The Annals of Probability
1976, Vol. 4, No. 2, 249–258

ASYMPTOTIC EXPANSIONS FOR SAMPLE QUANTILES

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This paper deals with an Edgeworth-type expansion for the distribution of a sample quantile. As the sample size n increases, these expansions establish a higher order approximation which holds uniformly for all Borel sets. If the underlying distribution function has $s+2$ left and right derivatives at the true quantile, the error of the approximation is of order $O(n^{-(s+1)})$. From this result asymptotic expansions for the distribution functions of sample quantiles and for percentage points are derived.

1. Introduction and preliminaries. Concerning the weak convergence of the distributions of sample quantiles, Smirnov (1949) found necessary and sufficient conditions. If the given distribution function has a second derivative, it is proved in Reiss (1974a) that the accuracy of the normal approximation is of order $n^{-\frac{1}{2}}$. This paper also includes "left and right differentiable" distribution functions (e.g. the double exponential distribution function). Under these weaker assumptions the limit distribution (or the leading term in the expansion) need not be a normal distribution.

Let $\mathbb{R}$ and $\mathbb{N}$ denote the sets of all real numbers and positive integers, respectively. Let $x_{1:n} \leqq \cdots \leqq x_{n:n}$ denote the components of $x \in \mathbb{R}^n$ arranged in increasing order. The ith order statistic $X_{i:n}: \mathbb{R}^n \to \mathbb{R}$ for the sample size n is defined by $X_{i:n}(x) = x_{i:n}$. $X_{[n\alpha]+1:n}$ is called the sample α-quantile (where $[n\alpha]$ denotes the integral part of $n\alpha$).

The distribution function F of a distribution P is defined by $F(t) = P(-\infty, t)$ for $t \in \mathbb{R}$. For $\alpha \in (0, 1)$, let ξ_α be a solution of $F(\xi_\alpha) = \alpha$; that is, ξ_α is an α-quantile of P. Denote by P^n the independent product of n identical components P. $\mathscr{B}$ is the Borel algebra on $\mathbb{R}$. $P^n * \psi$ denotes the distribution induced by P^n and the measurable function $\psi: \mathbb{R}^n \to \mathbb{R}$ (i.e. $P^n * \psi(B) = P^n(\psi^{-1}(B))$ for $B \in \mathscr{B}$). Let N_a denote the normal distribution with mean zero and variance a^{-2}. Φ is the distribution function and φ is the density function of the standard normal distribution N_1. For $\alpha \in (0, 1)$ let $\sigma_\alpha = (\alpha(1-\alpha))^{\frac{1}{2}}$.

Section 2 contains the main results. Some auxiliary results and the proofs of Theorem 2.1 and Theorem 2.7 can be found in Section 3.

2. The results. The versions (A) and (A′) of the following theorem correspond to the different assumptions that the given distribution function F is left and right differentiable, respectively, at some α-quantile ξ_α. The assumptions of (A) (of (A′)) imply that ξ_α is the smallest (largest) α-quantile. These assumptions

Received January 14, 1975; revised July 16, 1975.

AMS 1970 *subject classifications.* Primary 60F05, 62G35.

Key words and phrases. Sample quantiles, Edgeworth expansions, percentage points.

together imply that the α-quantile is uniquely determined. Note that the assumptions of (A′) imply that F is continuous at ξ_α.

THEOREM 2.1. *Assume that* $\lim_{n\in\mathbb{N}} n^{-\frac{1}{2}}(r_n - n\alpha) = 0$.

(A) *Assume that*

(2.2) *for some* $\varepsilon > 0$ *the restriction of* F *to* $(\xi_\alpha - \varepsilon, \xi_\alpha]$ *has a derivative on* $(\xi_\alpha - \varepsilon, \xi_\alpha]$ *which is left continuous at* ξ_α, *and*

(2.3) *the left derivative of* F *at* ξ_α, *say* p_α^-, *is positive.*

Then

$$\lim_{n\in\mathbb{N}} \sup_{B\in\mathscr{B}\cap(-\infty,0)} \left|P^n * \left(\frac{n^{\frac{1}{2}}p_\alpha^-}{\sigma_\alpha}(X_{r_n:n} - \xi_\alpha)\right)(B) - N_1(B)\right| = 0 . \tag{2.4}$$

(A′) *Assume that*

(2.2′) *for some* $\varepsilon > 0$ *the restriction of* F *to* $[\xi_\alpha, \xi_\alpha + \varepsilon)$ *has a derivative on* $[\xi_\alpha, \xi_\alpha + \varepsilon)$ *which is right continuous at* ξ_α, *and*

(2.3′) *the right derivative of* F *at* ξ_α, *say* p_α^+, *is positive.*

Then

$$\lim_{n\in\mathbb{N}} \sup_{B\in\mathscr{B}\cap[0,\infty)} \left|P^n * \left(\frac{n^{\frac{1}{2}}p_\alpha^+}{\sigma_\alpha}(X_{r_n:n} - \xi_\alpha)\right)(B) - N_1(B)\right| = 0 . \tag{2.4′}$$

If the assumptions of (A) and (A′) are fulfilled simultaneously then it is obvious that the sequence $P^n * (n^{\frac{1}{2}}/\sigma_\alpha)(X_{r_n:n}-\xi_\alpha)$, $n\in\mathbb{N}$, converges uniformly for all Borel sets to the distribution $M_{p_\alpha^-,p_\alpha^+}$ defined by $M_{p_\alpha^-,p_\alpha^+}(B) = N_{p_\alpha^-}(B \cap (-\infty, 0]) + N_{p_\alpha^+}(B \cap [0, \infty))$ for $B\in\mathscr{B}$.

The following example shows that the conditions which are sufficient for the weak convergence of the distributions of sample quantiles do not imply the strong convergence.

EXAMPLE 2.5. Let the distribution P be defined by its density function $p = 1_{[-\frac{1}{2},0]} + \sum_{i\in\mathbb{N}}((2i+1)/(i+1))1_{[1/(2i+1),1/2i]}$. Because $\sum_{i=1}^n((2i+1)/(i+1)) \times (1/2i - 1/(2i+1)) = \frac{1}{2}\sum_{i=1}^n(1/i - 1/(i+1)) = n/2(n+1)$ we easily derive that $\int p(x)\,dx = 1$ and $F(1/(2n+1)) = \frac{1}{2} + 1/2(n+1)$ for $n\in\mathbb{N}$. Therefore, $F(x) - \frac{1}{2} = 1/2(n+1)$ for $x\in[1/2(n+1), 1/(2n+1))$ and $F(x) - \frac{1}{2} = 1/2(n+1) + ((2n+1)/(n+1))(x - 1/(2n+1))$ for $x\in[1/(2n+1), 1/2n)$ whence $x - x^2 \leqq F(x) - \frac{1}{2} \leqq x$ for $x\in[-\frac{1}{2},\frac{1}{2}]$. Hence F is differentiable at zero with $F^{(1)}(0) = 1$, and $F(0) = \frac{1}{2}$. From a result of Smirnov (1949) (see Smirnov (1962), page 116), it follows that $\lim_{n\in\mathbb{N}} P^n * (2n^{\frac{1}{2}}X_{[n/2]:n})(-\infty, t) = \Phi(t)$ for all $t\in\mathbb{R}$. The convergence does not hold uniformly for all Borel sets since $P^n * (2n^{\frac{1}{2}}X_{[n/2]:n})(B_n) = 0$, $n\in\mathbb{N}$, and $\limsup_{n\in\mathbb{N}} N_1(B_n) > 0$ for the sets $B_n = \bigcup_{i\in\mathbb{N}}(2n^{\frac{1}{2}}/2(i+1), 2n^{\frac{1}{2}}/(2i+1))$, $n\in\mathbb{N}$. To prove this we use the explicit form of the density of $P^n * (2n^{\frac{1}{2}}X_{[n/2]:n})$ (see e.g. David (1970), page 9).

To obtain expansions with terms of order $n^{-i/2}$, $i = 1, \cdots, s$, we shall always assume hereafter that the following holds true:

ASSUMPTION 2.6. $\delta_n = n\alpha - r_n + 1 = O(1)$.

It is evident that the sequence $r_n = [n\alpha] + 1$, $n \in \mathbb{N}$, which determines the sample α-quantiles for the sample size n, fulfills this assumption. Let, furthermore, $p_{\alpha,i}^- = F^{(i)}(\xi_\alpha^-)$ and $p_{\alpha,i}^+ = F^{(i)}(\xi_\alpha^+)$ for $i \in \mathbb{N}$.

THEOREM 2.7. (A) *Assume that F has a bounded $(s+2)$th derivative (for some $s \in \{0, 1, \cdots\}$) on $(\xi_\alpha - \varepsilon, \xi_\alpha]$ for some $\varepsilon > 0$ and (2.3) is fulfilled. Then there exist polynomials $Q_{\delta_n,i}^-$, $i = 1, \cdots, s$ (see Remark 2.9) such that*

$$(2.8)\qquad \sup_{B \in \mathscr{B} \cap (-\infty,0)} \left| P^n * \left(\frac{n^{\frac{1}{2}} p_\alpha^-}{\sigma_\alpha} (X_{r_n:n} - \xi_\alpha) \right)(B) - \int_B \varphi(x)\left(1 + \sum_{i=1}^s n^{-i/2} Q_{\delta_n,i}^-(x)\right) dx \right| = O(n^{-(s+1)/2}) .$$

(A′) *Assume that F has a bounded $(s+2)$th derivative $(s \in \{0, 1, \cdots\})$ on $[\xi_\alpha, \xi_\alpha + \varepsilon)$ for some $\varepsilon > 0$ and (2.3′) is fulfilled. Then there exist polynomials $Q_{\delta_n,i}^+$, $i = 1, \cdots, s$, such that*

$$(2.8')\qquad \sup_{B \in \mathscr{B} \cap [0,\infty)} \left| P^n * \left(\frac{n^{\frac{1}{2}} p_\alpha^+}{\sigma_\alpha} (X_{r_n:n} - \xi_\alpha) \right)(B) - \int_B \varphi(x)\left(1 + \sum_{i=1}^s n^{-i/2} Q_{\delta_n,i}^+(x)\right) dx \right| = O(n^{-(s+1)/2}) .$$

REMARK 2.9. $Q_{\delta_n,i}^-$ and $Q_{\delta_n,i}^+$ are polynomials of degree $\leqq 3i$ for each $i \in \mathbb{N}$. The coefficients of $Q_{\delta_n,i}^-$ and $Q_{\delta_n,i}^+$ only depend on $p_{\alpha,j}^-$ and $p_{\alpha,j}^+$, respectively, for $j \in \{1, \cdots, i+1\}$ and on α and δ_n (see also (2.11)).

REMARK 2.10. Let F have a bounded $(s+2)$th derivative on a neighborhood U of $[F^{-1}(c_1), F^{-1}(c_2)]$, $0 < c_1 < c_2 < 1$, and let $F^{(1)}$ be bounded away from zero on U. Then (2.8) and (2.8′) hold true uniformly for all sequences δ_n, $n \in \mathbb{N}$, with $\delta_n \leqq C$ ($C > 0$ fixed) and $\alpha \in [c_1, c_2]$.

Let

$$a_1 = (1 - 2\alpha)/3\sigma_\alpha , \qquad a_2 = -(\alpha^3 + (1-\alpha)^3)/4\sigma_\alpha^2 ,$$
$$b_{\delta_n,1} = (\alpha - \delta_n)/\sigma_\alpha \quad \text{and} \quad b_{\delta_n,2} = ((1-\delta_n)\alpha^2 + \delta_n(1-\alpha)^2)/2\sigma_\alpha^2 .$$

For $-$ and $+$, respectively, define

$$d_1 = \sigma_\alpha p_{\alpha,2}/p_\alpha^2 \quad \text{and} \quad d_2 = \sigma_\alpha^2 p_{\alpha,3}/p_\alpha^3 .$$

Then for $-$ and $+$, respectively,

$$(2.11)\qquad \begin{aligned} Q_{\delta_n,1}(x) &= (a_1 - \tfrac{1}{2}d_1)x^3 + (b_{\delta_n,1} + d_1)x , \\ Q_{\delta_n,2}(x) &= \tfrac{1}{2}(a_1 - \tfrac{1}{2}d_1)^2 x^6 \\ &\quad + (a_1 b_{\delta_n,1} + \tfrac{5}{2}a_1 d_1 + a_2 - \tfrac{1}{2}b_{\delta_n,1}d_1 - \tfrac{5}{8}d_1^2 - \tfrac{1}{6}d_2)x^4 \\ &\quad + (\tfrac{1}{2}b_{\delta_n,1}^2 + \tfrac{3}{2}b_{\delta_n,1}d_1 + b_{\delta_n,2} + \tfrac{1}{2}d_2)x^2 \\ &\quad - (\tfrac{15}{2}a_1^2 + 3a_1 b_{\delta_n,1} + 3a_2 + \tfrac{1}{2}b_{\delta_n,1}^2 + b_{\delta_n,2}) . \end{aligned}$$

Hereafter we shall always assume that the assumptions of Theorem 2.7, (A) and (A′), are fulfilled simultaneously for some $s \in \{0, 1, \cdots\}$.

From Theorem 2.7 together with (2.11) we easily derive the following three corollaries.

COROLLARY 2.12. *Let* $s = 0$. *If* $p_\alpha^- = p_\alpha^+$ *then*

$$\sup_{B \in \mathscr{B}} \left| P^n * \left(\frac{n^{\frac{1}{2}} p_\alpha}{\sigma_\alpha} (X_{r_n:n} - \xi_\alpha)\right)(B) - N_1(B)\right| = O(n^{-\frac{1}{2}}) .$$

Concerning sample medians we obtain:

COROLLARY 2.13. *Let* $\alpha = \frac{1}{2}$ *and* $s = 1$. *If* $p_{\frac{1}{2},2}^- = p_{\frac{1}{2},2}^+ = 0$ *then for n odd*

$$\sup_{B \in \mathscr{B}} |P^n * (2n^{\frac{1}{2}}(X_{[n/2]+1:n} - \xi_{\frac{1}{2}})(B) - N_{p_{\frac{1}{2}}^-}(B \cap (-\infty, 0]) - N_{p_{\frac{1}{2}}^+}(B \cap [0, \infty))| = O(n^{-1}) .$$

Notice that the additional assumption of Corollary 2.12 is fulfilled if e.g. P is a symmetrical distribution and F has a second derivative at the median. If P is only symmetrical (e.g., the double exponential distribution is symmetrical but its distribution function is not twice differentiable at the median) then the approximation need not be of order n^{-1}.

Denote by $\mathscr{S}$ the set of all symmetrical Borel sets, that is $\mathscr{S} = \{B \in \mathscr{B}: x \in B$ implies $-x \in B\}$.

COROLLARY 2.14. *Let* $s = 1$. *If the second derivative of F at* ξ_α *exists then*

$$\sup_{S \in \mathscr{S}} \left| P^n * \left(\frac{n^{\frac{1}{2}} p_\alpha}{\sigma_\alpha} (X_{r_n:n} - \xi_\alpha)\right)(S) - N_1(S)\right| = O(n^{-1}) .$$

Hereafter we shall additionally assume that the $(s+1)$th derivative of F at ξ_α exists.

COROLLARY 2.15. *There exist polynomials* $R_{\delta_n,i}$ *with coefficients only depending on* $Q_{\delta_n,i}$ *for* $i = 1, \cdots, s$ *such that*

$$\sup_{t \in \mathbb{R}} \left| P^n \left\{x \in \mathbb{R}^n : \frac{n^{\frac{1}{2}} p_\alpha}{\sigma_\alpha} (x_{r_n:n} - \xi_\alpha) < t\right\} - \left(\Phi(t) + \varphi(t) \sum_{i=1}^{s} n^{-i/2} R_{\delta_n,i}(t)\right)\right| = O(n^{-(s+1)/2}) .$$

In particular

$$\begin{aligned} R_{\delta_n,1}(x) &= -(a_1 - \tfrac{1}{2}d_1)x^2 - (b_{\delta_n,1} + 2a_1) , \\ R_{\delta_n,2}(x) &= -\tfrac{1}{2}(a_1 - \tfrac{1}{2}d_1)^2 x^5 - (\tfrac{5}{2}a_1^2 + a_2 + a_1 b_{\delta_n,1} - \tfrac{1}{2}b_{\delta_n,1}d_1 - \tfrac{1}{6}d_2)x^3 \\ &\quad - (\tfrac{15}{2}a_1^2 + 3a_1 b_{\delta_n,1} + 3a_2 + \tfrac{1}{2}b_{\delta_n,1}^2 + b_{\delta_n,2})x . \end{aligned} \tag{2.16}$$

For $i = 1, 2$ we can use the explicit form of $Q_{\delta_n,1}$ and $Q_{\delta_n,2}$ given in (2.11) to find $R_{\delta_n,1}$ and $R_{\delta_n,2}$ directly (which fulfill the equation $R_{\delta_n,i}^{(1)}(x) - xR_{\delta_n,i}(x) = Q_{\delta_n,i}(x)$ for $i = 1, 2$).

We introduce the $(3s+1)$-dimensional inner product space $\mathscr{H}$ of all polynomials with degree $\leqq 3s$ equipped with the inner product

$$\langle h, g\rangle = \int h(x)g(x)\varphi(x)\,dx \qquad \text{for } h, g \in \mathscr{H}.$$

The Tschebyscheff–Hermite polynomials H_i, $i = 0, \cdots, 3s$ defined by

$$\varphi^{(i)}(x) = (-1)^i H_i(x)\varphi(x)\,, \qquad i = 0, \cdots, 3s$$

(see Kendall and Stuart (1958), (6.21), page 155), establish an orthogonal base of $\mathscr{H}$. Furthermore,

$$\langle H_i, H_i\rangle = i!$$

(see Kendall and Stuart (1958), (6.28), page 159).

PROOF. By Remark 2.9

$$\sum_{i=1}^{s} n^{-i/2}Q_{\delta_n,i} = \sum_{j=0}^{3s} \frac{1}{j!}\langle \sum_{i=1}^{s} n^{-i/2}Q_{\delta_n,i}, H_j\rangle H_j\,. \tag{2.17}$$

Because $(\varphi H_{i-1})^{(1)} = -\varphi H_i$ for $i \geqq 1$ and $H_0 \equiv 1$ we obtain by (2.17)

$$\begin{aligned}\int_{-\infty}^{t} \varphi(\eta)(1 + \textstyle\sum_{i=1}^{s} n^{-i/2}Q_{\delta_n,i}(\eta))\,d\eta \\ &= \Phi(t)(1 + \textstyle\sum_{i=1}^{s} n^{-i/2}\int Q_{\delta_n,i}(\eta)\varphi(\eta)\,d\eta) \\ &\quad - \varphi(t)\left(\sum_{j=1}^{3s}\frac{H_{j-1}(t)}{j!}\,(\textstyle\sum_{i=1}^{s} n^{-i/2}\langle Q_{\delta_n,i}, H_j\rangle)\right).\end{aligned}$$

By Theorem 2.7 we know that $\sum_{i=1}^{s} n^{-i/2}\int Q_{\delta_n,i}(\eta)\varphi(\eta)\,d\eta = O(n^{-(s+1)/2})$. Therefore, by collecting all terms of order $n^{-i/2}$ for $i = 1, \cdots, s$ the assertion follows.

Using a result stated in Pfanzagl (1973, see Lemma 7), we easily derive from Corollary 2.15:

COROLLARY 2.18. *There exist polynomials $R^*_{\delta_n,i}$, $i = 1, \cdots, s$, such that uniformly for all $|t| \leqq \log n$*

$$P^n\left\{x \in \mathbb{R}^n : \frac{n^{\frac{1}{2}}p_\alpha}{\sigma_\alpha}(x_{r_n:n} - \xi_\alpha) < t + \sum_{i=1}^{s} n^{-i/2}R^*_{\delta_n,i}(t)\right\} = \Phi(t) + O(n^{-(s+1)/2})\,.$$

In particular,

$$\begin{aligned}R^*_{\delta_n,1}(t) &= -R_{\delta_n,1}(t)\,, \\ R^*_{\delta_n,2}(t) &= R_{\delta_n,1}(t)R^{(1)}_{\delta_n,1}(t) - \frac{t}{2}R^2_{\delta_n,1}(t) - R_{\delta_n,2}(t)\,.\end{aligned} \tag{2.19}$$

By Corollary 2.18 we can justify a result of David and Johnson (1954), page 230, concerning a formal expansion for percentage points of the distribution function of medians. For $\alpha = \frac{1}{2}$ and $s = 2$ we obtain the following: Denote by λ_γ the $100\gamma\%$ point of N_1. Then for n odd

$$\begin{aligned}P^n\Big\{x \in \mathbb{R}^n : x_{[n/2]+1:n} \geqq \xi_{\frac{1}{2}} + \frac{1}{2n^{\frac{1}{2}}p_{\frac{1}{2}}}\Big(\lambda_\gamma - \frac{\lambda_\gamma^2 p_{\frac{1}{2},2}}{4n^{\frac{1}{2}}p_{\frac{1}{2}}^2} \\ - \frac{1}{4n}\Big(\lambda_\gamma + \lambda_\gamma^3\Big(1 - \frac{p^2_{\frac{1}{2},2}}{2p_{\frac{1}{2}}^4} + \frac{p_{\frac{1}{2},3}}{6p_{\frac{1}{2}}^3}\Big)\Big)\Big)\Big\} \\ = \gamma + O(n^{-\frac{3}{2}})\,.\end{aligned} \tag{2.20}$$

Finally, we remark that asymptotic expansions for the moments of the distribution of a sample quantile can be derived from Lemma 3.9. With the help of these expansions we can find an asymptotic expansion for the distribution of a sample quantile which is based on the Tschebyscheff–Hermite polynomials H_i instead of the polynomials $Q_{\delta_n,i}$. In other words, we obtain an asymptotic expansion of a Gram–Charlier form (see Kendall and Stuart (1958), (6.32), page 157). These results can be found in Reiss (1974b). Notice that for the considerations made above the existence of moments is needed whereas the expansions given in Theorem 2.7 exist without any assumptions on the moments.

3. Auxiliary results and proofs. We shall only prove the version (A) of Theorem 2.1 and Theorem 2.7. It will always be apparent in which way the lemmas of this section have to be reformulated and proved to get the corresponding versions (A′).

In essentially the same way as in Reiss (1974a), (2.8), we may prove

LEMMA 3.1. *Under the assumptions of Theorem* 2.1 (A) *there exists some constant* $c > 0$ *for each* $k \in \mathbb{N}$ *such that*

$$P^n\left\{x \in \mathbb{R}^n : \frac{n^{\frac{1}{2}} p_\alpha^-}{\sigma_\alpha}(x_{r_n:n} - \xi_\alpha) < -c(\log n)^{\frac{1}{2}}\right\} = O(n^{-k}) . \tag{3.2}$$

In Lemma 3.3 we get an asymptotic expansion for the Lebesgue-density $\hat{g}_n$ of $E^n * ((n^{\frac{1}{2}}/\sigma_\alpha)(X_{r_n:n} - \alpha))$ where E denotes the uniform distribution on $(0, 1)$.

LEMMA 3.3. *Under Assumption* 2.6 *there exist polynomials* $\hat{Q}_{\delta_n,i}$, $i = 1, \cdots, k$ *for* $k \in \{0, 1, \cdots\}$, *such that uniformly for all* $|y| \leqq \log n$

$$\hat{g}_n(y) - \varphi(y)(1 + \textstyle\sum_{i=1}^k n^{-i/2}\hat{Q}_{\delta_n,i}(y)) = O(\exp(-y^2/3)n^{-(k+1)/2}) . \tag{3.4}$$

The polynomials $\hat{Q}_{\delta_n,i}$ are identical with those given in Theorem 2.7 for $p_{\alpha,1}^- = p_{\alpha,1}^+ = 1$ and $p_{\alpha,j}^- = p_{\alpha,j}^+ = 0$ if $j > 1$. Furthermore, (3.4) holds true uniformly for all sequences δ_n, $n \in \mathbb{N}$, with $\delta_n \leqq C$ and $\alpha \in [a_1, a_2] \subset (0, 1)$ (where $C > 0$ and $a_1 < a_2$ are fixed constants).

PROOF. Let

$$h_n(y) = \left(1 + \frac{(1-\alpha)y}{\sigma_\alpha n^{\frac{1}{2}}}\right)^{r_n-1}\left(1 - \frac{\alpha y}{\sigma_\alpha n^{\frac{1}{2}}}\right)^{n-r_n}$$

for

$$y \in \left(-\frac{\alpha}{\sigma_\alpha}n^{\frac{1}{2}}, \frac{1-\alpha}{\sigma_\alpha}n^{\frac{1}{2}}\right)$$

and $h_n(y) = 0$ otherwise. Let

$$C_n^{-1} = \frac{\sigma_\alpha \alpha^{r_n-1}(1-\alpha)^{n-r_n}}{B(r_n, n - r_n + 1)n^{\frac{1}{2}}} .$$

From the well-known formula for the density of $E^n * X_{r_n:n}$ (see e.g. David (1970), page 9) we easily derive that

$$\hat{g}_n(y) = h_n(y)/C_n .$$

First we shall prove that uniformly for all $y \in (-\log n, \log n)$

$$(3.5)\qquad h_n(y) - \exp\left(-\frac{y^2}{2}\right)(1 + \sum_{i=1}^{k} n^{-i/2} H_{\delta_n,i}(y)) = O(\exp(-y^2/3)n^{-(k+1)/2})$$

where the polynomials $H_{\delta_n,i}$ only depend on α and δ_n.

Let

$$a_i = \frac{1}{i+2}((-1)^{i-1}(1-\alpha)^{(i+2)/2}\alpha^{-i/2} - \alpha^{(i+2)/2}(1-\alpha)^{-i/2})$$

and

$$b_{\delta_n,i} = \frac{1}{i}\left((-1)^i\left(\frac{1-\alpha}{\alpha}\right)^{i/2}\delta_n + \left(\frac{\alpha}{1-\alpha}\right)^{i/2}(1-\delta_n)\right).$$

Then for $i = 1, 2$ we have in particular

$$(3.6)\qquad \begin{aligned} H_{\delta_n,1}(y) &= a_1 y^3 + b_{\delta_n,1} y\,, \\ H_{\delta_n,2}(y) &= \tfrac{1}{2}a_1^2 y^6 + (a_1 b_{\delta_n,1} + a_2)x^4 + (\tfrac{1}{2}b_{\delta_n,1}^2 + b_{\delta_n,2})x^2\,. \end{aligned}$$

Expanding the log function around one and the exp function around $-y^2/2$ and collecting all terms of order $n^{-i/2}$ for $i = 1, \cdots, k$, we obtain uniformly for all $|y| \leqq \log n$

$$\begin{aligned} h_n(y) &= \exp\left[(r_n - 1)\log\left(1 + \frac{(1-\alpha)y}{\sigma_\alpha n^{\frac{1}{2}}}\right) + (n - r_n)\log\left(1 - \frac{\alpha y}{\sigma_\alpha n^{\frac{1}{2}}}\right)\right] \\ &= \exp\Bigg[(r_n - 1)\left(\sum_{i=1}^{k+2}(-1)^{i-1}\frac{1}{i}\left(\frac{(1-\alpha)y}{\sigma_\alpha n^{\frac{1}{2}}}\right)^i\right) \\ &\qquad - (n - r_n)\left(\sum_{i=1}^{k+2}\frac{1}{i}\left(\frac{\alpha y}{\sigma_\alpha n^{\frac{1}{2}}}\right)^i\right) + O(n^{-(k+1)/2}|y|^{k+3})\Bigg] \\ &= \exp\left[-\frac{y^2}{2} + \sum_{i=1}^{k}(a_i y^{i+2} + b_{\delta_n,i}y^i)n^{-i/2} + O(n^{-(k+1)/2}(|y|^{k+1} + |y|^{k+3}))\right] \\ &= \exp\left(-\frac{y^2}{2}\right)\left(1 + \sum_{j=1}^{k}\frac{1}{j!}\left(\sum_{i=1}^{k}(a_i y^{i+2} + b_{\delta_n,i}y^i)n^{-i/2}\right)^j\right) \\ &\qquad + O\left(\exp\left(-\frac{y^2}{3}\right)n^{-(k+1)/2}\right) \\ &= \exp\left(-\frac{y^2}{2}\right)(1 + \sum_{i=1}^{k} H_{\delta_n,i}(y)n^{i/2}) + O\left(\exp\left(-\frac{y^2}{3}\right)n^{-(k+1)/2}\right). \end{aligned}$$

Because $C_n(1 - \int_{|y|>\log n}\hat{g}(y)\,dy = \int_{-\log n}^{\log n} h_n(y)\,dy$ we obtain with the help of Lemma 3.1 and (3.5)

$$(3.7)\qquad C_n(2\pi)^{-\frac{1}{2}} - (1 + \sum_{i=1}^{k} d_{\delta_n,i}n^{-i/2}) = O(n^{-(k+1)/2})$$

where

$$d_{\delta_n,i} = \int \varphi(y)H_{\delta_n,i}(y)\,dy\,.$$

Especially,

$$d_{\delta_n,1} = 0$$

and

$$d_{\delta_n,2} = \tfrac{15}{2}a_1^2 + 3a_1 b_{\delta_n,1} + 3a_2 + \tfrac{1}{2}b_{\delta_n,1}^2 + b_{\delta_n,2}\,.$$

Using a Taylor expansion of the function $1/x$ the assertion follows from (3.5) and (3.7).

It is easy to see that the distribution function G_n of

$$P^n * \left(\frac{n^{\frac{1}{2}}p_\alpha^-}{\sigma_\alpha}(X_{r_n:n} - \xi_\alpha)\right)$$

is given by

$$G_n(y) = \sum_{i=r_n}^n \binom{n}{i} F(l_n(y))^i(1 - F(l_n(y)))^{n-i}$$

where $l_n(y) = \xi_\alpha + \sigma_\alpha y/(n^{\frac{1}{2}}p_\alpha^-)$. Assumption (2.2) implies that G_n has a derivative g_n on $(-\frac{1}{2}\log n, 0)$ for n sufficiently large. This implies that G_n is absolutely continuous on $(-\frac{1}{2}\log n, 0)$ (see Hewitt–Stromberg (1965), Theorem 18.14 (i) and Exercise 18.41 (d)).

(3.8) PROOF OF THEOREM 2.1. Denote by p the derivative of F on $(\xi_\alpha - \varepsilon, \xi_\alpha]$. Let $\alpha_n = r_n/n$. Let $w_n(y) = (n^{\frac{1}{2}}/\sigma_{\alpha_n})(F(l_n(y)) - \alpha_n)$. We obtain

$$g_n(y) = \frac{\sigma_\alpha p(l_n(y))}{\sigma_{\alpha_n} p_\alpha^-}\hat{g}_n(w_n(y)) \qquad \text{for } y \in (-\tfrac{1}{2}\log n, 0)$$

and sufficiently large n where $\hat{g}_n$ is defined as in Lemma 3.3 with α_n in place of α. Since $\lim_{n\in\mathbb{N}} n^{\frac{1}{2}}(\alpha - \alpha_n) = 0$, and p is left continuous at ξ_α, a Taylor expansion of w_n at ξ_α yields that $\lim_{n\in\mathbb{N}} w_n(y) = y$ for $y \leqq 0$ and $|w_n(y)| \leqq \log n$ for all $y \in (-\frac{1}{2}\log n, 0)$. Since Lemma 3.3 is applicable uniformly for all α_n, $n \in \mathbb{N}$, we obtain for $k = 0$ that $\lim_{n\in\mathbb{N}} \hat{g}_n(w_n(y)) = \varphi(y)$ for all $y < 0$. Hence from the continuity condition on p it follows that $g_n(y)$ tends to $\varphi(y)$ for $y \in (-\frac{1}{2}\log n, 0)$. Since $P^n * X_{r_n:n}(-\infty, \xi_\alpha) = E^n * X_{r_n:n}(-\infty, \alpha)$ we know that $\lim_{n\in\mathbb{N}} P^n * ((n^{\frac{1}{2}}p_\alpha^-/\sigma_\alpha)(X_{r_n:n} - \xi_\alpha))(-\infty, 0) = \Phi(0)$. Hence the assertion follows from (3.2) by an argument similar to that which leads to Scheffé's lemma (see Billingsley (1968), page 224).

LEMMA 3.9. *Under the assumptions of Theorem* 2.7 (A) *for* $s \in \{0, 1, \cdots\}$

(3.10) $$g_n(y) - \varphi(y)(1 + \textstyle\sum_{i=1}^s n^{-i/2}Q_{\delta_n,i}^-(y)) = O(\exp(-y^2/5)n^{-(s+1)/2})$$

uniformly for all $y \in (-\frac{1}{2}\log n, 0)$ *where the polynomials* $Q_{\delta_n,i}^-$ *are described in Remark* 2.9 *and* (2.11).

PROOF. Let $\hat{g}_n$ be defined as in Lemma 3.3. Let $w_n(y) = (n^{\frac{1}{2}}/\sigma_\alpha)(F(l_n(y)) - \alpha)$. Then

$$g_n(y) = \frac{p(l_n(y))}{p_\alpha}\hat{g}_n(w_n(y)) \qquad \text{for } y \in (-\tfrac{1}{2}\log n, 0)\,.$$

The following relations hold true uniformly for all $y \in (-\frac{1}{2}\log n, 0)$:

$$w_n(y) - (y + S_n(y)) = O(|y|^{s+2}n^{-(s+1)/2})$$

and

$$p(l_n(y))/p_\alpha - (1 + S_n^{(1)}(y)) = O(|y|^{s+1}n^{-(s+1)/2})$$

where

$$S_n(y) = \sum_{i=1}^{s} \frac{1}{(i+1)!} \frac{p_{\alpha,i+1}}{p_\alpha^{\,i+1}} \sigma_\alpha{}^i y^{i+1} n^{-i/2} .$$

Since $|w_n(y)| \leqq \log n$ for all $y \in (-\frac{1}{2}\log n, 0)$ Lemma 3.3 implies

$$g_n(y) - \frac{p(l_n(y))}{p_\alpha} \varphi(w_n(y))(1 + \sum_{i=1}^{s} n^{-i/2}\hat{Q}_{\delta_n,i}(w_n(y))) \\ = O(\exp(-w_n(y)^2/3)n^{-(s+1)/2}) .$$

We have

$$w_n(y)^2/2 - (y^2/2 + yS_n(y) + S_n(y)^2/2) = O(L_s(|y|)n^{(s+1)/2})$$

where L_s is some polynomial.

Therefore, expanding the exp function,

$$\exp(-w_n(y)^2/2) - \exp(-y^2/2)\left(1 + \sum_{j=1}^{s} \frac{1}{j!}(-yS_n(y) - S_n(y)^2/2)^j\right) \\ = O(\exp(-y^2/4)n^{-(s+1)/2}) .$$

Hence

$$g_n(y) - \varphi(y)(1 + \sum_{j=1}^{s}(-1)^j(\sum_{i=1}^{s} d_{\delta_n,i} n^{-i/2})^j)(1 + S_n^{(1)}(y)) \\ \times \left(1 + \sum_{j=1}^{s} \frac{1}{j!}(-yS_n(y) - S_n(y)^2/2)^j\right) \\ \times (1 + \sum_{j=1}^{s} n^{-j/2}\hat{Q}_{\alpha_n,j}(y + S_n(y))) \\ = O(\exp(-y^2/4)n^{-(s+1)/2}) .$$

Writing the factor of $\varphi(y)$ as a polynomial and collecting all terms of order $n^{-i/2}$ for $i = 1, \cdots, s$, the assertion follows.

(3.11) PROOF OF THEOREM 2.7. Integrating g_n on measurable subsets of $(-\frac{1}{2}\log n, 0)$ the assertion follows from Lemma 3.9 and Lemma 3.1.

Acknowledgments. The author wishes to thank Professor J. Pfanzagl for valuable suggestions. Thanks are also due to Professor H. E. Daniels for a question concerning the double exponential distribution which stimulated an extension of an earlier version of the paper (Reiss (1974b)) to "left and right differentiable" distribution functions.

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