18}$$

converges in law to a random discrete Gaussian vector, with covariance $[(F_{\star,\beta}^{\{-2\}})'']^{-1}$. We observe that, when $\beta = 2$, $F_{\star,\beta}^{\{-1\}} = 0$ so that $\boldsymbol{N} - N\boldsymbol{\epsilon}_\star$ converges to a centered discrete Gaussian vector.

A Elementary properties of the equilibrium measure with fixed filling fractions

A.1 Smooth dependence

In this section, we prove Lemma 7.2, i.e. we establish under some assumptions that the equilibrium measure $\mu_{\mathrm{eq},\boldsymbol{\epsilon}}^{V}$ depends smoothly on the potential V and the filling fractions $\boldsymbol{\epsilon}$. Let V be at least C^2 and confining on A. We know that the support consists of a finite union of pairwise disjoint segments:

$$\mathsf{S}_{\boldsymbol{\epsilon}}^{V} = \bigcup_{h=0}^{g} \mathsf{S}_{\boldsymbol{\epsilon},h}^{V}, \qquad \mathsf{S}_{\boldsymbol{\epsilon},h}^{V} = [\alpha_{\boldsymbol{\epsilon},h}^{-;V}, \alpha_{\boldsymbol{\epsilon},h}^{+;V}], \qquad \alpha_{\boldsymbol{\epsilon},h}^{-;V} < \alpha_{\boldsymbol{\epsilon},h}^{+;V}. \tag{A.1}$$

Upon squeezing A, we can always assume that it is the disjoint union of $(g+1)$ pairwise disjoint segments $\mathsf{A} = \bigcup_{h=0}^{g} \mathsf{A}_h$, which are neighborhoods of $\mathsf{S}_{\boldsymbol{\epsilon},h}^{V}$ in $\mathbb{R}$. The Stieltjes transform $W_{\boldsymbol{\epsilon}}^{\{-1\};V}$ of this equilibrium measure satisfies:

$$\forall h \in [\![0, g]\!], \qquad \forall x \in \mathsf{S}_{\boldsymbol{\epsilon},h}^{V}, \qquad W_{1,\boldsymbol{\epsilon}}^{\{-1\};V}(x + \mathrm{i}0) + W_{1,\boldsymbol{\epsilon}}^{\{-1\};V}(x + \mathrm{i}0) = V'(x), \tag{A.2}$$

and

$$\forall h \in [\![0, g]\!], \qquad \oint_{\mathsf{A}_h} \frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\, W_{1,\boldsymbol{\epsilon}}^{\{-1\};V}(\xi) = \epsilon_h. \tag{A.3}$$

The general solution of (A.2) takes the form:

$$W_{1,\boldsymbol{\epsilon}}^{\{-1\};V}(x) = \frac{1}{2}\Big(\frac{P_{\boldsymbol{\epsilon}}^{V}(x)}{\sigma(x)} + \oint_{\mathsf{A}} \frac{V'(x) - V'(\xi)}{\xi - x}\,\frac{\sigma(\xi)}{\sigma(x)} d\xi\Big), \tag{A.4}$$

where $P_{\boldsymbol{\epsilon}}^V$ is a polynomial of degree g which should be determined by (A.3), and we recall the notation:

$$\sigma(x) = \prod_{h=0}^{g} \sqrt{(x - \alpha_{\boldsymbol{\epsilon},h}^{-;V})(x - \alpha_{\boldsymbol{\epsilon},h}^{+;V})}. \tag{A.5}$$

This implies that the equilibrium measure has a density, which can be written in the form:

$$\mathrm{d}\mu_{\boldsymbol{\epsilon}}^V(x) = \mathrm{d}x\, S_{\boldsymbol{\epsilon}}^V(x) \prod_{h=0}^{g} |x - \alpha_{\boldsymbol{\epsilon},h}^{\bullet}|^{\rho_h^{\bullet}}. \tag{A.6}$$

for some $\rho_h^{\bullet} = \pm 1$, and $S_{\boldsymbol{\epsilon}}^V(x)$ is a smooth function in a neighborhood of the support. This rewriting assumes $S_{\boldsymbol{\epsilon}}^V(\alpha_h^{\bullet}) \neq 0$ when $\rho_h^{\bullet} = -1$.

Let $\mathcal{I} = [\![0, g]\!] \times \{\pm 1\}$, and $\mathcal{H} = \{I \in \mathcal{I} \quad \rho_I = -1\}$. We assume strong off-criticality as defined in Hypothesis 1.2:

Hypothesis A.1 *The initial data $V[0]$ and $\boldsymbol{\epsilon}[0]$ is strongly off-critical, in the sense that, for any $I \in \mathcal{I}\backslash\mathcal{H}$, $S(\alpha_I) \neq 0$ and $S'(\alpha_I) \neq 0$.*

For any $(h, \bullet) \in \mathcal{H}$, we consider the values α_I fixed and equal to $\alpha_{\boldsymbol{\epsilon}[0],h}^{\bullet;V[0]}$. We introduce the open set:

$$\mathcal{U} = \Big\{\boldsymbol{\alpha} \in \prod_{I \in \mathcal{I}\backslash\mathcal{H}} \mathbb{R} \ \Big|\ \text{all } \alpha_I \text{ , for } I \in \mathcal{I} \text{ are pairwise distinct}\Big\}, \tag{A.7}$$

and the map:

$$(\boldsymbol{F}, \boldsymbol{T}) \,:\, \mathcal{U} \times \mathbb{R}_g[X] \times \mathcal{C}^2(\mathsf{A}) \times \mathbb{R}^{g+1} \longrightarrow \mathbb{R}^{2g+2-|H|} \times \mathbb{R}^{g+1} \tag{A.8}$$

defined by:

$$\begin{aligned} \forall I \in \mathcal{H} \qquad & F_I[\boldsymbol{\alpha}, P, V, \boldsymbol{\epsilon}] &&= \frac{P(\alpha_I)}{\prod_{J \in \mathcal{I}\backslash\{I\}} \sqrt{\alpha_h^{\bullet} - \alpha_{h'}^{\bullet'}}} + \oint_A \frac{\sigma[\boldsymbol{\alpha}](\xi)\,\mathrm{d}\xi}{2\mathrm{i}\pi}\, \frac{V'(\alpha_I) - V'(\xi)}{\alpha_I - x}, \\ \forall h \in [\![0, g]\!] \qquad & T_h[\alpha, P, V, \boldsymbol{\epsilon}] &&= -\epsilon_h + \oint_{\mathsf{A}_h} \frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\, w[\boldsymbol{\alpha}, P, V](\xi), \end{aligned} \tag{A.9}$$

where:

$$w[\boldsymbol{\alpha}, P, V](x) = \frac{1}{2}\Big(\frac{P(x)}{\sigma[\boldsymbol{\alpha}](x)} + \oint_{\mathsf{A}} \frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\, \frac{V'(x) - V'(\xi)}{x - \xi}\, \frac{\sigma[\boldsymbol{\alpha}](\xi)}{\sigma[\boldsymbol{\alpha}](x)}\Big), \tag{A.10}$$

$$\sigma[\boldsymbol{\alpha}](x) = \prod_{I \in \mathcal{I}} \sqrt{x - \alpha_I}. \tag{A.11}$$

By construction, the data of the equilibrium measure $\mu_{\boldsymbol{\epsilon}}^V$ satisfies:

$$(\boldsymbol{F}, \boldsymbol{T})[\boldsymbol{\alpha}_{\boldsymbol{\epsilon}}^V, P_{\boldsymbol{\epsilon}}^V, V, \boldsymbol{\epsilon}] = \mathbf{0}. \tag{A.12}$$

We would like to apply the implicit function theorem to show that:

Lemma A.1 *If Hypothesis A.1 holds and $V[0]$ is $\mathcal{C}^r$ (resp. analytic) with $r \geqslant 2$, $\boldsymbol{\alpha}_{\boldsymbol{\epsilon}}^V$ and $P_{\boldsymbol{\epsilon}}^V$ are $\mathcal{C}^{r-1}$ (resp. analytic) functions of $(\boldsymbol{\epsilon}, V)$ close enough to $(\boldsymbol{\epsilon}[0], V[0])$.*

The contour integrals $\oint_{\mathsf{A}}$ used in this Appendix A.1 can be rewritten as integrals over the segment S. Thus, these manipulations do not assume that V is analytic in a neighborhood of A, and it does makes sense to consider only V of class $\mathcal{C}^r$ in the statement of Lemma A.1.

Proof. To achieve this we need to show that $(\mathrm{d}_{\boldsymbol{\alpha}}, \mathrm{d}_P)(F, T)$ when evaluated at a point such that $(\boldsymbol{F}, \boldsymbol{T})(\boldsymbol{\alpha}, P, V, \boldsymbol{\epsilon}) = 0$. We first compute:

$$\frac{\partial T_h}{\partial X^{h'}}[\boldsymbol{\alpha}, P, V, \boldsymbol{\epsilon}] = \oint_{\mathsf{A}_h} \frac{\mathrm{d}\xi}{2\mathrm{i}\pi} \frac{\xi^{h'}}{2\sigma[\boldsymbol{\alpha}](\xi)}. \tag{A.13}$$

It is well-known that the $g \times g$ submatrix of (A.13) consisting of indices $h, h' \in [\![1, g]\!]$ is invertible when $\boldsymbol{\alpha} \in \mathcal{U}$, and on top of that, we find:

$$\forall h' \in [\![0, g]\!], \qquad \sum_{h=0}^{g} \frac{\partial T_h}{\partial X^{h'}}[\boldsymbol{\alpha}, P, V, \boldsymbol{\epsilon}] \quad = \quad \oint_{\mathsf{A}} \frac{\mathrm{d}\xi}{2\mathrm{i}\pi} \frac{\xi^{h'}}{\sigma[\boldsymbol{\alpha}](\xi)} = -\operatorname*{Res}_{\xi\to\infty} \frac{\xi^{h'}\,\mathrm{d}\xi}{\sigma[\boldsymbol{\alpha}](\xi)} = \frac{\delta_{h',g}}{2}. \tag{A.14}$$

In particular, this expression does not vanish for $h' = g$, hence (A.13) is invertible. Then, we compute:

$$\begin{aligned}
\frac{\partial F_I}{\partial \alpha_J}[\boldsymbol{\alpha}, P, V] \quad &= \quad \delta_{I,J} \oint_{\mathsf{A}} \frac{\sigma[\boldsymbol{\alpha}](\xi)\,\mathrm{d}\xi}{2\mathrm{i}\pi} \frac{V''(\alpha_I)(\xi - \alpha_I) - (V'(\alpha_I) - V'(\xi))}{(\alpha_I - \xi)^2} \\
&\quad + \sum_{J \in \mathcal{I}} \frac{F_I[\boldsymbol{\alpha}, P, V] - F_J[\boldsymbol{\alpha}, P, V]}{2(\alpha_I - \alpha_J)} \\
&= \quad \delta_{I,J}\Big(\frac{P'(\alpha_I)}{\prod_{J \in \mathcal{I}\setminus\{I\}} \sqrt{\alpha_I - \alpha_J}} + \oint_{\mathsf{A}} \frac{\sigma[\boldsymbol{\alpha}](\xi)\,\mathrm{d}\xi}{2\mathrm{i}\pi} \frac{V''(\alpha_I)(\xi - \alpha_I) - (V'(\alpha_I) - V'(\xi))}{(\alpha_I - \xi)^2}\Big) \\
&= \quad \delta_{I,J} \lim_{x \to a_I} \sqrt{x - \alpha_I}\, \partial_x\big(w[\boldsymbol{\alpha}, P, V](x)\big),
\end{aligned} \tag{A.15}$$

and the latter does not vanish when evaluated at $(\boldsymbol{\alpha}_{\boldsymbol{\epsilon}[0]}^{V[0]}, P_{\boldsymbol{\epsilon}}^{V[0]}, V[0], \boldsymbol{\epsilon}[0])$ thanks to Hypothesis A.1. So, $(\mathrm{d}_{\boldsymbol{\alpha}}, \mathrm{d}_P)(\boldsymbol{F}, \boldsymbol{T})$ is invertible at this point. Besides, if V is $\mathcal{C}^r$ for $r \geqslant 2$, then $\mathrm{d}(\boldsymbol{F}, \boldsymbol{T})$ is $\mathcal{C}^{r-2}$. Therefore, by the implicit function theorem, for $(\boldsymbol{\epsilon}, V)$ close enough to $(\boldsymbol{\epsilon}[0], V[0])$, there is a unique function $\mathcal{C}^{r-1}$ function $(\boldsymbol{\alpha}_{\boldsymbol{\epsilon}}^V, P_{\boldsymbol{\epsilon}}^V)$ so that $(\boldsymbol{F}, \boldsymbol{T})(\boldsymbol{\alpha}_{\boldsymbol{\epsilon}}^V, P_{\boldsymbol{\epsilon}}^V, V, \boldsymbol{\epsilon}) = \boldsymbol{0}$. By uniqueness, it must correspond to the data of $\mu_{\mathrm{eq},\boldsymbol{\epsilon}}^V$. □

Corollary A.2 *If Hypothesis 1.2 holds for $(V[0], \boldsymbol{\epsilon}[0])$ with $V[0]$ not necessarily real-analytic but $\mathcal{C}^r$ with $r \geqslant 2$, they hold also for $(V, \boldsymbol{\epsilon})$ close enough to $(V[0], \boldsymbol{\epsilon})$ and $V[0]$ $\mathcal{C}^r$, and the density (A.6), once multiplied by $\sigma(x)$, is $\mathcal{C}^{r-1}$ for such data.*

Indeed, (A.4) shows that $W_{1,\boldsymbol{\epsilon}}^{\{-1\};V}(x)$ is a linear function of $P_{\boldsymbol{\epsilon}}^{V[0]}(x)/\sigma(x)$ and hence, once multiplied by $\sigma(x)$, it is a smooth function of $\boldsymbol{\epsilon}$. □

A.2 Hessian of the value of the energy functional

We are now in position to prove:

Proposition A.3 *If Hypothesis 1.2 holds for $(V[0], \boldsymbol{\epsilon}[0])$ with $V[0]$ not necessarily real-analytic but at least $\mathcal{C}^2$, $F_{\boldsymbol{\epsilon}}^{\{-2\};V}$ is $\mathcal{C}^2$ for $(\boldsymbol{\epsilon}, V)$ close enough to $(\boldsymbol{\epsilon}[0], V[0])$, and the $g \times g$ matrix $\boldsymbol{\tau}_{\boldsymbol{\epsilon}}^V$ with purely imaginary entries:*

$$\forall h, h' \in [\![1, g]\!], \qquad (\boldsymbol{\tau}_{\boldsymbol{\epsilon}}^V)_{h,h'} = \frac{1}{2\mathrm{i}\pi} \frac{\partial^2 F_{\boldsymbol{\epsilon}}^{\{-2\};V}}{\partial \epsilon_h \partial \epsilon_{h'}} \tag{A.16}$$

is such that $\operatorname{Im} \boldsymbol{\tau}_{\boldsymbol{\epsilon}}^V > 0$.

Proof. We are going to justify that $F_{\boldsymbol{\epsilon}}^{\{-2\}}$ is a concave function of $\boldsymbol{\epsilon}$ in the domain:

$$\mathcal{E}_g = \Big\{\boldsymbol{\epsilon} \in]0, 1[^{g+1}, \qquad \sum_{h=0}^{g} \epsilon_h = 1\Big\}, \tag{A.17}$$

and within Hypothesis 1.2, which will imply the result. Let $\boldsymbol{e}$ a vector of $\mathbb{R}^{g+1}$ such that $\sum_{h=0}^{g} e_h = 0$ and $\boldsymbol{e} \neq \mathbf{0}$. For any $\boldsymbol{\epsilon} \in \mathcal{E}_g$, we define $\boldsymbol{\epsilon}^t = \boldsymbol{\epsilon} + t\boldsymbol{e}$, which belongs to $\mathcal{E}_g$ for t small enough, and $\mu^t = \mu^V_{\mathrm{eq},\boldsymbol{\epsilon}^t}$. Its characterization as an equilibrium measure imply that :

$$U^t(x) = V(x) - 2\int \ln|x-y|\mathrm{d}\mu^t(y) \tag{A.18}$$

is locally constant on the support of μ^t, and $\mu^t(\mathsf{A}_h) = \epsilon_h + te_h$. Let us denote U_h^t the value of $U^t(x)$ when $x \in \mathsf{S}_h$.

Let us denote $F_{\boldsymbol{e}}(t) = F^{\{-2\};V}_{\boldsymbol{\epsilon}^t}$. After a classical result (see e.g. [AG97, Joh98]):

$$F_{\boldsymbol{e}}(t) = \frac{\beta}{2}\Big[\iint \ln|x-y|\mathrm{d}\mu^t(x)\mathrm{d}\mu^t(y) - \int V(x)\mathrm{d}\mu^t(x)\Big] \tag{A.19}$$

It follows from Corollary A.2 that the density of μ^t is $\mathcal{C}^1$ away from the edges, and its derivative is integrable at the edges, thus define a measure that we denote ν^t. It has same support as μ^t, and we deduce from the characterization of μ^t that $x \mapsto -2\int \ln|x-y|\mathrm{d}\nu^t(y)$ is locally constant on the support of μ^t, and $\nu^t[\mathsf{A}_h] = e_h$. We deduce that $F_{\boldsymbol{e}}(t)$ is $\mathcal{C}^1$ and:

$$F'_{\boldsymbol{e}}(t) = \frac{\beta}{2}\int_{\mathsf{A}}\Big(2\int \ln|x-y|\mathrm{d}\mu^t(y) - V(x)\Big)\mathrm{d}\nu^t(x) = -\frac{\beta}{2}\sum_{h=0}^{g} U_h^t\, e_h \tag{A.20}$$

e_h does not depend on t, whereas $t \mapsto U_h^t$ is $\mathcal{C}^1$ as one can deduce from the expression (A.18). Thus, $F_{\boldsymbol{e}}$ is $\mathcal{C}^2$ and:

$$F''_{\boldsymbol{e}}(t) = \beta\sum_{h=0}^{g}\int_{\mathsf{A}_h} \ln|x_h - y|\mathrm{d}\nu^t(y)\, e_h \tag{A.21}$$

where x_h is any point in the interior of A_h. This can be rewritten:

$$F''_{\boldsymbol{e}}(t) = \beta\sum_{h=0}^{g}\iint 2\ln|x-y|\mathrm{d}\nu^t(x)\mathbf{1}_{\mathsf{A}_h}(x)\mathrm{d}\nu^t(y)\mathbf{1}_{\mathsf{A}_h}(y) = \beta\sum_{h=0}^{g} Q[\nu^t\mathbf{1}_{\mathsf{A}_h}, \nu^t\mathbf{1}_{\mathsf{A}_h}] \tag{A.22}$$

It is well-known property (see e.g. [AG97, Dei99]) that, for any signed measure ν with total mass 0, $Q[\nu,\nu] \geqslant 0$, with equality iff $\nu = 0$. Since we chose $\boldsymbol{e} \neq \mathbf{0}$, the vector of measures $(\nu^t\mathbf{1}_{\mathsf{A}_h})_{0\leqslant h\leqslant g}$ is not identically zero, hence $F''_{\boldsymbol{e}}(t) < 0$. In other words, $F_{\boldsymbol{e}}$ is strictly concave for any direction $\boldsymbol{e}$, hence $F^{\{-2\}}_{\boldsymbol{\epsilon}}$ is a strictly concave function of $\boldsymbol{\epsilon} \in \mathcal{E}_g$ so that Hypothesis 1.2 holds. □

B Model Selberg integrals

B.1 Non decaying terms in correlators

Let us denote W_1^{ρ,γ_m} the first point correlator in the model with potential V_{ρ,γ_m} described in § 7.1.1. Let us denote $\Delta = (\gamma^+ - \gamma^-)/4$. It admits a $1/N$ expansion: $W_1\rho,\gamma_m = \sum_{k\geqslant -1} N^{-k}\, W_1^{\{k\};\gamma_m,\rho}$.The expression for the equilibrium measure gives access to $W_1^{\{-1\};\gamma_m,\rho}$,

$$W_1^{\{-1\};\gamma_m,++}(x) = \frac{x - \sqrt{(x-\gamma^-)(x-\gamma^+)}}{2\Delta^2},$$

$$W_1^{\{-1\};\gamma_m,-+}(x) = \frac{1}{2\Delta}\Big(1 - \sqrt{\frac{x-\gamma^+}{x-\gamma^-}}\Big), \tag{B.1}$$

$$W_1^{\{-1\};\gamma_m,+-}(x) = \frac{1}{2\Delta}\Big(1 - \sqrt{\frac{x-\gamma^-}{x-\gamma^+}}\Big), \tag{B.2}$$

$$W_1^{\{-1\};\gamma_m,--}(x) = \frac{1}{\sqrt{(x-\gamma^-)(x-\gamma^+)}}, \tag{B.3}$$

and then we deduce from (5.49):

$$\begin{aligned} W_1^{\{0\};\gamma_m,++}(x) &= W_1^{\{0\};\gamma_m,--}(x) &= \frac{1}{2}\Big(1-\frac{2}{\beta}\Big)\Big(\frac{1}{\sqrt{(x-\gamma^-)(x-\gamma^+)}} - \frac{x-\frac{\gamma^++\gamma^-}{2}}{(x-\gamma^-)(x-\gamma^+)}\Big), \\ W_1^{\{0\};\gamma_m,-+}(x) &= W_1^{\{0\};\gamma_m,+-}(x = &-\Big(1-\frac{2}{\beta}\Big)\,\frac{\Delta}{(x-\gamma^-)(x-\gamma^+)}. \end{aligned} \tag{B.4}$$

Besides, we find from (5.51) that all these models share the same $W_2^{\{0\}}$:

$$W_2^{\{0\};\gamma_m,\rho}(x_1,x_2) = \frac{2}{\beta}\,\frac{1}{2(x_1-x_2)^2}\Big(-1+\frac{x_1x_2-(x_1+x_2)\frac{\gamma^-+\gamma^+}{2}+\gamma^-\gamma^+}{\sqrt{(x_1-\gamma^-)(x_1-\gamma^+)(x_2-\gamma^-)(x_2-\gamma^+)}}\Big). \tag{B.5}$$

B.2 Exact formulas for partition function

Let us denote:

$$\begin{aligned} \mathcal{Z}_{N,\beta}^{++} &= \int_{\mathbb{R}^N}\prod_{1\leqslant i<j\leqslant N}|\lambda_i-\lambda_j|^{\beta}\prod_{i=1}^{N}e^{-\frac{\beta N}{2}\,V_{++}(\lambda_i)}\mathrm{d}\lambda_i, \qquad V_{++}(\lambda)=\frac{\lambda^2}{2} \\ &= \exp\Big\{-\Big[\frac{\beta N^2}{4}+\Big(1-\frac{\beta}{2}\Big)\frac{N}{2}\Big]\ln\Big(\frac{\beta N}{2}\Big)\Big\}\,(2\pi)^{N/2}\prod_{j=1}^{N}\frac{\Gamma(1+j\beta/2)}{\Gamma(1+\beta/2)}. \end{aligned} \tag{B.6}$$

$$\begin{aligned} \mathcal{Z}_{N,\beta}^{-+} &= \int_{\mathbb{R}_+^N}\prod_{1\leqslant i<j\leqslant N}|\lambda_i-\lambda_j|^{\beta}\prod_{i=1}^{N}e^{-\frac{\beta N}{2}\,V_{-+}(\lambda_i)}\mathrm{d}\lambda_i, \qquad V_{-+}(\lambda)=\lambda \\ &= \exp\Big\{-\Big[\frac{\beta N^2}{2}+\Big(1-\frac{\beta}{2}\Big)N\Big]\ln\Big(\frac{\beta N}{2}\Big)\Big\}\prod_{j=1}^{N}\frac{\Gamma(1+j\beta/2)\Gamma(1+(j-1)\beta/2)}{\Gamma(1+\beta/2)}. \end{aligned} \tag{B.7}$$

$$\begin{aligned} \mathcal{Z}_{N,\beta}^{--} &= \int_{[-2,2]^N}\prod_{1\leqslant i<j\leqslant N}|\lambda_i-\lambda_j|^{\beta}\prod_{i=1}^{N}\mathrm{d}\lambda_i \\ &= 2^{N^2\beta+(2-\beta)N}\prod_{j=1}^{N}\frac{\big(\Gamma(1+(j-1)\beta/2)\big)^2\Gamma(1+j\beta/2)}{\Gamma(2+(N-2+j)\beta/2)\Gamma(1+\beta/2)}. \end{aligned} \tag{B.8}$$

These are the values of the reference partition functions given in (B.6)-(B.8). To emphasize that they can be defined for N not restricted to be an integer by analytic continuation, we introduce a function related to the Barnes double Gamma function:

$$\Gamma_2(N\,;\,b_1,b_2) = \exp\Big(\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\zeta_2(s\,;\,b_1,b_2,x)\Big), \tag{B.9}$$

where:

$$\zeta_2(s\,;\,b_1,b_2,x) = \frac{1}{\Gamma(s)}\int_0^{\infty}\frac{e^{-tx}\,t^{s-1}\,\mathrm{d}t}{(1-e^{-b_1t})(1-e^{-b_2t})}. \tag{B.10}$$

Its properties are reviewed in [Spr09], in particular it solves the functional equation:

$$\Gamma_2(x+b_2\,;\,b_1,b_2) = \frac{\Gamma_2(x)}{\Gamma(x/b_1)}\,\sqrt{2\pi}\,b_1^{1/2-x/b_1}, \qquad \Gamma_2(1\,;\,b_1,b_2)=1. \tag{B.11}$$

We deduce from (B.11) the representation:

$$\prod_{j=1}^{N}\Gamma(1+j\beta/2) = (2\pi)^{N/2}(\beta/2)^{\beta N^2/4+N(1/2+\beta/4)}\,\frac{\Gamma(N+1)}{\Gamma_2(N+1\,;\,2/\beta,1)} \tag{B.12}$$

Therefore, we can recast the Selberg integrals as:

$$\begin{aligned}
\mathcal{Z}_{N,\beta}^{++} &= \exp\Big(-(\beta/4)N^2\ln N + (\beta/4-1/2)N\ln N + N\big[(\beta/2)\ln(\beta/2)+\ln(2\pi)-\ln\Gamma(1+\beta/2)\big]\Big) \\
&\quad\times\frac{\Gamma(N+1)}{\Gamma_2(N+1\,;\,2/\beta,1)} \\
\mathcal{Z}_{N,\beta}^{+-} &= \exp\Big(-(\beta/2)N^2\ln N + (\beta/2-1)N\ln N + N\big[\beta\ln(\beta/2)+\ln(2\pi)-\ln\Gamma(1+\beta/2)\big]\Big) \\
&\quad\times\frac{\Gamma^2(N+1)}{\Gamma(1+N\beta/2)\Gamma_2^2(N+1\,;\,2/\beta,1)} \\
\mathcal{Z}_{N,\beta}^{--} &= \exp\Big(\beta N^2\ln 2 + N\big[(2-\beta)\ln 2 + (3\beta/2)\ln(\beta/2)+\ln(2\pi)-\ln\Gamma(1+\beta/2)\big]\Big) \\
&\quad\times\frac{\Gamma(2/\beta+N-1)\Gamma(N-1)}{\Gamma(2/\beta+2N-1)\Gamma(2N-1)}\,\frac{\Gamma^3(N+1)\Gamma_2(2N-1\,;\,2/\beta,1)}{\Gamma^2(1+N\beta/2)\Gamma_2^3(N+1\,;\,2/\beta,1)\Gamma_2(N-1\,;\,2/\beta,1)}
\end{aligned}$$

B.3 Large N asymptotics of the partition function

We need the asymptotic expansion of Barnes double Gamma function [Spr09]:

$$\begin{aligned}
\ln\Gamma_2(x\,;2/\beta,1) \quad &=_{x\to\infty} \quad -\frac{\beta x^2\ln x}{4} + \frac{3\beta x^2}{8} + \frac{1}{2}\Big(1+\frac{\beta}{2}\Big)(x\ln x - x) - \frac{3+\beta/2+2/\beta}{12}\ln x \\
&\quad -\chi'(0\,;\,b_1,b_2) + \sum_{k\geqslant 1}(k-1)!\,E_k(b_1,b_2)\,x^{-k} + O(x^{-\infty}),
\end{aligned} \tag{B.13}$$

$E_k(b_1,b_2)$ are the polynomials in two variables appearing as coefficients in the expansion:

$$\frac{1}{(1-e^{-b_1t})(1-e^{-b_2t})} \underset{t\to 0}{=} \sum_{k\geqslant -2} E_k(b_1,b_2)\,t^k \tag{B.14}$$

which are expressible in terms of Bernoulli numbers. $\chi(s\,;\,b_1,b_2)$ is the analytic continuation to the complex plane of the series defined for $\mathrm{Re}\,s>2$:

$$\chi(s\,;\,b_1,b_2) = \sum_{(m_1,m_2)\in\mathbb{N}^2\setminus\{(0,0)\}} \frac{1}{(m_1b_1+m_2b_2)^s}. \tag{B.15}$$

For instance:

$$\chi'(0\,;\,1,1) = -\frac{\ln(2\pi)}{2} + \zeta'(-1) \tag{B.16}$$

We remind also the Stirling formula for the asymptotic expansion of the Gamma function:

$$\ln\Gamma(x) \underset{x\to\infty}{=} x\ln x - x - \frac{\ln x}{2} + \frac{\ln(2\pi)}{2} + \sum_{k\geqslant 1}\frac{B_{k+1}}{k(k+1)}\,x^{-k}, \tag{B.17}$$

where B_k are the Bernoulli numbers. We deduce the asymptotic expansions:

Lemma B.1

$$\begin{aligned}
\ln\mathcal{Z}_{N,\beta}^{++} &= -(3\beta/8)N^2 + (\beta/2)N\ln N + \big(-1/2-\beta/4+(\beta/2)\ln(\beta/2)+\ln(2\pi)-\ln\Gamma(1+\beta/2)\big)N \\
&\quad+\frac{3+\beta/2+2/\beta}{12}\ln N + \chi'(0\,;\,2/\beta,1) + \frac{\ln(2\pi)}{2} + o(1)
\end{aligned} \tag{B.18}$$

$$\begin{aligned}
\ln\mathcal{Z}_{N,\beta}^{+-} &= -(3\beta/4)N^2 + (\beta/2)N\ln N + \big(-1+(\beta/2)\ln(\beta/2)+\ln(2\pi)-\ln\Gamma(1+\beta/2)\big)N \\
&\quad+\frac{\beta/2+2/\beta}{6} + 2\chi'(0\,;\,2/\beta,1) - \frac{\ln(\beta/2)}{2} + \frac{\ln(2\pi)}{2} + o(1)
\end{aligned} \tag{B.19}$$

$$\begin{aligned}
\ln\mathcal{Z}_{N,\beta}^{--} &= (\beta/2)N\ln N + \big(-\beta/2+(\beta/2)\ln(\beta/2)+(\beta/2-1)\ln(2)+\ln(2\pi)-\ln\Gamma(1+\beta/2)\big)N \\
&\quad+\frac{-1+2/\beta+\beta/2}{4}\ln N + 3\chi'(0\,;\,2/\beta,1) + \frac{27-13(2/\beta)-13(\beta/2)}{12}\ln(2) \\
&\quad-\ln(\beta/2) + \frac{\ln(2\pi)}{2} + o(1)
\end{aligned} \tag{B.20}$$

where the $o(1)$ have an asymptotic expansion in powers of $1/N$.

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References

[AG97] G. Ben Arous and A. Guionnet, *Large deviations for Wigner's law and Voiculescu's non-commutative entropy*, Probab. Theory Related Fields **108** (1997), no. 4, 517–542.

[AGZ10] G. Anderson, A. Guionnet, and O. Zeitouni, *An introduction to random matrices*, Cambridge University Press, 2010, http://www.wisdom.weizmann.ac.il/~zeitouni/.

[Ake96] G. Akemann, *Higher genus correlators for the hermitian matrix model with multiple cuts*, Nucl. Phys. B **482** (1996), 403–430, hep-th/9606004.

[APS01] S. Albeverio, L. Pastur, and M. Shcherbina, *On the $1/N$ expansion for some unitary invariant ensembles of random matrices*, Commun. Math. Phys. **224** (2001), 271–305, doi:10.1007/s002200100531.

[BDE00] G. Bonnet, F. David, and B. Eynard, *Breakdown of universality in multi-cut matrix models*, J. Phys. A **33** (2000), 6739–6768, cond-mat/0003324.

[BE12] G. Borot and B. Eynard, *All-order asymptotics of hyperbolic knot invariants from non-perturbative topological recursion of A-polynomials*, math-ph/1205.2261.

[BG11] G. Borot and A. Guionnet, *Asymptotic expansion of β matrix models in the one-cut regime*, to appear in Commun. Math. Phys. (2011), math-PR/1107.1167.

[BM09] M. Bertola and M.Y. Mo, *Commuting difference operators, spinor bundles and the asymptotics of orthogonal polynomials with respect to varying complex weights*, Adv. Math. (2009), no. 1, 154–218, math-ph/0605043.

[Bor11] G. Borot, *Quelques problèmes de géométrie énumérative, de matrices aléatoires, d'intégrabilité, étudiés via la géometrie des surfaces de Riemann*, 2011, Thèse de Doctorat, Université d'Orsay. math-ph/1110.1493.

[CE06] L.O. Chekhov and B. Eynard, *Matrix eigenvalue model: Feynman graph technique for all genera*, JHEP (2006), no. 0612:026, math-ph/0604014.

[CG12] T. Claeys and T. Grava, *Critical asymptotic behavior for the Korteweg-de Vries equation and in random matrix theory*, Proc. MSRI (2012), math-ph/1210.8352.

[DE02] I. Dumitriu and A. Edelman, *Matrix models for beta ensembles*, J. Math. Phys. **43** (2002), no. 11, 5830–5847, math-ph/0206043.

[Dei99] P. Deift, *Orthogonal polynomials and random matrices: a Riemann-Hilbert approach*, Courant Lecture Notes in Mathematics, vol. 3, New York University Courant Institute of Mathematical Sciences, New York, 1999.

[DKM+97] P. Deift, T. Kriecherbauer, K.T.-R. McLaughlin, S. Venakides, and X. Zhou, *Asymptotics for polynomials orthogonal with respect to varying exponential weights*, Int. Math. Res. Notices **16** (1997), 759–782.

[DKM+99a] ———, *Strong asymptotics of orthogonal polynomials with respect to exponential weights via Riemann-Hilbert techniques*, Comm. Pure Appl. Math. **52** (1999), no. 12, 1491–1552.

[DKM+99b] ———, *Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory*, Comm. Pure Appl. Math. **52** (1999), no. 11, 1335–1425.

[dMPS95] A. Boutet de Monvel, L. Pastur, and M. Shcherbina, *On the statistical mechanics approach in the random matrix theory. Integrated density of states*, J. Stat. Phys. **79** (1995), no. 3-4, 585–611.

[Dub08] B. Dubrovin, *On universality of critical behaviour in Hamiltonian PDEs*, Amer. Math. Soc. Transl. **224** (2008), 59–109, math.AP/0804.3790.

[DZ95] P. Deift and X. Zhou, *A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the mKdV equation*, Ann. Math. **137** (1995), 295–368.

[EM03] N.M. Ercolani and K.T.-R. McLaughlin, *Asymptotics of the partition function for random matrices via Riemann-Hilbert techniques, and applications to graphical enumeration*, Int. Math. Res. Not. **14** (2003), 755–820, math-ph/0211022.

[Eyn01] B. Eynard, *Asymptotics of skew orthogonal polynomials*, J. Phys. A. **34** (2001), 7591–7605, [cond-mat.mes-hall/0012046.

[Eyn06] B. Eynard, *Large N asymptotics of orthogonal polynomials, from integrability to algebraic geometry*, Applications of random matrices in physics (É. Brézin, V. Kazakov, D. Serban, P. Wiegmann, and A. Zabrodin, eds.), Nato Science Series II, École d'été des Houches, Juin 2006, math-ph/0503052.

[Eyn09] ———, *Large N expansion of convergent matrix integrals, holomorphic anomalies, and background independence*, JHEP (2009), no. 0903:003, math-ph/0802.1788.

[Fay70] J. Fay, *Theta functions on Riemann surfaces*, Lecture Notes in Mathematics, vol. 352, Springer, Berlin, 1970.

[For10] P.J. Forrester, *Log-gases and random matrices*, Princeton University Press, 2010.

[GMS07] A. Guionnet and E. Maurel-Segala, *Second order asymptotics for matrix models*, Ann. Probab. **35** (2007), 2160–2212, math.PR/0601040.

[Joh98] K. Johansson, *On fluctuations of eigenvalues of random Hermitian matrices*, Duke Math. J. **91** (1998), no. 1, 151–204.

[KS10] T. Kriecherbauer and M. Shcherbina, *Fluctuations of eigenvalues of matrix models and their applications*, math-ph/1003.6121.

[Meh04] M.L. Mehta, *Random matrices*, third ed., Pure and Applied Mathematics, vol. 142, Elsevier/Academic, Amsterdam, 2004.

[MMS12] M. Maïda and E. Maurel-Segala, *Free transport-entropy inequalities for non-convex potentials and application to concentration for random matrices*, math.PR/1204.3208.

[Mum84] D. Mumford, *Tata lectures on Theta*, Modern Birkhäuser Classics, Birkhäuser, Boston, 1984, Volume I (no. 28), II (no. 43), III (no. 97).

[NN22] F. Nevanlinna and R. Nevanlinna, *Über die Eigenschaften einer analytischen Funktion in der Umgebung einer singuären Stelle oder Linie*, Acta Soc. Sci. Fennica **5** (1922), no. 5.

[Pas06] L. Pastur, *Limiting laws of linear eigenvalue statistics for Hermitian matrix models*, J. Math. Phys. **47** (2006), no. 10, math.PR/0608719.

[PS11] L. Pastur and M. Shcherbina, *Eigenvalue distribution of large random matrices*, Mathematical Survives and Monographs, vol. 171, American Mathematical Society, Providence, Rhode Island, 2011.

[Shc11] M. Shcherbina, *Orthogonal and symplectic matrix models: universality and other properties*, Commun. Math. Phys. **307** (2011), no. 3, 761–790, math-ph/1004.2765.

[Shc12] ———, *Fluctuations of linear eigenvalue statistics of β matrix models in the multi-cut regime*, math-ph/1205.7062.

[Spr09] M. Spreafico, *On the Barnes double zeta and Gamma functions*, J. Number Theory **129** (2009), 2035–2063.

[Sze39] G. Szegö, *Orthogonal polynomials*, Amer. Math. Soc., 1939, reprinted with corrections (2003).

[WZ06] P. Wiegmann and A. Zabrodin, *Large N expansion for the $2D$ Dyson gas*, J. Phys. A **39** (2006), 8933–8964, hep-th/0601009.