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ASYMPTOTIC EXPANSIONS FOR EIGENVALUES OF THE STEKLOV PROBLEM IN SINGULARLY PERTURBED DOMAINS

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Dedicated to Vladimir Andreevich Steklov

Abstract. Full asymptotic expansions are constructed and justified for two series of eigenvalues and the corresponding eigenfunctions of the spectral Steklov problem in a domain with a singular boundary perturbation having the form of a small cavity. The terms of those series are of type $\lambda_k + o(1)$ and $\varepsilon^{-1}(\mu_m + o(1))$, where λ_k and μ_m are the eigenvalues of the Steklov problem in a bounded domain without cavity and the exterior Steklov problem for a cavity of unit size. A similar problem of the surface wave is also treated. The smoothness requirements on the boundary are discussed and unsolved problems are stated.

§1. Introduction

1.1. Motivation. In this paper, we deal with the spectral Steklov problem

$$-\Delta u^\varepsilon(x) = 0, \qquad x \in \Omega(\varepsilon), \tag{1.1}$$

$$\partial_n u^\varepsilon(x) = \lambda^\varepsilon u^\varepsilon(x), \qquad x \in \partial\Omega(\varepsilon), \tag{1.2}$$

and with its versions pertaining to the linear theory of surface waves, in a singularly perturbed domain (see Figure 1)

$$\Omega(\varepsilon) = \Omega \setminus \overline{\omega_\varepsilon}. \tag{1.3}$$

Here Δ is the Laplace operator, Ω and ω are domains in the Euclidean space $\mathbb{R}^d$, $d \geq 2$, with smooth (of class C^∞, see Subsection 4.4 below) boundaries $\partial\Omega$ and $\partial\omega$ and with compact closures $\bar{\Omega} = \Omega \cup \partial\Omega$, $\bar{\omega} = \omega \cup \partial\omega$, and containing the origin $\mathcal{O}$ of the Cartesian coordinates $x = (x_1, \dots, x_d)$. Also in (1.1)–(1.3),

$$\omega_\varepsilon = \{x = (x_1, \dots, x_d) \in \mathbb{R}^d \,:\, \xi = \varepsilon^{-1} x \in \omega\}, \tag{1.4}$$

ε is a small parameter, and ∂_n stands for the derivative along the outer (relative to $\Omega(\varepsilon)$) normal on the surface $\partial\Omega(\varepsilon) = \partial\Omega \cup \partial\omega_\varepsilon$. In other words, the set (1.3) is obtained by removing from Ω a small inner cavity (1.4) (in Subsection 6.3 we also discuss the case of a small cavern, see Figure 2). The upper bound ε_0 for ε is fixed so that $\overline{\omega_\varepsilon} \subset \Omega$ for all $\varepsilon \in (0, \varepsilon_0]$; however, if needed, we reduce ε_0, keeping the notation.

We shall use the general approach [1, Chapters 9, 10] to the asymptotic analysis of spectral boundary-value problems in domains with singularly perturbed boundaries to construct the asymptotics of the eigenvalues

$$0 = \lambda_1^\varepsilon < \lambda_2^\varepsilon \leq \lambda_3^\varepsilon \leq \dots \leq \lambda_k^\varepsilon \leq \dots \to +\infty \tag{1.5}$$

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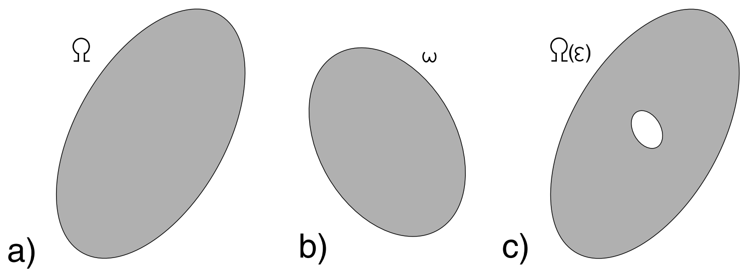

FIGURE 1. Initial (a and b) and singularly perturbed (c) domains.

of the Steklov problem (1.1), (1.2), and also the asymptotics of the corresponding eigenfunctions. Iteration processes and methods for justifying asymptotic expansions are fully known, and there are no obstructions for their realization. What turned out to be unexpected for the author is the result of [2] saying that the asymptotic series

$$\lambda_k^\varepsilon = \sum_{j=0}^{\infty} \varepsilon^j \lambda_k^{(j)} \tag{1.6}$$

converges, e.g., under the following conditions: $d \geq 3$, and $\lambda_k^0 = \lambda_k^{(0)}$ is a simple eigenvalue of the Steklov problem in the limiting domain Ω (if these conditions are violated, then the claim requires modification and loses elegance). The procedures of constructing and justifying the asymptotics presented in this paper and originating from the general algorithms [1] do not allow one to judge about the convergence of the series (1.6), indicating only an estimate $O(\varepsilon^{N+1})$ for the error occurring when the eigenvalue λ_k^ε is approximated by the partial sum

$$\sum_{j=0}^{N} \varepsilon^j \lambda_k^{(j)}, \tag{1.7}$$

however, for any natural N.

In the paper [2] and the preceding publications [3, 4] etc., an original approach was suggested, based on the analysis of the integral equation to which the Steklov boundary-value problem (1.1), (1.2) reduces. Unfortunately, this approach gives no explicit formulas for the terms in (1.6), proving only the real analyticity of the function

$$\varepsilon \mapsto \lambda_k^\varepsilon \tag{1.8}$$

for any k, e.g., under the conditions mentioned above. Being evidently significant, the result of [3] gives the impression that the spectrum of the singularly perturbed problem (1.1), (1.2) is simply obtained by an analytic perturbation of the spectrum

$$0 = \lambda_1^0 < \lambda_2^0 \leq \lambda_3^0 \leq \ldots \leq \lambda_k^0 \leq \ldots \to +\infty \tag{1.9}$$

of the Steklov problem in the intact domain Ω,

$$-\Delta v(x) = 0, \quad x \in \Omega, \quad \partial_n v(x) = \lambda v(x), \quad x \in \partial\Omega. \tag{1.10}$$

One of our goals in the present paper is to show that such an idyllic picture is alien for the entire spectrum, being only valid in the low-frequency part of the spectrum. Namely, in the sequence (1.5) we find another series of eigenvalues with regular asymptotics

$$\lambda_{M(\varepsilon)}^\varepsilon = \varepsilon^{-1} \mu_m^0 + O(1). \tag{1.11}$$

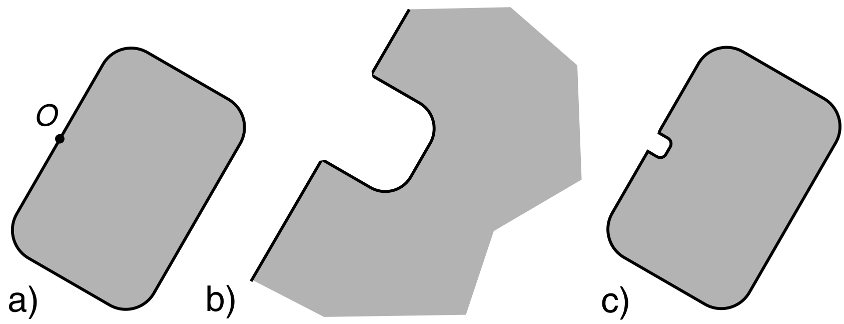

FIGURE 2. A singularly perturbed domain with a small cavity.

Here μ_m^0 is an element of the sequence

$$0 < \mu_1^0 < \mu_2^0 \le \mu_3^0 \le \ldots \le \mu_m^0 \le \ldots \to +\infty \tag{1.12}$$

of eigenvalues for the outer Steklov problem

$$\begin{aligned} -\Delta_\xi w(\xi) = 0, \quad \xi \in \mathbb{R}^d \setminus \bar{\omega}, \\ \partial_\nu w(\xi) = \mu w(\xi), \quad \xi \in \partial\omega, \end{aligned} \tag{1.13}$$

in which the dilated coordinates ξ as in (1.4) are used, and ∂_ν denotes the derivative along the outer (inner relative to the hole ω) normal in the same coordinates. The number $M(\varepsilon)$ of the eigenvalue (1.8) depends on the small parameter ε and grows unboundedly as $\varepsilon \to +0$. A fairly rough picture looks like this: when ε reduces from ε_0 to zero, the eigenvalue (1.11) runs to infinity, colliding alternately with the eigenvalues (1.9) and, thereby, increasing its number in the sequence (1.5), which leads to violating the stable nature of perturbations in the high-frequency range of the spectrum. This circumstance has two important consequences. First, we can get good estimates for the errors caused by the replacement of the functions (1.8) by the sums (1.7) only for $\varepsilon \in (0, \varepsilon_{kN}]$, and, as usual, $\varepsilon_{kN} \to +0$ as either of the natural indices k or N grows unboundedly. Second, it is usually hard to refine the asymptotic representations (1.11) in the high-frequency range.

The arising itself of different asymptotic series of eigenvalues is well knows and has been studied for many singularly perturbed problems (see, e.g., the survey [5]). In particular, a mathematical machinery was created for distinguishing the zones where various asymptotic formulas act; this techniques are based on *direct* and *inverse* reductions and involve individual and collective asymptotics, see [6] and, e.g., [7, 8]. Therefore, it is not hard to predict that the analyticity intervals $(0, \varepsilon_{k\infty})$ of the functions (1.8) will narrow rapidly when k grows (see the end of the preceding paragraph), but any estimates of their size remain unknown.

A new and, again, unexpected observation, which prompted the author to writing the present paper, is as follows: we can build and justify the full asymptotic expansion[1] for any eigenvalue (1.11) that remains in vicinity of a point $\varepsilon^{-1}\mu_m$ with a simple[2] eigenvalue μ_m of problem (1.13). As has already been mentioned, in high-frequency asymptotics, no explicit formulas are available for the main error terms (see the survey [5] once again);

[1]However, as usual, one can get an estimate for the discrepancy of approximation by a partial sum of the asymptotic series only when $\varepsilon \in (0, \varepsilon^{kN})$ and $\varepsilon^{kN} \to +0$ as $k \to \infty$ or $N \to \infty$.

[2]Some problems concerning full asymptotic expansions of the eigenvalues coming from *multiple* eigenvalues of limiting problems remain open in the low- as well as high-frequency parts of the spectrum of singularly (and even regularly) perturbed problems. These issues will be discussed in Subsections 2.4 and 3.4.

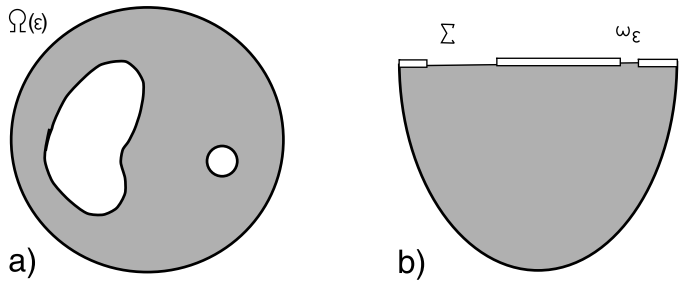

FIGURE 3. A pond with a piece of ice (Ω_ε), a clearing (Σ), and an ice-hole (ω_ε); view from above (a) and side view (b).

the author knows only one paper [9] where, in the case of the Steklov problem on the concatenation of a thin and massive domains (pond with a bank), the main discrepancies were found explicitly, but only for narrow ranges of the small parameter.

Possibly, the full asymptotic expansions described in §3 are the first result of this sort. Note that they are only obtained in the case where the outer boundary $\partial\Omega$ of the domain $\Omega(\varepsilon)$ is C^∞ (all iteration processes fit also for the Lipschitz boundary $\partial\omega$ of the hole ω; see Subsections 4.4 and 6.1). The reason is that, in accordance with the asymptotic *Anzatz* (1.11), under the change $\lambda^\varepsilon \mapsto \varepsilon^{-1}\mu^\varepsilon$ the boundary condition (1.2) on $\partial\Omega$ acquires the small parameter in the coefficient of the leading derivative and reshapes to the Dirichlet condition in the limit. But the term $\varepsilon^{-1}\partial_n$, treated formally as a small perturbation, is in fact not such, aggravating any singularity of the solution of the Dirichlet problem in Ω and expelling the asymptotic terms from the natural Sobolev class. Some consequences of the smoothness loss will be discussed in Subsections 4.4 and 5.3 by the example of a problem pertaining to the linear theory of waves on the surface of a weighty liquid; now we pass to the setting of such a problem.

1.2. Interaction of surface waves in clearings and ice-holes. One of physical interpretations of the Sobolev boundary conditions is related to propagation of surface waves within the framework of linear theory [10] (see the surveys [11, 12] and, e.g., the book [13]). We describe the setting of a problem about a pond covered with ice in which there are several large clearings and one (for simplicity) ice-hole of relatively small diameter.

Let Ω and ω be plane domains containing the origin of the coordinates $y = (y_1, y_2)$ and having smooth boundaries and compact closures. The domain Ω is assumed to be multiply connected, and ω is simply connected, i.e., the boundary $\partial\omega$ is a simple smooth closed contour, while the boundary $\partial\Omega$ consists of an outer contour Γ and several contours $\Gamma_1, \dots, \Gamma_J$ embracing the holes $\Sigma_1, \dots, \Sigma_J$; here $J \geq 1$. We denote by Ξ the 3-dimensional domain that lies in the lower half-space $\mathbb{R}^3_- = \{x = (y, z) : x_3 = z < 0\}$ and is bounded by the flat water surface $\Omega \cup \overline{\Sigma_1} \cup \dots \cup \overline{\Sigma_J} \subset \partial\mathbb{R}^3_-$, and also by a surface Υ (the bottom of the pond) lying in $\mathbb{R}^3_-$ and making nonzero angles[3] with the plane $\partial\mathbb{R}^3_- = \{x : z = 0\}$ along Γ. In what follows we shall non distinguish notationally between two-dimensional sets and their immersions lying on the plane $\partial\mathbb{R}^3_-$ in $\mathbb{R}^3$.

The volume Ξ is filled with water, and the area $\Omega(\varepsilon) = \Omega \setminus \overline{\omega_\varepsilon}$ of its surface is covered with an underformable piece of ice with clearings $\Sigma_1, \dots, \Sigma_J$ and an ice-hole ω_ε (see

[3]This requirement ensures the discreteness of the spectrum of problem (1.14)–(1.3): peak-like cusps can lead to the arising of a nonempty essential component in that spectrum (see [14, 15]).

Figure 3). The velocity potential u^ε satisfies the Laplace equation

$$-\Delta u^\varepsilon(x) = 0, \quad x \in \Xi, \tag{1.14}$$

the Neumann condition

$$\partial_n u^\varepsilon(x) = 0, \quad x \in \Upsilon \cup \Omega(\varepsilon) \tag{1.15}$$

(no normal flow), and the following kinematic condition on the free surfaces:

$$\partial_z u^\varepsilon(x) = \lambda^\varepsilon u^\varepsilon(x), \quad x \in \Sigma_1 \cup \cdots \cup \Sigma_J \cup \omega_\varepsilon, \tag{1.16}$$

where $\lambda^\varepsilon = \kappa^2/g$, κ is the frequency of harmonic (in time) oscillations of the liquid, and $g > 0$ is the free fall acceleration due to gravity.

Introducing the "trace" operator $\mathcal{T}^\varepsilon$ (see [16] and Subsection 2.5 below), we can reduce problem (1.14)–(1.16) to the study of the spectrum of a compact positive selfadjoint operator; this allows us to indicate the sequence (1.5) of eigenvalues the asymptotics of which will be studied in §4.

In Subsection 5.3 we also deal with a planar version of problem (1.14)–(1.16). Its physical interpretation causes doubt: the middle piece of ice in Figure 3,b is not connected with shores and must flow freely. Nevertheless, this two-dimensional problem allows us to discuss some new boundary layer phenomena.

1.3. Structure of the paper. In the next section, for $d \geq 3$, we describe in detail the procedure of constructing the coefficients of the asymptotic series for λ_k^ε and u_k^ε in the cases of a simple or a multiple eigenvalue λ_k^0 of the limiting problem (1.10), and obtain explicit formulas for the leading terms (Lemmas 2.1 and 2.2). Also, we prove Theorem 2.1 on the convergence $\lambda_k^\varepsilon \to \lambda_k^0$, together with an estimate for the rate of convergence (Theorem 2.2). Finally, in Theorem 2.3, for a simple eigenvalue λ_k^0, we estimate the errors of approximation of λ_k^ε and u_k^ε by partial sums of our formal asymptotic series. Note once again that the methods employed do not allow us to establish the convergence of the series (1.6).

In §3, we study the asymptotics of the eigenvalues (1.11) in the high-frequency range of the spectrum of problem (1.1), (1.2), again in the case where $d \geq 3$. The main attention is paid to the case of a simple eigenvalue μ_m^0 of problem (1.13), for which we can build full asymptotic series for $\lambda_{M^\varepsilon(m)}^\varepsilon$ and for $u_{M^\varepsilon(m)}^\varepsilon$; Theorem 3.2 justifies these asymptotic constructions. Also, we find the leading terms of the asymptotics corresponding to perturbation of a multiple eigenvalue (Theorem 3.1).

§4 is devoted to the 3-dimensional problem (1.14)–(1.16) on surface waves. The presentation of asymptotic procedures here is independent of the preceding sections, being based largely on arguments of physical nature.

Therefore, the reader interested only in the formal setting may restrict himself to the reading of §4 and Subsection 5.3. We only find the main asymptotic discrepancies in both the low- and high-frequency ranges, and Theorems 4.1 and 4.2 provide approximation error estimates. In the first case, we ignore the lower terms for simplicity; on the contrary, in the second case it is impossible to determine the second order errors without modification of the asymptotic constructions, for the reasons explained in Subsections 4.4 and 5.3.

In §5 we discuss the specifics of asymptotic expansions in planar ($d = 2$) problems (1.1), (1.2) and (1.14)–(1.16) and make an attempt (however, unsuccessful) to compensate for the singularities arising near an irregular point and the contours of the boundary.

The last §6 contains comments on the presented asymptotic analysis and discussion of similar spectral problems in singularly perturbed domains.

§2. Asymptotics in the low-frequency range

2.1. Asymptotic Ansätze. For $d \geq 3$ (see Subsection 5.1 for the case of the planar problem), the standard formal asymptotic series (cf. [17, 18] and [1, Chapter 9]) for solutions of the spectral problem (1.1), (1.2) looks like this:

$$\lambda_k^\varepsilon \sim \sum_{j=0}^{\infty} \varepsilon^j \lambda_k^{(j)}, \tag{2.1}$$

$$u_k^\varepsilon(x) \sim \sum_{j=0}^{\infty} \varepsilon^j \big(v_k^{(j)}(x) + \varepsilon w_k^{(j)}(\xi)\big). \tag{2.2}$$

Here, $\lambda_k^0 = \lambda_k^{(0)}$ and $v_k^{(0)}$ are the eigenvalue and the corresponding eigenfunction of the first limiting problem (1.10) and $w_k^{(j)}$ is a boundary layer type term written in the dilated coordinates $\xi = \varepsilon^{-1}x$ (see formula (1.4)). The coefficients of the asymptotic series (2.1) and (2.2) are to be determined, and, as we shall see later on,

$$\lambda_k^{(1)} = \ldots = \lambda_k^{(d-2)} = 0, \quad v_k^{(1)} = \ldots = v_k^{(d-2)} = 0. \tag{2.3}$$

For the functions $v_k^{(j)}$ and $w_k^{(j)}$ we write the following expansions, also to be verified, near $x = 0$ and near infinity:

$$v_k^{(j)}(x) = \sum_{p=0}^{N} r^p V_{kp}^{(j)}(\vartheta) + \widetilde{v}_{kN}^{(j)}(x), \tag{2.4}$$

$$w_k^{(j)}(\xi) = \sum_{p=0}^{N} \rho^{2-d-p} W_{kp}^{(j)}(\vartheta) + \widetilde{w}_{kN}^{(j)}(\xi). \tag{2.5}$$

Here, $r = |x|$, $\rho = |\xi|$, and ϑ is a point of the unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$. The angular parts $V_{kp}^{(j)}$ and $W_{kp}^{(j)}$ are smooth functions on $\mathbb{S}^{d-1}$, and the remainder terms $\widetilde{v}_{kN}^{(j)}$ and $\widetilde{w}_{kN}^{(j)}$ satisfy the estimates

$$\big|\nabla_x^q \widetilde{v}_{kN}^{(j)}(x)\big| \leq c_{kN}^{jq} r^{N+1-q}, \quad x \in \Omega, \tag{2.6}$$

$$\big|\nabla_\xi^q \widetilde{w}_{kN}^{(j)}(\xi)\big| \leq c_{kN}^{jq} \rho^{2-d-N-1-q}, \quad \xi \in \mathbb{R}^d \setminus \bar{\omega}, \tag{2.7}$$

for all $q \in \mathbb{N}_0 = \{0\} \cup \mathbb{N} = \{0, 1, 2, \ldots\}$, where $\nabla_x^q v$ is the collection of all derivatives of v of order q, and the notation $\nabla_\xi^q w$ has a similar meaning. Recall that the point $\xi = 0$ is assumed to lie inside the domain ω, i.e., $\rho \geq \rho_\omega > 0$ if $\xi \in \mathbb{R}^d \setminus \omega$. In fact, the functions $r^p V_{kp}^{(j)}(\vartheta)$ are polynomials of degree p in the variables $x = (x_1, \ldots, x_d)$, and the functions $\rho^{2-d-p} W_{kp}^{(j)}(\vartheta)$ are linear combinations of the derivatives of order p of the fundamental solution

$$\Phi(\xi) = \big(\operatorname{meas}_{d-1} \mathbb{S}^{d-1}\big)^{-1} (d-2)^{-1} |\xi|^{2-d} \tag{2.8}$$

for the Laplace operator Δ_ξ in $\mathbb{R}^d$ for $d \geq 3$. This refinement will not be needed in what follows, with the exception of the important formula

$$r^0 V_{k0}^{(j)}(\vartheta) = v_k^{(j)}(0) \in \mathbb{R}. \tag{2.9}$$

Note that problem (1.10) has an unbounded monotone sequence of eigenvalues (1.9) (written with regard to multiplicity), and the corresponding eigenfunctions

$$v_1^0, \ v_2^0, \ v_3^0, \ \ldots, \ v_k^0, \ \ldots$$

can be chosen to obey the orthogonality and normalization conditions

$$(v_k^0, v_l^0)_{\partial\Omega} = \delta_{k,l}, \quad k, l \in \mathbb{N} := \{1, 2, 3, \ldots\}, \tag{2.10}$$

where $(\ ,\ )_{\partial\Omega}$ is the natural scalar product in the Lebesgue space $L^2(\partial\Omega)$, and $\delta_{k,l}$ is the Kronecker symbol. The first eigenvalue is simple, and the corresponding eigenfunction $v_1^0(x) = (\mathrm{meas}_{d-1}\,\partial\Omega)^{-1/2}$ is constant.

2.2. Algorithm of constructing the asymptotics. Let $\lambda_k^0 > 0$ be a simple eigenvalue of the Steklov problem (1.10) in Ω (of course, the case of $k = 0$ is not interesting, because the zero eigenvalue is not perturbed). We put $v_k^{(0)} = v_k^0$ and plug the formal asymptotic expansions (2.2) and (2.1) in problem (1.1), (1.2). We see immediately that the functions $v_k^{(j)}$ and $w_k^{(j)}$ are harmonic in the domains Ω and $\mathbb{R}^d \setminus \bar\omega$, respectively. In the boundary condition (1.2) restricted to the outer part $\partial\Omega$ of the boundary $\partial\Omega(\varepsilon)$, in the boundary layer type terms (2.5) we pass to the slow variables x and, recalling the relation $\rho = \varepsilon^{-1}r$, collect the coefficients of the same powers of ε. As a result, we get the boundary-value problem

$$-\Delta_x v_k^{(j)}(x) = 0, \quad x \in \Omega, \partial_n v_k^{(j)}(x) - \lambda_k^0 v_k^{(j)}(x) = f_k^{(j)}(x), \quad x \in \partial\Omega, \tag{2.11}$$

with the right-hand side

$$\begin{aligned} f_k^{(j)}(x) = \sum_{q=1}^{j} \lambda_k^{(q)} v_k^{(j-q)}(x) - \sum_{l=0}^{j-d+1} \partial_n\big(r^{2-d-l} W_{kl}^{(j-d+1-l)}(\vartheta)\big) \\ + \sum_{q=0}^{j-d+1} \lambda_k^{(q)} \sum_{l=0}^{j-d+1-q} r^{2-d-l} W_{kl}^{(j-d+1-q-l)}(\vartheta). \end{aligned} \tag{2.12}$$

Similar manipulations with the boundary condition (1.2) restricted to the inner part $\partial\omega_\varepsilon$ of $\partial\Omega(\varepsilon)$ and passage to the fast variables ξ in the smooth type solutions (2.4) lead to the exterior Neumann problem

$$-\Delta_\xi w_k^{(j)}(\xi) = 0, \quad \xi \in \mathbb{R}^d \setminus \bar\omega, \quad \partial_\nu w_k^{(j)}(\xi) = g_k^{(j)}(\xi), \quad \xi \in \partial\omega, \tag{2.13}$$

with the right-hand side

$$g_k^{(j)}(\xi) = \sum_{q=0}^{j-1} \lambda_k^{(q)} w_k^{(j-q-1)}(\xi) - \sum_{l=1}^{j+1} \partial_\nu\big(\rho^l V_{kl}^{(j+1-l)}(\vartheta)\big) + \sum_{q=0}^{j} \lambda_k^{(q)} \sum_{l=0}^{j-q} \rho^l V_{kl}^{(j-q-l)}(\vartheta). \tag{2.14}$$

It should be noted that the boundary condition has lost the term $\lambda_k^0 w_k^{(j)}$, which passed to formula (2.14) with $j \mapsto j+1$, because of arising of a large factor in the relation $\partial_n = \varepsilon^{-1}\partial_\nu$. Also, the index $l = 0$ disappeared from the second sum, due to (2.9). Thus, the expression (2.14) turns out to depend only on the following collections of quantities:

$$\lambda_k^0, \dots, \lambda_k^{(j)}; \quad v_k^{(0)}, \dots, v_k^{(j)}; \quad w_k^{(0)}, \dots, w_k^{(j-1)}. \tag{2.15}$$

Writing the representation

$$f_k^{(j)}(x) = \lambda_k^{(j)} v_k^{(0)}(x) + \mathbf{f}_k^{(j)}(x), \tag{2.16}$$

i.e., chipping off one ($q = j$) summand from the first sum on the right in (2.12), we see that the remainder $\mathbf{f}_k^{(j)}$ depends on the quantities

$$\lambda_k^0, \dots, \lambda_k^{(j-1)}; \quad v_k^{(0)}, \dots, v_k^{(j-1)}; \quad w_k^{(0)}, \dots, w_k^{(j-d+1)}. \tag{2.17}$$

We use induction to describe an iteration process.

Base of induction. We take a simple eigenvalue λ_k^0 and the corresponding eigenfunction $v_k^{(0)} = v_k^0$ normalized as in (2.10) and then find the boundary layer $w_k^{(0)}$ from the problem

$$\begin{aligned} -\Delta_\xi w_k^{(0)}(\xi) = 0, \quad \xi \in \mathbb{R}^d \setminus \bar\omega, \\ \partial_\nu w_k^{(0)}(\xi) = -\partial_\nu\big(\xi \cdot \nabla_x v_k^{(0)}(0)\big) + \lambda_k^0 v_k^{(0)}(0), \quad \xi \in \partial\omega. \end{aligned} \tag{2.18}$$

The right-hand side of the boundary condition is formed in accordance with relation (2.14) and the Taylor formula at the point $\mathcal{O}$ for the infinitely differentiable function $v_k^{(0)}$; we have

$$r^0 V_{k0}^{(0)}(\vartheta) = v_k^{(0)}(0), \quad r^1 V_{k1}^{(1)}(\vartheta) = x \cdot \nabla_x v_k^{(0)}(0) = \sum_{m=1}^{d} x_m \frac{\partial v_k^{(0)}}{\partial x_m}(0).$$

Since $d \geq 3$ throughout in this section, the exterior Neumann problem (2.18) has a unique solution decaying at infinity and admitting the expansion (2.5), (2.7).

Induction step. Suppose we have made the steps with the numbers $0, \dots, j-1$; we show how to make the jth step. Since λ_k^0 is a simple eigenvalue for the formally selfadjoint problem (2.11), the Fredholm alternative yields the solvability condition

$$(f_k^{(j)}, v_k^0)_{\partial\Omega} = 0, \tag{2.19}$$

which takes the form

$$\lambda_k^{(j)} = \lambda_k^{(j)}(v_k^0, v_k^0)_{\partial\Omega} = -(\mathbf{f}_k^{(j)}, v_k^0)_{\partial\Omega}, \tag{2.20}$$

due to (2.16) and (2.10). As has already been mentioned, the expression $\mathbf{f}_k^{(j)}$ only depends on the quantities (2.17), which have already been found by the inductive assumption. So, problem (2.11) has a solution. This solution admits the representation (2.4), (2.6) (the Taylor formula) and is determined uniquely up to the term $c_k^{(j)} v_k^0$, but this arbitrariness is eliminated by the orthogonality condition

$$(v_k^{(j)}, v_k^0)_{\partial\Omega} = 0, \quad j \in \mathbb{N}. \tag{2.21}$$

As a result, we can calculate all quantities occurring in the list (2.15), and we can find the solution $w_k^{(j)}$ of the exterior Neumann problem (2.13), which is known to satisfy (2.5), (2.7). This completes the induction step.

2.3. Specific formulas. Now we calculate the main correction term in the asymptotics of the eigenvalue λ_k^ε assuming that the eigenvalue λ_k^0 is simple.

Lemma 2.1. *Relations* (2.3) *are valid, and*

$$\lambda_k^{(d-1)} = -\lambda_k^0 \operatorname{meas}_{d-1}(\partial\omega) |v_k^0(0)|^2, \tag{2.22}$$

where $\operatorname{meas}_{d-1}(\partial\omega)$ *is the* $(d-1)$*-dimensional area of the surface* $\partial\omega$.

Proof. First we note that by (2.12) and (2.16) we have

$$\mathbf{f}_k^{(j)}(x) = \sum_{q=1}^{j-1} \lambda_k^{(q)} v_k^{(j-q)}(x) \ \text{ for } \ j = 1, \dots, d-2, \tag{2.23}$$

$$\mathbf{f}_k^{(d-1)}(x) = \sum_{q=1}^{d-2} \lambda_k^{(q)} v_k^{(d-1-q)}(x) - \partial_n\big(r^{2-d} W_{k0}^{(0)}(\vartheta)\big) + \lambda_k^0 r^{2-d} W_{k0}^{(0)}(\vartheta). \tag{2.24}$$

By (2.20) the orthogonality conditions (2.21) for the functions $v^{(1)}, \dots, v_k^{(d-2)}$ mean that the first collection of relations (2.3) is fulfilled, so that $\mathbf{f}_k^{(1)} = \ldots = \mathbf{f}_k^{(d-2)} = 0$, and the second group is also fulfilled for the same reason.

The solution $w_k^{(0)}$ of the exterior Neumann problem (2.18) behaves at infinity as the fundamental solution (2.8), i.e., the last expression in (2.23) involves the leading term

$$\rho^{2-d} W_{k0}^{(0)}(\vartheta) = B_k^{(0)} \Phi(\xi) \tag{2.25}$$

of the expansion (2.5), $j = 0$. To calculate the factor $B_k^{(0)}$, we substitute the functions $w_k^{(0)}$ and 1 in the Green formula on the set $\mathbb{B}_R^d \setminus \bar{\omega}$, where $\mathbb{B}_R^d = \{\xi : \rho < R\}$ is a ball of a large radius R. We have

$$\int_{\partial\omega} \Big(\lambda_k^0 v_k^{(0)}(0) - \partial_\nu\big(\xi \cdot \nabla_x v_k^{(0)}(0)\big)\Big)\, ds_\xi = \int_{\partial\omega} \partial_\nu w_k^{(0)}(\xi)\, ds_\xi$$
$$= -\lim_{R\to\infty} \int_{\partial\mathbb{B}_R^d} \frac{\partial w_k^{(0)}}{\partial\rho}(\xi)\, ds_\xi = -B_k^{(0)} \lim_{R\to\infty} \int_{\partial\mathbb{B}_R^d} \frac{\partial\Phi}{\partial\rho}(\xi)\, ds_\xi = B_k^{(0)}.$$

Since $\int_{\partial\omega} \partial_\nu \xi_m\, ds_\xi = 0$, $m = 1, \dots, d$, we get

$$B_k^{(0)} = \lambda_k^0 v_k^{(0)}(0)\, \mathrm{meas}_{d-1}(\partial\omega). \tag{2.26}$$

It remains to calculate the integral

$$\lambda_k^{(d-1)} = -B_k^{(0)} \int_{\partial\Omega} \big(\lambda_k^0 \Phi(x) - \partial_n \Phi(x)\big) v_k^0(x)\, ds_x.$$

We apply the Green formula in the domain $\Omega \setminus \overline{\mathbb{B}^d}_\delta$ with a small spherical cavity to the harmonic functions Φ and v_k^0. Recalling the Steklov boundary condition in problem (1.10) for v_n^0, we write

$$\begin{aligned} \lambda_k^{(d-1)} &= B_k^{(0)} \lim_{\delta\to 0} \int_{\partial\mathbb{B}_\delta^d} \big(v_k^0(x)\partial_r\Phi(x) - \Phi(x)\partial_r v_k^0(x)\big)\, ds_x \\ &= B_k^{(0)} v_k^0(0) \lim_{\delta\to 0} \int_{\partial\mathbb{B}_\delta^d} \frac{\partial\Phi}{\partial r}(x)\, ds_x = -B_k^{(0)} v_k^0(0), \end{aligned} \tag{2.27}$$

Combined with (2.26), this ensures the desired result (2.22). □

Lemma 2.2. *If $v_k^0(0) = 0$, then, besides formula* (2.3), *we have*

$$\lambda_k^{(d-1)} = 0, \quad v_k^{(d-1)} = 0, \tag{2.28}$$

$$\lambda_k^{(d)} = -\sum_{m,l=1}^{d} M_{ml}(\omega) \frac{\partial v_k^{(0)}}{\partial x_m}(0) \frac{\partial v_k^{(0)}}{\partial x_l}(0), \tag{2.29}$$

where $(M_{kl}(\omega))_{k,l=1}^d$ is a symmetric positive definite matrix (*see below*).

Proof. By (2.26), the requirement $v_k^0(0) = 0$ means that $B_k^{(0)} = 0$; consequently, the right-hand side of (2.24) vanishes, together with the solution $v_k^{(d-1)}$ of problem (2.11) satisfying (2.21). Moreover, the solution $w_k^{(0)}$ of (2.18) can be written as

$$w_k^{(0)}(\xi) = -\sum_{m=1}^{d} \frac{\partial v_k^{(0)}}{\partial x_m}(0) \mathbf{w}_m(\xi), \tag{2.30}$$

where the $\mathbf{w}_m$ are the solutions of the external Neumann problem with the specific right-hand sides

$$-\Delta_\xi \mathbf{w}_m(\xi) = 0,\ \ \xi \in \mathbb{R}^d \setminus \bar{\omega}, \quad \partial_\nu \mathbf{w}_m(\xi) = \partial_\nu \xi_m,\ \ \xi \in \partial\omega,$$

as described, e.g., in [19, Appendix G]. It is known that these solutions decay at infinity at the rate of $O(\rho^{1-d})$ and admit the representation

$$\mathbf{w}_m(\xi) = \sum_{l=1}^{d} M_{ml}(\omega) \frac{\partial\Phi}{\partial\xi_l}(\xi) + O(\rho^{-d}), \quad \rho \to +\infty, \tag{2.31}$$

in which the coefficients of the derivatives of the fundamental solution form a symmetric positive definite matrix $M(\omega)$ of size $d \times d$. It is called the *matrix associated with the* *tensor of virtual mass*, and we have $M(\omega) \geq \mathbb{I}\,\mathrm{meas}_d\,\omega$, where $\mathrm{meas}_d\,\omega$ is the volume of the cavity, and $\mathbb{I}$ is the unit $(d \times d)$-matrix.

Now we use relations (2.3) and (2.28) to find that the right-hand (2.12) in problem (2.11) with the number $j = d$ takes the form

$$\begin{aligned} f_k^{(d)}(x) &= \lambda_k^{(d)} v_k^{(0)}(x) + \mathbf{f}_k^{(d)}(x) \\ &= \lambda_k^{(d)} v_k^{(0)}(x) - \partial_n\big(r^{1-d} W_{k1}^{(0)}(\vartheta)\big) + \lambda_k^{(0)} r^{1-d} W_{k1}^{(0)}(\vartheta) \\ &\qquad - \partial_n\big(r^{2-d} W_{k0}^{(1)}(\vartheta)\big) + \lambda_k^{(0)} r^{2-d} W_{k0}^{(1)}(\vartheta). \end{aligned}$$

We have

$$r^{1-d} W_{k1}^{(0)}(\vartheta) = -\sum_{m,l=1}^{d} M_{ml}(\omega) \frac{\partial v^{(0)}}{\partial x_m}(0) \frac{\partial \Phi}{\partial x_l}(x)$$

in accordance with the representations (2.30), (2.31), and (2.5) of the function $w_k^{(0)}$. At the same time, we shall not need to know the coefficient $B_k^{(1)}$ in the leading term of the expansion (2.5) of the function $w_k^{(1)}$ similar to (2.25).

As a result, we rewrite formula (2.20) with the number $j = d$ as follows:

$$\begin{aligned} \lambda_k^{(d)} &= -(\mathbf{f}_k^{(d)}, v_k^0)_{\partial\Omega} = B_k^{(1)} (\partial_n \Phi - \lambda_k^0 \Phi, v_k^0)_{\partial\Omega} \\ &\qquad - \sum_{m,l=1}^{d} M_{ml}(\omega) \frac{\partial v_k^{(0)}}{\partial x_m}(0) \Big(\partial_n \frac{\partial \Phi}{\partial x_l} - \lambda_k^0 \frac{\partial \Phi}{\partial x_l}, v_k^0\Big)_{\partial\Omega}. \end{aligned} \tag{2.32}$$

The first scalar product is equal to $-v_k^0(0) = 0$ (see (2.27) and the assumptions of the lemma). Moreover, we obtain

$$\begin{aligned} \Big(\partial_n \frac{\partial \Phi}{\partial x_l} - \lambda_k^0 \frac{\partial \Phi}{\partial x_l}, v_k^0\Big)_{\partial\Omega} &= \lim_{\delta\to 0} \int_{\partial\mathbb{B}_\delta^d} \Big(v_k^0(x) \frac{\partial}{\partial r} \frac{\partial \Phi}{\partial x_l}(x) - \frac{\partial \Phi}{\partial x_l}(x) \frac{\partial v_k^0}{\partial r}(x)\Big)\, ds_x \\ = \frac{1}{\mathrm{meas}_{d-1}\,\mathbb{S}^{d-1}} &\lim_{\delta\to 0} \int_{\partial\mathbb{B}_\delta^d} \Big(\big(v_k^0(0) + x\cdot\nabla_x v_k^0(0)\big)(d-1)\frac{x_l}{r^{d+1}} + \frac{x_l}{r^d}\frac{1}{r}\, x\cdot\nabla_x v_k^0(0)\Big)\, ds_x \\ = \frac{d}{\mathrm{meas}_{d-1}\,\mathbb{S}^{d-1}} &\frac{\partial v_k^0}{\partial x_l}(0) \lim_{\delta\to 0} \int_{\partial\mathbb{B}_\delta^d} \frac{x_l^2}{r^{d+1}}\, ds_x = \frac{\partial v_k^0}{\partial x_l}(0). \end{aligned}$$

Substituting the result in the right-hand side of (2.32), we arrive at (2.29). □

It should be mentioned that Lemmas 2.1 and 2.2 prescribe negative perturbations $\varepsilon^{d-1}\lambda_k^{(d-1)} + O(\varepsilon^d)$ and $\varepsilon^d \lambda_k^{(d)} + O(\varepsilon^{d+1})$ of the simple eigenvalue λ_k^0 in the cases where $v_k^0(0) \neq 0$ and $v_k^0(0) = 0$, $\nabla_x v_k^0(0) \neq 0 \in \mathbb{R}^d$. The sign of the perturbation of a simple eigenvalue will be discussed in Subsection 6.2.

2.4. Perturbation of a multiple eigenvalue. Suppose that, in the sequence (1.9), we have

$$\lambda_{k-1}^0 < \lambda_k^0 = \ldots = \lambda_{k+\varkappa_k-1}^0 < \lambda_{k+\varkappa_k}^0, \tag{2.33}$$

i.e., the eigenvalue λ_k^0 of problem (1.10) has multiplicity $\varkappa_k > 1$.

To a large extent, the procedure of constructing the asymptotics for the eigenvalues and eigenfunctions of problem (1.1), (1.2) remains the same as above. In particular, the general form of the *Ansätze* (2.1) and (2.2) is preserved, as well as the boundary-value problems (2.11), (2.12) and (2.13), (2.14). We recall that these problems are satisfied by the smooth type and the boundary layer type summands, which still admit the expansions (2.4), (2.6) and (2.5), (2.7). The eigenvalues $\lambda_k^\varepsilon, \ldots, \lambda_{k+\varkappa_k-1}^\varepsilon$ and the corresponding eigenfunctions $u_k^\varepsilon, \ldots, u_{k+\varkappa_k-1}^\varepsilon$ are studied simultaneously. Moreover, the leading components $v_k^{(0)}, \dots, v_{k+\varkappa_k-1}^{(0)}$ of the latter functions are sought in the form of linear combinations

$$v_p^{(0)}(x) = a_{p,k}^{(0)} v_k^0(x) + \dots + a_{p,k+\varkappa_k-1}^{(0)} v_{k+\varkappa_k-1}^0(x) \tag{2.34}$$

of eigenfunctions of problem (1.10) corresponding to the eigenvalue λ_k^0 and satisfying the orthogonality and normalization conditions (2.10). The rows of coefficients

$$a_{(p)}^{(0)} = (a_{p,k}^{(0)}, \dots, a_{p,k+\varkappa_k-1}^{(0)}) \in \mathbb{R}^{\varkappa_k} \tag{2.35}$$

turn out to be new unknowns.

Yet another novelty is the fact that, by the Fredholm alternative, the limiting Steklov problem (2.11) acquires $\varkappa_k$ solvability conditions

$$(f_p^{(j)}, v_m^0)_{\partial\Omega} = 0, \quad m = k, \dots, k + \varkappa_k - 1, \tag{2.36}$$

in place of one condition (2.19). Here and below, the objects supplied with indices $p = k, \dots, k + \varkappa_k - 1$ are calculated by the corresponding formulas in the preceding subsections with the replacement $k \mapsto p$. Thus, conditions (2.36) involve the right-hand (2.12) of problem (2.11) for determining the function $v_p^{(j)}$.

Repeating the calculations made in the proof of Lemma 2.1, we see that, even under the modified conditions, relations (2.3) remain valid with the replacements $k \mapsto p = k, \dots, k + \varkappa_k - 1$. Also, (2.16), (2.24), and (2.27) show that the solvability conditions (2.36) take the form

$$\begin{aligned} \lambda_m^{(d-1)} a_{p,m}^{(0)} &= \lambda_m^{(d-1)} (v_p^{(0)}, v_m^0)_{\partial\Omega} = -(\mathbf{f}_p^{(d-1)}, v_m^0)_{\partial\Omega} \\ &= -B_p^{(0)} (\lambda_k^0 \Phi - \partial_n \Phi, v_m^0)_{\partial\Omega} = -B_p^{(0)} v_m^0(0), \\ m &= k, \dots, k + \varkappa_k - 1. \end{aligned} \tag{2.37}$$

Here $B_p^{(0)} = \lambda_k^0 v_p^{(0)}(0) \operatorname{meas}_{d-1}(\partial\omega)$ is the coefficient (2.26) in the leading term (2.25) of the summand $w_p^{(0)}$ of the boundary layer type. As a result, equations (2.37) turn into the following linear system for the row (2.35):

$$P^{(k)} a_{(p)}^{(0)} = \lambda_p^{(d-1)} a_{(p)}^{(0)}. \tag{2.38}$$

The matrix $P^{(k)}$ of size $\varkappa_k \times \varkappa_k$ is composed of the elements

$$P_{mq}^{(k)} = -\lambda_k^0 \operatorname{meas}_{d-1}(\partial\omega) v_m^0(0) v_q^0(0) \tag{2.39}$$

and is symmetric and negative, but its rank cannot be greater than 1. Thus, it has $\varkappa_k - 1$ zero eigenvalues

$$\lambda_{k+1}^{(d-1)} = \ldots = \lambda_{k+\varkappa_k-1}^{(d-1)} = 0$$

and one nonpositive eigenvalue $\lambda_k^{(d-1)} \le 0$, which vanishes only if

$$v_k^0(0) = \ldots = v_{k+\varkappa_k-1}^0(0) = 0. \tag{2.40}$$

The corresponding eigenvectors-rows $a_{(k)}^{(0)}, \dots, a_{(k+\varkappa_k-1)}^{(0)} \in \mathbb{R}^{\varkappa_k}$ can be chosen to obey the orthogonality and normalization conditions

$$\sum_{l=k}^{k+\varkappa_k-1} a_{p,l}^{(0)} a_{m,l}^{(0)} = \delta_{p,m}, \quad p, m = k, \dots, k + \varkappa_k - 1. \tag{2.41}$$

Precisely the elements of these rows are the coefficients of the linear combinations (2.34). However, only under the additional condition

$$\varkappa_k = 2, \quad v_k^0(0) \neq 0 \tag{2.42}$$

are the rows $a^{(0)}_{(k)}$ and $a^{(0)}_{(k+1)}$ determined uniquely (up to the sign), i.e., the leading terms in (2.2) become fixed.

Assuming (2.40) and repeating the calculations of Lemma 2.2, we arrive at the formulas

$$\lambda_k^{(d-1)} = \ldots = \lambda_{k+\varkappa_k-1}^{(d-1)} = 0, \quad v_k^{(d-1)} = \ldots = v_{k+\varkappa_k-1}^{(d-1)} = 0,$$

similar to (2.28), and the compatibility conditions (2.36) of problem (2.11) for the functions $v_p^{(d)}, p = k, \ldots, k + \varkappa_k - 1$, give rise to the linear system

$$Q^{(k)} a^{(0)}_{(p)} = \lambda_p^{(d)} a^{(0)}_{(p)}, \tag{2.43}$$

involving a symmetric negative $(\varkappa_k \times \varkappa_k)$-matrix $Q^{(k)}$ with the entries

$$Q^{(k)}_{mq} = -\nabla_x v_m^0(0) \cdot M(\omega) \nabla_x v_q^0(0) = -\sum_{j,l=1}^{d} M_{j,l}(\omega) \frac{\partial v_m^0}{\partial x_j}(0) \frac{\partial v_q^0}{\partial x_l}(0) \tag{2.44}$$

(cf. the right-hand side of (2.29)). Since $M(\omega)$ is a symmetric positive definite matrix of size $(d \times d)$, the rank of $Q^{(k)}$ cannot exceed d, and

$$\operatorname{rank} Q^{(k)} = \dim \mathcal{L}\{\nabla_x v_k^0(0), \ldots, \nabla_x v^0_{k+\varkappa_k-1}(0)\},$$

where $\mathcal{L}\{\ldots\}$ is the linear span of the rows $\nabla_x v_m^0(0) \in \mathbb{R}^d$, $m = k, \ldots, k + \varkappa_k - 1$.

The raise of the rank of the matrix in (2.43) provides more possibilities for fixing the rows (2.35) of coefficients of the linear combinations (2.34). For instance, if the matrix $Q^{(k)}$ happens to have $\varkappa_k$ distinct eigenvalues $\lambda_k^{(d)}, \ldots, \lambda_{k+\varkappa_k-1}^{(d)}$, then the corresponding eigenvectors $a^{(0)}_{(k)}, \ldots, a^{(0)}_{(k+\varkappa_k-1)}$ satisfying (2.41) are determined up to the sign, and the leading terms (2.34) of the asymptotic *Ansatz* (2.2) are specified fully.

Note that in the situation described above the eigenvalues $\lambda_k^\varepsilon, \ldots, \lambda_{k+\varkappa_k-1}^\varepsilon$ of the singularly perturbed Steklov problem (1.1), (1.2) become simple.

On the other hand, if we have a multiple eigenvalue in the algebraic problem (2.43) (no matter zero or negative), then, as before, the leading terms of the *Ansatz* (2.2) needs refining via constructing lower asymptotic terms. The corresponding algorithms are known in principle, but they do not answer the following question: having constructed the full formal asymptotic series (2.1), can we judge whether a multiple eigenvalue $\lambda_k^0 = \ldots = \lambda_{k+\varkappa_k-1}^0$ splits into simple ones, or the collection $\lambda_k^\varepsilon, \ldots, \lambda_{k+\varkappa_k-1}^\varepsilon$ may contain multiple eigenvalues.

Of course, if the domain $\Omega(\varepsilon)$ has a geometric symmetry, then the Steklov problem (1.1), (1.2) is sure to possess multiple eigenvalues for all $\varepsilon \in (0, \varepsilon_0]$, but the above question is related to the possible existence of overpower (decaying as $o(\varepsilon^N)$ for any $N \in \mathbb{N}$) distances between two neighbors λ_p^ε and $\lambda_{p+1}^\varepsilon$ in the sequence (1.5). It is fairly plausible that the method of [2] can give a negative answer.

However, the lack of information on the leading term (2.34) makes no obstruction to justifying the constructed asymptotics in the next subsections. Therefore, now we restrict ourselves to a commentary on the situation where

$$v_k^0(0) \neq 0 \quad \text{and} \quad \lambda_k^{(d-1)} < 0, \quad \text{but} \quad \varkappa_k > 2, \tag{2.45}$$

which supplements the situation (2.42), (2.40).

Keeping the orthogonality conditions (2.10), we can choose a basis in the eigenspace of problem (1.10) corresponding to $\lambda = \lambda_k^0$ so that

$$v_k^0(0) \neq 0, \quad v_{k+1}^0(0) = \ldots = v_{k+\varkappa_k-1}^0(0) = 0. \tag{2.46}$$

Then in the matrix $P^{(k)}$ only one entry

$$P^{(k)}_{kk} = -\lambda_k^0 |v_k^0(0)|^2 \operatorname{meas}_{d-1}(\partial\omega) \quad (\text{equal to } \lambda_k^{(d-1)}),$$

is nonzero, whence $a^{(k)}_{(k)} = (1, 0, \dots, 0) \in \mathbb{R}^{\varkappa_k}$. We put $a^{(k)}_{pk} = 0$ for $p = k+1, \dots, k+\varkappa_k-1$ and then repeat the calculations of the proof of Lemma 2.2. It is easily seen that the truncated rows

$$a^{(k)\prime}_{(p)} = (a^{(k)}_{p,k+1}, \dots, a^{(k)}_{p,k+\varkappa_k-1}) \in \mathbb{R}^{\varkappa_k-1}$$

satisfy the linear system

$$Q^{(k)\prime\prime} a^{(k)\prime}_{(p)} = \lambda^{(d)}_p a^{(k)\prime}_{(p)}$$

with the matrix $Q^{(k)\prime\prime}$ of size $(\varkappa_k - 1) \times (\varkappa_k - 1)$ composed of the elements (2.44) with $m, q = k+1, \dots, k+\varkappa_k-1$. Again, the rank of $Q^{(k)\prime\prime}$ is at most d, but a possibility arose to detect new nontrivial perturbations of order of ε^d for the eigenvalue λ^0_k in the situation (2.45).

2.5. Operator setting of the Steklov problem. In the Sobolev space $H^1(\Omega(\varepsilon))$, we introduce the scalar product

$$\langle u^\varepsilon, v^\varepsilon\rangle_\varepsilon = (\nabla u^\varepsilon, \nabla v^\varepsilon)_{\Omega(\varepsilon)} + (u^\varepsilon, v^\varepsilon)_{\partial\Omega(\varepsilon)}. \tag{2.47}$$

Lemma 2.3. *Uniformly with respect to $\varepsilon \in (0, \varepsilon_0]$, the weighted Kondrat′ev*[4] *norm*

$$\|u^\varepsilon; V^1_0(\Omega(\varepsilon))\| = \left(\|\nabla u^\varepsilon; L^2(\Omega(\varepsilon))\|^2 + \|r^{-1}u^\varepsilon; L^2(\Omega(\varepsilon))\|^2\right)^{1/2}$$

is equivalent to the usual Sobolev norm $\|u^\varepsilon; H^1(\Omega(\varepsilon))\|$ and to the norm $\langle u^\varepsilon, u^\varepsilon\rangle^{1/2}_\varepsilon$ introduced above. We have

$$\|u^\varepsilon; L^2(\partial\omega_\varepsilon)\|^2 \le c\varepsilon\langle u^\varepsilon, u^\varepsilon\rangle_\varepsilon \tag{2.48}$$

where c is independent of $u^\varepsilon \in H^1(\Omega(\varepsilon))$ and of $\varepsilon \in (0, \varepsilon_0]$.

Proof. Let $\mathbb{B}^d_{R_\Omega}$ be a ball containing ω_ε for $\varepsilon \in (0, \varepsilon_0]$ and lying inside Ω together with the sphere $\partial\mathbb{B}^d_{R_\Omega}$. We recall the Steklov–Poincaré inequality

$$\|u^\varepsilon; L^2(\Omega \setminus \mathbb{B}^d_{R_\Omega})\|^2 \le c\left(\|\nabla u^\varepsilon; L^2(\Omega \setminus \mathbb{B}^d_{R_\Omega})\|^2 + \|u^\varepsilon; L^2(\partial\Omega)\|^2\right) \le c\langle u^\varepsilon, u^\varepsilon\rangle_\varepsilon$$

and the one-dimensional Hardy inequality

$$\int_\delta^{+\infty} r^{\alpha-1}|U(r)|^2\, dr \le \frac{4}{\alpha^2}\int_\delta^{+\infty} r^{\alpha+1}\left|\frac{dU}{dr}(r)\right|^2 dr, \tag{2.49}$$

valid for any $\delta > 0$ and $\alpha > 0$ for all $U \in C^\infty_c[\delta, +\infty)$ (continuously differentiable functions with compact support). We put $\alpha = d-2 > 0$ and $U(r, \vartheta) = \chi(x)u^\varepsilon(x)$, where $\chi \in C^\infty_c(\Omega)$ is a cut-off function equal to 1 on $\mathbb{B}^d_{R_\Omega}$. Assuming for simplicity[5] that the set ω is star-like relative to the coordinate origin, we pick an appropriate $\delta = \delta(\vartheta)$ and integrate inequality (2.49) over the angular variables $\vartheta \in \mathbb{S}^{d-1}$, obtaining the estimate

$$\begin{aligned}\|r^{-1}u^\varepsilon; L^2(\mathbb{B}^d_{R_\Omega} \setminus \omega_\varepsilon)\|^2 &\le c\|\nabla(\chi u^\varepsilon); L^2(\Omega(\varepsilon))\|^2\\ &\le c(\|\nabla u^\varepsilon; L^2(\Omega(\varepsilon))\|^2 + \|u^\varepsilon; L^2(\Omega \setminus \mathbb{B}^d_{R_\Omega}\|^2).\end{aligned}$$

Finally, we write the inequality

$$\varepsilon^{-1}\|u^\varepsilon; L^2(\partial\omega_\varepsilon)\|^2 \le c\left(\|\nabla u^\varepsilon; L^2(\mathbb{B}^d_{\varepsilon R_\omega} \setminus \omega_\varepsilon)\|^2 + \varepsilon^{-2}\|u^\varepsilon; L^2(\mathbb{B}^d_{\varepsilon R_\omega} \setminus \omega_\varepsilon)\|^2\right), \tag{2.50}$$

which is obtained from the standard trace inequality in $\mathbb{B}_{R_\omega} \setminus \bar\omega$ (see, e.g., [20, Chapter 1]) with the help of the coordinate compression $\xi \mapsto x = \varepsilon\xi$; here the radius R_ω is chosen so that $\bar\omega \in \mathbb{B}^d_{R_\omega}$. Since $\mathcal{O} \in \omega$ and $\varepsilon^{-1}r \ge c_\omega > 0$ for $x \in \mathbb{B}_{\varepsilon R_\omega} \setminus \omega_\varepsilon$, the right-hand side of (2.50) does not exceed $c\|u; V^1_0(\Omega(\varepsilon))\|^2$.

Combining the above estimates, we get the desired statement. □

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[4]We use the commonly adopted notation V^l_β, but not explain its origin.

[5]In the general situation, we need the extension (2.61) of u^ε to the cavity ω_ε.

Let $\mathcal{T}^\varepsilon$ be the operator given on the Hilbert space $\mathcal{H}^\varepsilon = H^1(\Omega(\varepsilon))$ with the scalar product (2.47) by the formula

$$\langle \mathcal{T}^\varepsilon u^\varepsilon, v^\varepsilon\rangle_\varepsilon = (u^\varepsilon, v^\varepsilon)_{\partial\Omega(\varepsilon)}, \quad u^\varepsilon, v^\varepsilon \in \mathcal{H}^\varepsilon. \tag{2.51}$$

This operator is symmetric, positive, and continuous, hence selfadjoint. The variational form of problem (1.1), (1.2) consists of finding a number λ^ε and a nontrivial function $u^\varepsilon \in \mathcal{H}^\varepsilon$ that satisfy the integral identity (see [20])

$$(\nabla u^\varepsilon, \nabla v^\varepsilon)_{\Omega(\varepsilon)} = \lambda^\varepsilon (u^\varepsilon, v^\varepsilon)_{\partial\Omega(\varepsilon)}, \quad v^\varepsilon \in \mathcal{H}^\varepsilon. \tag{2.52}$$

By the definitions (2.47) and (2.51), identity (2.52) is equivalent to the abstract equation

$$\mathcal{T}^\varepsilon u^\varepsilon = \tau^\varepsilon u^\varepsilon \quad \text{on} \quad \mathcal{H}^\varepsilon \tag{2.53}$$

with the new spectral parameter

$$\tau^\varepsilon = (1+\lambda^\varepsilon)^{-1}. \tag{2.54}$$

The properties of $\mathcal{T}^\varepsilon$ listed above show that, in accordance with Theorems 10.1.5 and 10.2.2 in [21], its spectrum $\sigma(\mathcal{T}^\varepsilon)$ is formed by the essential spectrum $\sigma_e(\mathcal{T}^\varepsilon) = \{\tau^\varepsilon = 0\}$ and the discrete spectrum $\sigma_d(\mathcal{T}^\varepsilon)$ consisting of an infinitely small sequence of eigenvalues

$$\tau_1^\varepsilon \ge \tau_2^\varepsilon \ge \tau_3^\varepsilon \ge \ldots \ge \tau_k^\varepsilon \ge \ldots \to +0, \tag{2.55}$$

Due to (2.54), this sequence is transformed into the sequence (1.5) of eigenvalues of the boundary-value problem (1.1), (1.2). The point $\tau^\varepsilon = 0$ is an eigenvalue of infinite multiplicity with the eigenspace

$$\{u^\varepsilon \in \mathcal{H}^\varepsilon : u^\varepsilon = 0 \text{ on } \partial\Omega(\varepsilon)\}.$$

The eigenvectors $u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon, \ldots, u_k^\varepsilon, \ldots \in \mathcal{H}^\varepsilon$ of the operator $\mathcal{T}^\varepsilon$, i.e., the eigenfunctions of the boundary-value problem (1.1), (1.2) or the variational problem (2.52) can be chosen so as to satisfy the orthogonality and normalization conditions

$$\langle u_k^\varepsilon, u_l^\varepsilon\rangle_\varepsilon = (\nabla u_k^\varepsilon, \nabla u_l^\varepsilon)_{\Omega(\varepsilon)} + (u_k^\varepsilon, u_l^\varepsilon)_{\partial\Omega(\varepsilon)} = \delta_{k,l}, \quad k, l \in \mathbb{N}. \tag{2.56}$$

The role of the main tool for justifying the asymptotics will be played by the following classical lemma about "near-eigenvalues" and "near-eigenvectors" (see the paper [22] and also the spectral resolvent expansion in Chapter 6 of the book [21]).

Lemma 2.4. *Suppose that, for some $t^\varepsilon \in \mathbb{R}_+$ and $\mathcal{U}^\varepsilon \in \mathcal{H}^\varepsilon$, we have*

$$\|\mathcal{U}^\varepsilon; \mathcal{H}^\varepsilon\| = 1, \quad \|\mathcal{T}^\varepsilon \mathcal{U}^\varepsilon - t^\varepsilon \mathcal{U}^\varepsilon; \mathcal{H}^\varepsilon\| =: \delta^\varepsilon \in (0, t^\varepsilon), \tag{2.57}$$

Then there exists an element τ_n^ε of the sequence (2.55) *such that*

$$|\tau_n^\varepsilon - t^\varepsilon| \le \delta^\varepsilon. \tag{2.58}$$

Moreover, for any $\delta_*^\varepsilon \in (\delta^\varepsilon, t^\varepsilon)$ we can find coefficients $b_n^\varepsilon, \ldots, b_{n+\kappa-1}^\varepsilon$ for which

$$\left\| \mathcal{U}^\varepsilon - \sum_{m=n}^{n+\kappa-1} b_n^\varepsilon u_n^\varepsilon; \mathcal{H}^\varepsilon \right\| \le 2\frac{\delta^\varepsilon}{\delta_*^\varepsilon}, \quad \sum_{j=n}^{n+\kappa-1} |b_m^\varepsilon|^2 = 1. \tag{2.59}$$

Here $\tau_n^\varepsilon, \ldots, \tau_{n+\kappa-1}^\varepsilon$ is the complete list of the eigenvalues of $\mathcal{T}^\varepsilon$ on the segment $[t^\varepsilon - \delta_*^\varepsilon, t^\varepsilon + \delta_*^\varepsilon]$, and the corresponding eigenvectors $u_n^\varepsilon, \ldots, u_{n+\kappa-1}^\varepsilon$ obey (2.56).

2.6. Convergence theorem. Let λ_k^ε be an eigenvalue of problem (1.1), (1.2), and let the corresponding eigenfunction u_k^ε be normalized as in (2.56). In the next subsection, it will be checked that for any fixed index $k \in \mathbb{N}$ there exist quantities $\varepsilon^{(k)} \in (0, \varepsilon_0]$ and $c^{(k)}$ such that

$$\lambda_k^\varepsilon \le c^{(k)} \quad \text{for} \quad \varepsilon \in (0, \varepsilon^{(k)}). \tag{2.60}$$

We extend the function u_k^ε to the cavity ω_ε by the formula

$$\widehat{u}_k^\varepsilon(x) = \begin{cases} u_k^\varepsilon(x) & \text{if } \ x \in \Omega(\varepsilon), \\ \overline{u}_k^\varepsilon + \widehat{u}_k^{\varepsilon\perp}(x) & \text{if } \ x \in \omega_\varepsilon, \end{cases} \tag{2.61}$$

where we have used the representation

$$\begin{gathered} u_k^\varepsilon(x) = \overline{u}_k^\varepsilon + u_k^{\varepsilon\perp}(x), \quad x \in \mathbb{B}^d_{\varepsilon R_\omega} \setminus \bar{\omega}_\varepsilon, \\ \overline{u}_k^\varepsilon = \big(\operatorname{meas}_d(\mathbb{B}^d_{\varepsilon R_\omega} \setminus \omega_\varepsilon)\big)^{-1} \int_{\mathbb{B}^d_{\varepsilon R_\omega} \setminus \omega_\varepsilon} u_k^\varepsilon(x)\, dx, \\ \int_{\mathbb{B}^d_{\varepsilon R_\omega} \setminus \omega_\varepsilon} u_k^{\varepsilon\perp}(x)\, dx = 0, \end{gathered} \tag{2.62}$$

and an extension $\widehat{\mathbf{u}}_k^{\varepsilon\perp}(\xi) = \widehat{u}_k^{\varepsilon\perp}(x)$ of the function

$$\mathbb{B}^d_{R_\omega} \setminus \omega \ni \xi \mapsto \mathbf{u}_k^{\varepsilon\perp}(\xi) = u_k^{\varepsilon\perp}(x)$$

to the set ω in the class H^1. Due to the last orthogonality condition in (2.62), we have the Poincaré inequality

$$\|u_k^{\varepsilon\perp}; L^2(\mathbb{B}^d_{\varepsilon R_\omega} \setminus \omega_\varepsilon)\|^2 \le c\varepsilon^2 \|\nabla u_k^{\varepsilon\perp}; L^2(\mathbb{B}^d_{\varepsilon R_\omega} \setminus \omega_\varepsilon)\|^2 = c\varepsilon^2 \|\nabla u_k^\varepsilon; L^2(\mathbb{B}^d_{\varepsilon R_\omega} \setminus \omega_\varepsilon)\|^2,$$

and the extension in question satisfies the estimate

$$\begin{aligned} &\varepsilon^{-2}\|\widehat{u}_k^{\varepsilon\perp}; L^2(\mathbb{B}^d_{\varepsilon R_\omega})\|^2 + \|\nabla \widehat{u}_k^{\varepsilon\perp}; L^2(\mathbb{B}^d_{\varepsilon R_\omega})\|^2 \\ &\quad = \varepsilon^{d-2}\|\widehat{\mathbf{u}}_k^{\varepsilon\perp}; H^1(\mathbb{B}^d_{R_\omega})\|^2 \le c h^{d-2}\|\mathbf{u}_k^{\varepsilon\perp}; H^1(\mathbb{B}^d_{R_\omega} \setminus \omega)\|^2 \\ &\quad = c\big(h^{-2}\|u_k^{\varepsilon\perp}; L^2(\mathbb{B}^d_{\varepsilon R_\omega} \setminus \omega_\varepsilon)\|^2 + \|\nabla u_k^{\varepsilon\perp}; L^2(\mathbb{B}^d_{\varepsilon R_\omega} \setminus \omega_\varepsilon)\|^2\big) \\ &\quad \le C\|\nabla u_k^\varepsilon; L^2(\mathbb{B}^d_{\varepsilon R_\omega} \setminus \omega_\varepsilon)\|^2. \end{aligned} \tag{2.63}$$

Combined with (2.56), (2.63), this implies

$$\|\nabla \widehat{u}_k^\varepsilon; L^2(\Omega)\| + \|\widehat{u}_k^\varepsilon; L^2(\partial\Omega)\| \le C^{(k)},$$

hence, by Lemma 2.3 applied in the domain $\Omega(0) = \Omega$, the Sobolev norms $\|\widehat{u}_k^\varepsilon; H^1(\Omega)\|$ are uniformly bounded. Thus, recalling (2.60), we can find an infinitely small sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ along which we have the convergences

$$\lambda_k^\varepsilon \to \lambda_k^\bullet, \ \ \widehat{u}_k^\varepsilon \to u_k^\bullet \ \text{ weakly in } H^1(\Omega) \text{ and strongly in } L^2(\Omega) \text{ and } L^2(\partial\Omega). \tag{2.64}$$

For any test function $v \in C_c^\infty(\bar{\Omega} \setminus 0)$ there exists a number $\varepsilon_v > 0$ depending on v such that $v = 0$ on $\overline{\omega_\varepsilon}$ for $\varepsilon \in (0, \varepsilon_v)$. Since $\widehat{u}_k^\varepsilon = u_k^\varepsilon$ outside ω_ε by (2.61), the integral identity (2.52) can be rewritten as

$$(\nabla \widehat{u}_k^\varepsilon, \nabla v)_\Omega = \lambda_k^\varepsilon (\widehat{u}_k^\varepsilon, v)_{\partial\Omega},$$

and then we can pass to the limit as $\varepsilon_j \to 0$, using (2.64). This results in the new integral identity

$$(\nabla u_k^\bullet, \nabla v)_\Omega = \lambda_k^\bullet (u_k^\bullet, v)_{\partial\Omega}, \tag{2.65}$$

in which, by closure, the test functions can be taken in the Sobolev class $H^1(\Omega)$, because the linear set $C_c^\infty(\bar{\Omega} \setminus \mathcal{O})$ is dense in that class.

Note that $\|u_k^\varepsilon; L^2(\partial\omega_\varepsilon)\|$ tends to zero as $\varepsilon \to +0$ by (2.48). Therefore, the integral identities (2.52) and (2.65), the convergences (2.64), and the normalization condition (2.56) show that

$$\begin{aligned}1 &= \langle u_k^\varepsilon, u_k^\varepsilon\rangle_\varepsilon = (1+\lambda_k^\varepsilon)\big(\|u_k^\varepsilon; L^2(\partial\Omega)\|^2 + \|u_k^\varepsilon; L^2(\partial\omega_\varepsilon)\|^2\big)\\ &= (1+\lambda_k^\varepsilon)\big(\|\widehat{u}_k^\varepsilon; L^2(\partial\Omega)\|^2 + \|u_k^\varepsilon; L^2(\partial\omega_\varepsilon)\|^2\big)\\ &\to\ (1+\lambda_k^\bullet)\|u_k^\bullet; L^2(\partial\Omega)\|^2 = \|\nabla u_k^\bullet; L^2(\Omega)\|^2 + \|u_k^\bullet; L^2(\partial\Omega)\|^2.\end{aligned}$$

Thus, $u_k^\bullet$ is a normalized eigenfunction of the limiting problem (1.10) corresponding to its eigenvalue $\lambda_k^\bullet$. The proof of the next statement will be finished in Subsection 2.7, because it remains to check estimate (2.60) and the fact that $\lambda_k^\bullet = \lambda_k^0$.

Theorem 2.1. *For the sequences* (1.5) *and* (1.9) *we have*

$$\lambda_k^\varepsilon \to \lambda_k^0, \quad k\in\mathbb{N}. \tag{2.66}$$

It should be noted that the index k in (2.66) is assumed to be fixed, so that, for sufficiently small $\varepsilon > 0$, "growing" eigenvalues (1.11) are excluded from consideration.

2.7. Justification of the eigenvalue asymptotics. For the role of approximate solutions of the abstract equation (2.53) we take

$$t_p^\varepsilon = \big(1+\lambda_p^0 + \varepsilon^{d-1}\lambda^{(d-1)}\big)^{-1}, \quad \mathcal{U}_p^\varepsilon = \|U_p^\varepsilon; \mathcal{H}^\varepsilon\|^{-1} U_p^\varepsilon, \tag{2.67}$$

$$U_p^\varepsilon(x) = v_p^{(0)}(x) + \varepsilon^{d-1} v_p^{(d-1)}(x) + \varepsilon\big(w_p^{(0)}(\varepsilon^{-1}x) + \widetilde{w}_p^\varepsilon(\varepsilon^{-1}x)\big). \tag{2.68}$$

Here, $v_p^{(0)}$ is the eigenfunction v_p^0 itself if the eigenvalue λ_p^0 of problem (1.10) is simple, and the linear combination (2.34) of the eigenfunctions $v_k^0, \dots, v_{k+\varkappa_k-1}^0$ if λ_p^0 is multiple (see condition (2.33)). The coefficient rows (2.35) are some eigenvectors of the matrix $P^{(k)}$ with entries (2.39) corresponding to its eigenvalues $\lambda_k^{(d-1)}, \dots, \lambda_{k+\varkappa_k-1}^{(d-1)}$. Also, $w_p^{(0)}$ and $v_p^{(d-1)}$ are solutions of problems (2.18) and (2.11), (2.16), (2.24), and the solvability of the latter problem is ensured by condition (2.36) (transformed into system (2.38)). Finally, $\widetilde{w}_p^\varepsilon$ is a function harmonic in $\mathbb{R}^d\setminus\bar\omega$ that satisfies

$$\big|\widetilde{w}_p^\varepsilon(\xi)\big| + \rho\big|\nabla_\xi \widetilde{w}_p^\varepsilon(\xi)\big| \le c_p\varepsilon\rho^{2-d}, \tag{2.69}$$

this function will be fixed later on. The factor c_p does not depend on $\varepsilon\in(0,\varepsilon_0]$.

First, we calculate the scalar products $\langle U_p^\varepsilon, U_q^\varepsilon\rangle_\varepsilon$ and check that

$$\big|\langle U_p^\varepsilon, U_q^\varepsilon\rangle_\varepsilon - (1+\lambda_p^0)\delta_{p,q}\big| \le c_{pq}\varepsilon^{d/2}. \tag{2.70}$$

For this, we observe that, by (2.10), (2.41), and Lemma 2.3,

$$\begin{gathered}(\nabla v_p^{(0)}, \nabla v_q^{(0)})_\Omega + (v_p^{(0)}, v_q^{(0)})_{\partial\Omega} = (1+\lambda_p^0)\delta_{p,q},\\ \big|(\nabla v_p^{(0)}, \nabla v_q^{(0)})_{\omega_\varepsilon}\big| \le c_{pq}\varepsilon^d, \quad \big|(v_p^{(0)}, v_q^{(0)})_{\partial\omega_\varepsilon}\big| \le c_{pq}\varepsilon^{d-1}.\end{gathered}$$

It is also clear that

$$\big|\langle v_p^{(0)}, \varepsilon^{d-1} v_q^{(d-1)}\rangle_\varepsilon\big| \le c\varepsilon^{d-1}, \quad \big|\langle v_p^{(0)}, \varepsilon(w_q^{(0)} + \widetilde{w}_q^\varepsilon)\rangle_\varepsilon\big| \le c\varepsilon^{d/2},$$

because

$$\begin{aligned}&\varepsilon^2\|w_q^{(0)}; \mathcal{H}^\varepsilon\|^2\\ &\quad\le c\varepsilon^2\left(\varepsilon^{d-2}\int_{\mathbb{R}^d\setminus\omega} |\nabla_\xi w_q^{(0)}(\xi)|^2\,d\xi + \varepsilon^{d-1}\int_{\partial\omega}|w_q^{(0)}(\xi)|^2\,ds_\xi + \int_{\partial\Omega}|\xi|^{2(2-d)}\,ds_x\right) \le c_q\varepsilon^d\end{aligned}$$

and a similar estimate is valid for $\varepsilon\widetilde{w}_p^\varepsilon$ by the assumption (2.69). Besides the coordinate dilation $x\mapsto\xi$, we have used formulas (2.5), (2.7) with $j=0$, $k=p$, $N=1$, and $q=0$.

Combined with elementary considerations, the above relations lead to inequality (2.70), which implies, in particular, that for any small ε we have

$$\|U_p^\varepsilon;\mathcal{H}^\varepsilon\| \geq c_p > 0. \tag{2.71}$$

We estimate the quantities δ_p^ε arising in Lemma 2.4. Applying formulas (2.47), (2.51), and (2.67), (2.68), and also recalling one of the definitions of the norm in Hilbert space, we obtain

$$\begin{aligned}\delta_p^\varepsilon &= \|\mathcal{T}^\varepsilon\mathcal{U}_p^\varepsilon - t_p^\varepsilon\mathcal{U}_p^\varepsilon;\mathcal{H}^\varepsilon\| = \sup|\langle\mathcal{T}^\varepsilon\mathcal{U}_p^\varepsilon - t_p^\varepsilon\mathcal{U}_p^\varepsilon, V^\varepsilon\rangle_\varepsilon| \\ &= \|U_p^\varepsilon;\mathcal{H}^\varepsilon\|^{-1}t_p^\varepsilon\sup\big|(\nabla U_p^\varepsilon,\nabla V^\varepsilon)_{\Omega(\varepsilon)} - \big(\lambda_p^0 + \varepsilon^{d-1}\lambda_p^{(d-1)}\big)(U_p^\varepsilon, V^\varepsilon)_{\partial\Omega(\varepsilon)}\big|.\end{aligned} \tag{2.72}$$

Here the supremum is over all $V^\varepsilon \in \mathcal{H}^\varepsilon$ such that $\|V^\varepsilon;\mathcal{H}^\varepsilon\| = 1$, and hence, by Lemma 2.3,

$$\|V^\varepsilon;H^1(\Omega)\| + \|V^\varepsilon;L^2(\partial\Omega)\| + \varepsilon^{-1/2}\|V^\varepsilon;L^2(\partial\omega_\varepsilon)\| \leq c.$$

The expression under the last supremum sign is equal to

$$\begin{aligned}-(\Delta_x U_p^\varepsilon, V^\varepsilon)_{\Omega(\varepsilon)} &+ \big(\partial_n U_p^\varepsilon - \big(\lambda_p^0 + \varepsilon^{d-1}\lambda_p^{(d-1)}\big)U_p^\varepsilon, V^\varepsilon\big)_{\partial\Omega} \\ &+ \big(\partial_n U_p^\varepsilon - \big(\lambda_p^0 + \varepsilon^{d-1}\lambda_p^{(d-1)}\big)U_p^\varepsilon, V^\varepsilon\big)_{\partial\omega_\varepsilon}.\end{aligned} \tag{2.73}$$

The first scalar product in (2.73) vanishes, because all terms of the sum in (2.68) are harmonic functions. On the external boundary $\partial\Omega$ we have

$$\begin{aligned}&\partial_n U_p^\varepsilon(x) - (\lambda_p^0 + \varepsilon^{d-1}\lambda_p^{(d-1)})U_p^\varepsilon(x) = \partial_n v_p^{(0)}(x) - \lambda_p^0 v_p^{(0)}(x) \\ &\quad + \varepsilon^{d-1}\big(\partial_n v_p^{(d-1)}(x) - \lambda_p^0 v_p^{(d-1)}(x) - \lambda_p^{(d-1)}v_p^{(0)}(x) + B_p^{(0)}(\partial_n\Phi(x) - \lambda_p^0\Phi(x))\big) \\ &\quad + \varepsilon\partial_n\big(w_p^{(0)}(\xi) - B_p^{(0)}\Phi(\xi) + \widetilde{w}_p^\varepsilon(\xi)\big) - \varepsilon\lambda_p^0\big(w_p^{(0)}(\xi) - B_p^{(0)}\Phi(\xi) + \widetilde{w}_p^\varepsilon(\xi)\big) \\ &\quad - \varepsilon^d\lambda_p^{(d-1)}\big(w_p^{(0)}(\xi) + \widetilde{w}_p^\varepsilon(\xi)\big) - \varepsilon^{2(d-1)}\lambda_p^{(d-1)}v_p^{(d-1)}(x).\end{aligned}$$

The first two summands on the right-hand side vanish in accordance with the definition of the asymptotic terms $v_p^{(0)}$ and $v_p^{(d-1)}$ of regular type, and the moduli of the other three terms are at most $c\varepsilon^d$ by estimates (2.7) and (2.69). Thus,

$$\big|\big(\partial_n U_p^\varepsilon - \big(\lambda_p^0 + \varepsilon^{d-1}\lambda_p^{(d-1)}\big)U_p^\varepsilon, V^\varepsilon\big)_{\partial\Omega}\big| \leq c_p\varepsilon^d.$$

On the inner boundary $\partial\omega_\varepsilon$ we write the identity

$$\begin{aligned}&\partial_n U_p^\varepsilon(x) - \big(\lambda_p^0 + \varepsilon^{d-1}\lambda_p^{(d-1)}\big)U_p^\varepsilon(x) \\ &\qquad = \partial_\nu w_p^{(0)}(\xi) + \partial_\nu\big(\xi\cdot\nabla_x v^{(0)}(0)\big) - \lambda_p^0 v_p^{(0)}(0) \\ &\qquad\qquad + \partial_\nu\widetilde{w}_p^\varepsilon(\xi) - \varepsilon\big(\lambda_p^0 + \varepsilon^{d-1}\lambda_p^{(d-1)}\big)\widetilde{w}_p^\varepsilon(\xi) - \widetilde{g}_p^\varepsilon(\xi),\end{aligned} \tag{2.74}$$

where

$$\begin{aligned}\widetilde{g}_p^\varepsilon(\xi) &= \varepsilon\big(\lambda_p^0 + \varepsilon^{d-1}\lambda_p^{(d-1)}\big)w_p^{(0)}(\xi) + \lambda_p^0\big(v_p^{(0)}(x) - v_p^{(0)}(0) + \varepsilon^{d-1}v_p^{(d-1)}(x)\big) \\ &\quad + \varepsilon^{d-1}\lambda_p^{(d-1)}\big(v_p^{(0)}(x) + \varepsilon^{d-1}v_p^{(d-1)}(x)\big) \\ &\quad - \varepsilon^{-1}\partial_\nu\big(v_p^{(0)}(x) - v_p^{(0)}(0) - x\cdot\nabla_x v_p^{(0)}(0) + \varepsilon^{d-1}v_p^{(d-1)}(x)\big).\end{aligned} \tag{2.75}$$

The first group of terms on the right in (2.74) vanishes due to the boundary conditions in problem (2.18). By smoothness, the Taylor formula for $v_p^{(0)}$ shows that the function (2.75) admits the estimates

$$\big|\nabla_{s(\xi)}^j\widetilde{g}_p^\varepsilon(\xi)\big| \leq c_j\varepsilon, \quad j\in\mathbb{N}_0.$$

Consequently, for small ε there exists a function satisfying (2.69) and solving the exterior problem

$$\begin{aligned}&-\Delta_\xi\widetilde{w}_p^\varepsilon(\xi) = 0, \quad \xi\in\mathbb{R}^d\setminus\bar{\omega}, \\ &\partial_\nu\widetilde{w}_p^\varepsilon(\xi) - \varepsilon\big(\lambda_p^0 + \varepsilon^{d-1}\lambda_p^{(d-1)}\big)\widetilde{w}_p^\varepsilon(\xi) = \widetilde{g}_p^\varepsilon(\xi), \quad \xi\in\partial\omega,\end{aligned} \tag{2.76}$$

which differs from the Neumann problem (2.13) only by a small perturbation of the boundary condition. Under such a choice of the extra summand $\varepsilon\widetilde{w}_p^\varepsilon$ in the *Ansatz* (2.68), the expression (2.74) and the last scalar product (2.73) turn out to be zero. As a result, using (2.67) and (2.71), we get the following estimate for the quantity (2.72):

$$\delta_p^\varepsilon \le c_p \varepsilon^d.$$

Since condition (2.57) of Lemma 2.4 is fulfilled, there exists an eigenvalue $\tau_{n(p)}^\varepsilon = (1+\lambda_{n(p)}^\varepsilon)^{-1}$ of the operator $\mathcal{T}^\varepsilon$ that satisfies inequality (2.58),

$$\left|\tau_{n(p)}^\varepsilon - \left(1+\lambda_p^0 + \varepsilon^{d-1}\lambda_p^{(d-1)}\right)^{-1}\right| \le c_p\varepsilon^d.$$

For ε small, simple transformations convert this inequality into the estimate

$$\left|\lambda_{n(p)}^\varepsilon - \lambda_p^0 - \varepsilon^{d-1}\lambda_p^{(d-1)}\right| \le C_p\varepsilon^d$$

for the eigenvalue of problem (1.1), (1.2). Now our nearest goal is to show that the indices p and $n(p)$ coincide.

If the eigenvalue λ_k^0 is simple or satisfies condition (2.46) (cf. (2.42) and (2.45)), then in the $C_k\varepsilon^d$-neighborhood of the point $\lambda_k^0 + \varepsilon^{d-1}\lambda_k^{(d-1)}$ we find at least one eigenvalue of problem (1.1), (1.2), but we cannot state so far that in a small neighborhood of an eigenvalue λ_k^0 of multiplicity $\varkappa_k$ there are at least $\varkappa_k$ distinct eigenvalues $\lambda_{n(k)}^\varepsilon, \dots, \lambda_{n(k)+\varkappa_k-1}^\varepsilon$. Note that the verification of this fact allows us to obtain relation (2.60), already used in Subsection 2.6: indeed, looking at the points $\lambda_1^0, \dots, \lambda_k^0$ we establish that $n(k) \ge k$, whence

$$\lambda_k^\varepsilon \le \lambda_{n(k)}^\varepsilon \le \lambda_k^0 + \varepsilon^{d-1}\lambda_k^{(d-1)} + C_k\varepsilon^d \le c_k. \tag{2.77}$$

Since the situations (2.40) and (2.46) are analyzed similarly, suppose for definiteness that we deal with the first of them (this case is more complicated), i.e., $\lambda_k^{(d-1)} = \dots = \lambda_{k+\varkappa_k-1}^{(d-1)} = 0$. Using the second part of Lemma 2.4 and taking

$$\delta_*^\varepsilon = \varepsilon^d D, \tag{2.78}$$

where $D > 0$ is large but fixed, we get normalized rows of coefficients

$$b^{(p)} = \left(b_{n(k)}^{(p)}, \dots, b_{n(k)+\kappa(k)-1}^{(p)}\right), \quad p = k, \dots, k+\varkappa_k - 1, \tag{2.79}$$

for which

$$\left\| \mathcal{U}_p^\varepsilon - \sum_{j=n(k)}^{n(k)+\kappa(k)-1} b_j^{(p)} u_j^\varepsilon; \mathcal{H}^\varepsilon \right\| \le 2\frac{\delta^\varepsilon}{\delta_*^\varepsilon} = \frac{2}{D}\max\{C_k, \dots, C_{k+\varkappa_k-1}\}. \tag{2.80}$$

Let $\mathcal{S}_p^\varepsilon$ denote the sum on the left in (2.80). Taking conditions (2.56) into account, we see that, first, $\|\mathcal{S}_p^\varepsilon; \mathcal{H}^\varepsilon\| = 1$ and, second,

$$\sum_{j=n(k)}^{n(k)+\kappa(k)-1} b_j^{(p)} b_j^{(q)} = \langle \mathcal{S}_p^\varepsilon, \mathcal{S}_q^\varepsilon\rangle_\varepsilon = \langle \mathcal{S}_p^\varepsilon - \mathcal{U}_p^\varepsilon, \mathcal{S}_q^\varepsilon\rangle_\varepsilon + \langle \mathcal{U}_p^\varepsilon, \mathcal{S}_q^\varepsilon - \mathcal{U}_q^\varepsilon\rangle_\varepsilon + \langle \mathcal{U}_p^\varepsilon, \mathcal{U}_q^\varepsilon\rangle_\varepsilon. \tag{2.81}$$

Formulas (2.70), (2.72), and (2.10) imply

$$|\langle \mathcal{U}_p^\varepsilon, \mathcal{U}_q^\varepsilon\rangle_\varepsilon - \delta_{p,q}| \le C_{pq}\varepsilon^{d/2}. \tag{2.82}$$

Together with formula (2.80), this estimate shows that the right-hand side in (2.81) is equal to $\delta_{p,q} + O(\varepsilon^{d/2} + D^{-1})$, that is, if ε is small and D is large, then the rows (2.79) are "almost orthonormal", which is possible only if

$$\varkappa_k \le \kappa(k).$$

Since $\tau^\varepsilon_{n(k)},\dots,\tau^\varepsilon_{n(k)+\kappa(k)-1}$ is the collection of eigenvalues of $\mathcal{T}^\varepsilon$ that lie on the segment $\big[(1+\lambda^0_k)^{-1}-D\varepsilon^d,(1+\lambda^0_k)^{-1}+D\varepsilon^d\big]$, it is easy to show that in the $C_k\varepsilon^d$-neighborhood of λ^0_k there are the eigenvalues $\lambda^\varepsilon_{n(k)},\dots,\lambda^\varepsilon_{n(k)+\kappa(k)-1}$. It remains to check that $n(k)=k$ and $\kappa(k)=\varkappa_k$.

Suppose that

$$n(k)+\kappa(k)>k+\varkappa_k. \tag{2.83}$$

By what was proved in this and the preceding subsections, each normalized eigenfunction u^ε_j of problem (1.1), (1.2) converges strongly in $L^2(\partial\Omega)$ to an eigenfunction $u^0_{m(j)}$ of problem (1.10), and $m(j)\neq m(k)$ if $j\neq k$. Thus, (2.83) implies that there is an eigenvalue $\lambda^\varepsilon_{m^\bullet(\varepsilon)}$ and an $\mathcal{H}^\varepsilon$-normalized eigenfunction $u^\varepsilon_{m^\bullet(\varepsilon)}$ such that

$$\begin{aligned}
&\lambda^\varepsilon_{m^\bullet(\varepsilon)}\to\lambda^\bullet_{m^\bullet}\in[0,\lambda^0_k],\\
&\|u^\varepsilon_{m^\bullet(\varepsilon)};L^2(\partial\omega_\varepsilon)\|\to 0,\quad u^\varepsilon_{m^\bullet(\varepsilon)}\to u^\bullet_{m^\bullet}\neq 0\quad\text{strongly in}\quad L^2(\partial\Omega),\\
&0=(u^\varepsilon_{m^\bullet(\varepsilon)},u^\varepsilon_{m(l)})_{\partial\Omega(\varepsilon)}\to(u^\bullet_{m^\bullet},u^0_l)_{\partial\Omega}=0,\quad l=1,\dots,k+\varkappa_k-1
\end{aligned}$$

(these convergences are along some infinitely small sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$). The last orthogonality conditions contradict the way of constructing the monotone sequence (1.9), i.e., in (2.83) we should replace the inequality sign $>$ by $\le$. Looking through the eigenvalues $\lambda^0_{k-1},\lambda^0_{k-2},\dots,\lambda^0_1$ one by one, we conclude that $n(k)=k$ and $\kappa(k)=\varkappa_k$.

Thus, the eigenvalues located in the $c\varepsilon^{d-1}$-neighborhood of an eigenvalue λ^0_k of multiplicity $\varkappa_k$ (the case of $\varkappa_k=1$ is not excluded) are precisely $\lambda^\varepsilon_k,\dots,\lambda^\varepsilon_{k+\varkappa_k-1}$. This completes the proof of Theorem 2.1. Moreover, the following statement has been established.

Theorem 2.2. *The rate of convergence in* (2.66) *is* $O(\varepsilon^{d-1})$, *and in the case of* (2.33) *we have*

$$|\lambda^\varepsilon_k-\lambda^0_k|\le C_k\varepsilon^{d-1},\quad |\lambda^\varepsilon_p-\lambda^0_k|\le c_k\varepsilon^d,\quad p=k+1,\dots,k+\varkappa_k-1. \tag{2.84}$$

2.8. Full asymptotic expansions. If λ^0_k is a simple eigenvalue of problem (1.10), then the role of approximate solutions of the abstract equations (2.53) will be played by

$$t^\varepsilon_{kN}=\Big(1+\lambda^0_k+\sum_{j=d-1}^N\varepsilon^j\lambda^{(j)}_k\Big)^{-1}\quad\text{and}\quad\mathcal{U}^\varepsilon_{kN}=\|U^\varepsilon_{kN};\mathcal{H}^\varepsilon\|^{-1}U^\varepsilon_{kN},$$

where N is an arbitrary fixed integer, and

$$U^\varepsilon_{kN}(x)=\sum_{j=1}^N\varepsilon^j(v^{(j)}_k(x)+\varepsilon w^{(j)}_k(\varepsilon^{-1}x))+\varepsilon^{1+N}\widetilde{w}^\varepsilon_{kN}(\varepsilon^{-1}x).$$

Here, the $v^{(j)}_k$ and $w^{(j)}_k$ are the smooth type and boundary layer type summands constructed in Subsections 2.2 and 2.3, and $\widetilde{w}^\varepsilon_{kN}$ is a small correction (cf. formulas (2.76) and (2.69)). Repeating with clear modifications the calculations made in the preceding subsection, we deduce the estimate

$$\delta^\varepsilon_{kN}=\|\mathcal{T}^\varepsilon\mathcal{U}^\varepsilon_{kN}-t^\varepsilon_{kN}\mathcal{U}^\varepsilon_{kN};\mathcal{H}^\varepsilon\|\le C_{kN}\varepsilon^{N+1},$$

which means, by Lemma 2.4, that the segment $\big[t^\varepsilon_{kN}-C_{kN}\varepsilon^{N+1},t^\varepsilon_{kN}+c_{kN}\varepsilon^{N+1}\big]$ contains an eigenvalue of the operator $\mathcal{T}^\varepsilon$. By Theorem 2.2, this eigenvalue is $\tau^\varepsilon_k=(1+\lambda^\varepsilon_k)^{-1}$. Consequently,

$$|(1+\lambda^\varepsilon_k)^{-1}-t^\varepsilon_{kN}|\le C_{kN}\varepsilon^{N+1}$$

or

$$\left|\lambda_k^\varepsilon - \lambda_k^0 - \sum_{j=d-1}^{N} \varepsilon^j \lambda_k^{(j)}\right| \le C_{kN}\varepsilon^{N+1} \quad \text{for} \quad \varepsilon \in (0, \varepsilon_{kN}]. \tag{2.85}$$

Now we put

$$\delta_* = \frac{1}{2}\min\left\{\frac{1}{1+\lambda_k^0} - \frac{1}{1+\lambda_{k+1}^0}, \frac{1}{1+\lambda_{k-1}^0} - \frac{1}{1+\lambda_k^0}\right\}$$

and apply the second part of Lemma 2.4. Again by Theorem 2.2, for small ε the segment $[t_{kN}^\varepsilon - \delta_*, t_{kN}^\varepsilon + \delta_*]$ contains precisely one eigenvalue τ_k^ε of $\mathcal{T}^\varepsilon$, i.e., formula (2.59) reshapes to

$$\|\mathcal{U}_{kN}^\varepsilon - b_k^\varepsilon u_k^\varepsilon; \mathcal{H}^\varepsilon\| \le 2\delta_*^{-1}\delta_{kN}^\varepsilon \le c_{kN}^*\varepsilon^{N+1},$$

where $|b_k^\varepsilon| = 1$. Thus, for $\mathbf{u}_k^\varepsilon = b_k^\varepsilon \|\mathcal{U}_{kN}^\varepsilon; \mathcal{H}^\varepsilon\| u_k^\varepsilon$ we obtain the estimate

$$\left\|\mathbf{u}_k^\varepsilon - \sum_{j=0}^{N} \varepsilon^j (v_k^{(j)} + \varepsilon w_k^{(j)}); H^1(\Omega(\varepsilon))\right\| \le C_{kN}\varepsilon^{N+1}. \tag{2.86}$$

Note that the smallness of the error term $\varepsilon^{N+1}\widetilde{w}_{kN}^\varepsilon$ has allowed us to remove it from the asymptotic construction.

We state the result

Theorem 2.3. *Let λ_k^0 be a simple eigenvalue of problem* (1.10). *For any natural $N \ge d-1$, there exist positive quantities ε_{kN} and c_{kN}, C_{kN} such that for $\varepsilon \in (0, \varepsilon_{kN}]$ the eigenvalue λ_k^ε of problem* (1.1), (1.2) *and the corresponding eigenfunction $\mathbf{u}_k^\varepsilon$, which is, in general, not normalized, but is such that*

$$\|\mathbf{u}_k^\varepsilon; L^2(\partial\Omega)\| = \|v_k^0; L^2(\partial\Omega)\| + O(\varepsilon) = 1 + O(\varepsilon),$$

satisfy inequalities (2.85) *and* (2.86).

Under condition (2.33), suppose that N terms of the series (2.1) for the eigenvalues $\lambda_k^\varepsilon, \dots, \lambda_{k+\varkappa_k-1}^\varepsilon$ have been constructed; then relations of the form (2.85) are fulfilled for them. The situation with eigenfunctions is more complicated because of the possible uncertainty in the choice of the coefficients of the linear combinations (2.34), $p = k, \dots, k+\varkappa_k - 1$, see Subsection 2.4. However, in the case where (2.42) is supplemented by the requirement $v_{k+1}^0(0) = 0$ (cf. the restriction (2.46)), estimates of the form (2.86) are fulfilled, but to completely identify the partial sums of the series involved one needs to calculate the summands $v_p^{(N+1)}$, $v_p^{(N+2)}$, and $w_p^{(N+1)}$, $p = k, k+1$.

We do not present precise statements concerning the case of multiple eigenvalues of the limiting problem (1.10): these statements are bulky, but fairly traditional.

§3. Asymptotics in the high-frequency range

3.1. External Steklov problem. The variational version of problem (1.13) appeals to the integral identity [20]

$$(\nabla w, \nabla v)_{\mathbb{R}^d\setminus\omega} = \mu(w, v)_{\partial\omega}, \tag{3.1}$$

in which, under the adopted restriction $d \ge 3$, the test functions can be taken in the space $\mathcal{H}$ obtained by completion from the linear set $C_c^\infty(\mathbb{R}^d \setminus \omega)$ (infinitely differentiable functions with compact support) in the Dirichlet norm $\|\nabla w; L^2(\mathbb{R}^d\setminus\omega)\|$. It is well known and not hard to check with the help of the one-dimensional Hardy inequality (2.49) that the space $\mathcal{H}$ consists of all functions $w \in H^1_{\mathrm{loc}}(\mathbb{R}^d \setminus \omega)$ with finite weighted Sobolev norm

$$\left(\|\nabla w; L^2(\mathbb{R}^d \setminus \omega)\|^2 + \|\rho^{-1} w; L^2(\mathbb{R}^d \setminus \omega)\|^2\right)^{1/2}. \tag{3.2}$$

Recall that $\mathcal{O} \in \omega$, i.e., the weight factor ρ^{-1} in the last summand is bounded in $\mathbb{R}^d \setminus \omega$.

In the Hilbert space $\mathcal{H}$ we introduce the scalar product[6]

$$\langle w, v \rangle = (\nabla w, \nabla v)_{\mathbb{R}^d \setminus \omega} + (w, v)_{\partial\omega} \tag{3.3}$$

and the "trace" operator T,

$$\langle Tw, v \rangle = (w, v)_{\partial\omega}, \quad w, v \in \mathcal{H}. \tag{3.4}$$

Obviously, T is a positive and continuous symmetric operator; hence, T is selfadjoint. Since $\partial\omega$ is a bounded surface, the operator T is compact, because so is the embedding $\mathcal{H} \subset L^2(\partial\omega)$ (see, e.g., [20, Chapter 1]). Thus, in accordance with Theorems 10.1.5 and 10.2.2 in [21], the spectrum of T consists of the essential spectrum $\sigma_e(T) = \{\tau = 0\}$ and the discrete spectrum $\sigma_d(T)$ forming an infinitely small positive sequence

$$\tau_1 \geq \tau_2 \geq \tau_3 \geq \ldots \geq \tau_k \geq \ldots \to +0, \tag{3.5}$$

where the eigenvalues are listed with regard to multiplicity. The root subspace of T is $\{w \in \mathcal{H} : w = 0 \quad \partial\omega\}$, i.e., $\tau = 0$ is an eigenvalue of infinite multiplicity. The eigenvectors $w_1, w_2, w_3, \ldots, w_k, \ldots$ corresponding to the normal eigenvalues (3.5) can be chosen so as to satisfy the orthogonality and normalization conditions

$$\langle w_k, w_l \rangle = \delta_{k,l}, \quad k, l \in \mathbb{N}. \tag{3.6}$$

By the definitions (3.4) and (3.3), the integral identity (3.1) is equivalent to the abstract equation

$$Tw = \tau w \quad \text{in} \quad \mathcal{H}, \tag{3.7}$$

and the spectral parameters are related to each other by the formula

$$\tau = (1 + \mu)^{-1}. \tag{3.8}$$

In accordance with (3.8), the spectrum of the variational problem (3.1) (or (1.13) in the differential form) coincides with a shift of the image of the set $\sigma_d(T) \setminus \{0\}$ under inversion: the point $\tau = 0$ is taken to infinity and, of course, cannot affect the spectrum of the external Steklov problem. The sequence (3.5) is transformed to the unbounded monotone positive sequence (1.12), and $\mu_k^0 = \tau_k^{-1} - 1$. For the role of the eigenfunctions $w_1^0, w_2^0, w_3^0, \ldots, w_k^0, \ldots \in \mathcal{H}$ we take $w_l^0 = \tau_l^{-1/2} w_l$, $l \in \mathbb{N}$. Relations (3.6)–(3.8), (3.4), (3.1) show that

$$(w_k^0, w_l^0)_{\partial\omega} = \langle T w_k^0, w_l^0 \rangle = \tau_k^{1/2} \tau_l^{-1/2} \langle w_k^0, w_l^0 \rangle = \delta_{k,l}, \quad k, l \in \mathbb{N}. \tag{3.9}$$

Being solutions of an elliptic boundary-value problem in a domain with smooth boundary, the eigenfunctions are infinitely differentiable everywhere in $\mathbb{R}^d \setminus \omega$. The harmonic functions w_k^0 decay at infinity by the finiteness of the weight norm (3.2); therefore, they can be represented as in (2.5), (2.7). Finally, the strong maximum principle implies that the first eigenvalue μ_1^0 is simple and that the corresponding eigenfunction can be taken positive.

[6]For $d \geq 3$, the form $(\nabla w, \nabla v)_{\mathbb{R}^d \setminus \omega}$ (without the last term in (3.3)) remains a scalar product in $\mathcal{H}$, but it is not so in the planar case, see Subsection 5.2.

3.2. Algorithm of constructing the asymptotics. We adopt the following asymptotic *Ansätze*:

$$\lambda^\varepsilon_{\kappa(\varepsilon)} \sim \frac{1}{\varepsilon}\sum_{j=0}^{\infty} \varepsilon^j \mu_k^{(j)}, \tag{3.10}$$

$$u^\varepsilon_{\kappa(\varepsilon)}(x) \sim \sum_{j=0}^{\infty} \varepsilon^j \big(w_k^{(j)}(\xi) + \varepsilon^{d-2} v_k^{(j)}(x)\big), \tag{3.11}$$

where the $\mu_k^{(0)} = \mu_k^0$ are the eigenvalues of problem (1.13), and the $w_k^{(0)} = w_k^0$ are the corresponding eigenfunctions. The remaining ingredients in (3.10), (3.11) are to be determined. It should be emphasized that, since the factor ε^{-1} is large, the eigenvalue (3.10) lies in the high-frequency range of the spectrum and, therefore, has no permanent order number in (1.5). Note that as the main component in the asymptotics of the eigenvalue (3.10), we have a summand of boundary layer type, but, in essence, this does not influence iteration processes, because the limiting problems on the domains Ω and $\mathbb{R}^d \setminus \bar\omega$, finite and infinite, are in fact of equal value within the framework of the method of matching asymptotic expansions (see explanations in Subsection 6.1).

Assuming that μ_k^0 is a simple eigenvalue, we repeat with minor modifications the asymptotic procedure described in Subsection 2.2 and form the boundary-value problems for the functions $w_k^{(j)}$ and $v_k^{(j)}$ that satisfy relations (2.4), (2.6) and (2.5), (2.7) (these expansions will need *a posteriori* verification). We plug the *Ansätze* (3.10) and (3.11) in problem (1.1), (1.2) and collect the coefficients of the same powers of the small parameter ε, written in the slow variables x and the fast variables ξ. The boundary layer type terms solve the interior Steklov problems

$$-\Delta_\xi w_k^{(j)}(\xi) = 0, \quad \xi \in \mathbb{R}^d \setminus \bar\omega, \partial_\nu w_k^{(j)}(\xi) - \mu_k^0 w_k^{(j)}(\xi) = g_k^{(j)}(\xi), \quad \xi \in \partial\omega, \tag{3.12}$$

with the right-hand sides

$$\begin{aligned} g_k^{(j)}(\xi) = \sum_{q=1}^{j} \mu_k^{(q)} w_k^{(j-q)}(\xi) - \sum_{l=0}^{j+2-d} \partial_\nu\big(\rho^l V_{kl}^{(j-l+2-d)}(\vartheta)\big) \\ + \sum_{q=0}^{j+2-d} \mu_k^{(q)} \sum_{l=0}^{j-q+2-d} \rho^l V_{kl}^{(j-q-l+2-d)}(\vartheta). \end{aligned} \tag{3.13}$$

Note that the factor ε^{-1} in (3.10) makes the orders of the expressions $\partial_n = \varepsilon^{-1}\partial_\nu$ and $\lambda^\varepsilon = \varepsilon^{-1}\mu^\varepsilon$ equal, i.e., it is responsible for the arising of the Steklov boundary conditions in problems (1.13) and (3.12). For the same reason, the summand $\lambda^\varepsilon_{\kappa(\varepsilon)} u^\varepsilon_{\kappa(\varepsilon)}(x)$ dominates $\partial_n u^\varepsilon_{\kappa(\varepsilon)}$ on the exterior part $\partial\Omega$ of the boundary $\partial\Omega(\varepsilon)$. Therefore, the regular type terms $v_k^{(j)}$ can be found as solutions of the following Dirichlet problem for the Laplace equation:

$$-\Delta_x v_k^{(j)}(x) = 0, \quad x \in \Omega, v_k^{(j)}(x) = f_k^{(j)}(x), \quad x \in \partial\Omega, \tag{3.14}$$

where

$$\begin{aligned} f_k^{(j)}(x) = \frac{1}{\mu_k^0}\Big(\partial_n v_k^{(j-1)}(x) - \sum_{q=1}^{j} \mu_k^{(q)} v_k^{(j-q)}(x) + \sum_{l=0}^{j-1} \partial_n\big(r^{2-d-l} W_{kl}^{(j-1-l)}(\vartheta)\big) \\ + \sum_{q=0}^{j} \mu_k^{(q)} \sum_{l=0}^{j-q} r^{2-d-l} W_{kl}^{(j-q-l)}(\vartheta)\Big). \end{aligned} \tag{3.15}$$

Writing (3.13) in the form

$$g_k^{(j)}(\xi) = \mu_k^{(j)} w_k^{(0)}(\xi) + \mathbf{g}_k^{(j)}(\xi), \tag{3.16}$$

we observe that, first, the remainder $\mathbf{g}_k^{(j)}$ depends on the collections of quantities

$$\mu_k^{(0)}, \ldots, \mu_k^{(j-1)}; \quad w_k^{(1)}, \ldots, w_k^{(j-1)}; \quad v_k^{(0)}, \ldots, v_k^{(j+2-d)}, \tag{3.17}$$

and, second, a single compatibility condition for problem (3.12) (simple eigenvalue plus the Fredholm alternative) looks like this:

$$\mu_k^{(j)} = -(\mathbf{g}_k^{(j)}, w_k^0)_{\partial\omega}. \tag{3.18}$$

The said above allows us to determine $\mu_k^{(j)}$ and $w_k^{(j)}$ in (3.10) and (3.11) as soon as the quantities (3.17) are known. Since the harmonic function $w_k^{(j)} \in \mathcal{H}$ decays at infinity, it admits the expansion (2.5), (2.7), and the arbitrariness in its choice is eliminated by the orthogonality condition

$$(w_k^{(j)}, w_k^0)_{\partial\omega} = 0, \quad j \in \mathbb{N}.$$

Now all the ingredients of the sum (3.15) have become known, and the solution $v_k^{(j)} \in C^\infty(\bar{\Omega})$ of problem (3.14) (it exists and is unique for clear reasons) satisfies (2.4) and (2.6).

We have finished the description of the asymptotics. It is quite transparent and needs no inductive decoration.

3.3. Specific formulas. Since the first eigenfunction w_1^0 of problem (1.13) is positive, so is the coefficient B_1^0 in the formula

$$w_1^0(\xi) = B_1^0\Phi(\xi) + O(|\xi|^{1-d}), \quad |\xi| \to +\infty, \tag{3.19}$$

(cf. (2.25)). Indeed, if $B_1^0 = 0$, then the harmonic function w_1^0 expands in a convergent series

$$\sum_{l=1}^{\infty} \rho^{2-d-l} W_{1l}^0(\vartheta) \tag{3.20}$$

in spherical functions, each of which (for $l \geq 1$) necessarily changes its sign on the unit sphere $\mathbb{S}^{d-1}$ (see, e.g., the book [23]). Thus, no leading term in (3.20) can ensure that $w_1^0(\xi)$ be positive for large $|\xi|$, so that all terms must be zero, which is surely impossible.

This allows us to think of the case where $B_k^0 \neq 0$ as typical, restricting to it when deducing specific formulas. Note that for a multiple eigenvalue we can always arrange that only one eigenfunction satisfy the above requirement (see Subsection 3.4).

Lemma 3.1. *We have*

$$\mu_k^{(1)} = \cdots = \mu_k^{(d-3)} = 0, \quad w_k^{(1)} = \cdots = w^{(d-3)} = 0, \tag{3.21}$$

$$\mu_k^{(d-2)} = -(B_k^0)^2 G_0(0,0) \geq 0, \tag{3.22}$$

where G_0 is the regular part of the Green function of the Dirichlet problem for the Laplace operator in Ω

$$G(x,y) = \Phi(x-y) + G_0(x,y). \tag{3.23}$$

Proof. Formulas (3.13) show that

$$\mathbf{g}_k^{(j)}(\xi) = \sum_{q=1}^{j-1} \mu_k^{(q)} w_k^{(j-q)}(\xi), \quad j = 1, \ldots, d-3,$$

and we get (3.21) by easy arguments (cf. the first part of the proof of Lemma 2.1). As a result, formulas (3.16) and (3.13) for $j = d-2$ take the form

$$\mathbf{g}_k^{(d-2)}(\xi) + \mu_k^{(d-2)} w_k^0(\xi) = g_k^{(d-2)}(\xi) = \mu_k^0 v_k^{(0)}(0) + \mu_k^{(d-2)} w_k^{(0)}(\xi) \tag{3.24}$$

in accordance with (2.9), $j = 0$, and $\partial_\nu(\rho^0 V_{k0}^{(0)}(\vartheta)) = 0$. As before (see the verification of formula (2.22) in Lemma 2.1), we apply the Green formula in the domain $\mathbb{B}_R^d \setminus \bar{\omega}$ and calculate the quantity (3.18) with the index $j = d - 2$:

$$\begin{aligned} \mu_k^{(d-2)} &= -\mu_k^0 v_k^{(0)}(0) \int_{\partial\omega} w_k^0(\xi)\, d\xi = v_k^{(0)}(0) \int_{\partial\omega} (\partial_\nu 1 - \mu_k^0 1) w_k^0(\xi)\, ds_\xi \\ &= v_k^{(0)}(0) \lim_{R\to\infty} \int_{\partial\mathbb{B}_R^d} \frac{\partial w_k^0}{\partial\rho}(\xi)\, ds_\xi = -v_k^{(0)}(0) B_k^0. \end{aligned} \tag{3.25}$$

The function $v_k^{(0)}$ itself is a solution of problem (3.14) with the right-hand side

$$f_k^{(0)}(x) = -(\mu_k^0)^{-1}\mu_k^0 r^{2-d} W_{k0}^{(0)}(\vartheta) = -B_k^0 \Phi(x). \tag{3.26}$$

Consequently, the representation (3.23) of the Green function ensures the identity

$$v_k^{(0)}(x) = B_k^0 G_0(x, 0). \tag{3.27}$$

Now, using (3.22), we get the desired relation (3.25). □

By the maximum principle, the regular part $G_0(x, 0) = G(x, 0) - \Phi(x)$ of the Green function is a negative smooth function on $\bar{\Omega}$; hence, the quantity (3.22) is positive. In other words, if the domain (1.3) is viewed as a result of truncation of the space $\mathbb{R}^d$ with a small hole (1.4) by the surface $\partial\Omega$, then under the condition $B_k^0 \neq 0$, the eigenvalue $\varepsilon^{-1}\mu_k$ of the external Steklov problem in the domain $\mathbb{R}^d \setminus \overline{\omega_\varepsilon}$ acquires a positive increment of order of ε^{d-3}.

3.4. A multiple eigenvalue. Suppose that for the sequence (1.12) we have

$$\mu_{m-1}^0 < \mu_m^0 = \ldots = \mu_{m+\varkappa_m-1}^0 < \mu_{m+\varkappa_m}^0, \tag{3.28}$$

i.e., μ_m^0 is an eigenvalue of multiplicity $\varkappa_m > 1$ for problem (1.13). The coefficients in expansions of the form (3.19) for the corresponding eigenfunctions $w_m^0, \ldots, w_{m+\varkappa_m-1}^0 \in \mathcal{H}$ will be denoted by $B_m^0, \ldots, B_{m+\varkappa_m-1}^0$. In the *Ansatz* (3.11) we put

$$w_p^{(0)}(\xi) = a_{p,m}^{(0)} w_m^0(\xi) + \ldots + a_{p,m+\varkappa_m-1}^{(0)} w_{m+\varkappa_m-1}^0(\xi) \tag{3.29}$$

and, accordingly, write the representations

$$B_p^{(0)} = a_{p,m}^{(0)} B_m^0 + \ldots + a_{p,m+\varkappa_m-1}^{(0)} B_{m+\varkappa_m-1}^0(\xi), \tag{3.30}$$

with unknown rows of coefficients (2.35). We assume the orthogonality and normalization conditions (2.41) and (3.9). Repeating the calculations of the preceding subsection, we see that, first, the regular type summand $v_p^{(0)}(x)$ can be found by formula (3.27) with the coefficient (3.30) and the regular part of the Green formula (3.23), and second, the compatibility conditions

$$(g_p^{(d-2)}, w_q^0)_{\partial w} = 0, \quad q = m, \ldots, m + \varkappa_m - 1,$$

for problem (3.12) with the right-hand side $g_p^{(d-2)}$ as in (3.24) turn into the algebraic system

$$\mu_p^{(d-2)} a_{(p)}^{(0)} = P^{(m)} a_{(p)}^{(0)} \quad \text{in} \quad \mathbb{R}^{\varkappa_m}. \tag{3.31}$$

Here, the symmetric positive matrix $P^{(m)}$ with the entries

$$P_{pq}^{(m)} = -B_p^{(0)} B_q^{(0)} G_0(0, 0) \tag{3.32}$$

has rank at most 1, so that its eigenvalues satisfy

$$0 = \mu_m^{(d-2)} = \ldots = \mu_{m+\varkappa_m-2}^{(d-2)} \leq \mu_{m+\varkappa_m-1}^{(d-2)}. \tag{3.33}$$

The corresponding eigenvectors provide the coefficients of the linear combinations (3.29). Preserving the orthogonality and normalization conditions (3.9), we can fix the eigenfunctions of problems (1.13) so that

$$B^0_m = \ldots = B^0_{m+\varkappa_m-2} = 0. \tag{3.34}$$

If $B^0_{m+\varkappa_m-1} \neq 0$, then on the list (3.33) we find the positive eigenvalue

$$\mu^{(d-2)}_{m+\varkappa_m-1} = -(B^0_{m+\varkappa_m-1})^2 G_0(0,0) > 0$$

(cf. (3.22)), but if $B^0_{m+\varkappa_m-1} = 0$, then the matrix $P^{(m)}$ and all its eigenvalues (3.33) are zero.

Finally, observe that, at that stage of the asymptotic procedure, we can fully fix the rows $a^{(m)} = (0,1)$ and $a^{(m)}_{(m+1)} = (1,0)$ only in the case where

$$\varkappa_m = 2, \quad \mu^{(d-2)}_{m+1} > 0 \tag{3.35}$$

(cf. (2.42)). If at least one of the conditions (2.23) is violated, then some arbitrariness in the choice of the vectors of the matrix $P^{(m)}$ survives. One can try to remedy this by constructing lower asymptotic terms (cf. Subsection 4.2), but we shall not go this way, because the information obtained above suffices for detecting the eigenvalues with stable asymptotics (1.11).

3.5. Justification of the asymptotics of eigenvalues. As in Subsection 2.7, we apply Lemma 2.4 about the "near-eigenvalues" and "near-eigenfunctions" of the operator $\mathcal{T}^\varepsilon$. Unfortunately, modification of the asymptotic structure requires the introduction of the new scalar product

$$\langle u^\varepsilon, v^\varepsilon\rangle_\varepsilon = (\nabla u^\varepsilon, \nabla v^\varepsilon)_{\Omega(\varepsilon)} + \varepsilon^{-1}(u^\varepsilon, v^\varepsilon)_{\partial\Omega(\varepsilon)} \tag{3.36}$$

in the same Hilbert space $\mathcal{H}^\varepsilon = H^1(\Omega(\varepsilon))$ (see Lemma 2.3). The corresponding changes touch the trace operator $\mathcal{T}^\varepsilon$ defined as in (2.51), the scalar product (3.36), and the sequence (1.7) of its eigenvalues, which are now related to the eigenvalues (1.5) of problem (1.1), (1.2) by the formula

$$\tau^\varepsilon_k = (\varepsilon^{-1} + \lambda^\varepsilon_k)^{-1} = \varepsilon(1 + \varepsilon\lambda^\varepsilon_k)^{-1}. \tag{3.37}$$

Combining the previous notation of §3 with the new content makes no ambiguity, but allows us to use some formulas from Subsections 2.5 and 2.7.

Assuming (3.28), we take the following approximate solutions of the abstract spectral equation (2.53):

$$t^\varepsilon_p = \varepsilon\big(1 + \mu^0_m + \varepsilon^{d-2}\mu^{(d-2)}_p\big)^{-1}, \quad \mathcal{U}^\varepsilon_p = \|U^\varepsilon_p; \mathcal{H}^\varepsilon\|^{-1} U^\varepsilon_p, \tag{3.38}$$

$$U^\varepsilon_p(x) = w^{(0)}_p(\varepsilon^{-1}x) + \varepsilon^{d-2} w^{(d-2)}_p(\varepsilon^{-1}x) + \varepsilon^{d-2} v^{(0)}_p(x) + \varepsilon^{d-2}\widetilde{v}^\varepsilon_p(x). \tag{3.39}$$

Here $p = m, \ldots, m+\varkappa_m - 1$, $w^{(0)}_p$ is the linear combination (3.29) the column of coefficients of which is the eigenvector $a^{(0)}_{(p)}$ of the matrix $P^{(m)}$ with the entries (3.32) that corresponds to its eigenvalue $\mu^{(d-2)}_p$. Also, $v^{(0)}_p$ is the solution of the Dirichlet problem (3.14) with the indices $j = 0$, $k = p$ and with the right-hand side (3.26), and $w^{(d-2)}_p$ is the solution of the exterior Steklov problem (3.12) with the indices $j = d-2$, $k = p$ and with the right-hand side (3.24) – the compatibility conditions for that problem were reshaped to the algebraic system (3.31). Finally, $\widetilde{v}^\varepsilon_p$ is a function harmonic in Ω, satisfying the estimate

$$|\widetilde{v}^\varepsilon_p(x)| + \big|\nabla\widetilde{v}^\varepsilon_p(x)\big| \le c_p\varepsilon \tag{3.40}$$

and fixed in what follows in such a way that

$$U_p^\varepsilon(x) = O(\varepsilon^{d-1}), \quad x \in \partial\Omega. \tag{3.41}$$

We repeat the arguments of Subsection 2.7 with some modifications and considerable simplifications. First, we check that

$$|\langle U_p^\varepsilon, U_q^\varepsilon\rangle_\varepsilon - \varepsilon^{d-2}(1+\mu_p^0)\delta_{p,q}| \le c_{pq}\varepsilon^{3(d-2)/2}, \quad p,q = m,\dots,m+\varkappa_m - 1. \tag{3.42}$$

For this, observe that, first, by (3.41), the expression $\varepsilon^{-1}|(U_p^\varepsilon, U_q^\varepsilon)_{\partial\Omega}|$ does not exceed $c_{pq}\varepsilon^{-1+2(d-1)} \le C_{pq}\varepsilon^{d-1}$, and second, the estimate (2.7) for the rate of decay of $w_l^{(0)}$ and also the orthogonality and normalization conditions (3.9), (2.41) show that

$$\begin{aligned}
&\big(\nabla_x w_p^{(0)}, \nabla_x w_q^{(0)}\big)_{\Omega(\varepsilon)} + \varepsilon^{-1}(w_p^{(0)}, w_q^{(0)})_{\partial\omega_\varepsilon} \\
&\quad = \varepsilon^{d-2}\big((\nabla_\xi w_p^{(0)}, \nabla_\xi w_q^{(0)})_{\mathbb{R}^d\setminus\omega} + (w_p^{(0)}, w_q^{(0)})_{\partial\omega}\big) + O\bigg(\int_{\mathbb{R}^d\setminus\Omega} \varepsilon^{-2}|\xi|^{2(1-d)}\,dx\bigg) \\
&\quad = \varepsilon^{d-2}(\mu_p^0+1)\delta_{p,q} + O(\varepsilon^{2(d-2)}).
\end{aligned}$$

Here, we have made the coordinate dilation $x \mapsto \xi$ and referred to the integral identity (3.1) with $w = w_p^{(0)}$, $v = w_q^{(0)}$, and $\mu = \mu_p^0$. Now inequality (3.42) is ensured by the simple estimates

$$\begin{gathered}
\big\|\varepsilon^{d-2}\nabla_x w_l^{(d-2)}; L^2(\Omega(\varepsilon))\big\|^2 + \big\|\varepsilon^{d-2} w_l^{(d-2)}; L^2(\partial\omega_\varepsilon)\big\|^2 \le c_l\varepsilon^{2(d-2)}, \\
\big\|\varepsilon^{d-2}\nabla_x (v_l^{(0)} + \widetilde{v}_l^\varepsilon); L^2(\Omega(\varepsilon))\big\|^2 + \big\|\varepsilon^{d-2} (v_l^{(0)} + \widetilde{v}_l^\varepsilon); L^2(\partial\omega_\varepsilon)\big\|^2 \le c_l\varepsilon^{2(d-2)}.
\end{gathered}$$

At the next step of justification of the asymptotics we deal with the quantities occurring in Lemma 2.4:

$$\begin{aligned}
\delta_p^\varepsilon &= \|\mathcal{T}^\varepsilon\mathcal{U}_p^\varepsilon - t_p^\varepsilon\mathcal{U}_p^\varepsilon; \mathcal{H}^\varepsilon\| = \sup|\langle \mathcal{T}^\varepsilon\mathcal{U}_p^\varepsilon - t_p^\varepsilon\mathcal{U}_p^\varepsilon, V^\varepsilon\rangle_\varepsilon| \\
&= \|U_p^\varepsilon; \mathcal{H}^\varepsilon\|^{-1} t_p^\varepsilon \sup\big|\varepsilon^{-1}\big(1+\mu_m^0 + \varepsilon^{d-2}\mu_p^{(d-2)}\big)(U_p^\varepsilon, V^\varepsilon)_{\partial\Omega(\varepsilon)} \\
&\qquad\qquad - (\nabla_x U_p^\varepsilon, \nabla_x V^\varepsilon)_{\Omega(\varepsilon)} - \varepsilon^{-1}(U_p^\varepsilon, V^\varepsilon)_{\partial\Omega(\varepsilon)}\big| \\
&= \|U_p^\varepsilon; \mathcal{H}^\varepsilon\|^{-1} t_p^\varepsilon \sup\big|(\nabla_x U_p^\varepsilon, \nabla_x V^\varepsilon)_{\Omega(\varepsilon)} + \varepsilon^{-1}\big(\mu_m^0 + \varepsilon^{d-2}\mu_p^{(d-2)}\big)(U_p^\varepsilon, V^\varepsilon)_{\partial\Omega(\varepsilon)}\big| \\
&\le c\varepsilon^{1-d/2}\varepsilon \sup\big|(\Delta_x U_p^\varepsilon, V^\varepsilon)_{\Omega(\varepsilon)} + \big(\partial_n U_p^\varepsilon - \varepsilon^{-1}(\mu_m^0 + \varepsilon^{d-2}\mu_p^{(d-2)})U_p^\varepsilon, V^\varepsilon\big)_{\partial\Omega(\varepsilon)}\big|.
\end{aligned} \tag{3.43}$$

Here the supremum is over all $V^\varepsilon \in \mathcal{H}^\varepsilon$ such that $\|V^\varepsilon; \mathcal{H}^\varepsilon\| = 1$, and for the last passage we have used (3.42), (3.38), and the Green formula on $\Omega(\varepsilon)$. Since all terms in (3.39) are harmonic functions, we see that $(\Delta_x U_p^\varepsilon, V^\varepsilon)_{\Omega(\varepsilon)} = 0$.

The correction term $\widetilde{v}_p^\varepsilon$ in (3.39) is chosen so that, besides (3.41), to ensure the relation

$$\partial_n U_p^\varepsilon(x) - \varepsilon^{-1}(\mu_m^0 + \varepsilon^{d-2}\mu_p^{(d-2)})U_p^\varepsilon(x) = O\big(\varepsilon^{(3d-5)/2}\big), \quad x \in \partial\Omega. \tag{3.44}$$

Then we shall have the following inequality, needed in the sequel:

$$\begin{aligned}
&\big|\big(\partial_n U_p^\varepsilon - \varepsilon^{-1}(\mu_m^0 + \varepsilon^{d-2}\mu^{(d-2)})U_p^\varepsilon, V^\varepsilon\big)_{\partial\Omega}\big| \\
&\qquad \le c_p\varepsilon^{(3d-5)/2}\|V^\varepsilon; L^2(\partial\Omega)\| \le c_p\varepsilon^{-2+3d/2}\|V^\varepsilon; \mathcal{H}^\varepsilon\| = c_p\varepsilon^{-2+3d/2}.
\end{aligned} \tag{3.45}$$

On the surface $\partial\Omega$ we have

$$\begin{aligned}\partial_n U_p^\varepsilon(x) - \varepsilon^{-1}\big(\mu_m^0 + \varepsilon^{d-2}\mu_p^{(d-2)}\big)U_p^\varepsilon(x) \\ = -\varepsilon^{-1}\mu_m^0\varepsilon^{d-2}\big(v_p^{(0)}(x) + B_p^{(0)}\Phi(x)\big) \\ + \varepsilon^{d-2}\big(\partial_n\widetilde{v}_p^\varepsilon(x) - \varepsilon^{-1}(\mu_m^0 + \varepsilon^{d-2}\mu_p^{(d-2)})\widetilde{v}_p^\varepsilon(x) - \widetilde{g}_p^\varepsilon(x)\big),\end{aligned} \tag{3.46}$$

$$\begin{aligned}\widetilde{g}_p^\varepsilon(x) = \varepsilon^{-1}\mu_m^0\varepsilon^{2-d}\big(w_p^{(0)}(\xi) - B_p^{(0)}\Phi(\xi)\big) \\ + \varepsilon^{-1}\mu_p^{(d-2)}\big(w_p^{(0)}(\xi) + \varepsilon^{d-2}w_p^{(d-2)}(\xi) + \varepsilon^{d-2}v_p^{(0)}(x)\big) \\ - \partial_n\big(\varepsilon^{2-d}w_p^{(0)}(\xi) + w_p^{(d-2)}(\xi) + v_p^{(0)}(x)\big) + \varepsilon^{-1}\mu_p^0 w_p^{(d-2)}(\xi).\end{aligned} \tag{3.47}$$

The first term on the right in (3.46) vanishes due to the boundary condition in problem (3.14), (3.26) for $v_p^{(0)}$. Thus, to eliminate the expression (3.46) completely, we need to solve the problem

$$\begin{aligned}-\Delta_x\widetilde{v}_p^\varepsilon(x) = 0, \quad x\in\Omega, \\ \varepsilon\partial_n\widetilde{v}_p^\varepsilon(x) - (\mu_m^0 + \varepsilon^{d-2}\mu_p^{(d-2)})\widetilde{v}_p^\varepsilon(x) = \varepsilon\widetilde{g}_p^\varepsilon(x), \quad x\in\partial\Omega.\end{aligned} \tag{3.48}$$

Because of the small parameter at the derivative ∂_n and the "incorrect" sign of the free term, we cannot claim that this problem is uniquely solvable. However, relation (3.44) only requires an approximate solution, which can be found easily with the help of iterated solutions of the Dirichlet problem for the Laplace equation in Ω, because the right-hand side of (3.47) is smooth and satisfies $|\nabla_s^j\widetilde{g}_p^\varepsilon(x)| \le c_{jp}$ for $j\in\mathbb{N}_0$, in accordance with formulas (2.5), (2.7), and (3.27) for $w_p^{(d-2)}$, $w_p^{(0)}$, and $w_p^{(0)}$. Note that inequality (3.40) for the approximate solution is ensured precisely by those estimates and by the coefficient ε of $\widetilde{g}_p^\varepsilon$ in the boundary condition (3.48).

So, the exponent of the power of ε occurring on the right in (3.44) can be made as large as we wish, but the arising of $O(\varepsilon^{d-1})$ in (3.41) is dictated by the requirement (3.40) and also the fact that $w^{(d-1)}(\xi) = O(\varepsilon^{d-2})$ and $w_p^{(0)}(\xi) + \varepsilon^{d-2}v_p^{(0)}(x) = w_p^{(0)}(\xi) - B_p^{(0)}\Phi(\xi) = O(\varepsilon^{d-1})$ on $\partial\Omega$.

Now we work with the last scalar product in the chain (3.43), restricted to the inner part $\partial\omega_\varepsilon$ of the surface $\partial\Omega(\varepsilon)$. Recalling the Steklov conditions in problems (1.13) and (3.12), (3.24) for $w_p^{(0)}$ and $w_p^{(d-2)}$, we see that on $\partial\omega_\varepsilon$ we have

$$\begin{aligned}\partial_n U_p^\varepsilon(x) - \varepsilon^{-1}\big(\mu_m^0 + \varepsilon^{d-2}\mu_p^{(d-2)}\big)U_p^\varepsilon(x) = \varepsilon^{-1}\big(\partial_\nu w_p^{(0)}(\xi) - \mu_m^0 w_p^{(0)}(\xi)\big) \\ + \varepsilon^{d-3}\big(\partial_\nu w_p^{(d-2)}(\xi) - \mu_m^0 w_p^{(d-2)}(\xi) - \mu_p^{(d-2)}w_p^{(0)}(\xi) - \mu_m^0 v_p^{(0)}(0)\big) \\ + \varepsilon^{d-2}\partial_n\big(v_p^{(0)}(x) + \widetilde{v}_p^\varepsilon(x)\big) - \varepsilon^{d-3}\mu_m^0\big(v_p^{(0)}(x) - v_p^{(0)}(0) + \widetilde{v}_p^\varepsilon(x)\big) \\ - \varepsilon^{2d-5}\mu_p^{(d-2)}w_p^{(d-2)}(\xi) - \varepsilon^{2d-5}\mu_p^{(d-2)}\big(v_p^{(0)}(x) + \widetilde{v}_p^\varepsilon(x)\big).\end{aligned}$$

Here, the first two expressions on the right-hand side vanish, and the moduli of the other terms are dominated by $c_p\varepsilon^{d-2}$, which follows from the Taylor formula for $v_p^{(0)}$ and estimate (3.40) for $\widetilde{v}_p^\varepsilon$. As a result,

$$\begin{aligned}\big|\big(\partial_n U_p^\varepsilon - \varepsilon^{-1}(\mu_m^0 + \varepsilon^{d-2}\mu_p^{(d-2)})U_p^\varepsilon, V^\varepsilon\big)_{\partial\omega_\varepsilon}\big| \\ \le c_p\varepsilon^{d-2}(\mathrm{meas}_{d-1}\,\partial\omega_\varepsilon)^{1/2}\|V^\varepsilon; L^2(\partial\omega_\varepsilon)\| \\ \le c_p\varepsilon^{d-2}\varepsilon^{(d-1)/2}\varepsilon^{1/2}\|V^\varepsilon; \mathcal{H}^\varepsilon\| = c_p\varepsilon^{-2+3d/2}.\end{aligned} \tag{3.49}$$

We summarize. By (3.45) and (3.49), the quantity (3.43) admits the estimate

$$\delta_p^\varepsilon \le c_p\varepsilon^{1-d/2}\varepsilon\varepsilon^{-2+3d/2} = c_p\varepsilon^d.$$

Hence, by Lemma 2.4, there is an eigenvalue $\tau^\varepsilon_{M^\varepsilon(p)}$ of the operator $\mathcal{T}^\varepsilon$ such that

$$|\tau^\varepsilon_{M^\varepsilon(p)} - t^\varepsilon_p| \le c_p\varepsilon^d.$$

We transform this relation, substituting formulas (3.37) and (3.38) in it, to obtain

$$\begin{aligned}(3.50)\qquad &\big|\lambda^\varepsilon_{M^\varepsilon(p)} - \varepsilon^{-1}(\mu^0_m + \varepsilon^{d-2}\mu^{(d-2)}_p)\big| \\ &\le \varepsilon^d\varepsilon^{-1}\big(1+\mu^0_m + \varepsilon^{d-2}\mu^{(d-2)}_p\big)\big(\varepsilon^{-1}+\lambda^\varepsilon_{M^\varepsilon(p)}\big) \le K_p\varepsilon^{d-2}\big(1+\varepsilon\lambda^\varepsilon_{M^\varepsilon(p)}\big).\end{aligned}$$

In particular, it follows that

$$\lambda^\varepsilon_{M(p)} \le \varepsilon^{-1}(\mu^0_m + \varepsilon^{d-2}\mu^{(d-2)}_p) + K_p\varepsilon^{d-2}(1+\varepsilon\lambda^\varepsilon_{M^\varepsilon(p)}),$$

i.e., for $\varepsilon\in(0,\varepsilon_p]$, where $\varepsilon_p = (K_p/2)^{1/(d-1)}$, we have the inequality $\lambda^\varepsilon_{M^\varepsilon(p)} \le c_p\varepsilon^{-1}$, which brings formula (3.50) to the form

$$(3.51)\qquad \big|\lambda^\varepsilon_{M^\varepsilon(p)} - \varepsilon^{-1}\big(\mu^0_m + \varepsilon^{d-2}\mu^{(d-2)}_p\big)\big| \le C_m\varepsilon^{d-2} \quad \text{for} \quad \varepsilon\in(0,\varepsilon_p].$$

The next (and last) step in the justification procedure for our asymptotics consists of applying the second part of Lemma 2.4, and then verifying the fact that at least $\varkappa_m$ distinct eigenvalues of problem (1.1), (1.2) satisfy inequality (3.51), possibly, with a slightly larger majorant, i.e., with the change $C_m \mapsto D_m > C_m$. Note that this step is not needed in the case of a simple ($\varkappa_m = 1$) eigenvalue or in the case where (3.33) is fulfilled for $p = m+\varkappa_m - 1$. In the case of (3.33) for $p = m,\dots,m+\varkappa_m-2$, or if

$$(3.52)\qquad B^0_m = \ldots = B^0_{m+\varkappa_m-1} = 0 \quad \text{and} \quad \mu^{(d-2)}_m = \ldots = \mu^{(d-2)}_{m+\varkappa_m-2} = 0,$$

the arguments are similar; therefore, as in Subsection 2.7, we restrict ourselves to considering the harder case (3.52) of the complete degeneration of the matrix $P^{(m)}$.

We take the number (2.78) and use Lemma 2.4 to find normalized rows

$$b^{(p)} = (b^{(p)}_{N^\varepsilon(m)},\dots,b^{(p)}_{N^\varepsilon(m)+K^\varepsilon(m)-1}),$$

for which, by (2.59), we have

$$(3.53)\qquad \Bigg\|\mathcal{U}^\varepsilon_p - \sum_{j=N^\varepsilon(m)}^{N^\varepsilon(m)-K^\varepsilon(m)-1} b^{(p)}_j u^\varepsilon_j \,:\, \mathcal{H}^\varepsilon\Bigg\| \le 2\frac{\delta^\varepsilon_p}{\delta^\varepsilon_*} \le 2\frac{C_m}{D_m}.$$

Here, $u^\varepsilon_{N^\varepsilon(m)},\dots,u^\varepsilon_{N^\varepsilon(m)+K^\varepsilon(m)-1}$ are the eigenvectors of the operator (the eigenfunctions of problem (1.1), (1.2)) orthonormal in the space $\mathcal{H}^\varepsilon$ with the scalar product (3.36) and corresponding the eigenvalues τ^ε_j that satisfy

$$(3.54)\qquad \tau^\varepsilon_j \in \big[\varepsilon(1+\mu^0_m)^{-1} - \varepsilon^d D_m, \varepsilon(1+\mu^0_m)^{-1} + \varepsilon^d D_m\big]\,.$$

Now, we can calculate as in (2.81) and employ formula (3.42), which is implied by (2.82), to deduce that, for ε small and D_m large, the rows $b^{(m)},\dots,b^{(m+\varkappa_m-1)} \in \mathbb{R}^{\varkappa}_m$ are "almost orthonormal", i.e.,

$$\Bigg|\sum_{j=N^\varepsilon(m)}^{N^\varepsilon(m)+K^\varepsilon(m)-1} b^{(p)}_j b^{(q)}_j - \delta_{p,q}\Bigg| \le C_m\big(\varepsilon^{d/2} + D^{-1}_m\big),$$

which is possible only if $K^\varepsilon(m) \ge \varkappa_m$.

We have proved the following statement.

Theorem 3.1. *If μ^0_m is an eigenvalue of multiplicity $\varkappa_m$ for problem* (1.13) (*cf.* (3.28)), *then in the sequence* (1.5) *of eigenvalues of problem* (1.1), (1.2) *there exist elements $\lambda^\varepsilon_{M^\varepsilon(m)},\dots,\lambda^\varepsilon_{M^\varepsilon(m+\varkappa_m-1)}$ with strictly increasing indices that satisfy* (3.5).

Observe that formula (3.53) with large D_m shows that the projections of the approximate solutions $\mathcal{U}^\varepsilon_m, \dots, \mathcal{U}^\varepsilon_{m+\varkappa_m-1}$ to the linear hull of the eigenfunctions of problem (1.1), (1.2) with the eigenvalues (3.54) have norms $1 + O(D_m^{-1})$.

Finally, for a simple eigenvalue μ^0_k, the iteration processes of Subsection 3.2 have led to constructing the full formal asymptotic series (3.10). Repeating the calculations before Theorem 3.1 with minor compilations, we arrive at the central claim of this section.

Theorem 3.2. *Let μ^0_k be a simple eigenvalue of the exterior Steklov problem* (1.13) (*or* (3.1) *in the variational setting*). *For any $N \in \mathbb{N}$ we can find two quantities $\varepsilon_{kN} > 0$ and $c_{kN} > 0$ such that, for $\varepsilon \in (0, \varepsilon_{kN}]$, the singularly perturbed Steklov problem* (1.1), (1.2) *has an eigenvalue $\lambda^\varepsilon_{M^\varepsilon(k)}$ such that*

$$\left| \lambda^\varepsilon_{M^\varepsilon(k)} - \varepsilon^{-1}\mu^0_k - \sum_{j=d-2}^{N} \varepsilon^{j-1}\mu^{(j)}_k \right| \le c_{kN}\varepsilon^N.$$

Here the $\mu^{(j)}_k$ are the coefficients of the formal asymptotic series (3.10) *given by formulas* (3.18), *and the number $M^\varepsilon(k)$ of the element $\lambda^\varepsilon_{M^\varepsilon(k)}$ in the sequence* (1.5) *grows unboundedly as $\varepsilon \to +0$.*

§4. Asymptotics of eigenvalues of surface waves

4.1. Limiting problems. Since the construction of asymptotic expansions for solutions of the spectral problem (1.14)–(1.16) of the linear theory of surface waves, as stated in Subsection 1.2, does not differ in essence from the procedures described in §§2 and 3, we only outline the presentation in this section. In particular, below we indicate only the principal asymptotic correction terms in the case of simple eigenvalues. In the low-frequency range, the higher-order terms can be found as before without any complications, but in the high-frequency range we cannot even construct the higher-order terms, because new effects of boundary layer type arise (see Subsections 4.4 and 5.3).

As $\varepsilon \to +0$, the ice-hole in Ω disappears, giving rise to the limiting problem

$$-\Delta v(x) = 0, \quad x \in \Xi, \quad \partial_n v(x) = 0, \quad x \in \Upsilon \cup \Omega, \tag{4.1}$$

$$\partial_z v(x) = \lambda v(x), \quad x \in \Sigma := \Sigma_1 \cup \dots \cup \Sigma_J, \tag{4.2}$$

the spectrum of which is known to be discrete, this is a positive, nonbounded, and monotone sequence of eigenvalues of the form (1.9). The corresponding eigenfunctions $v^0_1, v^0_2, \dots, v^0_k, \dots \in H^1(\Omega)$ can be chosen so as to satisfy the orthogonality and normalization conditions

$$(v^0_k, v^0_l)_\Sigma = \delta_{k,l}, \quad k, l \in \mathbb{N}. \tag{4.3}$$

It should be mentioned especially that the eigenfunctions v^0_j are not smooth everywhere in $\bar{\Xi}$, but may have singularities on the contours $\Gamma_1, \dots, \Gamma_J$ where the type of boundary conditions changes.

The coordinate dilation

$$(y, z) = x \mapsto \xi = (\eta, \zeta) = (\varepsilon^{-1}y, \varepsilon^{-1}z)$$

and the formal passage to $\varepsilon = 0$ transform the domains Ξ and $\Omega(\varepsilon)$, ω_ε to the half-space $\mathbb{R}^3_-$ and the sets $\partial\mathbb{R}^3_- \setminus \bar{\omega}$ and ω on its surface.

The second limiting problem looks like this:

$$-\Delta_\xi w(\xi) = 0, \quad \xi \in \mathbb{R}^3_-, \quad \partial_z w(\eta, 0) = 0, \quad \eta \in \mathbb{R}^2 \setminus \bar{\omega}, \tag{4.4}$$

$$\partial_z w(\eta, 0) = \mu w(\eta, 0), \quad \eta = (\eta_1, \eta_2) \in \omega. \tag{4.5}$$

Since the spectral parameter μ is involved in the Steklov boundary condition (4.5) on a finite part of the boundary, we can repeat the arguments of Subsection 3.1 to make sure that the variational setting of problem (4.4), (4.5), i.e.,

$$(\nabla_\xi w, \nabla_\xi v)_{\mathbb{R}^3_-} = \mu(w,v)_\omega, \quad v \in \mathcal{H}, \tag{4.6}$$

is realized on the weighted space with the norm

$$\|w;\mathcal{H}\| = \big(\|\nabla_\xi w; L^2(\mathbb{R}^3_-)\|^2 + \|(1+\rho)^{-1} w; L^2(\mathbb{R}^3_-)\|^2\big)^{1/2}.$$

Moreover, problem (4.6) has discrete spectrum, the sequence (1.12) of eigenvalues, and the eigenfunctions $w_1^0, w_2^0, \dots, w_k^0, \dots \in \mathcal{H}$ can be chosen so as to satisfy the orthogonality and normalization conditions

$$(w_k^0, w_l^0)_\omega = \delta_{k,l}, \quad k, l \in \mathbb{N}.$$

4.2. Asymptotics of the low eigenfrequencies. Let $\lambda_k^0 > 0$ be a simple eigenvalue of problem (4.1), (4.2). The corresponding eigenfunction v_k^0 satisfies equation (1.14) and the Neumann condition (1.15), but leaves a discrepancy in the Steklov spectral condition (1.16) on the small ice-hole ω_ε. Since the function v_k^0 is locally smooth in a neighborhood of the coordinate origin $\mathcal{O}$, into which the sets ω_ε collapse as $\varepsilon \to +0$, we see that the function v_k is locally smooth and obeys the Taylor formula

$$v_k^0(x) = v_k^0(0) + x \cdot \nabla_x v_k^0(0) + O(|x|^2). \tag{4.7}$$

We have $\partial_z v_k^0(0) = 0$ by the Neumann condition at the point $\mathcal{O} \in \Omega$, i.e., the second summand on the right in (4.7) is equal to $y \cdot \nabla_y v_k^0$. Thus, the leading part of the discrepancy in the Steklov condition (1.16) on ω_ε takes the form $\lambda_k^0 v_k^0(0)$, and the summand $\varepsilon w_k^0(\xi)$ of the boundary layer type that compensates for this discrepancy can be found with the help of the following Neumann problem on the half-space:

$$-\Delta_\xi w_k^0(\xi) = 0, \quad \xi \in \mathbb{R}^3_-, \tag{4.8}$$

$$\partial_\zeta w_k^0(\eta, 0) = 0, \quad \eta \in \mathbb{R}^2 \setminus \bar{\omega}, \quad \partial_\zeta w_k^0(\eta,0) = \lambda_k^0 v_k^0(0), \quad \eta \in \omega. \tag{4.9}$$

Observe that the spectral parameter has disappeared from (4.9) because

$$\partial_z w_k^0(\varepsilon^{-1}x) - \lambda^\varepsilon w_k^0(\varepsilon^{-1}x) = \varepsilon^{-1}(\partial_\zeta w_k^0(\xi) - \varepsilon\lambda^\varepsilon w_k^0(\xi)),$$

which means, in accordance with the natural asymptotic *Ansatz*

$$\lambda_k^\varepsilon = \lambda_k^0 + \varepsilon^2 \lambda_k' + \dots, \tag{4.10}$$

that the derivative $\partial_\zeta w_k^0(\eta,0)$ dominates the term $\varepsilon \lambda^\varepsilon w_k^0(\eta, 0)$.

Problem (4.8), (4.9) has a unique solution, which decays at infinity and admits the representation

$$w_k^0(\xi) = B_k^0 (2\pi\rho)^{-1} + O(\rho^{-2}), \quad \rho \to +\infty. \tag{4.11}$$

The function w_k^0 is smooth everywhere except for the contour $\partial\omega$, on which the right-hand side of (4.9) has a first kind discontinuity, so that $w_k^0 \notin H^2_{\mathrm{loc}}\big(\overline{\mathbb{R}^3_-}\big)$. Nevertheless, we can differentiate (4.11) outside of a ball $\mathbb{B}^3_{R_\omega}$ of large radius, assuming that $\nabla_\xi O(\rho^{-l}) = O(\rho^{-l-1})$. The constant B_k^0 is calculated by the formula

$$\begin{aligned} \lambda_k^0 v_k^0(0) \operatorname{meas}_2 \omega &= \int_\omega \partial_\zeta w_k^0(\xi)\, ds_\xi \\ &= -\lim_{R\to\infty} \int_{\mathbb{S}_R^{2-}} \partial_\rho w_k^0(\xi)\, ds_\xi = \frac{B_k^0}{2\pi} \lim_{R\to\infty} \int_{\mathbb{S}_R^{2-}} \frac{ds_\xi}{\rho^2} = B_k^0, \end{aligned} \tag{4.12}$$

because the area of the half-sphere $\mathbb{S}_R^{2-} = \{\xi : \rho = R,\ \zeta < 0\}$ is equal to $2\pi R^2$.

Since

$$\varepsilon w_k^0(\varepsilon^{-1}x) = \varepsilon^2 B_k^0 (2\pi r)^{-1} + O(\varepsilon^3 r^{-2}), \tag{4.13}$$

the boundary layer leaves discrepancies of order of ε^2 in the Neumann boundary conditions (1.15) on the bottom Υ and the Steklov condition (1.16) on the clearings $\Sigma_1, \dots, \Sigma_J$. Note that the sets Υ and Σ_j are at a positive distance from the point $\mathcal{O}$, i.e., in (4.13) we have $r \ge r_0 > 0$, and that there is no discrepancy on the set Ω, which contains $\mathcal{O}$. As a result, using the asymptotic *Ansatz* (4.10), we can write the following boundary-value problem for determining the leading correction term $\varepsilon^2 v_k'(x)$ of regular type:

$$\begin{aligned} &-\Delta_x v_k'(x) = 0, \quad x \in \Xi, \\ &\partial_n v_k'(x) = -\frac{B_k^0}{2\pi}\partial_n \frac{1}{r}, \quad x \in \Upsilon, \quad \partial_z v_k'(y,0) = 0, \quad y \in \Omega, \\ &\partial_z v_k'(y,0) - \lambda_k^0 v_k'(y,0) = \lambda_k' v_k^0(y,0) - (\partial_z - \lambda_k^0)\frac{B_k^0}{2\pi r}, \quad y \in \Sigma. \end{aligned} \tag{4.14}$$

The eigenvalue λ_k^0 is simple, and the Fredholm alternative yields a compatibility condition for problem (4.14):

$$\lambda_k' \int_\Sigma |v_k^0(y,0)|^2\,dy - \int_\Sigma v_k^0(y,0)(\partial_z - \lambda_k^0)\frac{B_k^0}{2\pi r}\,dy - \int_\Upsilon v_k^0(x)\frac{\partial}{\partial n}\frac{B_k^0}{2\pi r}\,ds_x = 0.$$

Using the normalization (4.3) and the Green formula in the domain $\Xi \setminus \overline{\mathbb{B}_\delta^3}$ with a small spherical cavity, we get the identity

$$\begin{aligned} \lambda_k' &= \lambda_k' \int_\Sigma |v_k^0(y,0)|^2\,dy = \int_\Sigma v_k^0(y,0)(\partial_z - \lambda_k^0)\frac{B_k^0}{2\pi r}\,dy + \int_\Upsilon v_k^0(x)\frac{\partial}{\partial n}\frac{B_k^0}{2\pi r}\,ds_x \\ &= \frac{B_k^0}{2\pi}\lim_{\delta\to 0}\int_{\mathbb{S}_\delta^{2-}} \left(v_k^0(x)\frac{\partial}{\partial r}\frac{1}{r} - \frac{1}{r}\frac{\partial v_k^0}{\partial r}(x)\right) ds_x = -B_k^0 v_k^0(0). \end{aligned}$$

Now, relation (4.12) leads to the final formula

$$\lambda_k' = -\lambda_0 \operatorname{meas}_2 \omega |v_k^0(0)|^2 \le 0.$$

Similar calculations are also possible in the case where the eigenvalue λ_k^0 has multiplicity $\varkappa_k$ (cf. the requirement (2.33)); we need to distinguish the situations (2.46) and (2.40). We formulate a result on asymptotics, which is obtained in the same way as Theorems 2.1 and 2.2.

Theorem 4.1. *Suppose that the multiplicity of an eigenvalue λ_k^0 of the first limiting problem* (4.1), (4.2) *is equal to $\varkappa_k$ (see condition* (2.33)*). There exist two positive quantities ε_k and c_k such that, for $\varepsilon \in (0, \varepsilon_k]$, the eigenvalues $\lambda_k^\varepsilon, \dots, \lambda_{k+\varkappa_k-1}^\varepsilon$ of the singularly perturbed problem* (1.14)–(1.16) *satisfy the inequalities*

$$\begin{aligned} &\left|\lambda_k^\varepsilon - \lambda_k^0 + \varepsilon^2 \lambda_k^0 \operatorname{meas}_2 \omega |v_k^0(0)|^2\right| \le c_k \varepsilon^3, \\ &\left|\lambda_p^\varepsilon - \lambda_k^0\right| \le c_k\varepsilon^3, \quad p = k+1, \dots, k+\varkappa_k - 1. \end{aligned} \tag{4.15}$$

4.3. Asymptotics of the high eigenfrequences. We choose an eigenvalue μ_m^0 of the second limiting problem (4.4), (4.5) and, first, suppose that it is simple (e.g., take μ_1^0). The corresponding eigenfunction w_m^0 can be written as in (4.11) with some coefficient B_m^0. Thus, the discrepancy of the term $w_m^0(\varepsilon^{-1}x)$ in the Neumann condition (1.15) on Υ is $O(\varepsilon)$. Therefore, an asymptotic regular type correction term should be sought in the form $\varepsilon v_m'(x)$, and it must satisfy the relations

$$\begin{aligned} &-\Delta_x v_m'(x) = 0, \quad x \in \Xi, \\ &\partial_n v_m'(x) = -\frac{B_m^0}{2\pi}\partial_n \frac{1}{r}, \quad x \in \Upsilon, \quad \partial_z v_m'(y,0) = 0, \quad y \in \Omega. \end{aligned} \tag{4.16}$$

To find the boundary conditions on $\Sigma_1, \dots, \Sigma_J$, we take an asymptotic *Ansatz*

$$\lambda^{\varepsilon}_{M^{\varepsilon}(m)} = \varepsilon^{-1}\mu^0_m + \mu'_m + \dots, \tag{4.17}$$

similar to (1.11) and (3.10), for the eigenvalue in question. Since the large factor ε^{-1} is multiplied by μ^0_m, the Steklov condition (1.16) on Σ turns into the Dirichlet condition

$$v'_m(x) = -\frac{B^0_m}{2\pi r}, \quad x \in \Sigma_1 \cup \dots \cup \Sigma_J, \tag{4.18}$$

and the terms $\varepsilon(\mu^0_m)^{-1}\partial_z v'_m(x)$ and $\varepsilon(\mu^0_m)^{-1}B^0_m\partial_n(2\pi r)^{-1}$ are assumed to be small (see the discussion in the next subsection).

For the solution of problem (4.16), (4.18) we obtain the formula

$$v'_m(x) = B^0_m G_0(x, 0), \tag{4.19}$$

where $G_0(x, y') = G(x, y') - \frac{1}{2\pi}(|y - y'|^2 + z^2)^{-1/2}$ is the regular part of the generalized Green function (see [24]), i.e., the distributional solution of the mixed boundary-value problem

$$\begin{gathered} -\Delta_x G(x, y') = 0, \quad x \in \Xi, \quad \partial_n G(x, y') = 0, \quad x \in \Upsilon, \\ G(x, y') = 0, \quad y' \in \Sigma, \quad \partial_z G(y, 0, y') = \delta(y - y'), \quad y \in \Omega, \end{gathered}$$

with the Dirac δ-function on the flat part Ω of the boundary $\partial\Xi$. As is well known, G_0 is a negative function smooth everywhere except for the contours Γ and $\Gamma_1, \dots, \Gamma_J$. Therefore, the principal part of the discrepancy of the sum $w^0_m(\xi) + \varepsilon v'_m(x)$ in the boundary condition (1.16) on ω_ε is compensated by the correction term $\varepsilon w'_m(\xi)$ of the boundary layer type. Thus, taking the expansion (4.17) into account, we arrive at the problem

$$\begin{gathered} -\Delta_\xi w'_m(\xi) = 0, \quad \xi \in \mathbb{R}^3_-, \quad \partial_\zeta w'_m(\eta, 0) = 0, \quad \eta \in \mathbb{R}^2 \setminus \bar{\omega}, \\ \partial_\zeta w'_m(\eta, 0) - \mu^0_m w'_m(\eta, 0) = \mu'_m w^0_m(\eta, 0) + \mu^0_m v'_m(0), \quad \eta \in \omega. \end{gathered} \tag{4.20}$$

The only compatibility condition

$$\int_\omega w^0_m(\eta, 0)\left(\mu'_m w^0_m(\eta, 0) + \mu^0_m v'_m(0)\right) d\eta = 0$$

for problem (4.20) takes the form

$$\begin{aligned} \mu'_m &= \mu'_m \int_\omega |w^0_m(\eta, 0)|^2 \, d\eta = -\mu^0_m v'_m(0) \int_\omega w^0_m(\eta, 0) \, d\eta \\ &= \int_\omega w^0_m(\eta, 0)(\partial_\zeta - \mu^0_m) v'_m(0) \, d\eta \\ &= v'_m(0) \lim_{R \to \infty} \int_{\mathbb{S}^{2-}_R} \frac{\partial w^0_m}{\partial \rho}(\xi) \, ds_\xi = -v'_m(0) B^0_m = -(B^0_m)^2 G_0(0, 0). \end{aligned}$$

We have used identity (4.19). Thus, the correction $\mu'_m \geq 0$ in the asymptotics of a simple eigenvalue has been found.

Similar calculations are needed also in the case of an eigenvalue μ^0_m of multiplicity $\varkappa_m$ under the additional requirement $B^0_{m+\varkappa_m-1} \neq 0$. However, if $B^0_{m+\varkappa_m-1} = 0$, and, by (3.34), the coefficients B^0_p in the expansions (4.11) of the eigenfunctions w^0_p, $p = m, \dots, m+\varkappa_m-1$, vanish, then all the correction terms μ'_p in the *Ansätze* (4.17) become zero. We formulate a result that can be obtained by an approach similar to that in Subsection 3.5, but with modifications described in the next subsection and caused by singularities of solutions on the contours where the boundary condition changes its type.

Theorem 4.2. *Suppose that an eigenvalue* μ_m^0 *of problem* (4.4), (4.5) *has multiplicity* $\varkappa_m$ (*see* (3.28)). *There exist two positive quantities* ε_m *and* C_m *such that, for* $\varepsilon \in (0, \varepsilon_k]$, *in the sequence* (1.5) *of eigenvalues of the singularly perturbed problem* (1.14)–(1.16) *there are terms* $\lambda^\varepsilon_{M^\varepsilon(m)}, \dots, \lambda^\varepsilon_{M^\varepsilon(m+\varkappa_m-1)}$ *for which*

$$\begin{aligned} &\big|\lambda^\varepsilon_{M^\varepsilon(p)} - \varepsilon^{-1}\mu_m^0\big| \le C_m\varepsilon^{-1/2}, \quad p = m, \dots, m+\varkappa_m-2, \\ &\big|\lambda^\varepsilon_{M^\varepsilon(m+\varkappa_m-1)} - \varepsilon^{-1}\mu_m^0 + (B_m^0)^2 G_0(0,0)\big| \le C_m\varepsilon^{-1/2}, \end{aligned} \tag{4.21}$$

where B_m^0 *is the coefficient in* (4.11), $G_0(0,0) < 0$ *is the value taken by the regular part of the Green function at the origin, and the indices* $M^\varepsilon(m) < M^\varepsilon(m+1) < \dots < M^\varepsilon(m+\varkappa_m-2) < M^\varepsilon(m+\varkappa_m-1)$ *of the eigenvalues occurring in the asymptotic formulas* (4.21) *grow unboundedly as* $\varepsilon \to +0$.

4.4. On the smoothness of asymptotic terms. As has already been mentioned, the eigenfunctions v_k^0 and w_m^0 are not infinitely differentiable everywhere in $\bar{\Xi}$ and $\overline{\mathbb{R}^3_-}$, but acquire singularities on the contours $\Gamma, \Gamma_1, \dots, \Gamma_J$ and $\gamma = \partial\omega$, respectively. For example, the results of [25, 26, 27, 28] (see also [29, Chapter 9]) give the following asymptotic formulas near Γ_j, $j = 1, \dots, J$, for the solution of the first limiting problem:

$$\begin{aligned} v_k^0(x) = {} & v_k^{(00)}(s_j) + v_k^{(01)}(s_j)\mathbf{r}_j\cos\varphi_j \\ & - \pi^{-1}\lambda_k^0 v_k^{(00)}(s)\mathbf{r}_j\big(\ln\mathbf{r}_j\cos\varphi_j + (\varphi_j - \pi)\sin\varphi_j\big) + O(\mathbf{r}_j^2|\ln\mathbf{r}_j|^2), \\ & \mathbf{r}_j \to +0, \end{aligned} \tag{4.22}$$

where s_j is the arc length on the smooth contour Γ_j, $(\mathbf{r}_j, \varphi_j) \in \mathbb{R}_+ \times (0, \pi)$ is the polar coordinate system in the planes orthogonal to Γ_j, and the functions $v_k^{(00)}$ and $v_k^{(01)}$ are of class $C^\infty(\Gamma_j)$. Formula (4.22) shows that v_k^0 does not belong to the Sobolev class H^2 because of the logarithmic singularities of the gradient. The same function v_k^0 has weaker singularities also on the edge Γ. Namely, denoting by $\theta(s) \in (0, \pi)$ the angle under which the surface Υ intersects the plane $\partial\mathbb{R}^3_-$ at the point $s \in \Gamma$ and using the same results as above, we get

$$v_k^0(x) = v_k^{(00)}(s) + v_k^{(01)}(s)\mathbf{r}^{\pi/\theta(s)}\cos\big(\pi\theta(s)^{-1}\varphi\big) + O(\mathbf{r}^{1+\pi/\theta(s)}), \quad \mathbf{r} \to +0.$$

The notation here is similar to that in (4.21). It is easily seen that the second derivatives $\nabla_x^2 v_k^0$ belong to the Lebesgue class L^2 in a neighborhood of Γ, but not to the Sobolev class H^1 for $2\theta(s) > \pi$.

All the listed singularities make no obstruction to the realization of the asymptotic procedure of Subsection 3.2 for problem (1.14)–(1.16), because it is based exclusively on the *local* smoothness of solutions – the expansions (2.4), (2.6) are written near the point $\mathcal{O}$, i.e., at a positive distance from Γ and $\Gamma_1, \dots, \Gamma_J$. Moreover, the solutions of the Neumann problem (4.8), (4.9) and of similar problems turn out to be smooth outside of the ball $\mathbb{B}^3_{R_\omega} \ni \xi$ and therefore give rise to smooth discrepancies in the Steklov boundary conditions on Σ_j and in the Neumann ones on Υ. Finally, the summands $\lambda_k^{(j)} v_k^{(j-q)}$ occurring in (2.12) belong to the Sobolev–Slobodetskiĭ space $H^{1/2}(\Sigma)$, and hence, the mixed – with the Steklov and Neumann boundary conditions — problems of the form (4.14) have solutions in $H^1(\Xi)$ that are infinitely differentiable far from Γ and $\Gamma_1, \dots, \Gamma_J$.

The facts mentioned above show that, in the case of a simple eigenvalue λ_k^0 of problem (4.1), (4.2), the direct repetition of the procedure described in Subsection 2.2 ensures the construction of formal asymptotic series (2.1) and (2.2) for the eigenvalue λ_k^ε and the corresponding eigenfunctions of problem (1.14)–(1.16).

The situation is different for the full asymptotic series (3.10) and (3.11). On the one hand, the eigenfunction w_m^0 of problem (4.4), (4.5) admits a representation similar to

(4.22) near Γ, which does not affect the asymptotic procedure, because w_m^0 is infinitely differentiable outside the ball $\mathbb{B}^3_{R_\omega} \supset \bar{\omega}$, and, consequently, the right-hand sides of problem (4.16), (4.15) are smooth. On the other hand, the solution v_m' of this mixed problem, with the Neumann and Dirichlet boundary conditions, acquires a singularity,

$$v_m'(x) = v_m^{(1/2)}(s_j)\mathbf{r}_j^{1/2}\sin(\varphi_j/2) + O(\mathbf{r}_j^{3/2}), \quad \mathbf{r}_j \to +0, \tag{4.23}$$

on the contours $\Gamma_1, \dots, \Gamma_J$, where the type of the boundary condition changes. The Dirichlet condition (4.18) was obtained from the Steklov condition (1.16), by using the *Ansatz* (4.17) and under the assumption that the expression $\varepsilon(\mu_m^0)^{-1}\partial_z v_m'(y,0)$ is small, but the formula

$$(\mu_m^0)^{-1}\partial_z v_m'(y,0) = (2\mu_m^0)^{-1} v_m^{(1/2)}(s_j)\mathbf{r}_j^{-1/2} + O(\mathbf{r}_j^{1/2}), \quad \mathbf{r}_j \to +0, \tag{4.24}$$

implied by (4.23), means that the above expression grows unboundedly near Γ_j. Thus, the next regular type asymptotic term $\varepsilon^2 v_m''(x)$ cannot belong to $H^1(\Xi)$, because it satisfies the Dirichlet conditions

$$v_m''(y,0) = (\mu_m^0)^{-1}\partial_z v_m'(y,0) + f_m''(y), \quad y \in \Sigma_1 \cup \dots \cup \Sigma_J,$$

with a smooth component f_m''. Moreover, we cannot fix the summand v_m'' even allowing for the growth $O(\mathbf{r}_j^{-1/2})$, because the homogeneous problem (4.16), (4.18) has an uncountable collection of solutions with such singularities (see [30] and, e.g., [29, Chapter 12]).

This situation can be remedied only by the investigation of a new boundary layer, see Subsection 5.3.

The lack of lower asymptotic terms does not prevent us from justifying the leading term; however, the error estimate becomes worse because of the $O(\mathbf{r}_j^{-1/2})$-singularities of $\nabla_x v_m'(x)$, which is reflected in the statement of Theorem 4.2.

Once again, we turn the reader's attention to the fact that in §3 the outer part $\partial\Omega$ of the boundary $\partial\Omega(\varepsilon)$ is assumed to be C^∞-smooth. The full asymptotic series (3.10) and (3.11) are not available without this assumption. However, the full asymptotic series (2.1) and (2.2) can be constructed as in §2 also for a Lipcshitz surface $\partial\Omega$. The smoothness of the inner part $\partial\omega_\varepsilon$ of the boundary $\partial\Omega(\varepsilon)$ is not of principal value for both asymptotic constructions.

§5. Specifics of asymptotics in the planar case

5.1. Low-frequency range of the Steklov problem spectrum. For $d = 2$, the Neumann problem (2.13) loses unique solvability in the class of function that decay at infinity. Note that there is a solution $w_k^{(j)}$ growing logarithmically as $\rho \to +\infty$ and determined up to a constant summand. In other words, problem (2.13) with the right-hand side $g_k^{(j)} \in C^\infty(\partial\omega)$ has a unique solution

$$w_k^{(j)}(\xi) = C_k^{(j)}\frac{1}{2\pi}\ln\frac{1}{\rho} + \sum_{p=1}^{N}\rho^{-p}W_{kp}^{(j)}(\vartheta) + \widetilde{w}_{kN}^{(j)}(\xi), \tag{5.1}$$

where N is an arbitrary positive integer, and the remainder term $\widetilde{w}_{kN}^{(j)}$ satisfies estimate (2.7). Observe the restriction $p \geq 1$, implying the absence of the term $O(1)$. The functions (5.1) fail to possess a natural property of a boundary layer: they do not decay at infinity, but the slow (compared to power-like) growth of the first summand on the right in (5.1) presents no obstruction to the realization of the asymptotic procedure described in §2. However, some modifications are needed.

First, the identity $\ln\rho = \ln r - \ln\varepsilon$, which accompanies the coordinate change $\xi \mapsto x$, leads to a polynomial dependence of the formal asymptotic terms on $\ln\varepsilon$, i.e., in formulas (2.1) and (2.2) we should make the changes

$$\lambda_k^{(j)} \mapsto \lambda_k^{(j)}(\ln\varepsilon), \quad v_k^{(j)}(x) \mapsto v_k^{(j)}(x;\ln\varepsilon), \quad w_k^{(j)}(\xi) \mapsto w_k^{(j)}(\xi;\ln\varepsilon).$$

Second, the right-hand sides $f_k^{(j)}$ and $g_k^{(j)}$ of problems (2.11) and (2.13) take the form

$$\begin{aligned} f_k^{(j)}(x;\ln\varepsilon) &= \sum_{q=1}^{j} \lambda_k^{(q)}(\ln\varepsilon) v_k^{(j-q)}(x;\ln\varepsilon) - \sum_{l=1}^{j-1} \partial_n\big(r^{-1} W_{kl}^{(j-1-l)}(\vartheta;\ln\varepsilon)\big) \\ &+ \sum_{q=0}^{j-1} \lambda_k^{(q)}(\ln\varepsilon) \sum_{l=1}^{j-1-q} r^{-l} W_{kl}^{(j-1-q-l)}(\vartheta;\ln\varepsilon) \\ &- \frac{C_k^{(j-1)}}{2\pi}\partial_n \ln\frac{1}{r} + \sum_{q=0}^{j-1}\lambda_k^{(q)}(\ln\varepsilon)\frac{C_k^{(j-1-q)}}{2\pi}\Big(\ln\frac{1}{r} + \ln\varepsilon\Big), \end{aligned} \tag{5.2}$$

$$\begin{aligned} g_k^{(j)}(\xi;\ln\varepsilon) &= \sum_{q=0}^{j-1} \lambda_k^{(q)}(\ln\varepsilon) w_k^{(j-1-1)}(\xi;\ln\varepsilon) \\ &+ \sum_{q=0}^{j-1} \lambda_k^{(q)}(\ln\varepsilon) \sum_{l=0}^{j-q} \rho^l V_{kl}^{(j-q-l)}(\vartheta;\ln\varepsilon) - \sum_{l=1}^{j+1} \partial_\nu\left(\rho^l V_{kl}^{(j+1-l)}(\vartheta;\ln\varepsilon)\right). \end{aligned} \tag{5.3}$$

As before, the function $\mathbf{f}_k^{(j)}$ given by (2.16) and (5.2) depends on the quantities (2.17), and the function (5.3) depends on the quantities (2.15). As a result, this order of finding the terms of the series (2.1) and (2.2), as described in Subsection 2.2, remains the same. The polynomial dependence of the above functions on $\ln\varepsilon$ is specified in the next statement.

Lemma 5.1. *If $\lambda^0 > 0$ is a simple eigenvalue and $v_k^0(0) \neq 0$, then the degrees of the polynomials $\ln\varepsilon \mapsto w_k^{(j)}(\xi;\ln\varepsilon)$, $\ln\varepsilon \mapsto v_k^{(j)}(x;\ln\varepsilon)$, and $\ln\varepsilon \mapsto \lambda_k^{(j)}(\ln\varepsilon)$ are equal to j, $j-1$, and $j-2$, respectively.*

Proof. For $d = 2$, the constant $C_k^{(0)}$ in the expansion (5.1) of the solution $w_k^{(0)}$ of problem (2.18) is calculated by the formula

$$\begin{aligned} \lambda_k^0 v_k^0(0)\,\mathrm{meas}_1\,\partial\omega &= \int_{\partial\omega}\big(\lambda_k^0 v_k^0(0) - \partial_\nu(\xi\cdot\nabla_x v_k^0(0))\big)\,ds_\xi \\ &= \frac{C_k^{(0)}}{2\pi}\lim_{R\to\infty}\int_{\partial\mathbb{B}_R^2}\frac{\partial}{\partial\rho}\ln\frac{1}{\rho}\,ds_\xi = C_k^{(0)}, \end{aligned}$$

so that the function $\ln\varepsilon \mapsto f_k^{(1)}(x;\ln\varepsilon)$ is linear, and

$$f_k^{(1)}(x;\ln\varepsilon) - \lambda_q^{(1)}(\ln\varepsilon)v_k^0(x) = \mathbf{f}_k^{(1)}(x;\ln\varepsilon) = \widehat{\mathbf{f}}_k^{(1)}(x) + \lambda_k^0(2\pi)^{-1}C_k^{(0)}\ln\varepsilon.$$

On the other hand, invoking the orthogonality condition (2.19) with $l = 0$ and $v_1^0(x) = (\mathrm{meas}_1\,\partial\Omega)^{-1/2}$, we see that that the quantity (2.20) with $j = 1$ does not depend on $\ln\varepsilon$, i.e., $\lambda_k^{(1)}(\ln\varepsilon) = \lambda_k^{(1)}$. However, $\ln\varepsilon \mapsto v_k^{(1)}(x;\ln\varepsilon)$ is a linear function. Therefore, so is the right-hand side of (5.3) with $j = 1$, because of the summand $\partial_\nu\big(\rho^1 V_{k1}^{(1)}(\vartheta;\ln\varepsilon)\big)$. The calculations made in the proof of Lemma 2.2 (they remain valid also for $d = 2$; cf. [19, Appendix G]) demonstrate that the factor $C_k^{(1)}$ in (5.1) with $j = 1$ does not depend on $\ln\varepsilon$, but the angle part $W_{k1}^{(1)}(\vartheta;\ln\varepsilon)$ does depend, provided the gradient $\nabla_x v_k^{(1)}(0;\ln\varepsilon)$ possesses this property. Thus, the previous argument establishes that $\lambda_k^{(2)}$

does not depend on the logarithm, but the right-hand side $f_k^{(3)}$ involves the expression $\lambda_k^0 r^{-1} W_{k1}^{(1)}(\vartheta; \ln\varepsilon)$ and if no geometrical symmetry is present, then we have no reason to think that the expression is $L^2(\partial\Omega)$-orthogonal to the trace of the eigenfunction v_k^0.

The proof can be finished by induction the base of which has been prepared above. □

Note that, for many reasons, in particular, if the requirement $v_k^0(0) \neq 0$ is violated, the degrees of the polynomials in question may happen to be smaller than in Lemma 5.1. We leave the discussion of this topic incomplete.

The third modification of the material of §2 caused by the passage to the case of $d = 2$ concerns the justification procedure. Note that the norms

$$\|u^\varepsilon; V_0^1(\Omega(\varepsilon))\| \quad \text{and} \quad \|u^\varepsilon; H^1(\Omega(\varepsilon))\|$$

mentioned in Lemma 2.3 are no longer equivalent, because the Hardy inequality (2.49) fails for $\alpha = d - 2 = 0$, and we should replace it with the following new version of it[7]

$$\int_\delta^1 \frac{1}{t} |\ln t|^{-2} |U(t)|^2\, dt \leq 4 \int_\delta^1 t \Big|\frac{dU}{dt}(t)\Big|^2\, dt, \quad U \in C_c^\infty[\delta, 1). \tag{5.4}$$

This implies the weighted inequality

$$\|r^{-1}(1 + |\ln r|)^{-1} u^\varepsilon; L^2(\Omega(\varepsilon))\| \leq c\|u^\varepsilon; H^1(\Omega(\varepsilon))\|, \tag{5.5}$$

which can serve as a substitute of the estimate used in the proof of Lemma 2.3. The factor $(1 + |\ln r|)^{-1}$ arising on the left-hand side in (5.5), which vanishes as $r \to 0$, does not affect significantly the deduction of estimates in Subsections 2.5–2.7, but brings additional coefficients of the form $|\ln\varepsilon|^m$ because, since $\mathcal{O} \in \omega$, on the domain $\Omega(\varepsilon)$ we have $r^{-1}(1 + |\ln r|)^{-1} \leq cr^{-1}(1 + |\ln\varepsilon|)^{-1}$, i.e.,

$$\|u^\varepsilon; H^1(\Omega(\varepsilon))\| \leq c\|u^\varepsilon; V_0^1(\Omega(\varepsilon))\| \leq C(1 + |\ln\varepsilon|)\|u^\varepsilon; H^1(\Omega(\varepsilon))\|. \tag{5.6}$$

Now we formulate a theorem on the asymptotics of solutions of the spectral problem (1.1), (1.2) in a singularly perturbed planar domain (1.3); this theorem is obtained with the help of inequality (5.6), playing the role of a substitute of Lemma 2.3.

Theorem 5.1. 1) *The theorem on the relationship between the sequences* (1.5) *and* (1.9) *remains valid for $d = 2$, but the majorants $c_k\varepsilon^{d-1}$ and $c_k\varepsilon^d$ in* (2.84) *should be replaced with $c_k\varepsilon|\ln\varepsilon|$ and $c_k\varepsilon^2|\ln\varepsilon|$, respectively.*

2) *Theorem* 2.3 *on the perturbation of a simple eigenvalue of problem* (1.10) *remains valid, but the majorants $C_{kN}\varepsilon^{N+1}$ in* (2.85) *and* (2.86) *should be replaced by $C_{kN}(\beta)\varepsilon^{N+\beta}$, where β is an arbitrary number in $(0, 1)$, and $C_{kN}(\beta) \to +\infty$ as $\beta \to 1-0$.*

5.2. High-frequency range of the Steklov problem spectrum. It is well known that for $d = 2$ any constant function in $\mathbb{R}^d \setminus \bar\omega$ can be approximated, in the norm $\|\cdot; \mathcal{H}\|$ given by the scalar product (3.3), by functions of class $C_c^\infty(\mathbb{R}^d \setminus \omega)$; consequently, a constant lies in $\mathcal{H}$, and, in contrast to the many-dimensional situation, the external Steklov problem (3.1) acquires the zero eigenvalue $\mu_0^0 = 0$ with the eigenfunction $w_0^0(\xi) = (\operatorname{meas}_1 \partial\omega)^{1/2}$. The fact that $1 \in \mathcal{H}$ is confirmed by the version (5.4) of the Hardy inequality, yielding the following equivalent weighted norm in $\mathcal{H}$:

$$\left(\|\nabla w; L^2(\mathbb{R}^2 \setminus \omega)\|^2 + \|\rho^{-1}(1 + |\ln\rho|)^{-1} w; L^2(\mathbb{R}^2 \setminus \omega)\|^2\right)^{1/2}.$$

Moreover, the eigenfunctions $w_1^0, w_2^0, \dots, w_m^0, \dots \in \mathcal{H}$ corresponding to the positive eigenvalues (1.12) may lose the key property of decaying at infinity, because the bounded harmonic function w_m^0 admits the expansion

$$w_m^0(\xi) = B_m^0 + O(\rho^{-1}), \quad \rho \to +\infty, \tag{5.7}$$

[7] This is obtained from (2.49) with $\alpha = 0$ by the change $r \mapsto t = e^{1/r}$.

and it may happen that $B_m^0 \neq 0$.

The constant $B_m^0 \neq 0$ in (5.7) changes crucially the asymptotics of eigenvalues and eigenfunctions of problem (1.1), (1.2). Assuming that μ_m^0 is a simple eigenvalue, we follow [17, 31] (see also [1, §9.1]) to indicate a modified algorithm for constructing the asymptotics. In comparison with the original problem treated in [17], the limiting problems in the domains Ω and $\mathbb{R}^2 \setminus \bar{\omega}$ interchange their roles; the possibility itself of such an exchange without heavy consequences is discussed in Subsection 6.1.

The eigenvalues and eigenfunctions are sought in the form

$$\lambda^\varepsilon_{M^\varepsilon(m)} = \varepsilon^{-1}(\mu_m^0 + \mathfrak{m}_m(\mathfrak{z})) + \dots, \tag{5.8}$$

$$u^\varepsilon_{M^\varepsilon(m)}(x) = w_m^0(\xi) + \mathfrak{w}_m(\xi;\mathfrak{z}) + v_m'(x;\mathfrak{z}) + \dots, \tag{5.9}$$

where $\mathfrak{m}_m(\mathfrak{z})$ and $\mathfrak{w}_m(\mathfrak{z})$ are small corrections, of order of $\mathfrak{z} = |\ln\varepsilon|^{-1}$, and the dots replace the remainder terms of order of $O(\varepsilon^\beta)$ with $\beta \in (0,1)$. Thus, from the outset we associate the asymptotic *Ansätze* (5.8) and (5.9) with a power-like accuracy of approximation. At the same time, it is not hard to construct a series in the inverse powers of the large parameter $|\ln\varepsilon|$, but the approach suggested in [17] and employed in this section allows us to sum such series.

The smooth type solution $v_m'(x;\mathfrak{z})$ is designed to compensate for the discrepancy of the summand $w_m^0(\xi)$ in the boundary condition (1.2) on $\partial\Omega$, replaced by the Dirichlet condition for the reason mentioned in Subsection 3.2. As a result, we get the problem

$$-\Delta_x v_m'(x;\mathfrak{z}) = 0, \quad x \in \Omega \setminus \mathcal{O}, \quad v_m'(x;\mathfrak{z}) = -B_m^0, \quad x \in \partial\Omega. \tag{5.10}$$

Following the approach of [17], for the role of the solution of (5.10) we take the expression

$$v_m'(x;\mathfrak{z}) = -B_m^0 + 2\pi\mathfrak{z}B_m^0 G(x,0),$$

which needs comment. First, $G(x,y)$ is the Green function of the Dirichlet problem for the Laplace operator in Ω, admitting the representation (3.23) with the fundamental solution $\Phi(x) = -(2\pi)^{-1}\ln r$. In other words, problem (5.10) is posed in the punctured domain $\Omega \setminus \mathcal{O}$, and its solution is sought in the Lebesgue class $L^2(\Omega)$. Second, the coefficient of G, which cannot be arbitrary in general, is chosen so that

$$\begin{aligned} v_m'(x;\mathfrak{z}) &= -B_m^0 + B_m^0|\ln\varepsilon|^{-1}(-\ln r + 2\pi G_0(0,0)) + O(\mathfrak{z}r) \\ &= -B_m^0 + B_m^0|\ln\varepsilon|^{-1}(-\ln\rho - \ln\varepsilon + 2\pi G_0(0,0)) + O(\mathfrak{z}\varepsilon\rho) \\ &= B_m^0|\ln\varepsilon|^{-1}(-\ln\rho + 2\pi G_0(0,0)) + O(\varepsilon|\ln\varepsilon|^{-1}) \text{ for } x = \varepsilon\xi \in \partial\omega_\varepsilon, \end{aligned}$$

i.e., the function v_m' is made small on $\partial\omega_\varepsilon$. Finally, if we ensure that the form $\mathfrak{w}_m$ of the boundary layer type decays at infinity, then the smallness of the total discrepancy of the asymptotic summands in the *Ansätze* (5.8) and (5.9) acquires a power-like nature.

The problem for the corrections $\mathfrak{m}_m(\mathfrak{z})$ and $\mathfrak{w}_m(\xi;\mathfrak{z})$ looks like this:

$$\begin{aligned} &-\Delta_\xi \mathfrak{w}_m(\xi;\mathfrak{z}) = 0, \quad \xi \in \mathbb{R}^2 \setminus \bar{\omega}, \\ &\partial_\nu \mathfrak{w}_m(\xi;\mathfrak{z}) - \mu_m^0 \mathfrak{w}_m(\xi;\mathfrak{z}) = \mathfrak{f}_m(\xi;\mathfrak{z}) := \mathfrak{m}_m(\mathfrak{z})\mathfrak{w}_m(\xi;\mathfrak{z}) + \mathfrak{m}_m(\mathfrak{z})w_m^0(\xi) \\ &\qquad + B_m^0\mathfrak{z}\left(\partial_\nu \ln\rho - (\mu_m^0 + \mathfrak{m}_m(\mathfrak{z}))(\ln\rho - 2\pi G_0(0,0))\right), \quad \xi \in \partial\omega. \end{aligned} \tag{5.11}$$

Suppose for the moment that the right-hand side $\mathfrak{f}_m$ depending on $\mathfrak{m}_m(\mathfrak{z})$ and $\mathfrak{w}_m(\cdot;\mathfrak{z})$ is fixed. Since μ_m^0 is a simple eigenvalue of problem (1.13), the formulas

$$\mathfrak{f}_m(\cdot;\mathfrak{z}) \in H^{1/2}(\partial\omega), \quad \left(\mathfrak{f}_m(\cdot;\mathfrak{z}), w_m^0\right)_{\partial\omega} = 0 \tag{5.12}$$

guarantee the existence of a solution $\mathfrak{w}_m(\cdot;\mathfrak{z}) = \mathfrak{R}_m\mathfrak{f}_m(\cdot;\mathfrak{z}) \in \mathcal{H}$ of problem (5.11), which is determined up to a summand $C_m w_m^0$, and hence, is fixed uniquely by the requirement of decay at infinity:

$$\mathfrak{w}_m(\xi;\mathfrak{z}) = O(\rho^{-1}), \quad \rho \to +\infty. \tag{5.13}$$

The possibility to ensure (5.13) follows from two observations: a particular solution $\mathfrak{w}_m^0(\cdot;\mathfrak{z}) \in \mathcal{H}$, being a harmonic function, can be written as

$$\mathfrak{w}_m^0(\xi;\mathfrak{z}) = \mathfrak{c}_m^0(\mathfrak{z}) + O(\rho^{-1}), \quad \rho \to +\infty,$$

and taking $c_m(\mathfrak{z}) = -(B_m^0)^{-1}\mathfrak{c}_m^0(\mathfrak{z})$ we see that the sum $\mathfrak{w}_m^0(\xi;\mathfrak{z}) + c_m(\mathfrak{z})w_m^0(\xi)$ satisfies (5.13).

Observe that

$$\begin{aligned}\int_{\partial\omega} w_m^0(\xi)\big(\partial_\nu \ln\rho - \mu_m^0 \ln\rho\big)\, ds_\xi &= \lim_{R\to\infty}\int_{\partial\mathbb{B}_R^2}\left(\ln\rho\,\frac{\partial w_m^0}{\partial\rho}(\xi) - w_m^0(\xi)\frac{\partial}{\partial\rho}\ln\rho\right) ds_\xi \\ &= -B_m^0 \lim_{R\to\infty}\int_{\partial\mathbb{B}_R^2}\frac{1}{\rho}\, ds_\xi = -2\pi B_m^0.\end{aligned}$$

Now, by formulas (5.12) with $k = 0, m$ and $l = m$, the last identity in (3.9) takes the form

$$\mathfrak{m}_m(\mathfrak{z}) = 2\pi(B_m^0)^2\mathfrak{z} - B_m^0\mathfrak{z}\mathfrak{m}_m(\mathfrak{z})(\ln\rho, w_m^0)_{\partial\omega} - \mathfrak{m}_m(\mathfrak{z})(\mathfrak{w}_m(\cdot;\mathfrak{z}), w_m^0)_{\partial\omega}. \tag{5.14}$$

On the other hand, problem (5.11) transforms to the abstract nonlinear equation

$$\mathfrak{w}_m(\cdot;\mathfrak{z}) = \mathfrak{R}_m\mathfrak{F}_m(\mathfrak{z}, \mathfrak{m}_m(\mathfrak{z}), \mathfrak{w}_m(\cdot;\mathfrak{z})), \tag{5.15}$$

where $\mathfrak{F}_m(\mathfrak{z}, \mathfrak{m}_m(\mathfrak{z}), \mathfrak{w}_m(\cdot;\mathfrak{z}))$ is the new form of writing the right-hand side of problem (5.11). Relation (5.14) and (5.15) form a system of nonlinear equations for the pair $\{\mathfrak{m}_m(\mathfrak{z}), \mathfrak{w}_m(\cdot;\mathfrak{z})\} \in \mathbb{R}\times\mathcal{H}$. The presence of the small parameter $\mathfrak{z} = |\ln\varepsilon|^{-1}$ and the polynomial dependence on $\mathfrak{z}$, $\mathfrak{m}_m(\mathfrak{z})$, and $\mathfrak{w}_m(\cdot;\mathfrak{z})$ show that for $\mathfrak{z} \in (0, \mathfrak{z}_0]$ and some $\mathfrak{z}_0 > 0$ there exists a unique small solution of system (5.11),

$$|\mathfrak{m}_m(\mathfrak{z})| + \|\mathfrak{w}_m(\cdot;\mathfrak{z});\mathcal{H}\| \le c\mathfrak{z},$$

which, moreover, turns out to be a real analytic function of the variable $\mathfrak{z}$ (see, e.g., [32]), i.e., the series

$$\mathfrak{m}_m(\mathfrak{z}) = \sum_{j=1}^{\infty}\mathfrak{z}^j\mathfrak{m}_m^{(j)}, \quad \mathfrak{w}_m(\xi;\mathfrak{z}) = \sum_{j=1}^{\infty}\mathfrak{z}^j\mathfrak{w}_m^{(j)}(\xi) \tag{5.16}$$

converge. Substituting formulas (5.16) in (5.14) and collecting the coefficients of $\mathfrak{z} = |\ln\varepsilon|^{-1}$, we see that $\mathfrak{m}_m^{(1)} = 2\pi(B_m^0)^2 > 0$. Thus, $\mathfrak{m}_m(\mathfrak{z}) > 0$ if $\mathfrak{z} > 0$ is small.

The justification of the asymptotics is done along the standard lines. Before stating the results, observe that, in the case of an eigenvalue μ_m^0 of problem (1.13) of multiplicity $\varkappa_m$ (see (3.28)), the corresponding eigenfunctions $w_m^0, \dots, w_{m+\varkappa_m-1}^0$ can be fixed so as to satisfy (3.9) and (3.34). Consequently, these eigenfunctions decay at infinity, and their discrepancies in the boundary condition (1.2) are compensated by the smooth type summands $\varepsilon v_p'(x)$. As a result, the iteration processes of constructing the asymptotics fully follow the algorithm described in §3 and there is no need to introduce the logarithmically small correction $\mathfrak{m}_p(|\ln\varepsilon|^{-1})$ and $\mathfrak{w}_p(\xi;|\ln\varepsilon|^{-1})$. If the coefficient $B_{m+\varkappa_m-1}^0$ is also zero, then the above complications of asymptotic constructions are not needed any longer for $w_{m+\varkappa_m-1}^0$, but if $B_{m+\varkappa_m-1}^0 \ne 0$, then the *Ansätze* (5.8) and (5.9) are required indeed, with the replacement $m \mapsto m + \varkappa_m - 1$.

Theorem 5.2. *If $d = 2$ and μ_m^0 is a positive eigenvalue of multiplicity $\varkappa_m$ for problem* (1.13) (*cf.* (3.28)), *then in the sequence* (1.5) *of eigenvalues of problem* (1.1), (1.2) *there exist terms $\lambda^\varepsilon_{M^\varepsilon(m)}, \dots, \lambda^\varepsilon_{M^\varepsilon(m+\varkappa_m-1)}$ such that*

$$\begin{aligned} &\big|\lambda^\varepsilon_{M^\varepsilon(p)} - \varepsilon^{-1}\mu_m^0\big| \le c_m(\beta)\varepsilon^{-\beta}, \quad p = m, \dots, m+\varkappa_m - 2, \\ &\big|\lambda^\varepsilon_{M^\varepsilon(m+\varkappa_m-1)} - \varepsilon^{-1}\big(\mu_m^0 + \mathfrak{m}_{m+\varkappa_m-1}(|\ln\varepsilon|^{-1})\big)\big| \le c_m(\beta)\varepsilon^{-\beta}, \end{aligned} \tag{5.17}$$

where $\beta \in (0,1)$, and the factor $c_m(\beta)$ is independent of $\varepsilon \in (0, \varepsilon_m]$ for some $\varepsilon_m > 0$, but $c_m(\beta) \to +\infty$ as $\beta \to 1-0$. Moreover, $\mathfrak{z} \mapsto \mathfrak{m}_{m+\varkappa_m-1}(\mathfrak{z})$ is a real-analytic function in a neighborhood of the point $\mathfrak{z} = 0$, $\mathfrak{m}_{m+\varkappa_m-1}(0) = 0$, and the term $\mathfrak{m}_{m+\varkappa_m-1}(|\ln\varepsilon|^{-1})$ disappears from the second formula in (5.17) *whenever $B^0_{m+\varkappa_m-1} = 0$.*

5.3. On fast oscillation of surface waves in clearings. Under the passage to the case of $d = 2$, the asymptotic procedure of Subsection 4.1, pertaining to the low-frequency range of the spectrum of problem (1.14)–(1.16), requires only minor repair, which was already discussed in detail in Subsection 5.1. In contrast, in the case of high-frequency range of the spectrum of the two-dimensional problem (1.14)–(1.16), complications of two types arise, as described in Subsections 5.2 and 4.4. For the first of them, determining the corrections $\mathfrak{m}_m(|\ln\varepsilon|^{-1})$ and $\mathfrak{w}_m(\xi; |\ln\varepsilon|^{-1})$ in the *Ansätze* (5.8) does not meet any new difficulty. Therefore, we return to discussing the singularities found in the expansions (4.23) and (4.24). For an eigenvalue $\lambda^\varepsilon = \varepsilon^{-1}\mu^0 + o(\varepsilon^{-1})$, the Steklov boundary condition in the leading part takes the form

$$\varepsilon\partial_n u^\varepsilon(x) - \mu^0 u^\varepsilon(x) = \dots$$

thereby acquiring the small parameter at the highest derivative. Within the framework of the Vishik–Lyusternik method, modified in [33, 34] to cover the case of singularly perturbed elliptic boundary-value problems in domains with corner (conic) points on the boundary, the phenomenon of a two-dimensional boundary layer arises near a point where the boundary condition changes its type (we mean the Neumann and Steklov conditions in our case). In problem (1.14)–(1.16), the coordinate dilation in ε^{-1} times also leads to a new limiting problem in the lower half-plane:

$$\begin{aligned} &-\Delta_\xi \mathfrak{v}(\xi) = \mathfrak{g}(\xi), \quad \xi \in \mathbb{R}^2_-, \\ &\partial_\zeta \mathfrak{v}(\eta, 0) = 0, \quad \eta < 0, \quad \partial_\zeta \mathfrak{v}(\eta, 0) - \mu\mathfrak{v}(\eta, 0) = 0, \quad \eta > 0. \end{aligned} \tag{5.18}$$

Should the coefficient μ be negative, the results of [35] would show that the solution of problem (5.18) with the right-hand side $\mathfrak{g} \in C_c^\infty(\overline{\mathbb{R}^2_-})$ decays at infinity with the rate $O(\rho^{-1})$, so that there would be no obstruction to constructing the asymptotics (cf. [34, 35, 36, 37], etc.). However, $\mu = \mu_m^0 > 0$ and therefore the properties of the solutions of problem (5.18) change radically: no solution decays at infinity, and in the expansion of a bounded solution an *oscillating* wave arises. Namely, the following representation is known (see, e.g., the book [13]):

$$\mathfrak{v}(\xi) = \chi_+(\eta)\,\mathrm{Re}\big(\mathfrak{c}e^{\mu(\zeta + i\eta)}\big) + O((1+\rho)^{-1/2}), \quad \rho \to +\infty, \tag{5.19}$$

where $\mathfrak{c} \in \mathbb{C}$ is a constant, and $\chi_+ \in C^\infty(\mathbb{R})$ is a cut-off function, $\chi_+(\eta) = 1$ for $\eta > 2$ and $\chi_+(\eta) = 0$ for $\eta < 1$. The first summand on the right in (5.19) decays exponentially as $\zeta \to -\infty$, representing indeed a *surface wave*. In the initial coordinates (y, z) it takes the form

$$e^{\mu z/\varepsilon}\,\mathrm{Re}(\mathfrak{c}e^{i\mu y/\varepsilon}). \tag{5.20}$$

Since the first factor decays rapidly when we move down the free surface of liquid $\{x = (y, z) : z = 0\}$, the function (5.20) can be viewed as a boundary layer, but the fast oscillation of the second factor

$$\cos(\mu y/\varepsilon)\operatorname{Re}\mathfrak{c}-\sin(\mu y/\varepsilon)\operatorname{Im}\mathfrak{c}$$

does not allow us to apply the Vishik–Lyusternik method. The author knows of no method for treating the asymptotic components like (5.20) and leaves open the question how to construct a boundary layer for smoothing the above-mentioned singularities of solutions. It is plausible that such rapidly-oscillating asymptotic components will lead to new formulas for eigenvalues and eigenfunctions, which will differ from those studied in the present paper.

§6. Versions and generalizations

6.1. The Kelvin transformation. Suppose $R=1$, i.e., the set $\bar\omega$ is contained in the unit ball $\mathbb{B}_1^d$. The coordinate change

$$x\mapsto x'=\varepsilon|x|^{-2}x$$

(inversion plus contraction) transforms the domain $\Omega(\varepsilon)$ to the domain $\Omega'(\varepsilon)=\Omega'\setminus\overline{\omega'_\varepsilon}$ of the same type, with

$$\Omega'=\{x'\in\mathbb{R}^d : |x'|^{-2}x'\notin\bar\omega\},\quad \omega'=\left\{x' : |x'|^{-2}x'\notin\bar\Omega\right\}. \tag{6.1}$$

For $d\geq 3$, the Kelvin transformation

$$u(x)\mapsto u'(x')=|x'|^{2-d}u(\varepsilon|x'|^{-2}x') \tag{6.2}$$

(see, e.g., [38, §11.2]) leaves a function harmonic, i.e., the image u' of a solution u of problem (1.1), (1.2) satisfies the Laplace equation

$$-\Delta_{x'}u'(x')=0,\quad x'\in\Omega'(\varepsilon). \tag{6.3}$$

We find how the Steklov boundary conditions (1.2) are transformed.

Suppose that, locally, the boundary $\partial\Omega$ is given by the formula $F(x)=0$, so that the derivative along the normal takes the form

$$\partial_n=|\nabla_xF(x)|^{-1}\nabla_xF(x)\cdot\nabla_x.$$

We introduce the dilated coordinates $\xi'=\varepsilon^{-1}x'=|x|^{-2}x$. Since $x=|\xi'|^{-2}\xi'$, the equation for $\partial\omega'$ looks like this:

$$F(\xi'):=F(|\xi'|^{-2}\xi')=0.$$

Lemma 6.1. *We have*

$$\nabla_x=|\xi'|^2\mathcal{J}(\vartheta)\nabla_{\xi'}, \tag{6.4}$$

where $\mathcal{J}(\vartheta)=\mathbb{I}-2\Theta(\vartheta)$ *is an orthogonal matrix,* $\vartheta=|\xi'|^{-1}\xi'\in\mathbb{S}^{d-1}$, $\mathbb{I}$ *is the unit* $(d\times d)$*-matrix, and the matrix* $\Theta(\vartheta)$ *is formed by the products* $\vartheta_j\vartheta_k$, $j,k=1,\dots,d$.

Proof. Identity (6.4) is obvious, and the orthogonality of the matrix $\mathcal{J}(\vartheta)$ is checked as follows:

$$(1-2\vartheta_j^2)^2+4\biggl(\sum_{p=1}^d\vartheta_j^2\vartheta_p^2-\vartheta_j^4\biggr)=1,$$

$$-2\vartheta_j\vartheta_k(1-2\vartheta_j^2)-2\vartheta_j\vartheta_k(1-2\vartheta_k^2)+4\biggl(\sum_{p=1}^d\vartheta_j\vartheta_k\vartheta_p^2-4\vartheta_j\vartheta_k(\vartheta_j^2+\vartheta_k^2)\biggr)=0,\quad j\neq k.$$

□

Thus, the derivative along the inner (relative to ω') normal is calculated by the formula

$$|\nabla_{\xi'}F'(\xi')|^{-1}\nabla_{\xi'}F'(\xi')\cdot\nabla_{\xi'} = |\xi'|^2|\mathcal{J}(\vartheta)\nabla_x F(s)|^{-1}\nabla_x F(x)\cdot\nabla_x = a_\omega(\xi')\partial_n,$$

where

$$a_\omega(\xi') = |\xi'|^2|\mathcal{J}(|\xi'|^{-1}\xi')\nabla_x F(|\xi'|^{-2}\xi')|^{-1}|\nabla_x F(|\xi'|^{-2}\xi')|.$$

Consequently, the boundary condition (1.2) restricted to the outer surface $\partial\Omega$ reshapes to the following condition on the function u' (see (6.2)):

$$\varepsilon\partial_{n'}(|\xi'|^{d-2}u'(x')) = \lambda a_\omega(\xi')|\xi'|^{d-2}u'(x'), \quad x'\in\partial\omega'. \tag{6.5}$$

Similar calculations show that the boundary condition (1.2) restricted to the inner surface $\partial\omega_\varepsilon$ gives rise to the condition

$$\partial_{n'}(|x'|^{d-2}u'(x')) = \varepsilon\lambda' a_\Omega(x')|x'|^{d-2}u'(x'), \quad x\in\partial\Omega'. \tag{6.6}$$

In the resulting spectral problem (6.3), (6.5), (6.6), the Steklov boundary conditions on $\partial\omega'_\varepsilon$ and $\partial\Omega'_\varepsilon$ acquired respectively the variable densities $a_\omega(x')$ and also the free terms $(d-2)\partial_{n'(\xi)}|\xi'|$ and $(d-2)\partial_{n'(x')}|x'|$. This modification of the problem does not affect the general procedures of constructing the asymptotics, as presented in §§2 and 3, but now the limiting problems (1.10) and (1.13) interchange their roles: after the Kelvin transformation, problem (1.13) is posed in the bounded domain Ω' and describes the asymptotic terms of smooth type, while the transformed problem (1.10) acts in the unbounded domain $\mathbb{R}^d\setminus\bar{\omega'}$, typical of the boundary layer. Precisely this observation explains the equivalence of the two limiting problems for asymptotic constructions within the framework of composite expansions method, which was already mentioned in Subsections 3.2 and 5.2; see also the book [1], where this equivalence was used repeatedly.

Observe that the small factor ε is present on the left and right parts of the boundary conditions (6.5) and (6.6), respectively[8]. Precisely for this reason, the (positive) eigenvalues $\lambda_2^0, \lambda_3^0, \lambda_4^0, \dots$ of the modified outer Steklov problem in $\mathbb{R}^d\setminus\bar{\omega'}$ describe the asymptotics in the high-frequency range of the spectrum of the singularly perturbed problem in $\Omega'(\varepsilon)$, while the eigenvalues $0, \mu_1^0, \mu_2^0, \dots$ of the modified Steklov problem in Ω' describe the asymptotics in the low-frequency range. In the new interpretation of problem (1.1), (1.2), only the zero eigenvalue, which is of little interest, has changed its range.

6.2. Other boundary conditions on the cavity surface. As was already mentioned in Subsection 2.3, the increments

$$\lambda_k^\varepsilon - \lambda_k^0 = \varepsilon^{d-1}\lambda_k^{(d-1)} + O(\varepsilon^d) \tag{6.7}$$

and

$$\lambda_k^\varepsilon - \lambda_k^0 = \varepsilon^{d}\lambda_k^{(d)} + O(\varepsilon^{d+1}), \tag{6.8}$$

calculated by formulas (2.22) and (2.29), respectively, under the assumptions $v_k^0(0)\neq 0$ and $v_k^0(0)=0$, $\nabla_x v_k^0(0)\neq 0\in\mathbb{R}^d$, turn out to be negative. At the same time, in the high-frequency range of the spectrum the increment

$$\lambda_{M^\varepsilon(m)}^\varepsilon - \varepsilon^{-1}\mu_m^0 = \varepsilon^{d-2}\mu_k^{(d-2)} + O(\varepsilon^{d-1}) \tag{6.9}$$

[8]Note that in (6.5) the small parameter arose at the highest derivative, implying that the standard asymptotic procedures (see, e.g., [22]) require that precisely the boundary $\partial\omega'$ be smooth, i.e., require the smoothness of the initial boundary, $\partial\Omega$, by the definition (6.1).

becomes positive provided $B_k^0 \neq 0$ (see (3.22)). The different signs of the quantities (6.7), (6.8), and (6.9) mean that there are no *a priori* relationship between the terms of the sequences (1.5) and (1.9). However, the eigenvalues of the mixed spectral problem

$$-\Delta u^\varepsilon(x) = 0, \quad x \in \Omega(\varepsilon), \quad \partial_n u^\varepsilon(x) = \lambda^\varepsilon u^\varepsilon(x), \quad x \in \partial\Omega, \tag{6.10}$$

$$u^\varepsilon(x) = 0, \quad x \in \partial\omega_\varepsilon, \tag{6.11}$$

and the eigenvalues (1.9) of the Steklov limiting problem (1.10) satisfy

$$\lambda_k^\varepsilon > \lambda_k^0. \tag{6.12}$$

Inequality (6.12) can be deduced easily from the minimax principle (see, e.g., [21, Theorem 10.2.2]) applied to the trace operator $\mathcal{T}^\varepsilon$ in $\mathcal{H}^\varepsilon = H_0^1(\Omega(\varepsilon); \partial\omega_\varepsilon)$ defined by the formulas

$$\langle u^\varepsilon, v^\varepsilon \rangle_\varepsilon = (\nabla u^\varepsilon, \nabla v^\varepsilon)_{\Omega(\varepsilon)} + (u^\varepsilon, v^\varepsilon)_{\partial\Omega}, \quad \langle \mathcal{T}^\varepsilon u^\varepsilon, v^\varepsilon \rangle_\varepsilon = (u^\varepsilon, v^\varepsilon)_{\partial\Omega}, \quad u^\varepsilon, v^\varepsilon \in \mathcal{H}^\varepsilon \tag{6.13}$$

(cf. (2.47) and (2.51)); here $H_0^1(\Omega(\varepsilon); \partial\omega_\varepsilon)$ is the subspace of functions in the Sobolev class $H^1(\Omega(\varepsilon))$ that vanish on $\partial\omega_\varepsilon$.

Note that $\lambda_0^\varepsilon > 0$ due to the Dirichlet condition (6.11). For $d \geq 3$, the procedure of constructing the asymptotics for solutions of the spectral problem (1.10), (6.11) differs little from that presented in §2; in particular, it is not difficult to obtain the representation

$$\lambda_k^\varepsilon = \lambda_k^0 + \varepsilon^{d-2} \operatorname{cap}_d \omega |v_k^0(0)|^2 + O(\varepsilon^{d-1}),$$

where λ_k^0 is a simple eigenvalue, v_k^0 is the corresponding $L^2(\partial\Omega)$-normalized eigenfunction of problem (1.10), and $\operatorname{cap}_d \omega$ stands for the harmonic capacity of the set $\bar{\omega} \subset \mathbb{R}^d$.

In the planar case, the asymptotic constructions complicate, as was mentioned in Subsection 5.2 in another context. A full description of asymptotic procedures can be found in the paper [17] (see also [1, Chapter 9]), the material of which needs only minor modification when we pass to the Steklov spectral condition on $\partial\Omega$.

If we replace the Dirichlet conditions (6.11) with the Neumann conditions

$$\partial_n u^\varepsilon(x) = 0, \quad x \in \partial\omega_\varepsilon, \tag{6.14}$$

then, as before, we can apply the max-min principle (see [21, Theorem 10.2.2])

$$-\tau_k^\varepsilon = \max_{E_k^\varepsilon} \inf_{u^\varepsilon \in E_k^\varepsilon \setminus \{0\}} \frac{\langle -\mathcal{T}^\varepsilon u^\varepsilon, u^\varepsilon \rangle_\varepsilon}{\langle u^\varepsilon, u^\varepsilon \rangle_\varepsilon} \tag{6.15}$$

to the trace operator[9] $-\mathcal{T}^\varepsilon$ (with the minus sign) given by formulas (6.13) with $\mathcal{H}^\varepsilon = H^1(\Omega(\varepsilon))$. In (6.14) E_k^ε, is an arbitrary subspace in $H^1(\Omega(\varepsilon))$ of codimension $k-1$, i.e., $\dim(H^1(\Omega(\varepsilon)) \ominus E_k^\varepsilon) = k-1$; in particular, $E_1^\varepsilon = H^1(\Omega(\varepsilon))$.

For any $k \in \mathbb{N}$, there exists $\varepsilon_k > 0$ such that, for $\varepsilon \in (0, \varepsilon_k]$, the restrictions to $\Omega^\varepsilon = \Omega \setminus \overline{\omega_\varepsilon}$ of the eigenfunctions $v_1^0, \dots, v_k^0 \in C^\infty(\overline{\Omega})$ of problem (1.10) remain linearly independent, and therefore, every subspace E_k^ε as in (6.15) includes a nontrivial linear combination

$$v_k^\varepsilon = \sum_{l=1}^N a_l^{E_N^\varepsilon} v_l^0, \quad \sum_{l=1}^N \left|a_l^{E_N^\varepsilon}\right|^2 = 1.$$

[9]When deducing (6.12), we apply the minimax principle (6.15) for $\varepsilon = 0$ to the trace operator $-\mathcal{T}^0$ corresponding to the Steklov problem (1.10) in the limiting domain Ω.

Thus, using (6.13) and (2.10), we obtain

$$
\begin{aligned}
-\tau_k^\varepsilon &\le \max_{E^\varepsilon} \frac{-(v_k^\varepsilon, v_k^\varepsilon)_{\partial\Omega}}{(\nabla v_k^\varepsilon, \nabla v_k^\varepsilon)_{\Omega(\varepsilon)} + (v_k^\varepsilon, v_k^\varepsilon)_{\partial\Omega}} \\
&= \max_{E_k^\varepsilon} \frac{-1}{1 + (\nabla v_k^\varepsilon, \nabla v_k^\varepsilon)_{\Omega} - (\nabla v_k^\varepsilon, \nabla v_k^\varepsilon)_{\omega_\varepsilon}} \\
&= \max_{E_k^\varepsilon} \frac{-1}{1 + \sum_{l=1}^k \lambda_l^0 \big|a_l^{E_N^\varepsilon}\big|^2 - \|\nabla v_k^\varepsilon; L^2(\omega_\varepsilon)\|^2} \le \frac{-1}{1+\lambda_k^0}.
\end{aligned}
\tag{6.16}
$$

Recalling the relationship (2.54) between the eigenvalues of the operator $\mathcal{T}^\varepsilon$ and those of problem (6.10), (6.14), we see that (6.16) implies the relation

$$\lambda_k^0 \ge \lambda_k^\varepsilon. \tag{6.17}$$

It can be checked that inequality (6.17) is strict for $k > 0$.

It is easily seen that, in the case of the Steklov spectral problem (1.1), (1.2), no manipulations with the max-min principle can be a success, because both the numerator and denominator of the Rayleigh ratio are perturbed.

Following the general approach of [1, Chapter 9], applied in §§2 and 3 to the Steklov problem (1.1), (1.2), we can construct full asymptotic series (2.1) and (2.2) for the solutions of the spectral problem (6.10), (6.14). Note that asymptotic constructions for small cavities with the Neumann boundary conditions were presented, e.g., in [39]. The author knows of no asymptotic expansions in the high-frequency range of the spectrum of problems (6.10), (6.11) and (6.10), (6.14).

Typically, a continuous spectrum arises in problems concerning wave propagation on the surface of a liquid that fills an infinite channel with immersed or half-immersed obstacles. Nevertheless, the introduction of the trace operator $\mathcal{T}$ and transformation of problems in question into abstract equations of the form [16] made it possible to present elementary proofs of the known facts about the arising of trapped waves that decay exponentially at infinity and correspond to points of the discrete spectrum located below the continuous spectrum (see [10, 40, 41] and also the surveys [11, 12] and, e.g., the book [13]), as well as to give new sufficient conditions for the trapping of waves (see [42, 43, 44] and other publications). All these results together with various comparison principles are deduced from the general formula (6.15).

6.3. Other boundary conditions on the outer surface and on a cavern. The Laplace equation

$$-\Delta u^\varepsilon(x) = 0, \quad x \in \Omega(\varepsilon), \tag{6.18}$$

supplied by the Steklov spectral conditions

$$\partial_n u^\varepsilon(x) = \lambda^\varepsilon u^\varepsilon(x), \quad x \in \gamma_\varepsilon, \tag{6.19}$$

on the inner boundary$\gamma_\varepsilon = \partial\omega_\varepsilon$, and also by the Dirichlet conditions

$$u^\varepsilon(x) = 0, \quad x \in \Gamma, \tag{6.20}$$

or the Neumann conditions

$$\partial_n u^\varepsilon(x) = 0, \quad x \in \Gamma, \tag{6.21}$$

on the exterior boundary $\Gamma = \partial\Omega$, can be studied with the help of the procedure described in §3; in contrast to problem (1.1), (1.2), we can find the full asymptotic series (3.10) and (3.11) also in the case where $\partial\Omega$ and $\partial\omega$ are Lipschitz surfaces. The iteration processes still go well in the situation where the point $\mathcal{O}$ to which the sets (1.4) are shrinking is located on $\partial\Omega$ (see Figure 2a), i.e., the domain $\Omega(\varepsilon) = \Omega\backslash\overline{\omega_\varepsilon}$ in equation (6.18) has a small cavern (Figure 2c), the boundary conditions (6.19) are posed on its surface $\gamma_\varepsilon = \partial\omega_\varepsilon \cap \Omega$, and the boundary conditions (6.20) and (6.21) are restricted to $\Gamma(\varepsilon) = \partial\Omega \setminus \overline{\omega_\varepsilon}$.

In essence, the asymptotic analysis of the low-frequency range of the spectrum of the Steklov problem (1.1), (1.2) reduces to the iteration processes of §2 (cf. [45, 46]), but the high-frequency range requires absolutely new ideas, because the continuous spectrum of the Steklov problem in the infinite domain $\mathbb{R}^d_- \setminus \bar{\omega}$ (Figure 2b) fills the entire half-axis $[0, +\infty)$.

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