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ASYMPTOTIC EXPANSIONS FOR THE POWER OF DISTRIBUTION FREE TESTS IN THE ONE-SAMPLE PROBLEM[1]

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Asymptotic expansions are established for the power of distribution free tests in the one-sample problem. These expansions are then used to obtain deficiencies in the sense of Hodges and Lehmann (1970) for distribution free tests with respect to their parametric competitors and for the estimators of location associated with these tests.

1. Introduction. Let $X_1, \cdots, X_N$ be independent and identically distributed random variables with a common absolutely continuous distribution. For $N = 1, 2, \cdots$, consider the problem of testing the hypothesis that this distribution is symmetric about zero against a sequence of alternatives that is contiguous to the hypothesis as $N \to \infty$. The level α of the sequence of tests is fixed in (0, 1). Standard tests for this problem are linear rank tests and linear permutation tests and expressions for the limiting powers of such tests are of course well-known. In this paper we shall be concerned with obtaining asymptotic expansions to order N^{-1} for the powers π_N of these tests, i.e. expressions of the form $\pi_N = c_0 + c_1 N^{-\frac{1}{2}} + c_{2,N} N^{-1} + o(N^{-1})$. Of course this involves establishing similar expansions for the distribution function of the test statistic under the hypothesis as well as under contiguous alternatives. For simplicity we shall eventually limit our discussion to contiguous location alternatives and in this case terms of order $N^{-\frac{1}{2}}$ do not occur in the expansions.

One reason to consider these problems would be to obtain better numerical approximations for the critical value of the test statistic and the power of the test than can be provided by the usual normal approximation. A number of authors have investigated this possibility, usually dealing only with the hypothesis in order to obtain critical values and more often for the two-sample case than for the one-sample tests we are concerned with here. For an account of this work we refer to a review paper of Bickel (1974), which incidentally also contains a preview of the present study. Here we merely note that, with the exception of a recent paper of Rogers (1971), all previous work is based on formal Edgeworth

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expansions. One of the purposes of the present paper is to give a rigorous proof of the validity of such expansions. Rogers (1971) has given such a proof for the two-sample Wilcoxon test under the hypothesis. In a companion paper (Bickel and van Zwet (1975)) expansions will be derived for the general two-sample linear rank test under the hypothesis as well as under contiguous location alternatives.

Here we shall not dwell on the numerical aspects of the expansions we obtain. Numerical results are contained in the Ph. D. thesis of Albers (1974). We only mention that the expansions for the power seem to behave as might be expected. In those cases where the normal approximation already produces reasonably good results, the expansions perform even better and often much better. On the other hand, in cases where the normal approximation is known to be disastrous—the Wilcoxon test for Cauchy alternatives for instance—the expansion is as bad or even worse.

We shall concentrate on a different aspect of the expansions for the power. Consider two sequences of tests $\{T_N\}$ and $\{T_N'\}$ for the same hypothesis at the same fixed level α. Let $\pi_N(\theta_N)$ and $\pi_N'(\theta_N)$ denote the powers of these tests against the same sequence of contiguous alternatives parametrized by a parameter θ. If T_N is more powerful than T_N' we search for a number $k_N = N + d_N$ such that $\pi_N(\theta_N) = \pi'_{k_N}(\theta_N)$. Here k_N and d_N are treated as continuous variables, the power π_N' being defined for real N by linear interpolation between consecutive integers. The quantity d_N was named the deficiency of $\{T_N'\}$ with respect to T_N by Hodges and Lehmann (1970), who introduced this concept and initiated its study. Of course, in many cases of interest, d_N is analytically intractable and one can only study its asymptotic behavior as N tends to infinity.

Suppose that for $N \to \infty$, the ratio N/k_N tends to a limit e, the asymptotic relative efficiency of $\{T_N'\}$ with respect to $\{T_N\}$. If $0 < e < 1$, we have $d_N \sim (e^{-1} - 1)N$ and further asymptotic information about d_N is not particularly revealing. On the other hand, if $e = 1$, the asymptotic behavior of d_N, which may now be anything from $o(1)$ to $o(N)$, does provide important additional information. Of special interest is the case where d_N tends to a finite limit, the asymptotic deficiency of $\{T_N'\}$ with respect to $\{T_N\}$ (cf. Hodges and Lehmann (1970)).

Of course, an asymptotic evaluation of d_N is a more delicate matter than showing that $e = 1$. What is needed is an expansion for the power of the type we discussed above. With the aid of such expansions we arrive at the following results. Let F be a distribution function with a density f that is symmetric about zero and let b be a positive real number. Consider the problem of testing the hypothesis F against the sequence of alternatives $F(x - bN^{-\frac{1}{2}})$ at level α. Let d_N denote the deficiency of the locally most powerful rank test with respect to the most powerful test for this problem. Under certain regularity conditions on F we establish an expression for d_N with remainder $o(1)$ and show that this expression remains unchanged if the exact scores in the locally most powerful rank test are replaced by the corresponding approximate scores. The asymptotic behavior of d_N is found to be governed by that of

$$I_N = \int_{1/N}^{1-1/N} \left(\frac{d^2}{dt^2} f\left(F^{-1}\left(\frac{1+t}{2}\right)\right)\right)^2 t(1-t)\,dt \tag{1.1}$$

in the sense that $d_N = O(I_N)$ as $N \to \infty$. By taking F to be the normal distribution we find that the deficiency of both Fraser's normal scores test and van der Waerden's test with respect to the $\bar{X}$-test for contiguous normal alternatives tends to ∞ at the rate of $\frac{1}{2} \log\log N$. For logistic alternatives the deficiency of Wilcoxon's signed rank test with respect to the most powerful parametric test tends to a finite limit. Another typical result is that for contiguous normal alternatives the deficiency of the permutation test based on $\sum X_i$ with respect to Student's test tends to zero for $N \to \infty$.

Combining numerical and Monte Carlo methods, Albers (1974) has evaluated the deficiency of the normal scores test with respect to the $\bar{X}$-test for $N = 5 - (1) - 10$, 20 and 50. The results agree reasonably well with the asymptotic expression for d_N.

To every linear rank test with nonnegative and nondecreasing scores, there corresponds an estimator of location due to Hodges and Lehmann (1963). A similar correspondence exists between the locally most powerful parametric test and the maximum likelihood estimator. We shall exploit this correspondence to obtain asymptotic expansions for the distribution functions of these estimators. We shall show that, when suitably defined, the deficiency of the Hodges–Lehmann estimator associated with the locally most powerful rank test with respect to the maximum likelihood estimator is asymptotically equivalent to the deficiency of the parent tests.

In Section 2 we establish an asymptotic expansion for the distribution function of the general linear rank statistic for the one-sample problem under the hypothesis as well as under alternatives. We specialize to contiguous location alternatives in Section 3 and derive an expansion for the power of the linear rank test. In Section 4 we deal with the important case where the scores are exact or approximate scores generated by a smooth function J. Linear permutation tests are discussed in Section 5. The results on deficiencies of distribution free tests are contained in Section 6. Finally, Section 7 is devoted to estimators.

Although the basic ideas underlying this paper are simple, the proofs are a highly technical matter. The most laborious parts are dealt with in two appendices. We have omitted the proofs of Theorem 5.1 and Lemma 6.1 because we felt that their inclusion would entail much repetition without essentially new ideas. Some relevant results have been left out altogether for much the same reasons. We are referring to a treatment of contiguous alternatives other than location alternatives for linear rank tests, to expansions for the power of locally most powerful parametric tests, most powerful permutation tests and randomized rank score tests. These missing parts may all be found in the Ph. D. thesis of Albers (1974).

2. The basic expansion. Let $X_1, \cdots, X_N$ be independent and identically distributed (i.i.d.) random variables (rv's) with common distribution (df) G and density g, and let $0 < Z_1 < Z_2 < \cdots < Z_N$ denote the order statistics of the absolute values of $X_1, \cdots, X_N$. If $|X_{R_j}| = Z_j$, define

$$(2.1) \qquad \begin{aligned} V_j &= 1 \quad \text{if} \quad X_{R_j} > 0 \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

We introduce a vector of scores $a = (a_1, \cdots, a_N)$ and define the statistic

$$(2.2) \qquad T = \textstyle\sum_{j=1}^N a_j V_j .$$

We shall be concerned with obtaining an asymptotic expansion for the distribution of T as $N \to \infty$.

Our notation strongly suggests that we are considering a fixed underlying df G and perhaps also a fixed infinite sequence of scores as $N \to \infty$. However, this is merely a matter of notational convenience and our main concern will in fact be the case where the df depends on N and the scores form a triangular array $a_{j,N}, j = 1, \cdots, N, N = 1, 2, \cdots$. Since we are suppressing the index N throughout our notation we shall formally present our results in terms of error bounds for a fixed, but arbitrary, value of N. However, as we shall point out following the proof of Theorem 2.2, these results are really asymptotic expansions in disguise.

The rv T is of course the general linear rank statistic for testing the hypothesis that g is symmetric about zero. Under this hypothesis, $V_1, \cdots, V_N$ are i.i.d. with $P(V_j = 1) = \frac{1}{2}$. For general G, $V_1, \cdots, V_N$ are not independent. However, one easily verifies that, conditional on $Z = (Z_1, \cdots, Z_N)$, the rv's $V_1, \cdots, V_N$ are independent with

$$(2.3) \qquad P_j = P(V_j = 1 \mid Z) = \frac{g(Z_j)}{g(Z_j) + g(-Z_j)} .$$

As independence allows us to obtain expansions of Edgeworth type, we shall carry out the following program to arrive at an expansion for the distribution of T. First we obtain an Edgeworth expansion for the distribution of $\sum a_j W_j$, where $W_1, \cdots, W_N$ are independent with $p_j = P(W_j = 1) = 1 - P(W_j = 0)$. Having done this we substitute the random vector $P = (P_1, \cdots, P_N)$ defined in (2.3) for $p = (p_1, \cdots, p_N)$ in this expansion. The expected value of the resulting expression will then give us an expansion for the distribution of T.

In carrying out the first part of this program we shall indicate any dependence on $p = (p_1, \cdots, p_N)$ in our notation. Consider the rv

$$(2.4) \qquad \frac{\sum_{j=1}^N a_j(W_j - p_j)}{\tau(p)} ,$$

where

$$(2.5) \qquad \tau^2(p) = \textstyle\sum_{j=1}^N p_j(1 - p_j)a_j^2$$

denotes the variance of $\sum a_j W_j$. Obviously (2.4) has expectation 0 and variance 1; its third and fourth cumulants, multiplied by $N^{\frac{1}{2}}$ and N respectively, are

$$\kappa_3(p) = -N^{\frac{1}{2}} \frac{\sum p_j(1-p_j)(2p_j-1)a_j^3}{\tau^3(p)}, \tag{2.6}$$

$$\kappa_4(p) = N \frac{\sum p_j(1-p_j)(1-6p_j+6p_j^2)a_j^4}{\tau^4(p)}. \tag{2.7}$$

Let R and ρ denote the df and the characteristic function (ch.f.) of (2.4), thus

$$R(x, p) = P\left(\frac{\sum a_j(W_j - p_j)}{\tau(p)} \leq x\right), \tag{2.8}$$

$$\rho(t, p) = \prod_{j=1}^N \left[p_j \exp\left\{i(1-p_j)\frac{a_j t}{\tau(p)}\right\} + (1-p_j)\exp\left\{-ip_j \frac{a_j t}{\tau(p)}\right\}\right]. \tag{2.9}$$

A formal Edgeworth expansion to order N^{-1} for the df R is given by (Cramér (1946), page 229)

$$\tilde{R}(x, p) = \Phi(x) + \phi(x)\{N^{-\frac{1}{2}}Q_1(x, p) + N^{-1}Q_2(x, p)\}, \tag{2.10}$$

where Φ and ϕ denote the df and the density of the standard normal distribution, and

$$Q_1(x, p) = -\frac{\kappa_3(p)}{6}(x^2 - 1), \tag{2.11}$$

$$Q_2(x, p) = -\frac{\kappa_4(p)}{24}(x^3 - 3x) - \frac{\kappa_3^2(p)}{72}(x^5 - 10x^3 + 15x).$$

Let $\tilde{r}(x, p)$ be the derivative of $\tilde{R}(x, p)$ with respect to x. In what follows we shall need an expression for the Fourier transform $\tilde{\rho}(t, p) = \int \exp(itx)\tilde{r}(x, p)\,dx$ of $\tilde{r}$ and one easily verifies that

$$\tilde{\rho}(t, p) = e^{-\frac{1}{2}t^2}\left\{1 - \frac{\kappa_3(p)it^3}{6N^{\frac{1}{2}}} + \frac{3\kappa_4(p)t^4 - \kappa_3^2(p)t^6}{72N}\right\}. \tag{2.12}$$

To justify a formal Edgeworth expansion like (2.10), i.e. to show that $|\tilde{R} - R|$ is indeed $o(N^{-1})$, one usually invokes the following result (Feller (1966), page 512).

LEMMA 2.1. *Let R be a* df *with vanishing expectation and* ch.f. ρ. *Suppose that $R - \tilde{R}$ vanishes at $\pm\infty$ and that $\tilde{R}$ has a derivative $\tilde{r}$ such that $|\tilde{r}| \leq m$. Finally, suppose that $\tilde{r}$ has a continuously differentiable Fourier transform $\tilde{\rho}$ such that $\tilde{\rho}(0) = 1$ and $\tilde{\rho}'(0) = 0$. Then for all x and $T > 0$,*

$$|R(x) - \tilde{R}(x)| \leq \frac{1}{\pi}\int_{-T}^{T}\left|\frac{\rho(t) - \tilde{\rho}(t)}{t}\right|dt + \frac{24m}{\pi T}. \tag{2.13}$$

To prove that $|R - \tilde{R}| = o(N^{-1})$, it therefore suffices to show that e.g. for $T = bN^{\frac{3}{2}}$, the integral in (2.13) is $o(N^{-1})$. For the case we are considering this may be done in the standard manner (Feller (1966), Chapter 16) with one important modification at the point where it is shown that $|\rho(t, p)/t|$ is sufficiently small when $|t|$ is of the order $\tau(p)$ or larger. Here one usually makes what Feller calls the extravagantly luxurious assumption that the ch.f.'s of all summands are uniformly bounded away from 1 in absolute value outside every neighborhood of 0. Obviously this condition is not satisfied in our case where the summands $a_j W_j$ are lattice rv's. Weaker sufficient conditions of this type are known, but all seem to imply at the very least that the sum itself is nonlattice. In our case this would exclude for instance both the sign test and the Wilcoxon test.

Although the assumptions mentioned above may be unnecessarily strong, it is clear that one has to exclude cases where the sum (2.4) can only assume relatively few different values. As $\tilde{R}$ is continuous, one can not allow R to have jumps of order N^{-1} or larger. Thus the sign test where jumps of order $N^{-\frac{1}{2}}$ occur, will certainly have to be excluded. However, it is exactly the simple lattice character of this statistic that makes it easily amenable to other methods of expansion (see for instance Albers (1974)). For the Wilcoxon statistic on the other hand, all jumps are $O(N^{-\frac{3}{2}})$ and the assumptions we shall make will not rule out this case.

For $0 < \varepsilon < \frac{1}{2}$ and $\zeta > 0$ consider the set of those a_j for which the corresponding p_j satisfies $\varepsilon \leqq p_j \leqq 1 - \varepsilon$, and let $\gamma(\varepsilon, \zeta, p)$ denote the Lebesgue measure λ of the ζ-neighborhood of this set, thus

$$\gamma(\varepsilon, \zeta, p) = \lambda\{x \mid \exists_j |x - a_j| < \zeta, \varepsilon \leqq p_j \leqq 1 - \varepsilon\}. \tag{2.14}$$

LEMMA 2.2. *Suppose that positive numbers c, C, δ and ε exist such that*

$$\frac{1}{N} \sum_{j=1}^N p_j(1 - p_j)a_j^2 \geqq c, \qquad \frac{1}{N} \sum_{j=1}^N a_j^4 \leqq C, \tag{2.15}$$

$$\gamma(\varepsilon, \zeta, p) \geqq \delta N\zeta \qquad \textit{for some} \quad \zeta \geqq N^{-\frac{3}{2}} \log N. \tag{2.16}$$

Then there exist positive numbers b, B and β depending on N, a and p only through c, C, δ and ε, such that

$$\int_{\log(N+1) \leqq |t| \leqq bN^{\frac{3}{2}}} \left| \frac{\rho(t, p) - \tilde{\rho}(t, p)}{t} \right| dt \leqq BN^{-\beta \log N}.$$

PROOF. Since (2.15) implies that $|\kappa_3(p)| \leqq (Cc^{-2})^{\frac{3}{4}}$ and $|\kappa_4(p)| \leqq Cc^{-2}$,

$$\int_{|t| \geqq \log(N+1)} \left| \frac{\tilde{\rho}(t, p)}{t} \right| dt \leqq B_1 N^{-\beta_1 \log N},$$

where B_1, $\beta_1 > 0$ depend only on c and C. Also, for all t,

$$\begin{aligned} |\rho(t, p)| &= \prod_{j=1}^N \left\{1 - 2p_j(1 - p_j)\left(1 - \cos \frac{a_j t}{\tau(p)}\right)\right\}^{\frac{1}{2}} \\ &\leqq \exp\left\{-\sum p_j(1 - p_j)\left[\frac{1}{2}\left(\frac{a_j t}{\tau(p)}\right)^2 - \frac{1}{24}\left(\frac{a_j t}{\tau(p)}\right)^4\right]\right\} \\ &\leqq \exp\left\{-\tfrac{1}{2}t^2 + \frac{Ct^4}{96c^2N}\right\}. \end{aligned} \tag{2.17}$$

For $|t| \leqq 4cC^{-\frac{1}{4}}N^{\frac{1}{2}}$ this is $\leqq \exp(-t^2/3)$. Hence, if $b' = 4cC^{-\frac{1}{4}}$, there exist positive constants B_2 and β_2 such that

$$\int_{\log(N+1)\leqq|t|\leqq b'N^{\frac{1}{2}}} \left|\frac{\rho(t,p)}{t}\right| dt \leqq B_2 N^{-\beta_2 \log N} .$$

As $\gamma(\varepsilon, \zeta, p)/\zeta$ is nonincreasing in ζ, we may assume that $\zeta \leqq 1$ in (2.16). Because of (2.15), for any $M > \zeta$ the number of $|a_j| \geqq M - \zeta$ can be at most $CN(M-\zeta)^{-4}$; choosing $M = (8C/\delta)^{\frac{1}{4}} + 1$ we have $CN(M-\zeta)^{-4} \leqq \delta N/8 \leqq \gamma(\varepsilon, \zeta, p)/8\zeta$. It follows that

$$\lambda\{x \mid \exists_j |a_j| \geqq M - \zeta, |x - a_j| < \zeta\} \leqq 2\zeta \frac{\gamma(\varepsilon, \zeta, p)}{8\zeta} = \frac{\gamma(\varepsilon, \zeta, p)}{4} .$$

Together with (2.16) this implies that for every real t

$$\lambda\left\{z \mid \exists_j |a_j| \leqq M - \zeta, \left|z - \frac{a_j t}{\tau(p)}\right| < \frac{\zeta|t|}{\tau(p)}, \varepsilon \leqq p_j \leqq 1 - \varepsilon\right\} \geqq \frac{3|t|\gamma(\varepsilon, \zeta, p)}{4\tau(p)} .$$

Take $b = \delta[(32M/\pi c^{\frac{1}{2}}) + (16/b')]^{-1}$. Then, for every $|t| \in [b'N^{\frac{1}{2}}, bN^{\frac{3}{2}}]$

$$\lambda\left\{z \mid |z| \leqq \frac{M|t|}{\tau(p)}, |z - k\pi| \leqq \frac{2\zeta b N^{\frac{3}{2}}}{\tau(p)} \text{ for some integer } k\right\}$$
$$\leqq \left(\frac{2M|t|}{\pi\tau(p)} + 1\right)\frac{4\zeta b N^{\frac{3}{2}}}{\tau(p)} \leqq \left(\frac{2M|t|}{\pi(cN)^{\frac{1}{2}}} + \frac{|t|}{b'N^{\frac{1}{2}}}\right)\frac{4bN^{\frac{3}{2}}}{\tau(p)}\frac{\gamma(\varepsilon, \zeta, p)}{\delta N} = \frac{|t|\gamma(\varepsilon, \zeta, p)}{4\tau(p)} ,$$

and hence

$$\lambda\left\{z \mid |z| \leqq \frac{M|t|}{\tau(p)}, \exists_j |a_j| \leqq M - \zeta, \left|z - \frac{a_j t}{\tau(p)}\right| < \frac{\zeta|t|}{\tau(p)}, \varepsilon \leqq p_j \leqq 1 - \varepsilon;\right.$$
$$\left. |z - k\pi| > \frac{2\zeta b N^{\frac{3}{2}}}{\tau(p)} \text{ for every integer } k\right\} \geqq \frac{|t|\gamma(\varepsilon, \zeta, p)}{2\tau(p)} .$$

As $\zeta|t| \leqq \zeta b N^{\frac{3}{2}}$, this implies that the number of indices j for which $|(a_j t/\tau(p)) - k\pi| > \zeta b N^{\frac{3}{2}}/\tau(p)$ for every integer k and $\varepsilon \leqq p_j \leqq 1 - \varepsilon$, is at least equal to

$$\frac{\tau(p)}{2\zeta|t|} \cdot \frac{|t|\gamma(\varepsilon, \zeta, p)}{2\tau(p)} \geqq \frac{\delta N}{4} .$$

For such an index j we have for all $|t| \in [b'N^{\frac{1}{2}}, bN^{\frac{3}{2}}]$,

$$\left\{1 - 2p_j(1-p_j)\left(1 - \cos\frac{a_j t}{\tau(p)}\right)\right\}^{\frac{1}{2}} \leqq \left\{1 - 2\varepsilon(1-\varepsilon)\frac{\zeta^2 b^2 N^3}{(\pi\tau(p))^2}\right\}^{\frac{1}{2}}$$
$$\leqq \exp\left\{-\frac{\varepsilon(1-\varepsilon)\zeta^2 b^2 N^3}{(\pi\tau(p))^2}\right\}$$

and hence, as $4\tau^2(p) \leqq C^{\frac{1}{2}}N$ and $\zeta \geqq N^{-\frac{3}{2}} \log N$,

$$|\rho(t, p)| \leqq \exp\left\{-\frac{\delta\varepsilon(1-\varepsilon)b^2 N^4 \zeta^2}{4\pi^2\tau^2(p)}\right\} \leqq \exp\left\{-\frac{\delta\varepsilon(1-\varepsilon)b^2}{\pi^2 C^{\frac{1}{2}}}(\log N)^2\right\} .$$

This implies that for some B_3, $\beta_3 > 0$ depending on c, C, δ and ε,

$$\int_{b'N^{\frac{1}{4}} \leqq |t| \leqq bN^{\frac{3}{2}}} \left| \frac{\rho(t, p)}{t} \right| dt \leqq B_3 N^{-\beta_3 \log N} ,$$

which completes the proof. □

We now justify expansion (2.10).

THEOREM 2.1. *Suppose that positive numbers c, C, δ and ε exist such that* (2.15) *and* (2.16) *are satisfied. Then there exists $A > 0$ depending on N, a and p only through c, C, δ and ε such that*

$$\sup_x |R(x, p) - \tilde{R}(x, p)| \leqq AN^{-\frac{5}{4}} . \tag{2.18}$$

PROOF. For $0 \leqq y \leqq 1$ and $-\pi/2 \leqq z \leqq \pi/2$, $\operatorname{Re}[y \exp\{i(1-y)z\} + (1-y)\exp\{-iyz\}] \geqq \frac{1}{2}$, and hence we have the following Taylor expansion (mod. $2\pi i$)

$$\begin{aligned} \log (ye^{i(1-y)z} &+ (1-y)e^{-iyz}) \\ &= -\tfrac{1}{2}y(1-y)z^2 + \tfrac{1}{6}y(1-y)(2y-1)iz^3 \\ &\quad + \tfrac{1}{24}y(1-y)(1-6y+6y^2)z^4 + M_1(y, z) , \end{aligned} \tag{2.19}$$

where $|M_1(y, z)| \leqq C_1|z|^5$ for some fixed $C_1 > 0$. If $|a_j t/\tau(p)| \leqq \pi/2$ for all j, we can apply this expansion to the logarithm of every factor in (2.9) which yields

$$\rho(t, p) = \exp\left\{-\tfrac{1}{2}t^2 - \frac{\kappa_3(p)it^3}{6N^{\frac{1}{2}}} + \frac{\kappa_4(p)t^4}{24N} + M_2(t, p)\right\} , \tag{2.20}$$

where $|M_2(t, p)| \leqq C_1|t/\tau(p)|^5 \sum |a_j|^5$.

Condition (2.15) implies that $\max |a_j| \leqq (CN)^{\frac{1}{4}}$ and hence that $|a_j t/\tau(p)| \leqq (Cc^{-2})^{\frac{1}{4}}N^{-\frac{1}{4}}|t|$ for all j. We have already seen that $|\kappa_3(p)| \leqq (Cc^{-2})^{\frac{3}{4}}$ and $|\kappa_4(p)| \leqq Cc^{-2}$; because $\max |a_j| \leqq (CN)^{\frac{1}{4}}$ we also have $\tau^{-5}(p) \sum |a_j|^5 \leqq (Cc^{-2})^{\frac{5}{4}}N^{-\frac{5}{4}}$. It follows from these remarks that there exists $c_1 > 0$, depending only on c and C, such that for $|t| \leqq c_1 N^{\frac{1}{4}}$ expansion (2.20) is valid and also

$$\left| -\frac{\kappa_3(p)it^3}{6N^{\frac{1}{2}}} \right| + \left| \frac{\kappa_4(p)t^4}{24N} \right| + |M_2(t, p)| \leqq \tfrac{1}{4}t^2 .$$

Hence, for $|t| \leqq c_1 N^{\frac{1}{4}}$, Taylor expansion of (2.20) yields

$$\rho(t, p) = \tilde{\rho}(t, p) + M_3(t, p) , \tag{2.21}$$

where $\tilde{\rho}$ is given by (2.12), $|M_3(t, p)| \leqq (N^{-\frac{3}{2}} + N^{-\frac{5}{4}} \sum |a_j|^5)|t|^5 Q(|t|) \exp(-t^2/4)$, and Q is a polynomial with coefficients depending on c and C. This implies the existence of $A_1 > 0$ depending on c and C and such that

$$\int_{|t| \leqq c_1 N^{\frac{1}{4}}} \left| \frac{\rho(t, p) - \tilde{\rho}(t, p)}{t} \right| dt \leqq A_1 N^{-\frac{5}{4}} . \tag{2.22}$$

As c_1 depends only on c and C we may assume without loss of generality that N is so large that $\log (N + 1) \leqq c_1 N^{\frac{1}{4}}$. The theorem is now proved by combining

(2.22) and Lemma 2.2, noting that $\tilde{r}(x, t) = (\partial/\partial x)\tilde{R}(x, t)$ is bounded by a number depending only on c and C and applying Lemma 2.1. □

It will be clear that by requiring that $\sum |a_j|^5 \leqq CN$ in Theorem 2.1 one obtains $|R - \tilde{R}| \leqq AN^{-\frac{3}{2}}$ which is the "natural" order of the remainder.

Before we replace p by the random vector $P = (P_1, \cdots, P_N)$ defined in (2.3) and compute the unconditional distribution of T by taking the expected value, we first have to change the standardization of $\sum a_j W_j$ into one that does not involve p. As before, let $W_1, \cdots, W_N$ be independent with $P(W_j = 1) = 1 - P(W_j = 0) = p_j$, let $\bar{p} = (\bar{p}_1, \cdots, \bar{p}_N)$ be a vector with $0 \leqq \bar{p}_j \leqq 1$ for all j, and consider the df $R^*(x, p, \bar{p})$ of the rv $\tau^{-1}(\bar{p}) \sum a_j(W_j - \bar{p}_j)$, thus

$$R^*(x, p, \bar{p}) = P\left(\frac{\sum a_j(W_j - \bar{p}_j)}{\tau(\bar{p})} \leqq x\right). \tag{2.23}$$

Here $\tau^2(\bar{p}) = \sum \bar{p}_j(1 - \bar{p}_j)a_j^2$ in accordance with (2.5); similarly $\kappa_3(\bar{p})$, $\kappa_4(\bar{p})$, $Q_1(x, \bar{p})$, $Q_2(x, \bar{p})$ and $\tilde{R}(x, \bar{p})$ are defined by replacing p by $\bar{p}$ in (2.6), (2.7), (2.11) and (2.10).

For reasons that will become clear in the sequel we shall also at this stage expand $\tau(\bar{p})/\tau(p)$ in powers of $(\tau^2(p) - \tau^2(\bar{p}))/\tau^2(\bar{p})$; at the same time the numerators of $\kappa_3(p)$ and $\kappa_4(p)$ will be expanded about the point $p = \bar{p}$. Later on, when p_j is replaced by P_j, we shall e.g. take $\bar{p}_j = EP_j$ thus ensuring that $P_j - \bar{p}_j$ is roughly speaking a rv of order $N^{-\frac{1}{2}}$. At the moment, however, we do not make any assumptions about $p - \bar{p}$ and as a result Lemma 2.3 provides only a formal expansion in the sense that we do not claim that the remainder term is at all small.

The expansion for $R^*(x, p, \bar{p})$ that we shall establish is

$$\begin{aligned} \tilde{R}^*(x, p, \bar{p}) = \tilde{R}(x - u, \bar{p}) - \phi(x - u) \Big\{ & \frac{1}{2} \frac{\tau^2(p) - \tau^2(\bar{p})}{\tau^2(\bar{p})} (x - u) \\ & + \frac{1}{6} \frac{\sum (p_j - \bar{p}_j)(1 - 6\bar{p}_j + 6\bar{p}_j^2)a_j^3}{\tau^3(\bar{p})} [(x - u)^2 - 1] \\ & + \frac{1}{8} \left(\frac{\tau^2(p) - \tau^2(\bar{p})}{\tau^2(\bar{p})}\right)^2 [(x - u)^3 - 3(x - u)] \\ & + \frac{\kappa_3(\bar{p})}{12N^{\frac{1}{2}}} \frac{\tau^2(p) - \tau^2(\bar{p})}{\tau^2(\bar{p})} [(x - u)^4 - 6(x - u)^2 + 3] \Big\}, \end{aligned} \tag{2.24}$$

where $\tilde{R}$ is given by (2.10) and

$$u = \frac{\sum (p_j - \bar{p}_j)a_j}{\tau(\bar{p})}. \tag{2.25}$$

LEMMA 2.3. *Let $\bar{p} = (\bar{p}_1, \cdots, \bar{p}_N)$ be a vector of real numbers in* [0, 1] *and suppose that positive numbers c, C, δ and ε exist such that* (2.15) *and* (2.16) *are satisfied and that*

$$\frac{1}{N} \sum_{j=1}^N \bar{p}_j(1 - \bar{p}_j)a_j^2 \geqq c. \tag{2.26}$$

Then there exists $A > 0$ *depending on* N, a, p *and* $\tilde{p}$ *only through* c, C, δ *and* ε *and such that*

$$(2.27)\qquad \sup_x |R^*(x, p, \tilde{p}) - \tilde{R}^*(x, p, \tilde{p})| \leqq A\{N^{-\frac{5}{4}} + N^{-\frac{3}{2}} \sum (p_j - \tilde{p}_j)^2 |a_j|^3 + N^{-3}|\tau^2(p) - \tau^2(\tilde{p})|^3\} .$$

PROOF. Changing the standardization in Theorem 2.1 we find

$$(2.28)\qquad \sup_x \left| R^*(x, p, \tilde{p}) - \tilde{R}\left((x-u)\frac{\tau(\tilde{p})}{\tau(p)}, p\right)\right| \leqq AN^{-\frac{5}{4}} .$$

The assumptions of the lemma ensure that $\tau^2(\tilde{p})/\tau^2(p) \geqq cC^{-\frac{1}{2}}$, $\tau^2(p)/\tau^2(\tilde{p}) \geqq cC^{-\frac{1}{2}}$, $|\kappa_3(p)| \leqq (c^{-2}C)^{\frac{3}{4}}$, $|\kappa_3(\tilde{p})| \leqq (c^{-2}C)^{\frac{3}{4}}$, $|\kappa_4(p)| \leqq c^{-2}C$ and $|\kappa_4(\tilde{p})| \leqq c^{-2}C$. It follows that the derivatives of $\tilde{R}((x-u)y, p)$ with respect to y are bounded for $y^2 \geqq cC^{-\frac{1}{2}}$ and all $x - u$, and hence

$$\begin{aligned}(2.29)\qquad \tilde{R}\left((x-u)\frac{\tau(\tilde{p})}{\tau(p)}, p\right) &= \tilde{R}(x-u, p) + \tilde{R}'(x-u, p)\left(\frac{\tau(\tilde{p})}{\tau(p)} - 1\right)(x-u) \\ &\quad + \tfrac{1}{2}\tilde{R}''(x-u, p)\left(\frac{\tau(\tilde{p})}{\tau(p)} - 1\right)^2 (x-u)^2 + O\left(\left(\frac{\tau(\tilde{p})}{\tau(p)} - 1\right)^3\right),\end{aligned}$$

where $\tilde{R}'(x, p)$ and $\tilde{R}''(x, p)$ denote first and second derivatives of $\tilde{R}(x, p)$ with respect to x. Since $(\tau^2(p) - \tau^2(\tilde{p}))/\tau^2(\tilde{p}) \geqq -1 + cC^{-\frac{1}{2}}$,

$$(2.30)\qquad \frac{\tau(\tilde{p})}{\tau(p)} = 1 - \frac{1}{2}\,\frac{\tau^2(p) - \tau^2(\tilde{p})}{\tau^2(\tilde{p})} + \frac{3}{8}\left(\frac{\tau^2(p) - \tau^2(\tilde{p})}{\tau^2(\tilde{p})}\right)^2 - \cdots ,$$

where the remainder is of the order of the first term omitted. As $\kappa_3(\tilde{p})$ and $\kappa_4(\tilde{p})$ are bounded, we obtain the following one and two term expansions with remainder for $\kappa_3(p)$ and $\kappa_4(p)$.

$$\begin{aligned}\kappa_3(p) &= \left[\kappa_3(\tilde{p}) - N^{\frac{1}{2}}\,\frac{\sum\{p_j(1-p_j)(2p_j-1) - \tilde{p}_j(1-\tilde{p}_j)(2\tilde{p}_j-1)\}a_j^3}{\tau^3(\tilde{p})}\right]\left(\frac{\tau(\tilde{p})}{\tau(p)}\right)^3 \\ &= \kappa_3(\tilde{p}) + O(N^{-1}|\tau^2(p) - \tau^2(\tilde{p})| + N^{-1}\textstyle\sum |p_j - \tilde{p}_j|\,|a_j|^3) \\ (2.31)\qquad &= \kappa_3(\tilde{p})\left[1 - \frac{3}{2}\,\frac{\tau^2(p) - \tau^2(\tilde{p})}{\tau^2(\tilde{p})}\right] + N^{\frac{1}{2}}\,\frac{\sum (p_j - \tilde{p}_j)(1 - 6\tilde{p}_j + 6\tilde{p}_j^2)a_j^3}{\tau^3(\tilde{p})} \\ &\quad + O(N^{-2}(\tau^2(p) - \tau^2(\tilde{p}))^2 + N^{-1}\textstyle\sum (p_j - \tilde{p}_j)^2|a_j|^3 \\ &\quad + N^{-2}|\tau^2(p) - \tau^2(\tilde{p})| \textstyle\sum |p_j - \tilde{p}_j|\,|a_j|^3) ,\end{aligned}$$

$$(2.32)\qquad \kappa_4(p) = \kappa_4(\tilde{p}) + O(N^{-1}|\tau^2(p) - \tau^2(\tilde{p})| + N^{-1}\textstyle\sum |p_j - \tilde{p}_j|a_j^4) .$$

In (2.29) we may now replace $\tilde{R}$, $\tilde{R}'$ and $\tilde{R}''$ by explicit expressions and substitute (2.32) and appropriate versions of (2.31) and (2.30). The algebra is straightforward and will be omitted. Combining the result with (2.28) we find that (2.27) holds if a term

$$\begin{aligned}&O(N^{-2}\textstyle\sum |p_j - \tilde{p}_j|(|a_j|^3 + a_j^4) + N^{-\frac{5}{2}}|\tau^2(p) - \tau^2(\tilde{p})| \textstyle\sum |p_j - \tilde{p}_j|\,|a_j|^3 \\ &\quad + N^{-2}|\tau^2(p) - \tau^2(\tilde{p})| + N^{-\frac{5}{2}}(\tau^2(p) - \tau^2(\tilde{p}))^2)\end{aligned}$$

is added to the right-hand side. Here, as well as above, the order symbol is uniform for fixed c and C. The lemma is now proved by noting that

$$N^{-2} \sum |p_j - \tilde{p}_j| |a_j|^3 \leqq N^{-\frac{5}{2}} \sum |a_j|^3 + N^{-\frac{3}{2}} \sum (p_j - \tilde{p}_j)^2 |a_j|^3 ,$$
$$N^{-2} \sum |p_j - \tilde{p}_j| a_j^4 \leqq N^{-\frac{5}{2}} \sum |a_j|^5 + N^{-\frac{3}{2}} \sum (p_j - \tilde{p}_j)^2 |a_j|^3 ,$$
$$N^{-\frac{5}{2}} |\tau^2(p) - \tau^2(\tilde{p})| \sum |p_j - \tilde{p}_j| |a_j|^3 \leqq N^{-\frac{3}{2}} \sum (p_j - \tilde{p}_j)^2 |a_j|^3 + N^{-\frac{7}{2}} (\tau^2(p) - \tau^2(\tilde{p}))^2 \sum |a_j|^3 ,$$
$$N^{-2} |\tau^2(p) - \tau^2(\tilde{p})| + N^{-\frac{5}{2}} (\tau^2(p) - \tau^2(\tilde{p}))^2 \leqq N^{-\frac{3}{2}} + N^{-3} |\tau^2(p) - \tau^2(\tilde{p})|^3 ,$$

and that $\sum |a_j|^3 \leqq C^{\frac{3}{4}} N$ and $\sum |a_j|^5 \leqq (CN)^{\frac{5}{4}}$. □

We shall now replace p by $P = (P_1, \cdots, P_N)$ in $\tilde{R}^*(x, p, \tilde{p})$ and take expectations. Define the vector $\pi = (\pi_1, \cdots, \pi_N)$ by

$$\pi_j = EP_j , \qquad j = 1, \cdots, N; \tag{2.33}$$

it will play the role of $\tilde{p}$. Furthermore, for $\zeta > 0$ we let $\gamma(\zeta)$ denote the Lebesgue measure λ of the ζ-neighborhood of the set $\{a_1, \cdots, a_N\}$, thus

$$\gamma(\zeta) = \lambda\{x \mid \exists_j |x - a_j| < \zeta\} . \tag{2.34}$$

THEOREM 2.2. *Let $X_1, \cdots, X_N$ be* i.i.d. *with common* df *G and density g, and let T, P and π be defined by* (2.2), (2.3) *and* (2.33). *Suppose that positive numbers c, C, δ, δ' and ε exist with $\delta' <$* min $(\delta/2, c^2C^{-1})$ *and such that*

$$\frac{1}{N} \sum_{j=1}^N a_j^2 \geqq c , \qquad \frac{1}{N} \sum_{j=1}^N a_j^4 \leqq C , \tag{2.35}$$

$$\gamma(\zeta) \geqq \delta N \zeta \qquad \textit{for some} \quad \zeta \geqq N^{-\frac{3}{2}} \log N , \tag{2.36}$$

$$P\left(\varepsilon \leqq \frac{g(X_1)}{g(X_1) + g(-X_1)} \leqq 1 - \varepsilon\right) \geqq 1 - \delta' . \tag{2.37}$$

Then there exists $A > 0$ depending on N, a and G only through c, C, δ, δ' and ε, and such that

$$\sup_x \left| P\left(\frac{T - \sum a_j \pi_j}{\tau(\pi)} \leqq x\right) - E\tilde{R}^*(x, P, \pi) \right| \leqq A\{N^{-\frac{5}{4}} + N^{-\frac{3}{4}}[\sum \{E(P_j - \pi_j)^2\}^{\frac{5}{2}}]^{\frac{2}{5}} + N^{-\frac{3}{2}}[\sum \{E|P_j - \pi_j|^3\}^{\frac{2}{3}}]^{\frac{3}{2}}\} . \tag{2.38}$$

PROOF. We start by showing that a, P and π satisfy the conditions for a, p and $\tilde{p}$ in Lemma 2.3 with large probability.

The number of P_j that lie in $[\varepsilon, 1 - \varepsilon]$ is equal to the number of $g(X_j)/(g(X_j) + g(-X_j))$ in that interval. Applying an exponential bound for binomial probabilities (Okamoto (1958)) we find that for $\delta'' \in (\delta', \min(\delta/2, c^2C^{-1}))$, (2.37) implies

$$P(\varepsilon \leqq P_j \leqq 1 - \varepsilon \text{ for at least } (1 - \delta'')N \text{ indices } j) \geqq 1 - e^{-2N(\delta'' - \delta')^2} .$$

Suppose that $\varepsilon \leqq P_j \leqq 1 - \varepsilon$ for at least $(1 - \delta'')N$ values of j. It then follows from (2.36) that a and P satisfy condition (2.16) if δ is replaced by $\delta - 2\delta'' > 0$.

For $\eta \in (0, 1)$, suppose that $a_j^2 \leqq \eta c$ for exactly k indices j and let $\sum'$ indicate summation over the remaining $N - k$ indices. Because of (2.35)

$$c \leqq \frac{1}{N} \sum a_j^2 \leqq \frac{k}{N} \eta c + \frac{1}{N} \sum' a_j^2 \leqq \eta c + \frac{N-k}{N} \left(\frac{1}{N-k} \sum' a_j^4 \right)^{\frac{1}{2}}$$
$$\leqq \eta c + \left(\frac{N-k}{N} C \right)^{\frac{1}{2}},$$

and hence the number of $a_j^2 > \eta c$ is at least $(1-\eta)^2 c^2 C^{-1} N$. By choosing η sufficiently small we can ensure that $(1-\eta)^2 c^2 C^{-1} > \delta''$. This implies that $N^{-1}\tau^2(P) \geqq \tilde{c}$, where $\tilde{c} = ((1-\eta)^2 c^2 C^{-1} - \delta'')\varepsilon(1-\varepsilon)\eta c > 0$. This in turn ensures that $N^{-1}\tau^2(\pi) \geqq N^{-1}E\tau^2(P) \geqq c^*$, where $c^* = \tilde{c}(1 - \exp\{-2(\delta'' - \delta')^2\}) > 0$.

Thus we have shown that if c, C, δ and ε are replaced by positive numbers c^*, C, $\delta - 2\delta''$ and ε depending only on c, C, δ, δ' and ε, then a and π satisfy (2.26) and the second part of (2.15), whereas a and P satisfy (2.16) and the first part of (2.15) except on a set E with $P(E) \leqq \exp\{-2N(\delta'' - \delta')^2\} = O(N^{-\frac{5}{4}})$. Hence a, P and π satisfy the assumptions of Lemma 2.3 on the complement of E. In dealing with the set E it will suffice to note that $\tilde{R}^*(x, P, \pi)$ is bounded since (2.26) and the second part of (2.15) ensure the boundedness of $\kappa_3(\pi)$, $\kappa_4(\pi)$, $(\tau^2(P) - \tau^2(\pi))/\tau^2(\pi)$ and $\sum |a_j|^3/\tau^3(\pi)$. Of course $R^*(x, P, \pi)$, being a probability, is also bounded.

As

$$P\left(\frac{T - \sum a_j \pi_j}{\tau(\pi)} \leqq x \right) = ER^*(x, P, \pi),$$

the left-hand side of (2.38) is bounded above by

$$E \sup_x |R^*(x, P, \pi) - \tilde{R}^*(x, P, \pi)|. \tag{2.39}$$

Applying Lemma 2.3 on the complement of E and using the boundedness of $|R^*(x, P, \pi) - \tilde{R}^*(x, P, \pi)|$ together with $P(E) = O(N^{-\frac{5}{4}})$ we find that (2.39) is

$$O(N^{-\frac{5}{4}} + N^{-\frac{3}{2}} \sum E(P_j - \pi_j)^2 |a_j|^3 + N^{-3} E|\tau^2(P) - \tau^2(\pi)|^3),$$

where the order symbol is uniform for fixed c, C, δ, δ' and ε. Now

$$N^{-\frac{3}{2}} \sum E(P_j - \pi_j)^2 |a_j|^3 \leqq N^{-\frac{3}{2}} [\sum \{E(P_j - \pi_j)^2\}^{\frac{5}{2}}]^{\frac{2}{5}} (\sum |a_j|^5)^{\frac{3}{5}},$$
$$N^{-3}E|\tau^2(P) - \tau^2(\pi)|^3 \leqq N^{-3}E[\sum |P_j - \pi_j| a_j^2]^3 \leqq N^{-3}[\sum \{E|P_j - \pi_j|^3\}^{\frac{1}{3}} a_j^2]^3$$
$$\leqq N^{-3}[\sum \{E|P_j - \pi_j|^3\}^{\frac{2}{3}}]^{\frac{3}{2}} (\sum a_j^4)^{\frac{3}{2}},$$

and since $\sum |a_j|^5 \leqq (CN)^{\frac{5}{4}}$ and $\sum a_j^4 \leqq CN$, this completes the proof. □

We note that the boundedness of $\tilde{R}^*(x, P, \pi)$ on E plays an important role in the above proof. Because $\tau(P)$ may be arbitrarily small on E, this explains why we had to remove $\tau(p)$ from the denominator of the expansion in Lemma 2.3 by means of (2.30).

Although Theorem 2.2 is formally stated as a result for a fixed, but arbitrary value of N, it is of course meaningless for fixed N because we do not investigate the way in which A depends on c, C, δ, δ' and ε. In fact the theorem is a purely asymptotic result. Let us for a moment indicate dependence on N by a superscript. Thus, for $N = 1, 2, \cdots$, consider the distribution of the statistic $T^{(N)}$ based on a vector of scores $a^{(N)} = (a_1^{(N)}, \cdots, a_N^{(N)})$ when the underlying df is $G^{(N)}$. Fix positive values of c, C, δ, δ' and ε with $\delta' < \min(\delta/2, c^2C^{-1})$. The theorem asserts that if for every N, $a^{(N)}$ and $G^{(N)}$ satisfy (2.35)—(2.37) for these fixed c, C, δ, δ' and ε, then the error of the approximation $E\tilde{R}^*(x, P^{(N)}, \pi^{(N)})$ is

$$O(N^{-\frac{5}{4}} + N^{-\frac{3}{4}}[\sum \{E(P_j^{(N)} - \pi_j^{(N)})^2\}^{\frac{5}{2}}]^{\frac{2}{5}} + N^{-\frac{3}{2}}[\sum \{E|P_j^{(N)} - \pi_j^{(N)}|^3\}^{\frac{2}{3}}]^{\frac{3}{2}})$$

as $N \to \infty$. Moreover, the order of the remainder is uniform for all such sequences $a^{(N)}$, $G^{(N)}$, $N = 1, 2, \cdots$.

Assumption (2.36) may need some clarification. It is clear from the proof of Lemma 2.2 that the role of conditions (2.16) and (2.36) in Theorems 2.1 and 2.2 is to ensure that the a_j do not cluster too much around too few points. Assumption (2.36) is certainly satisfied if for some $k \geqq \delta N/2$, indices $j_1, j_2, \cdots, j_k$ exist such that $a_{j_{i+1}} - a_{j_i} \geqq 2N^{-\frac{3}{2}} \log N$ for $i = 1, \cdots, k-1$. Under condition (2.35) this will typically be the case. Consider for instance the important case $a_j = EJ(U_{j:N})$, where $U_{1:N} < U_{2:N} < \cdots < U_{N:N}$ are order statistics from the uniform distribution on (0, 1) and J is a continuously differentiable, nonconstant function on (0, 1) with $\int J^4 < \infty$. Here both (2.35) and (2.36) are satisfied for all N with fixed c, C and δ. The same is true if $a_j = J(j/(N+1))$ provided that J is monotone near 0 and 1.

For a large class of underlying df's G, the right-hand side of (2.38) is uniformly $o(N^{-1})$. Still Theorem 2.2 does not yet provide an explicit expansion to order N^{-1} for the distribution of T since we are still left with the task of computing the expected value of $\tilde{R}^*(x, P, \pi)$. This is of course a trivial matter under the hypothesis that g is symmetric about zero and, more generally, in the case where, for some $\eta > 0$, $g(x)/g(-x) = \eta$ for all $x > 0$. In this case $P_j = \eta(1+\eta)^{-1}$ with probability 1 for all j and an expansion for the distribution of T is already contained in Theorem 2.1. For fixed alternatives in general, however, the computation of $E\tilde{R}^*(x, P, \pi)$ presents a formidable problem that we shall not attempt to solve here. It would seem that what is needed, is an expansion for the distribution of a linear combination of functions of order statistics.

In the remaining part of this paper we shall restrict attention to sequences of alternatives that are contiguous to the hypothesis. Heuristically the situation is now as follows. Since $g(x)/(g(x) + g(-x)) = \frac{1}{2} + O(N^{-\frac{1}{2}})$, $P_j - \frac{1}{2}$ and $\pi_j - \frac{1}{2}$ will be $O(N^{-\frac{1}{2}})$, whereas $P_j - \pi_j$ will be $O(N^{-1})$ instead of $O(N^{-\frac{1}{2}})$ as before. In the first place this allows us to simplify $E\tilde{R}^*(x, P, \pi)$ considerably as a number of terms may now be relegated to the remainder and functions of π_j may be expanded about the point $\pi_j = \frac{1}{2}$. Much more important, however, is the fact that $U^* = \tau^{-1}(\pi) \sum (P_j - \pi_j)a_j$ will now be $O(N^{-\frac{1}{2}})$ and that we may therefore expand $\tilde{R}^*(x, P, \pi)$ in powers of U^*. This means that we shall be dealing with low moments of linear combinations of functions of order statistics rather than with their distributions. We need hardly point out that a heuristic argument like this can be entirely misleading and that the actual order of the remainder in our expansion will of course have to be investigated. The unduly complicated form of the remainder terms in the preceeding theorem is, of course, preparatory to such further expansion.

Define

$$\tilde{K}(x) = \Phi(x) + \phi(x)\left\{\frac{\sum a_j^2 E(2P_j-1)^2 - 4\sigma^2(\sum a_j P_j)}{2\sum a_j^2}\, x + \frac{\sum a_j^3(2\pi_j - 1)}{3(\sum a_j^2)^{\frac{3}{2}}}(x^2-1) + \frac{\sum a_j^4}{12(\sum a_j^2)^2}(x^3-3x)\right\}, \tag{2.40}$$

where $\sigma^2(Z)$ denotes the variance of a rv Z. Carrying out the type of computation outlined above we arrive at the following simplified version of Theorem 2.2.

THEOREM 2.3. *Theorem* 2.2 *continues to hold if* (2.38) *is replaced by*

$$\sup_x \left| P\left(\frac{2T - \sum a_j}{(\sum a_j^2)^{\frac{1}{2}}} \leq x\right) - \tilde{K}\left(x - \frac{\sum a_j(2\pi_j - 1)}{(\sum a_j^2)^{\frac{1}{2}}}\right)\right| \leq A\{N^{-\frac{5}{4}} + \sum \{E(2P_j-1)^4\}^{\frac{5}{4}} + N^{-\frac{3}{4}}[\sum \{E|P_j - \pi_j|^3\}^{\frac{4}{9}}]^{\frac{9}{4}}\}. \tag{2.41}$$

PROOF. The proof of this theorem becomes somewhat shorter if we use a modification of Theorem 2.2 as a starting point rather than Theorem 2.2 itself. We recall that Theorem 2.2 was proved by an application of Lemma 2.3 for $\tilde{p} = \pi$. However, the proof clearly goes through for any other choice of $\tilde{p}$ that satisfies (2.26). Because of (2.35), we may therefore replace π in (2.38) by a vector $\tilde{p}$ with $\tilde{p}_j = \frac{1}{2}$ for all j. Noting that for this choice of $\tilde{p}$, $\kappa_3(\tilde{p}) = 0$, $\kappa_4(\tilde{p}) = -2N\sum a_j^4/(\sum a_j^2)^2$, $\tau^2(P) - \tau^2(\tilde{p}) = -\frac{1}{4}\sum (2P_j-1)^2 a_j^2$, and adding the last two terms in $\tilde{R}^*(x, P, \tilde{p})$ to the remainder, we obtain

$$\begin{aligned} P\left(\frac{2T - \sum a_j}{(\sum a_j^2)^{\frac{1}{2}}} \leq x\right) &= E\Phi(x - \tilde{U}) + E\phi(x - \tilde{U})\left\{\frac{\sum a_j^4}{12(\sum a_j^2)^2}[(x-\tilde{U})^3 - 3(x-\tilde{U})]\right. \\ &\quad + \frac{\sum a_j^2(2P_j-1)^2}{2\sum a_j^2}(x - \tilde{U}) \\ &\quad \left. + \frac{\sum a_j^3(2P_j-1)}{3(\sum a_j^2)^{\frac{3}{2}}}[(x-\tilde{U})^2 - 1]\right\} \\ &\quad + O(N^{-\frac{5}{4}} + N^{-\frac{3}{4}}[\sum \{E(2P_j-1)^2\}^{\frac{5}{2}}]^{\frac{2}{5}} + N^{-\frac{3}{2}}[\sum \{E|2P_j - 1|^3\}^{\frac{2}{3}}]^{\frac{3}{2}} \\ &\quad + N^{-2}E[\sum a_j^2(2P_j-1)^2]^2 + N^{-\frac{3}{2}}\sum a_j^2 E(2P_j-1)^2), \end{aligned} \tag{2.42}$$

where $\tilde{U} = \sum a_j(2P_j - 1)/(\sum a_j^2)^{\frac{1}{2}}$. All order symbols in this proof are uniform for fixed c, C, δ, δ' and ε. The remainder in (2.42) may be simplified by noting that

$$\begin{aligned} N^{-\frac{3}{4}}[\sum \{E(2P_j-1)^2\}^{\frac{5}{2}}]^{\frac{2}{5}} + N^{-\frac{3}{2}}[\sum \{E|2P_j-1|^3\}^{\frac{2}{3}}]^{\frac{3}{2}} &\leq N^{-\frac{5}{4}} + \sum \{E(2P_j-1)^2\}^{\frac{5}{2}} + N^{-1}\sum E|2P_j-1|^3 \\ &\leq N^{-\frac{5}{4}} + N^{-\frac{3}{2}} + 2\sum \{E(2P_j-1)^4\}^{\frac{5}{4}}, \end{aligned}$$

$$
\begin{aligned}
N^{-\frac{3}{2}}E[\sum a_j^2(2P_j-1)^2]^2 + N^{-\frac{3}{2}}\sum a_j^2 E(2P_j-1)^2
&\leqq 2N^{-\frac{3}{2}}E[\sum a_j^2(2P_j-1)^2]^2 + N^{-\frac{3}{2}} \\
&\leqq 2N^{-\frac{3}{2}}\sum a_j^4 \sum E(2P_j-1)^4 + N^{-\frac{3}{2}} \\
&\leqq 2C\sum \{E(2P_j-1)^4\}^{\frac{5}{4}} + (2C+1)N^{-\frac{3}{2}} .
\end{aligned}
$$

Define $U=\sum a_j(P_j-\pi_j)/(\sum a_j^2)^{\frac{1}{2}}$, so $x-\tilde{U}=x-\sum a_j(2\pi_j-1)/(\sum a_j^2)^{\frac{1}{2}}-2U$. By expanding in powers of U under the expectation sign in (2.42) we find

$$
\begin{aligned}
&P\left(\frac{2T-\sum a_j}{(\sum a_j^2)^{\frac{1}{2}}}\leqq x\right) \\
&\quad = \tilde{K}\left(x-\frac{\sum a_j(2\pi_j-1)}{(\sum a_j^2)^{\frac{1}{2}}}\right) + O(N^{-\frac{5}{4}} + \sum \{E(2P_j-1)^4\}^{\frac{5}{4}} + E|U|^3 \\
&\qquad + E|U|\{N^{-1} + N^{-1}\sum a_j^2(2P_j-1)^2 + N^{-\frac{3}{2}}\sum |a_j|^3|2P_j-1|\}) .
\end{aligned}
\tag{2.43}
$$

Now

$$
\begin{aligned}
N^{-\frac{3}{2}}\sum |a_j|^3|2P_j-1| &\leqq N^{-2}\sum a_j^4 + N^{-1}\sum a_j^2(2P_j-1)^2 , \\
N^{-1}E|U| &\leqq N^{-\frac{3}{2}} + E|U|^3 , \\
N^{-1}E|U|\sum a_j^2(2P_j-1)^2 &\leqq N^{-\frac{1}{2}}EU^2 + N^{-\frac{3}{2}}E[\sum a_j^2(2P_j-1)^2]^2 \\
&\leqq N^{-\frac{3}{2}} + E|U|^3 + C\sum \{E(2P_j-1)^4\}^{\frac{5}{4}} + CN^{-\frac{3}{2}} ,
\end{aligned}
$$

where the last inequality is based on a bound obtained earlier in this proof. It follows that the remainder in (2.43) is of the order of the sum of its first three terms. The proof is completed by noting that

$$
\begin{aligned}
E|U|^3 &\leqq (cN)^{-\frac{3}{2}}E[\sum |a_j||P_j-\pi_j|]^3 \leqq (cN)^{-\frac{3}{2}}[\sum |a_j|\{E|P_j-\pi_j|^3\}^{\frac{1}{3}}]^3 \\
&\leqq (cN)^{-\frac{3}{2}}(\sum a_j^4)^{\frac{3}{4}}[\sum \{E|P_j-\pi_j|^3\}^{\frac{4}{9}}]^{\frac{9}{4}} . \ \square
\end{aligned}
$$

Theorem 2.3 provides the basic expansion for the distribution of T under contiguous alternatives. In Section 3 we shall be concerned with a further simplification of this expansion and a precise evaluation of the order of the remainder term.

3. Contiguous location alternatives.

The analysis in this section will be carried out for contiguous location alternatives rather than for contiguous alternatives in general. The general case can be treated in much the same way as the location case but the conditions as well as the results become more involved. The interested reader is referred to Albers (1974).

Let F be a df with a density f that is positive on R^1, symmetric about zero and four times differentiable with derivatives $f^{(i)}$, $i=1,\cdots,4$. Define functions

$$
\psi_i = \frac{f^{(i)}}{f} , \qquad i=1,\cdots,4 , \tag{3.1}
$$

and suppose that positive numbers ε and C exist such that for

$$
\begin{aligned}
&m_1=6 , \quad m_2=3 , \quad m_3=\tfrac{4}{3} , \quad m_4=1 , \\
&\sup\{\textstyle\int_{-\infty}^{\infty}|\psi_i(x+y)|^{m_i}f(x)\,dx : |y|\leqq\varepsilon\}\leqq C , \qquad i=1,\cdots,4 .
\end{aligned}
\tag{3.2}
$$

Let $X_1, \cdots, X_N$ be i.i.d. with common df $G(x) = F(x-\theta)$ where

$$0 \leqq \theta \leqq CN^{-\frac{1}{2}} \tag{3.3}$$

for some positive C. Note that (3.2) and (3.3) together imply contiguity. Let $0 < Z_1 < Z_2 < \cdots < Z_N$ denote the order statistics of $|X_1|, \cdots, |X_N|$ and let T be defined by (2.2). Probabilities, expected values and variances under G will be denoted by P_θ, E_θ and σ_θ^2; under F they will be indicated by P_0, E_0 and σ_0^2. Define

$$\begin{aligned} K_\theta(x) = \Phi(x) + \phi(x) \bigg\{ &\frac{\sum a_j^4}{12(\sum a_j^2)^2}(x^3-3x) - \theta \frac{\sum a_j^3 E_0 \psi_1(Z_j)}{3(\sum a_j^2)^{\frac{3}{2}}}(x^2-1) \\ &+ \frac{\theta^2}{2\sum a_j^2}[\sum a_j^2 E_0 \psi_1^2(Z_j) - \sigma_0^2(\sum a_j \psi_1(Z_j))]x \\ &+ \frac{\theta^3}{6(\sum a_j^2)^{\frac{1}{2}}} \sum a_j E_0[3\psi_1^3(Z_j) - 6\psi_1(Z_j)\psi_2(Z_j) + \psi_3(Z_j)] \bigg\}, \end{aligned} \tag{3.4}$$

and

$$\eta = -\theta \frac{\sum a_j E_0 \psi_1(Z_j)}{(\sum a_j^2)^{\frac{1}{2}}}. \tag{3.5}$$

We shall show that $K_\theta(x-\eta)$ is an expansion to order N^{-1} for the df of $(2T - \sum a_j)/(\sum a_j^2)^{\frac{1}{2}}$. The expansion will be established in Theorem 3.1 and an evaluation of the order of the remainder will be given in Theorem 3.2.

Let $\pi(\theta)$ denote the power of the one-sided level α test based on T for the hypothesis of symmetry against the alternative $G(x) = F(x-\theta)$. Suppose that for some $\varepsilon > 0$,

$$\varepsilon \leqq \alpha \leqq 1 - \varepsilon. \tag{3.6}$$

We prove that an expansion for $\pi(\theta)$ is given by

$$\tilde{\pi}(\theta) = 1 - K_\theta(u_\alpha - \eta) + \phi(u_\alpha - \eta)\frac{\sum a_j^4}{12(\sum a_j^2)^2}(u_\alpha^3 - 3u_\alpha), \tag{3.7}$$

where $u_\alpha = \Phi^{-1}(1-\alpha)$ denotes the upper α-point of the standard normal distribution.

THEOREM 3.1. *Suppose that positive numbers, c, C, δ and ε exist such that* (2.35), (2.36), (3.2) *and* (3.3) *are satisfied. Then there exists $A > 0$ depending on N, a, F and θ only through c, C, δ and ε and such that*

$$\begin{aligned} \sup_x \left| P_\theta\left(\frac{2T - \sum a_j}{(\sum a_j^2)^{\frac{1}{2}}} \leqq x\right) - K_\theta(x-\eta)\right| \\ \leqq A\{N^{-\frac{5}{4}} + N^{-\frac{3}{4}}\theta^3[\sum \{E_0|\psi_1(Z_j) - E_0\psi_1(Z_j)|^3\}^{\frac{4}{3}}]^{\frac{3}{4}}\}, \end{aligned} \tag{3.8}$$

$$|\eta| \leqq A, \tag{3.9}$$

$$\begin{aligned} &\theta \frac{|\sum a_j^3 E_0 \psi_1(Z_j)|}{(\sum a_j^2)^{\frac{3}{2}}} \leqq AN^{-1}, \qquad \theta^2 \frac{\sum a_j^2 E_0 \psi_1^2(Z_j)}{\sum a_j^2} \leqq AN^{-1}, \\ &\frac{\theta^3}{(\sum a_j^2)^{\frac{1}{2}}} |\sum a_j E_0[3\psi_1^3(Z_j) - 6\psi_1(Z_j)\psi_2(Z_j) + \psi_3(Z_j)]| \leqq AN^{-1}. \end{aligned} \tag{3.10}$$

If, in addition, (3.6) *is satisfied there exists* $A' > 0$ *depending on* N, a, F, θ *and* α *only through* c, C, δ *and* ε *and such that*

$$|\pi(\theta) - \tilde{\pi}(\theta)| \leqq A'\{N^{-\frac{5}{4}} + N^{-\frac{3}{4}}\theta^3[\sum \{E_0|\psi_1(Z_j) - E_0\psi_1(Z_j)|^3\}^{\frac{4}{9}}]^{\frac{9}{4}}\} . \tag{3.11}$$

PROOF. We begin by checking assumption (2.37). One easily verifies that

$$\left|\frac{\partial}{\partial\theta}\frac{f(x-\theta) - f(x+\theta)}{f(x-\theta) + f(x+\theta)}\right| \leqq \tfrac{1}{2}|\psi_1(x-\theta)| + \tfrac{1}{2}|\psi_1(x+\theta)| .$$

Hence the symmetry of f and an application of Markov's inequality and Fubini's theorem yield

$$\begin{aligned} P_\theta\left(\varepsilon \leqq \frac{g(X_1)}{g(X_1) + g(-X_1)} \leqq 1 - \varepsilon\right) &= P_\theta\left(\left|\frac{f(X_1-\theta) - f(X_1+\theta)}{f(X_1-\theta) + f(X_1+\theta)}\right| \leqq 1 - 2\varepsilon\right) \\ &\geqq P_\theta\left(\textstyle\int_0^\theta \{|\psi_1(X_1 - t)| + |\psi_1(X_1 + t)|\}\, dt \leqq 2(1-2\varepsilon)\right) \\ &\geqq 1 - \frac{1}{2(1-2\varepsilon)} E_\theta \int_0^\theta \{|\psi_1(X_1 - t)| + |\psi_1(X_1+t)|\}\, dt \\ &\geqq 1 - \frac{\theta}{1-2\varepsilon} \sup_{|t|\leqq\theta} E_\theta|\psi_1(X_1 + t)| . \end{aligned}$$

Take $\varepsilon < \frac{1}{2}$ and choose $\delta' = \frac{1}{2}\min(\delta/2, c^2C^{-1})$. Because of (3.3) there exists $N_0 > 0$ depending only on c, C, δ and ε such that for $N \geqq N_0$, $2\theta \leqq \varepsilon$ and $\theta \leqq (1-2\varepsilon)C^{-\frac{1}{3}}\delta'$. Then (3.2) implies that (2.37) is satisfied for $N \geqq N_0$. This is of course sufficient to ensure that the conclusion of Theorem 2.3 holds.

The passage from (2.41) to (3.8) is achieved by Taylor expansion with respect to θ. Since this part of the proof is highly technical and laborious it will not be given in the body of the text. Instead we refer the interested reader to Appendix 1 where the results we shall need are stated in Corollary A1.1. Using parts (A1.27), (A1.31) and (A1.32) of Corollary A1.1 together with the inequality $\sum \{E_\theta(2P_j - 1)^4\}^{\frac{5}{4}} \leqq \sum E_\theta|2P_j - 1|^5$ we see that the left-hand side of (3.8) is bounded by the right-hand side of (3.8) plus a term

$$O(\theta^{\frac{19}{8}}\{E_0|\sum a_j(\psi_1(Z_j) - E_0\psi_1(Z_j))|^3\}^{\frac{1}{3}} + N^{-\frac{1}{2}}\theta^5\sigma_0^{\,2}(\sum a_j\psi_1(Z_j))) . \tag{3.12}$$

Here, and later in this proof all order symbols are uniform for fixed c, C, δ and ε. Now

$$\begin{aligned} \theta^{\frac{19}{8}}\{E_0|\sum a_j(\psi_1(Z_j) - E_0\psi_1(Z_j))|^3\}^{\frac{1}{3}} &+ N^{-\frac{1}{2}}\theta^5\sigma_0^{\,2}(\sum a_j\psi_1(Z_j)) \\ &\leqq \theta^{\frac{27}{16}} + \theta^6 E_0|\sum a_j(\psi_1(Z_j) - E_0\psi_1(Z_j))|^3 \\ &\quad + N^{-\frac{1}{2}}\theta^3 + N^{-\frac{1}{2}}\theta^6\sigma_0^{\,3}(\sum a_j\psi_1(Z_j)) \\ &= O(N^{-\frac{5}{4}} + N^{-\frac{3}{2}}\theta^3 E_0|\sum a_j(\psi_1(Z_j) - E_0\psi_1(Z_j))|^3) , \end{aligned}$$

$$\begin{aligned} E_0|\sum a_j(\psi_1(Z_j) - E_0\psi_1(Z_j))|^3 &\leqq [\sum |a_j|\{E_0|\psi_1(Z_j) - E_0\psi_1(Z_j)|^3\}^{\frac{1}{3}}]^3 \\ &\leqq (CN)^{\frac{3}{4}}[\sum \{E_0|\psi_1(Z_j) - E_0\psi_1(Z_j)|^3\}^{\frac{4}{9}}]^{\frac{9}{4}} , \end{aligned}$$

which proves (3.8). In view of (2.35) and (3.3) it is clear that (3.9) and (3.10) are merely restating parts (A1.28)—(A1.30) of Corollary A1.1.

The one-sided level α test based on T rejects the hypothesis if $(2T - \sum a_j)(\sum a_j^2)^{-\frac{1}{2}} \geqq \xi_\alpha$ with possible randomization if equality occurs. Taking $\theta = 0$ in (3.8) we find that

$$1 - \Phi(\xi_\alpha) - \phi(\xi_\alpha) \frac{\sum a_j^4}{12(\sum a_j^2)^2} (\xi_\alpha^3 - 3\xi_\alpha) = \alpha + O(N^{-\frac{5}{4}}) ,$$

and hence because of (2.35) and (3.6),

$$\xi_\alpha = u_\alpha - \frac{\sum a_j^4}{12(\sum a_j^2)^2} (u_\alpha^3 - 3u_\alpha) + O(N^{-\frac{5}{4}}) . \tag{3.13}$$

The power of this test against the alternative $F(x - \theta)$ is

$$\pi(\theta) = 1 - K_\theta(\xi_\alpha - \eta) + O(N^{-\frac{5}{4}} + N^{-\frac{3}{4}}\theta^3[\sum \{E_0|\psi_1(Z_j) - E_0\psi_1(Z_j)|^3\}^{\frac{4}{3}}]^{\frac{9}{4}}) . \tag{3.14}$$

In (3.14) we expand $K_\theta(\xi_\alpha - \eta)$ around $u_\alpha - \eta$. Noting that $|\xi_\alpha - u_\alpha| = O(N^{-1})$ and using (2.35) and (3.10) we arrive at the conclusion that the left-hand side of (3.11) is bounded by the right-hand side of (3.11) plus a term

$$O(N^{-2}\theta^2\sigma_0^2(\sum a_j\psi_1(Z_j))) = O(N^{-3} + N^{-\frac{3}{2}}\theta^3 E_0|\sum a_j(\psi_1(Z_j) - E_0\psi_1(Z_j))|^3) .$$

As we have already shown earlier in this proof that such a term does not change the order of the remainder in (3.11), the proof of Theorem 3.1 is completed. □

For $i = 1, 2, 3$, define functions Ψ_i on (0, 1) by

$$\Psi_i(t) = \psi_i\left(F^{-1}\left(\frac{1+t}{2}\right)\right) = \frac{f^{(i)}\left(F^{-1}\left(\frac{1+t}{2}\right)\right)}{f\left(F^{-1}\left(\frac{1+t}{2}\right)\right)} . \tag{3.15}$$

THEOREM 3.2. *Suppose that positive numbers C and δ exist such that* (3.3) *is satisfied and that* $|\Psi_1'(t)| \leqq C(t(1-t))^{-\frac{4}{3}+\delta}$ *for all* $0 < t < 1$. *Then there exists* $A'' > 0$ *depending on N, F and θ only through C and δ and such that*

$$N^{-\frac{3}{4}}\theta^3[\sum \{E_0|\psi_1(Z_j) - E_0\psi_1(Z_j)|^3\}^{\frac{4}{3}}]^{\frac{9}{4}} \leqq A''N^{-\frac{5}{4}} .$$

For the highly technical proof of this result the reader is referred to Appendix 2. Theorem 3.2 follows at once from Corollary A2.1 in this appendix by taking $h = \Psi_1$.

4. Exact and approximate scores. The expansions given in Section 3 can be simplified further if we make certain smoothness assumptions about the scores a_j. Consider a continuous function J on (0, 1) and let $U_{1:N} < U_{2:N} < \cdots < U_{N:N}$ denote order statistics of a sample of size N from the uniform distribution on (0, 1). For $N = 1, 2, \cdots$ we define the exact scores generated by J by

$$a_j = a_{j,N} = EJ(U_{j:N}) , \qquad j = 1, \cdots, N , \tag{4.1}$$

and the approximate scores generated by J by

$$a_j = a_{j,N} = J\left(\frac{j}{N+1}\right), \qquad j = 1, \cdots, N. \tag{4.2}$$

For almost all well-known linear rank tests the scores are of one of these two types. The locally most powerful rank test against location alternatives of type F is based on exact scores generated by the function $-\Psi_1$, where Ψ_1 is defined in (3.15).

So far, we have systematically kept the order of the remainder in our expansions down to $O(N^{-\frac{3}{2}})$. From this point on, however, we shall be content with a remainder that is $o(N^{-1})$, because otherwise we would have to impose rather restrictive conditions. In the previous sections we have also consistently stressed the fact that the remainder depends on a and F only through certain constants occurring in our conditions, thus in effect indicating classes of scores and distributions for which the expansion holds uniformly. As the number of these constants is becoming rather large, we prefer to formulate our results from here on for a fixed score function J and a fixed df F. The reader can easily construct uniformity classes for himself by using the results of Section 3 and tracing the development of Appendix 2.

DEFINITION 4.1. $\mathscr{J}$ is the class of functions J on (0, 1) that are twice continuously differentiable and nonconstant on (0, 1), and satisfy

$$\textstyle\int_0^1 J^4(t)\,dt < \infty . \tag{4.3}$$

$$\limsup_{t\to 0,1} t(1-t)\left|\frac{J''(t)}{J'(t)}\right| < \tfrac{3}{2}. \tag{4.4}$$

$\mathscr{F}$ is the class of df's F on R^1 with positive densities f that are symmetric about zero, four times differentiable and such that, for $\psi_i = f^{(i)}/f$, $\Psi_i(t) = \psi_i(F^{-1}((1+t)/2))$, $m_1 = 6$, $m_2 = 3$, $m_3 = \frac{4}{3}$, $m_4 = 1$,

$$\limsup_{y\to 0} \textstyle\int_{-\infty}^{\infty} |\psi_i(x+y)|^{m_i} f(x)\,dx < \infty , \qquad i = 1, \cdots, 4, \tag{4.5}$$

$$\limsup_{t\to 0,1} t(1-t)\left|\frac{\Psi_1''(t)}{\Psi_1'(t)}\right| < \tfrac{3}{2}. \tag{4.6}$$

For $J \in \mathscr{J}$ and $F \in \mathscr{F}$, let

$$\begin{aligned}\tilde{K}_\theta(x) = \Phi(x) + \phi(x)\Big\{& N^{-1}\frac{\int_0^1 J^4(t)\,dt}{12(\int_0^1 J^2(t)\,dt)^2}(x^3-3x) \\ &- N^{-\frac{1}{2}}\theta\frac{\int_0^1 J^3(t)\Psi_1(t)\,dt}{3(\int_0^1 J^2(t)\,dt)^{\frac{3}{2}}}(x^2-1) + \frac{\theta^2}{2\int_0^1 J^2(t)\,dt} \\ &\times [\textstyle\int_0^1 J^2(t)\Psi_1^2(t)\,dt - \int_0^1\int_0^1 J(s)\Psi_1'(s)J(t)\Psi_1'(t)(s\wedge t - st)\,ds\,dt]x \\ &+ \frac{N^{\frac{1}{2}}\theta^3}{6(\int_0^1 J^2(t)\,dt)^{\frac{1}{2}}}\textstyle\int_0^1 J(t)[3\Psi_1^3(t) - 6\Psi_1(t)\Psi_2(t) + \Psi_3(t)]\,dt\Big\},\end{aligned} \tag{4.7}$$

$$(4.8)\qquad K_{\theta,1}(x) = \tilde{K}_\theta(x) + \phi(x)\frac{N^{-\frac{1}{2}}\theta}{2(\int_0^1 J^2(t)\,dt)^{\frac{1}{2}}}\left\{\frac{\int_0^1 J(t)\Psi_1(t)\,dt}{\int_0^1 J^2(t)\,dt}\sum_{j=1}^N \sigma^2(J(U_{j:N})) - 2\sum_{j=1}^N \operatorname{Cov}(J(U_{j:N}), \Psi_1(U_{j:N}))\right\},$$

$$(4.9)\qquad K_{\theta,2}(x) = \tilde{K}_\theta(x) + \phi(x)\frac{N^{-\frac{1}{2}}\theta}{2(\int_0^1 J^2(t)\,dt)^{\frac{1}{2}}}\left\{\frac{\int_0^1 J(t)\Psi_1(t)\,dt}{\int_0^1 J^2(t)\,dt}\int_{1/N}^{1-1/N}(J'(t))^2 t(1-t)\,dt - 2\int_{1/N}^{1-1/N} J'(t)\Psi_1'(t)t(1-t)\,dt\right\},$$

$$(4.10)\qquad \tilde{\eta} = -N^{\frac{1}{2}}\theta\frac{\int_0^1 J(t)\Psi_1(t)\,dt}{(\int_0^1 J^2(t)\,dt)^{\frac{1}{2}}},$$

$$(4.11)\qquad \pi_i(\theta) = 1 - K_{\theta,i}(u_\alpha - \tilde{\eta}) + \phi(u_\alpha - \tilde{\eta})N^{-1}\frac{\int_0^1 J^4(t)\,dt}{12(\int_0^1 J^2(t)\,dt)^2}(u_\alpha^3 - 3u_\alpha),$$

for $i = 1, 2$. Then, in the notation of Section 3, we have for contiguous location alternatives and exact scores

THEOREM 4.1. *Let $F \in \mathscr{F}$, $J \in \mathscr{J}$, $a_j = EJ(U_{j:N})$ for $j = 1, \cdots, N$, and let $0 \leq \theta \leq CN^{-\frac{1}{2}}$, $\varepsilon \leq \alpha \leq 1 - \varepsilon$ for positive C and ε. Then, for every fixed J, F, C and ε, there exist positive numbers A, $\delta_1, \delta_2, \cdots$ such that $\lim_{N\to\infty}\delta_N = 0$ and for every N*

$$(4.12)\qquad \sup_x\left|P_\theta\left(\frac{2T - \sum a_j}{(\sum a_j^2)^{\frac{1}{2}}} \leq x\right) - K_{\theta,1}(x - \tilde{\eta})\right| \leq \delta_N N^{-1},$$

$$(4.13)\qquad \sup_x\left|P_\theta\left(\frac{2T - \sum a_j}{(\sum a_j^2)^{\frac{1}{2}}} \leq x\right) - K_{\theta,2}(x - \tilde{\eta})\right| \leq \delta_N N^{-1} + AN^{-\frac{3}{2}}\int_{1/N}^{1-1/N}|J'(t)|(|J'(t)| + |\Psi_1'(t)|)(t(1-t))^{\frac{1}{2}}\,dt,$$

$$(4.14)\qquad |\pi(\theta) - \pi_1(\theta)| \leq \delta_N N^{-1}$$

$$(4.15)\qquad |\pi(\theta) - \pi_2(\theta)| \leq \delta_N N^{-1} + AN^{-\frac{3}{2}}\int_{1/N}^{1-1/N}|J'(t)|(|J'(t)| + |\Psi_1'(t)|)(t(1-t))^{\frac{1}{2}}\,dt.$$

PROOF. For fixed $J \in \mathscr{J}$, positive constants c, C and δ exist for which (2.35) and (2.36) hold for all N (cf. one of the remarks following the proof of Theorem 2.2). Similarly, for fixed $F \in \mathscr{F}$, (3.2) is satisfied and it follows that the conclusions of Theorem 3.1 hold with A and A' depending only on F, J, C and ε. Also (4.5) ensures that Ψ_1^6 is summable and together with (4.6) and the second part of Corollary A2.1, this implies that the conclusion of Theorem 3.2 holds with A'' depending only on F and C.

To complete the proof we now apply the results collected in Corollary A2.2 to the expansions $K_\theta(x - \eta)$ and $\tilde{\pi}(\theta)$ in Theorem 3.1 and then expand these functions of η around the point $\eta = \tilde{\eta}$, while noting that $\eta - \tilde{\eta} = o(N^{-\frac{1}{2}})$ by (A2.22) and (A2.23). □

In general, the expansions given in Theorem 4.1 will not hold if the exact scores are replaced by approximate scores $a_j = J(j/(N+1))$, because $\eta - \tilde{\eta}$ will then give rise to a different term of order N^{-1}. If $J = -\Psi_1$, however, it is clear from Corollary A2.2 and the proof of Theorem 4.1 that expansions (4.13) and (4.15) are valid for approximate as well as exact scores. Also for $J = -\Psi_1$, these expansions may be simplified because $F \in \mathscr{F}$ implies that by partial integration

$$\int_0^1 \int_0^1 \Psi_1(s)\Psi_1'(s)\Psi_1(t)\Psi_1'(t)(s \wedge t - st)\, ds\, dt = \tfrac{1}{4} \int_0^1 \Psi_1^4(t)\, dt - \tfrac{1}{4}\left(\int_0^1 \Psi_1^2(t)\, dt\right)^2 ,$$
$$\int_0^1 \Psi_1(t)[6\Psi_1(t)\Psi_2(t) - \Psi_3(t)]\, dt = \tfrac{10}{3} \int_0^1 \Psi_1^4(t)\, dt + \int_0^1 \Psi_2^2(t)\, dt .$$

It follows that in this case $\tilde{\eta}$, $K_{\theta,2}(x - \tilde{\eta})$ and $\pi_2(\theta)$ reduce to

$$\eta^* = N^{\frac{1}{2}}\theta\left(\int_0^1 \Psi_1^2(t)\, dt\right)^{\frac{1}{2}} , \tag{4.16}$$

$$\begin{aligned} L_\theta(x) = \Phi(x - \eta^*) &+ \frac{\phi(x-\eta^*)}{72N} \\ &\times \left\{ \frac{\int_0^1 \Psi_1^4(t)\, dt}{(\int_0^1 \Psi_1^2(t)\, dt)^2} [6(x^3 - 3x) + 6\eta^*(x^2 - 1) - 3\eta^{*2}x - 5\eta^{*3}] \right. \\ &+ \frac{12 \int_0^1 \Psi_2^2(t)\, dt}{(\int_0^1 \Psi_1^2(t)\, dt)^2} \eta^{*3} + 9\eta^{*2}(x - \eta^*) \\ &+ \left. \frac{36 \int_{1/N}^{1-1/N} (\Psi_1'(t))^2 t(1-t)\, dt}{\int_0^1 \Psi_1^2(t)\, dt} \eta^* \right\} , \end{aligned} \tag{4.17}$$

$$\begin{aligned} \pi^*(\theta) = 1 - \Phi(u_\alpha - \eta^*) &+ \frac{\eta^*\phi(u_\alpha - \eta^*)}{72N} \\ &\times \left\{ \frac{\int_0^1 \Psi_1^4(t)\, dt}{(\int_0^1 \Psi_1^2(t)\, dt)^2} [-6(u_\alpha^2 - 1) + 3\eta^* u_\alpha + 5\eta^{*2}] \right. \\ &- \frac{12 \int_0^1 \Psi_2^2(t)\, dt}{(\int_0^1 \Psi_1^2(t)\, dt)^2} \eta^{*2} - 9\eta^*(u_\alpha - \eta^*) \\ &- \left. \frac{36 \int_{1/N}^{1-1/N} (\Psi_1'(t))^2 t(1-t)\, dt}{\int_0^1 \Psi_1^2(t)\, dt} \right\} . \end{aligned} \tag{4.18}$$

Finally we note that for $F \in \mathscr{F}$, $-\Psi_1$ can not be constant on (0, 1) because the density $f(x) = \frac{1}{2}\lambda e^{-\lambda|x|}$ of the double exponential distribution is not differentiable at zero. It follows that $-\Psi_1 \in \mathscr{J}$ for every $F \in \mathscr{F}$. We have proved

THEOREM 4.2. *Let $F \in \mathscr{F}$ and let either $a_j = -E\Psi_1(U_{j:N})$ for $j = 1, \cdots, N$ or $a_j = -\Psi_1(j/(N+1))$ for $j = 1, \cdots, N$. Suppose that $0 \leqq \theta \leqq CN^{-\frac{1}{2}}$ and $\varepsilon \leqq \alpha \leqq 1 - \varepsilon$ for positive C and ε. Then, for every fixed F, C and ε, there exist positive numbers $A, \delta_1, \delta_2, \cdots$ such that $\lim_{N\to\infty} \delta_N = 0$ and for every N*

$$\sup_x \left| P_\theta\left(\frac{2T - \sum a_j}{(\sum a_j^2)^{\frac{1}{2}}} \leqq x\right) - L_\theta(x) \right| \leqq \delta_N N^{-1} + AN^{-\frac{3}{2}} \int_{1/N}^{1-1/N} (\Psi_1'(t))^2 (t(1-t))^{\frac{1}{2}}\, dt , \tag{4.19}$$

$$|\pi(\theta) - \pi^*(\theta)| \leqq \delta_N N^{-1} + AN^{-\frac{3}{2}} \int_{1/N}^{1-1/N} (\Psi_1'(t))^2 (t(1-t))^{\frac{1}{2}}\, dt . \tag{4.20}$$

At this point it may be useful to make some remarks concerning the assumptions in Theorems 4.1 and 4.2. Conditions (4.4) and (4.6) ensure that J' and Ψ_1' do not oscillate too wildly near 0 and 1. They also limit the growth of these functions near 0 and 1, but in this respect conditions (4.3) and (4.5) for $i = 1$ are typically much stronger. Together with (4.4) and (4.6) they imply that $J'(t) = o((t(1-t))^{-\frac{5}{4}})$ and $\Psi_1'(t) = o((t(1-t))^{-\frac{7}{6}})$ near 0 and 1 (cf. the proof of Corollary A2.1).

For expansions (4.13), (4.15), (4.19) and (4.20) to be meaningful rather than just formally correct, even stronger growth conditions have to be imposed. Consider, for example, expansion (4.20) and suppose, as is typically the case, that Ψ_1' remains bounded near 0. If $\Psi_1'(t) = o((1-t)^{-1})$ near 1, then the right-hand side in (4.20) is $o(N^{-1})$ and the expansion makes sense. However, if $\Psi_1'(t)$ is of exact order $(1-t)^{-1}$, the expansion reduces to

$$\pi(\theta) = 1 - \Phi(u_\alpha - \eta^*) - \frac{\eta^*\phi(u_\alpha - \eta^*)}{2N} \frac{\int_0^{1-1/N} (\Psi_1'(t))^2 t(1-t)\,dt}{\int_0^1 \Psi_1^2(t)\,dt} + O(N^{-1}).$$

Finally, if $\Psi_1'(t) \sim (1-t)^{-1-\delta}$ for $t \to 1$ and some $0 < \delta < \frac{1}{6}$, then all we have left in (4.20) is $\pi(\theta) = 1 - \Phi(u_\alpha - \eta^*) + O(N^{-1+2\delta})$. Of course, in these cases too, more exact results can be obtained by paying careful attention to the behavior of the extreme order statistics.

We conclude this section with a few applications of Theorems 4.1 and 4.2. The tedious computations will be omitted. First we consider the power $\pi_{W,N}(\theta)$ and $\pi_{W,L}(\theta)$ of Wilcoxon's signed rank test (W) against normal (N) and logistic (L) location alternatives $G(x) = \Phi(x - \theta)$ and $G(x) = (1 + \exp\{-(x-\theta)\})^{-1}$ respectively, where $\theta = O(N^{-\frac{1}{2}})$. We find

$$\begin{aligned}\pi_{W,N}(\theta) = 1 - \Phi(u_\alpha - \tilde\eta) - \frac{\tilde\eta\phi(u_\alpha - \tilde\eta)}{N}\Big\{&\tfrac{26}{5} - 2^{\frac{1}{2}} - \tfrac{69}{20}u_\alpha^2 \\ &+ \Big(\tfrac{169}{20} - \frac{2(3)^{\frac{1}{2}}}{3}\Big)u_\alpha\tilde\eta - \Big(\tfrac{103}{20} - \frac{2(3)^{\frac{1}{2}}}{3} - \frac{\pi}{9}\Big)\tilde\eta^2 \\ &+ \frac{12 \operatorname{arctn} 2^{\frac{1}{2}}}{\pi}(-1 + u_\alpha^2 - 2u_\alpha\tilde\eta + \tilde\eta^2)\Big\} + o(N^{-1}),\end{aligned} \tag{4.21}$$

where $\tilde\eta = (3N/\pi)^{\frac{1}{2}}\theta$, and

$$\begin{aligned}\pi_{W,L}(\theta) = 1 - \Phi(u_\alpha - \eta^*) - \frac{\eta^*\phi(u_\alpha - \eta^*)}{20N}\{2 + 3u_\alpha^2 + u_\alpha\eta^* + \eta^{*2}\} \\ + o(N^{-1}),\end{aligned} \tag{4.22}$$

where $\eta^* = (N/3)^{\frac{1}{2}}\theta$.

As a second example we consider the one-sample normal scores test which is based on the scores $a_j = E\Phi^{-1}((1 + U_{j:N})/2)$. Its power $\pi_{NS,N}(\theta)$ and $\pi_{NS,L}(\theta)$ against the normal and logistic location alternatives described above satisfies

$$\begin{aligned}\pi_{NS,N}(\theta) = 1 - \Phi(u_\alpha - \eta^*) - \frac{\eta^*\phi(u_\alpha - \eta^*)}{4N}\Big\{&-1 + u_\alpha^2 \\ &+ 2\int_0^{\Phi^{-1}(1-1/2N)} \frac{(2\Phi(x) - 1)(1 - \Phi(x))}{\phi(x)}\,dx\Big\} + o(N^{-1}),\end{aligned} \tag{4.23}$$

where now $\eta^* = N^{\frac{1}{2}}\theta$, and

$$\begin{aligned}\pi_{NS,L}(\theta) = 1 - \Phi(u_\alpha - \tilde{\eta}) \\ - \frac{\tilde{\eta}\phi(u_\alpha - \tilde{\eta})}{12N}\Big\{23 - 12(2)^{\frac{1}{2}} + u_\alpha^2 + (2\pi - 5)u_\alpha\tilde{\eta} \\ + (72 \operatorname{arctn} 2^{\frac{1}{2}} - 22\pi + 1)\tilde{\eta}^2 \\ - 6 \int_0^{\Phi^{-1}(1-1/2N)} \frac{(2\Phi(x) - 1)(1 - \Phi(x))}{\phi(x)}\, dx\Big\} + o(N^{-1}),\end{aligned} \tag{4.24}$$

where now $\tilde{\eta} = (N/\pi)^{\frac{1}{2}}\theta$. We note that Theorem 4.2 ensures that (4.23) will also hold for van der Waerden's one-sample test which is based on the approximate scores $a_j = \Phi^{-1}((N + j + 1)/2(N + 1))$. To evaluate the integral in (4.23) and (4.24) we write

$$\begin{aligned}\int_0^{\Phi^{-1}(1-1/2N)} \frac{(2\Phi(x) - 1)(1 - \Phi(x))}{\phi(x)}\, dx \\ = \tfrac{1}{2} \log \log N + \tfrac{1}{2} \log 2 - 2 \int_0^\infty \log x\, \phi(x)\, dx \\ + \int_0^\infty \frac{(2\Phi(x) - 1)\{x(1 - \Phi(x)) - \phi(x)\}}{x\phi(x)}\, dx + o(1) \\ = \tfrac{1}{2} \log \log N + \tfrac{1}{2} \log 2 + 0.05832\cdots + o(1),\end{aligned} \tag{4.25}$$

where the final result is obtained by numerical integration.

5. Permutation tests. In this section we consider distribution free tests other than rank tests, viz. permutation tests. We limit our discussion to linear permutation tests that reject the hypothesis of symmetry if

$$\sum_{i=1}^N h(X_i) \geqq \xi_\alpha(Z) \tag{5.1}$$

with possible randomization if equality occurs. Here h is a function on R^1, $Z = (Z_1, \cdots, Z_N)$ denotes the vector of order statistics of $|X_1|, \cdots, |X_N|$ as before and ξ_α is chosen in such a way that under the hypothesis of symmetry

$$P(\sum_{i=1}^N h(X_i) \geqq \xi_\alpha(Z) \mid Z) = \alpha \quad \text{a.s.} \tag{5.2}$$

with an obvious modification if there is randomization.

Since (5.1) is equivalent to $\sum \{h(X_i) - h(-X_i)\} \geqq 2\xi_\alpha(Z) - \sum \{h(Z_j) + h(-Z_j)\}$, we assume without loss of generality that h is antisymmetric about the origin, i.e.

$$h(x) = -h(-x) \quad \text{for all } x. \tag{5.3}$$

But then, under G and conditional on Z, $\sum h(X_i)$ is distributed as $2 \sum a_j(V_j - \frac{1}{2})$ with V_j as in (2.3) and $a_j = h(Z_j)$. This means that we can obtain an expansion for this conditional distribution of $\sum h(X_i)$ if we can apply Theorem 2.1.

Under the hypothesis of symmetry, $P_j = \frac{1}{2}$ in (2.3) for all j. Hence in this case Theorem 2.1 yields an expansion for the conditional df of $\sum h(X_i)/(\sum h^2(Z_j))^{\frac{1}{2}}$ that holds uniformly on the set of all values of Z for which the $a_j = h(Z_j)$ satisfy

(2.35) and (2.36) for fixed c, C and δ. If α satisfies (3.6), this immediately leads to an expansion for $\xi_\alpha(Z)$. We find (cf. (3.13))

$$\frac{\xi_\alpha(Z)}{(\sum h^2(Z_j))^{\frac{1}{2}}} = u_\alpha - \frac{\sum h^4(Z_j)}{12(\sum h^2(Z_j))^2}(u_\alpha^3 - 3u_\alpha) + O(N^{-\frac{5}{4}}) \tag{5.4}$$

uniformly on the set E_0^c where, for fixed positive c, C and δ, $\sum h^2(Z_j) \geqq cN$, $\sum h^4(Z_j) \leqq CN$ and $\lambda\{x \mid \exists_j |x - h(Z_j)| < \zeta\} \geqq \delta N\zeta$ for some $\zeta \geqq N^{-\frac{3}{2}} \log N$.

Next we consider the contiguous location alternatives $G(x) = F(x - \theta)$ of Section 3. Under these alternatives, Theorem 2.1 yields an expansion for the conditional df of $\frac{1}{2}\{\sum h(X_i) - \sum (2P_j - 1)h(Z_j)\}/\{\sum P_j(1 - P_j)h^2(Z_j)\}^{\frac{1}{2}}$ uniformly on the set E_θ^c where, for fixed positive c, C and δ, $\sum P_j(1 - P_j)h^2(Z_j) \geqq cN$, $\sum h^4(Z_j) \leqq CN$ and $\lambda\{x \mid \exists_j |x - h(Z_j)| < \zeta, \varepsilon \leqq P_j \leqq 1 - \varepsilon\} \geqq \delta N\zeta$ for some $\zeta \geqq N^{-\frac{3}{2}} \log N$.

Since $E_0 \subset E_\theta$ it suffices to show that $P_\theta(E_\theta) = O(N^{-\frac{5}{4}})$ in order to obtain an expansion to $O(N^{-\frac{5}{4}})$ for the conditional power given Z of the permutation test. The unconditional power is then obtained by taking the expectation. This is done in very much the same way as in Sections 2 and 3 for linear rank tests, the only difference being that now not only the P_j but also the a_j depend on Z.

This program is carried out in Albers (1974) for the special case of the locally most powerful permutation test where $h = -\psi_1 = -f'/f$. In Theorem 5.1 we reproduce a version of this result without further proof. Of course a similar result may be obtained for the general linear permutation test (5.1) with $h \neq -\psi_1$.

Suppose that F is a df with a density f that is positive, symmetric about zero and five times differentiable. Define ψ_i and Ψ_i by (3.1) and (3.15) and take $h = -\psi_1$. Let $\pi_P(\theta)$ be the power of the permutation test (5.1) against the alternative $F(x - \theta)$ and define

$$\begin{aligned}\pi_P^*(\theta) = 1 &- \Phi(u_\alpha - \eta^*) \\ &+ \frac{\eta^*\phi(u_\alpha - \eta^*)}{72N}\left\{\frac{\int_0^1 \Psi_1^4(t)\,dt}{(\int_0^1 \Psi_1^2(t)\,dt)^2}[-6u_\alpha^2 - 3 + 3u_\alpha\eta^* + 5\eta^{*2}]\right. \\ &\left. - \frac{12\int_0^1 \Psi_2^2(t)\,dt}{(\int_0^1 \Psi_1^2(t)\,dt)^2}\eta^{*2} + 9(1 - u_\alpha\eta^* + \eta^{*2})\right\},\end{aligned} \tag{5.5}$$

where η^* is given by (4.16).

THEOREM 5.1. *Let F satisfy (4.5) for $i = 1, \cdots, 5$ and $m_1 = 10$, $m_2 = \frac{5}{2}$, $m_3 = \frac{5}{3}$, $m_4 = \frac{5}{4}$, $m_5 = 1$ and suppose that positive numbers C and ε exist such that $0 \leqq \theta \leqq CN^{-\frac{1}{2}}$ and $\varepsilon \leqq \alpha \leqq 1 - \varepsilon$. Take $h = -\psi_1$. Then there exists $A > 0$ depending on N, F, θ and α only through F, C and ε and such that*

$$|\pi_P(\theta) - \pi_P^*(\theta)| \leqq AN^{-\frac{5}{4}}.$$

For $F = \Phi$, we have $-\psi_1(x) = x$ and Theorem 5.1 provides an expansion for the power of the permutation test based on $\sum X_i$ against normal shift alternatives

$\Phi(x - \theta)$ with $0 \leqq \theta \leqq CN^{-\frac{1}{2}}$ and $\varepsilon \leqq \alpha \leqq 1 - \varepsilon$. We find that this power equals

$$1 - \Phi(u_\alpha - N^{\frac{1}{2}}\theta) - \frac{\theta u_\alpha^2 \phi(u_\alpha - N^{\frac{1}{2}}\theta)}{4N^{\frac{1}{2}}} + O(N^{-\frac{5}{4}}) . \tag{5.6}$$

But (5.6) is also the power of Student's one-sided one-sample test for Φ against $\Phi(x - \theta)$ (cf. Hodges and Lehmann (1970)). It follows that for testing the hypothesis Φ against contiguous normal shift alternatives for fixed $0 < \alpha < 1$, the powers of the permutation test based on $\sum X_i$ and of Student's test differ by only $O(N^{-\frac{5}{4}})$ as $N \to \infty$. In fact, this difference is $O(N^{-\frac{3}{2}})$, since Φ satisfies the stronger regularity conditions needed to replace $N^{-\frac{5}{4}}$ by $N^{-\frac{3}{2}}$ in Theorem 5.1.

The remainder of this section will be devoted to a further investigation of this rather striking phenomenon. Roughly speaking, we shall show that for testing any given symmetric distribution against near alternatives, the permutation test (5.1) is almost equivalent to Student's test applied to $h(X_1), \cdots, h(X_N)$ with the correct level of significance for the given null-distribution. Our proof differs from the one outlined above in that we do not use power expansions to establish the near equivalence of the two tests. Instead, we show that the critical regions of the tests are almost identical. This more direct approach has the additional advantage of providing a simple explanation of our result.

Let F be the df of a distribution that is symmetric about zero and consider the problem of testing the hypothesis that $X_1, \cdots, X_N$ have df F against the alternative that they have another df G. For this testing problem and an arbitrary h satisfying (5.3) we compare the permutation test (5.1) with Student's test applied to $h(X_1), \cdots, h(X_N)$ that rejects the hypothesis if

$$\tilde{T} = \frac{\sum h(X_i)}{[\sum h^2(X_i) - N^{-1}(\sum h(X_i))^2]^{\frac{1}{2}}} (1 - N^{-1})^{\frac{1}{2}} \geqq t_\alpha \tag{5.7}$$

with possible randomization if equality occurs. Here t_α depends on α, h, F and N and is chosen in such a way that the test (5.7) has level α.

THEOREM 5.2. *Suppose there exist positive numbers $c, C, \varepsilon, \eta, \delta_1, \delta_2, \cdots$ with $\lim_{N\to\infty} \delta_N = 0$ and $m > 8$, such that hF^{-1} and hG^{-1} are monotone and differentiable on intervals I_F and I_G of length at least η where*

$$\left|\frac{d}{dt} h(F^{-1}(t))\right| \geqq c , \qquad \left|\frac{d}{dt} h(G^{-1}(t))\right| \geqq c , \tag{5.8}$$

and such that $\varepsilon \leqq \alpha \leqq 1 - \varepsilon$, and

$$\int_{-\infty}^{\infty} |h(x)|^m \, dF(x) \leqq C , \qquad \int_{-\infty}^{\infty} |h(x)|^m \, dG(x) \leqq C , \tag{5.9}$$

$$\left|\int_{-\infty}^{\infty} h^{2k}(x) \, dF(x) - \int_{-\infty}^{\infty} h^{2k}(x) \, dG(x)\right| \leqq \delta_N \qquad \text{for } k = 1, 2 . \tag{5.10}$$

Then there exist $A > 0$ depending on N, F, G, h and α only through c, C, η and ε, and $\beta > 0$ depending only on m, such that the powers of the tests (5.1) and (5.7) for F against G differ by at most $A(N^{-\beta} + \delta_N)N^{-1}$.

PROOF. We denote probabilities and expected values under $G(F)$ by $P_G(P_F)$ and $E_G(E_F)$. By (5.9) and (5.8) we have

$$\sigma_G^2(h(X_1)) \leqq E_G h^2(X_1) \leqq [E_G h^4(X_1)]^{\frac{1}{2}} \leqq C^{2/m}, \tag{5.11}$$

$$\sigma_G^2(h(X_1)) \geqq 2 \int_0^{\eta/2} (ct)^2\, dt = \frac{c^2\eta^3}{12}, \tag{5.12}$$

so that these moments are bounded away from 0 and ∞. For positive integer $k \leqq 4$, Markov's inequality, the Marcinkievitz–Zygmund–Chung inequality (Chung (1951)) and (5.9) yield

$$\begin{aligned} P_G(|\sum (h^k(X_i) - E_G h^k(X_i))| \geqq \tau N) & \\ &\leqq \frac{E_G|\sum (h^k(X_i) - E_G h^k(X_i))|^{m/k}}{(\tau N)^{m/k}} \\ &\leqq B_m(\tau^2 N)^{-m/(2k)} E_G|h^k(X_1) - E_G h^k(X_1)|^{m/k} \\ &\leqq B_m C\left(\frac{2}{\tau}\right)^{m/k} N^{-m/(2k)}, \end{aligned} \tag{5.13}$$

where B_m depends only on m. Choose

$$\beta = \min\left(\frac{m-8}{2m+8}, \tfrac{1}{4}\right). \tag{5.14}$$

Taking $\tau = N^{-\beta}$ in (5.13) and using (5.3) we find that

$$\frac{1}{N} \sum h^{2k}(Z_j) = \frac{1}{N} \sum h^{2k}(X_i) = E_G h^{2k}(X_1) + O(N^{-\beta}), \qquad k = 1, 2, \tag{5.15}$$

$$\frac{1}{N} \sum h^2(X_i) - \left[\frac{1}{N} \sum h(X_i)\right]^2 = \sigma_G^2(h(X_1)) + O(N^{-\beta}), \tag{5.16}$$

uniformly on a set with probability $1 - O(N^{-1-\beta})$ under G.

Assumption (5.3) implies that

$$\lambda\{x \mid \exists_j |x - h(Z_j)| < \zeta\} \geqq \tfrac{1}{2}\lambda\{x \mid \exists_i |x - h(X_i)| < \zeta\},$$

and under G the right-hand side is distributed like

$$\tfrac{1}{2}\lambda\{x \mid \exists_j |x - h(G^{-1}(U_{j:N}))| < \zeta\},$$

where $U_{1:N} < \cdots < U_{N:N}$ are order statistics from a uniform distribution on $(0, 1)$. Now for $n \geqq 1$

$$\begin{aligned} P(U_{j+n:N} - U_{j:N} \leqq z) &= \iint_{0<s<t<1,\, t-s\leqq z} \frac{N!}{(j-1)!\,(n-1)!\,(N-j-n)!} s^{j-1}(t-s)^{n-1}(1-t)^{N-j-n}\, ds\, dt \\ &\leqq \frac{(Nz)^{n-1}}{(n-1)!} \iint_{0<s<t<1} \frac{(N-n+1)!}{(j-1)!\,(N-j-n)!} s^{j-1}(1-t)^{N-j-n}\, ds\, dt \\ &= \frac{(Nz)^{n-1}}{(n-1)!}. \end{aligned}$$

Taking $n = 6$ and $z = 2c^{-1}N^{-\frac{5}{6}} \log N$ we see that

$$P\left(U_{6(k+1):N} - U_{6k:N} \geqq 2c^{-1}N^{-\frac{5}{6}} \log N \quad \text{for all} \quad 1 \leqq k \leqq \left[\frac{N}{6}\right] - 1\right)$$
$$\geqq 1 - \frac{N}{6}(2c^{-1}N^{-\frac{5}{6}} \log N)^6 = 1 - O(N^{-1-\beta}) .$$

Together with (5.8) this implies that for $\zeta = N^{-\frac{5}{6}} \log N$

$$\lambda\{x \mid \exists_j |x - h(Z_j)| < \zeta\} \geqq \tfrac{1}{2}\eta N\zeta \tag{5.17}$$

with probability $1 - O(N^{-1-\beta})$ under G.

Now (5.11), (5.12), (5.15) and (5.17) ensure that expansion (5.4) holds uniformly except on a set E_0 with $P_G(E_0) = O(N^{-1-\beta})$. Simplifying this expansion by using (5.11), (5.12) and (5.15) once more, we arrive at the conclusion that the power against G of the test (5.1) is given by

$$\pi_P(G) = P_G\left(\frac{\sum h(X_i)}{(\sum h^2(X_i))^{\frac{1}{2}}} \geqq u_\alpha - \frac{E_G h^4(X_1)}{12N(E_G h^2(X_1))^2}(u_\alpha^3 - 3u_\alpha) + O(N^{-1-\beta})\right) + O(N^{-1-\beta}) . \tag{5.18}$$

Here the first remainder term depends on Z but may now be taken to be uniformly $O(N^{-1-\beta})$.

The inequality $\sum h(X_i)/(\sum h^2(X_i))^{\frac{1}{2}} \geqq a$ is algebraically equivalent with

$$\frac{\sum h(X_i)}{[\sum h^2(X_i) - N^{-1}(\sum h(X_i))^2]^{\frac{1}{2}}} \geqq \frac{a}{(1 - a^2/N)^{\frac{1}{2}}}$$

on the set where $\sum h^2(X_i) - N^{-1}(\sum h(X_i))^2 \neq 0$ and provided that $a^2 < N$. We may apply this to (5.18) in view of the condition $\varepsilon \leqq \alpha \leqq 1 - \varepsilon$, (5.11), (5.12) and (5.16). At the same time we may replace E_G by E_F in (5.18), and by (5.10) this only involves adding $O(\delta_N N^{-1})$ to the first remainder term in (5.18). In this way we obtain

$$\pi_P(G) = P_G\left(\tilde{T} \geqq u_\alpha + \frac{u_\alpha^3 - u_\alpha}{2N} - \frac{E_F h^4(X_1)}{12N(E_F h^2(X_1))^2}(u_\alpha^3 - 3u_\alpha) + O\left(\frac{N^{-\beta} + \delta_N}{N}\right)\right) + O(N^{-1-\beta}) , \tag{5.19}$$

where $\tilde{T}$ is the statistic in (5.7).

By (5.11), (5.12) and (5.16) we have for $B \geqq 0$,

$$\begin{aligned} &\sup_t P_G(t \leqq \tilde{T} \leqq t + BN^{-1}(N^{-\beta} + \delta_N)) \\ &\leqq \sup_t P_G\left(t \leqq \frac{N^{-\frac{1}{2}} \sum h(X_i)}{\sigma_G(h(X_1))} \leqq t + 2BN^{-1}(N^{-\beta} + \delta_N)\right) + O(N^{-1-\beta}) . \end{aligned} \tag{5.20}$$

Now (5.8) ensures that under G the distribution of $h(X_1)$ has an absolutely continuous part; in fact, this distribution may be written as a mixture $Q = \eta\tilde{Q}_1 + (1 - \eta)\tilde{Q}_2$ where $\tilde{Q}_1$ is an absolutely continuous distribution with density $\tilde{q}_1 \leqq (c\eta)^{-1}$. Moreover, (5.9) and Markov's inequality imply that $\tilde{Q}_1([-C_1, C_1]) \geqq \frac{1}{2}$

where $C_1 = \max(1, (2C/\eta)^{\frac{1}{2}})$. It follows that $Q = (\eta/2)Q_1 + (1 - \eta/2)Q_2$ where $Q_1([-C_1, C_1]) = 1$ and Q_1 is absolutely continuous with density $q_1 \leqq c_1 = 2(c\eta)^{-1}$.

Let ρ_1 be the ch.f. of Q_1. Obviously, for any fixed $t \neq 0$, $|\rho_1(t)| \leqq |\bar{\rho}_1(t)|$ where $\bar{\rho}_1$ is the ch.f. of the distribution with density

$$\bar{q}_1(y) = c_1 \qquad \text{for} \quad y \in \bigcup_{k=0}^{n}\left[-C_1 + \frac{2k\pi}{|t|}, -C_1 + \frac{2k\pi + 2\xi}{|t|}\right]$$
$$= 0 \qquad \text{elsewhere,}$$

with $n = [C_1|t|/\pi]$ and $(n+1)c_1 2\xi/|t| = 1$. An easy calculation yields $|\bar{\rho}_1(t)| = (\sin\xi)/\xi$; for $|t| \geqq \pi/C_1$ we have $\xi \geqq \pi/(4c_1C_1)$. It follows that there exists $b > 0$ depending only on η, c and C, such that the ch.f. of $h(X_1)$ under G satisfies

$$(5.21) \qquad |E_G e^{ith(X_1)}| \leqq 1 - b \qquad \text{for } |t| \geqq \pi\,.$$

Because of (5.9), (5.12), (5.21) and Lemma 1 in Cramér (1962), page 27, the df of $\sigma_G^{-1}(h(X_1))N^{-\frac{1}{2}} \sum (h(X_i) - E_G h(X_i))$ under G has an Edgeworth expansion; uniformly for all G satisfying (5.8) and (5.9) for fixed c, C and η, the derivative of this expansion is bounded and its remainder term is $O(N^{-3})$. Applying this result and (5.20) to (5.19) we find

$$(5.22) \qquad \pi_P(G) = P_G(\tilde{T} \geqq \tilde{t}_\alpha) + O(N^{-1}(N^{-\beta} + \delta_N))$$

uniformly for fixed c, C, η and ε, where

$$(5.23) \qquad \tilde{t}_\alpha = u_\alpha + \frac{u_\alpha^3 - u_\alpha}{2N} - \frac{E_F h^4(X_1)}{12N(E_F h^2(X_1))^2}(u_\alpha^3 - 3u_\alpha)\,.$$

Let t_α be as defined in (5.7). Since F satisfies all assumptions imposed on G, (5.22) will hold under F as well as under G. We have $\pi_P(F) = \alpha$ and hence $\tilde{t}_\alpha = t_{\tilde{\alpha}}$ where $|\tilde{\alpha} - \alpha| = O(N^{-1}(N^{-\beta} + \delta_N))$ uniformly for $\varepsilon \leqq \alpha \leqq 1 - \varepsilon$, but of course also uniformly for $\varepsilon/2 \leqq \alpha \leqq 1 - \varepsilon/2$. Because t_α is decreasing in α and $\tilde{t}_\alpha$ has a bounded derivative with respect to α for $\varepsilon/2 \leqq \alpha \leqq 1 - \varepsilon/2$, it follows that

$$(5.24) \qquad t_\alpha = \tilde{t}_\alpha + O(N^{-1}(N^{-\beta} + \delta_N))$$

uniformly for $\varepsilon \leqq \alpha \leqq 1 - \varepsilon$. In view of (5.22) and the preceding part of the proof this implies that

$$(5.25) \qquad \pi_P(G) = P_G(\tilde{T} \geqq t_\alpha) + O(N^{-1}(N^{-\beta} + \delta_N))$$

uniformly for fixed c, C, η and ε. This completes the proof. □

It may be useful to comment briefly on assumption (5.10) in Theorem 5.2. Of course this assumption is satisfied for a sequence of alternatives G_N that tends to F in an appropriate manner. It is easy to see, for instance, that if the sequence G_N^N is contiguous to F^N, (5.9) implies (5.10) with $\delta_N = O(N^{-\frac{1}{2}})$. Similarly, (5.9) will imply (5.10) for some sequence $\delta_N = o(1)$ if h is continuous and G_N converges weakly to F.

6. Deficiencies of distribution free tests. Let F be a fixed df with density f that is positive, symmetric about zero and five times differentiable. Consider the problem of testing, on the basis of $X_1, \cdots, X_N$, the hypothesis $G = F$ against the alternative $G(x) = F(x - \theta)$ at level α. For any particular θ, the maximum power $\pi^+(\theta)$ is attained by the test based on the statistic $\sum \{\log f(X_i - \theta) - \log f(X_i)\}$. This statistic is a sum of i.i.d. random variables and therefore its df admits an Edgeworth expansion under the usual conditions. By expanding the cumulants of the statistic Albers (1974) obtains an expansion for $\pi^+(\theta)$. Define Ψ_i by (3.15) and take

$$\tilde{\pi}^+(\theta) = 1 - \Phi(u_\alpha - \eta^*) + \frac{\eta^*\phi(u_\alpha - \eta^*)}{72N} \left\{ \frac{\int_0^1 \Psi_1^4(t)\,dt}{(\int_0^1 \Psi_1^2(t)\,dt)^2} [3(u_\alpha^2 - 1) - 3\eta^* u_\alpha + 2\eta^{*2}] - \frac{3\int_0^1 \Psi_2^2(t)\,dt}{(\int_0^1 \Psi_1^2(t)\,dt)^2} \eta^{*2} - 9[(u_\alpha^2 - 1) - \eta^* u_\alpha] \right\}, \tag{6.1}$$

where η^* is given by (4.16). Lemma 6.1 is a version of Albers' result.

LEMMA 6.1. *Let F satisfy (4.5) for $m_i = 5/i$, $i = 1, \cdots, 5$, and suppose that positive numbers C and ε exist such that $0 \leqq \theta \leqq CN^{-\frac{1}{2}}$ and $\varepsilon \leqq \alpha \leqq 1 - \varepsilon$. Then there exists $A > 0$ depending on N, F, θ and α only through F, C and ε and such that*

$$|\pi^+(\theta) - \tilde{\pi}^+(\theta)| \leqq AN^{-\frac{3}{2}}. \tag{6.2}$$

For the same testing problem Theorem 4.2 provides an expansion for the power $\pi(\theta)$ of the locally most powerful rank test. Together, Theorem 4.2 and Lemma 6.1 will enable us to find the deficiency d_N of the locally most powerful rank test with respect to the most powerful parametric test. To ensure that F satisfies the assumptions of both Theorem 4.2 and Lemma 6.1, we require that $F \in \mathscr{F}_1$, where

DEFINITION 6.1. $\mathscr{F}_1$ is the class of df's F on R^1 with positive densities f that are symmetric about zero, five times differentiable and such that (4.5) is satisfied for $i = 1, \cdots, 5$ with $m_1 = 6$, $m_2 = 3$, $m_3 = \frac{5}{3}$, $m_4 = \frac{5}{4}$, $m_5 = 1$, and such that (4.6) holds.

Furthermore, define

$$\tilde{d}_N = \frac{1}{12} \left\{ \frac{\int_0^1 \Psi_1^4(t)\,dt}{(\int_0^1 \Psi_1^2(t)\,dt)^2} [3(u_\alpha^2 - 1) - 2\eta^* u_\alpha - \eta^{*2}] + \frac{3\int_0^1 \Psi_2^2(t)\,dt}{(\int_0^1 \Psi_1^2(t)\,dt)^2} \eta^{*2} - 3[(u_\alpha^2 - 1) - 2\eta^* u_\alpha + \eta^{*2}] + 12\frac{\int_{1/N}^{1-1/N} (\Psi_1'(t))^2 t(1-t)\,dt}{\int_0^1 \Psi_1^2(t)\,dt} \right\}, \tag{6.3}$$

with η^* as in (4.16).

THEOREM 6.1. *Let d_N be the deficiency of the locally most powerful rank test* *with respect to the most powerful parametric test for testing $G = F$ against $G(x) = F(x - \theta)$ on the basis of $X_1, \cdots, X_N$ and at level α. Suppose that $F \in \mathscr{F}_1$ and that $cN^{-\frac{1}{2}} \leqq \theta \leqq CN^{-\frac{1}{2}}$, $\varepsilon \leqq \alpha \leqq 1 - \varepsilon$ for positive c, C and ε. Then, for every fixed F, c, C and ε, there exist positive numbers $A, \delta_1, \delta_2, \cdots$ such that $\lim_{N\to\infty} \delta_N = 0$ and for every N*

$$|d_N - \tilde{d}_N| \leqq \delta_N + AN^{-\frac{1}{2}} \int_{1/N}^{1-1/N} (\Psi_1'(t))^2(t(1-t))^{\frac{1}{2}}\, dt \,. \tag{6.4}$$

This result continues to hold if the locally most powerful rank test is replaced by the rank test with the corresponding approximate scores $a_j = -\Psi_1(j/(N+1))$.

PROOF. As $\mathscr{F}_1 \subset \mathscr{F}$, the remark following Theorem 4.2 shows that

$$\int_{1/N}^{1-1/N} (\Psi_1'(t))^2(t(1-t))^{\nu}\, dt = o(N^{\frac{3}{2}-\nu}) \qquad \text{for } \nu = 1, \tfrac{1}{2}\,. \tag{6.5}$$

Theorem 4.2 and Lemma 6.1 provide expansions for $\pi(\theta)$ and $\pi^+(\theta)$. In view of (6.5), the boundedness of u_α and the fact that $c \leqq N^{\frac{1}{2}}\theta \leqq C$, it is clear from these expansions that $d_N = o(N^{\frac{1}{2}})$. To find d_N we replace N by $N + d_N$ and η^* by $\eta^*(1 + d_N N^{-1})^{\frac{1}{2}}$ in the expansion for $\pi(\theta)$ and equate the result to the expansion for $\pi^+(\theta)$. Taylor expansion with respect to $d_N N^{-1}$ in (4.18) yields

$$\begin{aligned} &\frac{\eta^*\phi(u_\alpha - \eta^*)}{24N}\Big\{12d_N + \frac{\int_0^1 \Psi_1^4(t)\,dt}{(\int_0^1 \Psi_1^2(t)\,dt)^2}[-3(u_\alpha^2 - 1) + 2\eta^* u_\alpha + \eta^{*2}] \\ &\qquad - \frac{3\int_0^1 \Psi_2^2(t)\,dt}{(\int_0^1 \Psi_1^2(t)\,dt)^2}\eta^{*2} + 3(u_\alpha^2 - 1) - 6\eta^* u_\alpha + 3\eta^{*2} \\ &\qquad - \frac{12\int_{1/N}^{1-1/N} (\Psi_1'(t))^2 t(1-t)\,dt}{\int_0^1 \Psi_1^2(t)\,dt}\Big\} \\ &\qquad = o(N^{-1}) + O(N^{-\frac{3}{2}} \int_{1/N}^{1-1/N} (\Psi_1'(t))^2(t(1-t))^{\frac{1}{2}}\,dt)\,, \end{aligned} \tag{6.6}$$

uniformly for fixed $F \in \mathscr{F}_1$, c, C and ε. As $\eta^*\phi(u_\alpha - \eta^*)$ is bounded away from zero, (6.4) follows. The last assertion of the theorem is an immediate consequence of Theorem 4.2. □

Obviously (6.3) and (6.4) imply that under the conditions of Theorem 6.1

$$d_N = O(\int_{1/N}^{1-1/N} (\Psi_1'(t))^2 t(1-t)\,dt) \tag{6.7}$$

for $N \to \infty$. Hence d_N remains bounded as $N \to \infty$ if $\int_0^1 (\Psi_1'(t))^2 t(1-t)\,dt$ converges. Fortunately, in most cases of interest Theorem 6.1 provides more detailed information than (6.7) and remarks similar to those following Theorem 4.2 apply. Typically Ψ_1' will be bounded near 0 and the asymptotic behavior of d_N will be determined by the rate of growth of Ψ_1' near 1. If $\Psi_1'(t) = o((1-t)^{-1})$ near 1, then $d_N = \tilde{d}_N + o(1)$. If $\Psi_1'(t)$ is of exact order $(1-t)^{-1}$, then

$$d_N = \frac{\int_0^{1-1/N} (\Psi_1'(t))^2 t(1-t)\,dt}{\int_0^1 \Psi_1^2(t)\,dt} + O(1)$$

and d_N will be of the order $\log N$. Finally, if $\Psi_1'(t) \sim (1-t)^{-1-\delta}$ for $t \to 1$ and some $0 < \delta < \frac{1}{6}$, then the expansion (6.4) reduces to $d_N = O(N^{2\delta})$, which is nothing but (6.7).

We shall give two applications of Theorem 6.1. First we consider the problem of testing the hypothesis $G = \Phi$ against the alternative $G(x) = \Phi(x - \theta)$, where $cN^{-\frac{1}{2}} \leq \theta \leq CN^{-\frac{1}{2}}$. Let d_N be the deficiency of the normal scores test (or van der Waerden's test) with respect to the most powerful parametric test based on $\bar{X}$. Computations similar to those in Section 4 yield

$$(6.8) \qquad \begin{aligned} d_N &= \tfrac{1}{2}(u_\alpha^2 - 1) + \int_0^{\Phi^{-1}(1-1/2N)} \frac{(2\Phi(x) - 1)(1 - \Phi(x))}{\phi(x)}\, dx + o(1) \\ &= \tfrac{1}{2}\log\log N + \tfrac{1}{2}(u_\alpha^2 - 1) + \tfrac{1}{2}\log 2 + 0.05832\cdots + o(1)\,. \end{aligned}$$

In this case $d_N \sim \frac{1}{2}\log\log N \to \infty$ for $N \to \infty$. Note that there is no dependence on θ in this expansion for d_N and that the leading term is also independent of α.

As a second example we take the logistic df $F(x) = (1 + e^{-x})^{-1}$ and consider the testing problem $G = F$ against $G(x) = F(x - bN^{-\frac{1}{2}})$, where $b > 0$ is fixed. Now d_N is the deficiency of Wilcoxon's signed rank test with respect to the most powerful parametric test for this problem. We find

$$(6.9) \qquad d_N = \tfrac{1}{60}\{18 + 12u_\alpha^2 + 4(3)^{\frac{1}{2}}bu_\alpha + b^2\} + o(1)$$

and here d_N tends to a finite limit for $N \to \infty$.

Having shown that the deficiency of a distribution free test with respect to the best parametric test may tend to a finite limit, we now address ourselves to the intriguing question whether this limit can be zero. To answer this question we first have to decide what is meant by the best parametric test. So far, we have compared the performance of a distribution free test with that of the most powerful parametric test for known scale against a simple location alternative, thus in effect comparing with envelope power. Of course this comparison is not quite fair. Computed in this way, the deficiency of a distribution free test reflects the losses incurred by using (i) the same test against every location alternative $\theta > 0$; (ii) a scale invariant test; (iii) a distribution free test. Since our main interest is the deficiency due to (iii), it is more appropriate to compare with the uniformly most powerful scale invariant test, if such a test exists. Unfortunately, invariant tests are in general rather intractable, the main exception being Student's test for the normal location case. We note that Hodges and Lehmann (1970) have shown that the deficiency of Student's test with respect to the most powerful parametric test based on $\bar{X}$ tends to a finite but positive limit, so that it does indeed matter whether one compares with Student's test or with envelope power.

We are thus led to consider the normal location case with Student's test as the best parametric test. To establish the existence of a distribution free test with deficiency tending to zero, the obvious candidate is the permutation test based on $\sum X_i$. Theorem 6.2 is an immediate consequence of Theorem 5.1 and the remark following it.

THEOREM 6.2. *Let d_N be the deficiency of the permutation test based on $\sum X_i$ with respect to Student's test for testing $G = \Phi$ against $G(x) = \Phi(x - \theta)$ on the* *basis of* $X_1, \cdots, X_N$ *and at level* α. *Suppose that positive numbers* c, C *and* ε *exist such that* $cN^{-\frac{1}{2}} \leqq \theta \leqq CN^{-\frac{1}{2}}$ *and* $\varepsilon \leqq \alpha \leqq 1 - \varepsilon$. *Then there exists* $A > 0$ *depending on* N, θ *and* α *only through* c, C *and* ε *and such that*

$$d_N \leqq AN^{-\frac{1}{4}}\,. \tag{6.10}$$

Hence in this case we do find that d_N tends to zero for $N \to \infty$. Perhaps the most surprising thing about this example is that asymptotically one has to pay a certain price for scale invariance, but that once this price has been paid, there is no additional penalty for using a distribution free test. We note that the remark following Theorem 5.1 implies that (6.10) may be replaced by $d_N \leqq AN^{-\frac{1}{2}}$.

Theorem 6.2 may of course be generalized considerably by taking Theorem 5.2 for $h(x) \equiv x$ as a starting point instead of Theorem 5.1. For d_N as in Theorem 6.2, it is clear that $d_N = o(1)$ for a much larger class of testing problems than the normal location problem of Theorem 6.2. Although Student's test is generally not optimal for these problems, this shows how closely the two tests resemble one another.

7. Expansions and deficiencies for related estimators. Let $T = T(X_1, \cdots, X_N)$ be given by (2.2) and suppose that the scores a_j are nonnegative and nondecreasing in $j = 1, \cdots, N$. Define the statistic M by

$$M(X_1, \cdots, X_N) = \tfrac{1}{2} \sup \{t: 2T(X_1 - t, \cdots, X_N - t) > \textstyle\sum a_j\} \\ + \tfrac{1}{2} \inf \{t: 2T(X_1 - t, \cdots, X_N - t) < \textstyle\sum a_j\}\,. \tag{7.1}$$

Suppose that $X_1, \cdots, X_N$ are i.i.d. with common df $G(x) = F(x - \mu)$, where F has a density f that is symmetric about zero. Then M is the midpoint of the interval between the upper and lower 0.5 confidence bounds for μ induced by the statistic T. Hodges and Lehmann (1963) proposed M as an estimator for μ and studied its connection with T. They showed that the normal approximation to the power of the level $\frac{1}{2}$ test based on T for contiguous location alternatives could be used to establish asymptotic normality of M. We shall show that, similarly, power expansions for level $\frac{1}{2}$ yield expansions for the df of $N^{\frac{1}{2}}(M - \mu)$. We restrict attention to the case where the scores are generated by a smooth function J.

Let $\mathscr{J}$ and $\mathscr{F}$ be given by Definition 4.1, let $\pi(\theta, \frac{1}{2})$ denote the power of the level $\frac{1}{2}$ right-sided test based on T against the alternative $F(x - \theta)$ and define $K_{\theta,i}$ and $\tilde{\eta}$ as in (4.8)—(4.10).

THEOREM 7.1. *Let* $F \in \mathscr{F}$, $J \in \mathscr{J}$, *suppose that* J *is nonnegative and nondecreasing and let* $a_j = EJ(U_{j:N})$. *Take* $\theta = \xi N^{-\frac{1}{2}}$. *Then, for every fixed* J, F *and* $C > 0$,

$$\sup_{|\xi| \leqq C} |P_\mu(N^{\frac{1}{2}}(M - \mu) \leqq \xi) - \pi(\theta, \tfrac{1}{2})| = O(N^{-\frac{1}{2}})\,, \tag{7.2}$$

$$\sup_{|\xi| \leqq C} |P_\mu(N^{\frac{1}{2}}(M - \mu) \leqq \xi) - \{1 - K_{\theta,1}(-\tilde{\eta})\}| = o(N^{-1})\,, \tag{7.3}$$

$$\sup_{|\xi| \leqq C} |P_\mu(N^{\frac{1}{2}}(M - \mu) \leqq \xi) - \{1 - K_{\theta,2}(-\tilde{\eta})\}| \\ = o(N^{-1}) + O(N^{-\frac{3}{2}} \textstyle\int_{1/N}^{1-1/N} |J''(t)|(|J'(t)| + |\Psi_1'(t)|)(t(1 - t))^{\frac{1}{2}}\, dt)\,. \tag{7.4}$$

PROOF. It follows from Hodges and Lehmann (1963) that M is translation invariant and that its distribution is absolutely continuous and symmetric about μ. Thus, for $\theta = \xi N^{-\frac{1}{2}}$,

$$P_\mu(N^{\frac{1}{2}}(M-\mu) \leqq \xi) = P_\theta(M \geqq 0)\,, \tag{7.5}$$

and, in view of (7.1),

$$P_\theta(2T > \textstyle\sum a_j) \leqq P_\theta(M \geqq 0) \leqq P_\theta(2T \geqq \sum a_j)\,. \tag{7.6}$$

According to the proof of Theorem 4.1, the conclusions of Theorems 3.1 and 3.2 hold, which implies that $P_\theta(2T = \sum a_j) = O(N^{-\frac{5}{4}})$ uniformly for $|\theta| \leqq CN^{-\frac{1}{2}}$. This proves (7.2). The remaining part of Theorem 7.1 is now an immediate consequence of Theorem 4.1. □

The case where $J = -\Psi_1$, with Ψ_1 as in (3.15), is of course of special interest. Theorem 7.2 deals with this case for exact as well as approximate scores. Note that for $F \in \mathscr{F}$, the condition that $-\Psi_1$ is nonnegative and nondecreasing is equivalent to concavity of $\log f$, i.e. to strong unimodality of f.

THEOREM 7.2. *Let $F \in \mathscr{F}$, suppose that f is strongly unimodal and let either $a_j = -E\Psi_1(U_{j:N})$ for $j = 1, \cdots, N$ or $a_j = -\Psi_1(j/(N+1))$ for $j = 1, \cdots, N$. Then, for every fixed F and $C > 0$,*

$$\sup_{|\xi| \leqq C} |P_\mu(N^{\frac{1}{2}}(M-\mu) \leqq \xi) - \pi(\xi N^{-\frac{1}{2}}, \tfrac{1}{2})| = O(N^{-\frac{5}{4}})\,, \tag{7.7}$$

$$\begin{aligned} &P_\mu((N \textstyle\int_0^1 \Psi_1^2(t)\,dt)^{\frac{1}{2}}(M-\mu) \leqq x) \\ &\quad = \Phi(x) + \frac{x\phi(x)}{72N}\left\{x^2\left[\frac{5\int_0^1 \Psi_1^4(t)\,dt}{(\int_0^1 \Psi_1^2(t)\,dt)^2} - \frac{12\int_0^1 \Psi_2^2(t)\,dt}{(\int_0^1 \Psi_1^2(t)\,dt)^2} + 9\right]\right. \\ &\qquad \left. + \frac{6\int_0^1 \Psi_1^4(t)\,dt}{(\int_0^1 \Psi_1^2(t)\,dt)^2} - \frac{36\int_{1/N}^{1-1/N} (\Psi_1'(t))^2 t(1-t)\,dt}{\int_0^1 \Psi_1^2(t)\,dt}\right\} \\ &\qquad + o(N^{-1}) + O(N^{-\frac{3}{2}} \textstyle\int_{1/N}^{1-1/N} (\Psi_1'(t))^2 (t(1-t))^{\frac{3}{2}}\,dt) \end{aligned} \tag{7.8}$$

uniformly for $|x| \leqq C$.

PROOF. The proof of (7.7) is identical to the proof of (7.2) in Theorem 7.1. Expansion (7.8) follows from (7.7) and Theorem 4.2. □

The estimators in Theorem 7.2 are efficient and their natural competitor is the maximum likelihood estimator M' which solves

$$\textstyle\sum_{j=1}^N \psi_1(X_j - M') = 0 \tag{7.9}$$

with ψ_1 as in (3.1). The performance of M' is connected with that of the locally most powerful test for F against $F(x-\theta)$, which is based on the statistic $-\sum \psi_1(X_j)$. Let $\pi'(\theta, \frac{1}{2})$ be the power of the level $\frac{1}{2}$ right-sided test based on $-\sum \psi_1(X_j)$ for F against $F(x-\theta)$.

LEMMA 7.1. *Suppose that f is positive, symmetric about zero and strongly unimodal and that* (4.5) *is satisfied for $m_i = 5/i$, $i = 1, \cdots, 5$. Then, for every fixed F and*

$C > 0$,

$$\sup_{|\xi| \leqq C} |P_\mu(N^{\frac{1}{2}}(M' - \mu) \leqq \xi) - \pi'(\xi N^{-\frac{1}{2}}, \tfrac{1}{2})| = O(N^{-\frac{3}{2}}), \tag{7.10}$$

$$\begin{aligned} &P_\mu((N \textstyle\int_0^1 \Psi_1^2(t)\,dt)^{\frac{1}{2}}(M' - \mu) \leqq x) \\ &\quad = \Phi(x) + \frac{x\phi(x)}{72N}\left\{x^2\left[\frac{5 \int_0^1 \Psi_1^4(t)\,dt}{(\int_0^1 \Psi_1^2(t)\,dt)^2} - \frac{12 \int_0^1 \Psi_2^2(t)\,dt}{(\int_0^1 \Psi_1^2(t)\,dt)^2} + 9\right]\right. \\ &\qquad \left. - \frac{3 \int_0^1 \Psi_1^4(t)\,dt}{(\int_0^1 \Psi_1^2(t)\,dt)^2} + 9\right\} + O(N^{-\frac{3}{2}}) \end{aligned} \tag{7.11}$$

uniformly for $|x| \leqq C$.

PROOF. The estimator M' is translation invariant and its distribution is symmetric about μ. Thus, for $\theta = \xi N^{-\frac{1}{2}}$, (7.5) holds with M replaced by M', and in view of (7.9),

$$P_\theta(-\textstyle\sum \psi_1(X_j) > 0) \leqq P_\mu(N^{\frac{1}{2}}(M' - \mu) \leqq \xi) \leqq P_\theta(-\sum \psi_1(X_j) \geqq 0). \tag{7.12}$$

Since f is everywhere positive and ψ_1 is everywhere differentiable, the distribution of $\psi_1(X_1)$ under θ contains a fixed absolutely continuous component for all θ in a neighborhood of zero. Together with (4.5) for $m_1 = 5$, this ensures that the df of $\sum \psi_1(X_j)$ under θ possesses an Edgeworth expansion with remainder $O(N^{-\frac{3}{2}})$ uniformly for $|\theta| \leqq CN^{-\frac{1}{2}}$. This implies that $P_\theta(-\sum \psi_1(X_j) = 0) = O(N^{-\frac{3}{2}})$ uniformly for $|\theta| \leqq CN^{-\frac{1}{2}}$, which proves (7.10).

The expansion for the df of $\sum \psi_1(X_j)$ is used in Albers (1974) to establish an expansion for the power of the locally most powerful test under the conditions of Lemma 6.1. Specializing to the case where $\alpha = \frac{1}{2}$ and using (7.10) we obtain (7.11). □

There is no unique natural measure of scale to assess the performance of an estimator $\hat\mu$ admitting an expansion of the form (7.8) or (7.11). One possibility is to consider a family of measures determined by the quantiles of $\hat\mu$. We can define $\sigma(\hat\mu, s)$ to be the s-quantile of $(\hat\mu - \mu)$ divided by $u_{1-s} = \Phi^{-1}(s)$. As we are only considering estimators that are distributed symmetrically about μ, $\sigma(\hat\mu, s)$ may serve as a measure of scale for any $\frac{1}{2} < s < 1$. If we fix a value of s, we can define the deficiency $D_N(s)$ of a sequence of estimators $\{\hat\mu_{2,N}\}$ with respect to an estimator $\hat\mu_{1,N}$ by equating $\sigma(\hat\mu_{2,N+D_N}, s)$ and $\sigma(\hat\mu_{1,N}, s)$, with the usual convention that σ is determined by linear interpolation for nonintegral values of $N + D_N$. Similarly, for two sequences of level α tests, $d_N(\alpha, s)$ will denote the deficiency as defined in Section 1 for the case where the alternative θ is chosen in such a way that the common power equals s.

Let $\mathscr{F}_1$ be given by Definition 6.1.

THEOREM 7.3. *Let $d_N(\frac{1}{2}, s)$ be the deficiency for level $\frac{1}{2}$ and power s of the locally most powerful rank test with respect to the locally most powerful test for testing F against $F(x - \theta)$. Let $D_N(s)$ be the deficiency of the Hodges–Lehmann estimator associated with the locally most powerful rank test with respect to the maximum likelihood estimator for estimating μ in $F(x - \mu)$. Suppose that $F \in \mathscr{F}_1$ and that f is* *strongly unimodal. Then, for fixed F and $\frac{1}{2} < s < 1$,*

$$|D_N(s) - d_N(\tfrac{1}{2}, s)| = O(N^{-\frac{1}{4}})\,, \tag{7.13}$$

$$D_N(s) = \frac{\int_{1/N}^{1-1/N} (\Psi_1'(t))^2 t(1-t)\,dt}{\int_0^1 \Psi_1^2(t)\,dt} - \frac{1}{4}\,\frac{\int_0^1 \Psi_1^4(t)\,dt}{(\int_0^1 \Psi_1^2(t)\,dt)^2} + \frac{1}{4} + o(1) + O(N^{-\frac{1}{2}} \textstyle\int_{1/N}^{1-1/N} (\Psi_1'(t))^2 (t(1-t))^{\frac{1}{2}}\,dt)\,. \tag{7.14}$$

This result continues to hold if in the locally most powerful rank test and the associated estimator, the exact scores are replaced by the approximate scores $a_j = -\Psi_1(j/(N+1))$.

PROOF. The conditions of Theorem 7.2 and Lemma 7.1 are satisfied. Writing M_N and M_N' for M and M', we see that for some ξ

$$P_\mu(N^{\frac{1}{2}}(M_N' - \mu) \leqq \xi) = s + O(N^{-\frac{3}{2}})\,, \tag{7.15}$$

$$P_\mu(N^{\frac{1}{2}}(M_{N+d_N} - \mu) \leqq \xi) = s + O(N^{-\frac{5}{4}})\,. \tag{7.16}$$

By the remark following Theorem 4.2 we have $\Psi_1'(t) = o((t(1-t))^{-\frac{7}{6}})$ near 0 and 1, and combining this with (7.8) and (7.11) we find that (7.15) and (7.16) imply (7.13). The proof of (7.14) is now the same as that of Theorem 6.1. □

An interesting property of the expansion (7.14) is that it is independent of s. Thus, to the order considered, the deficiency $D_N(s)$ is asymptotically independent of the particular choice of the quantile used to measure the performance of the estimators. Of course, this reflects the fact that the deficiency $d_N(\frac{1}{2}, s)$ is independent of the power in the same asymptotic sense. Algebraically, the reason for this phenomenon is that the term involving $x^3\phi(x)$ is the same in (7.8) and (7.11).

We also note that upon formal substitution of $\alpha = \frac{1}{2}$ and $\theta = 0$ in (6.3), the expansion for d_N in Theorem 6.1 reduces to the expansion for $D_N(s)$ in Theorem 7.3. This shows that if the remainder in (7.14) is $o(1)$, then $D_N(s)$ will tend to a nonnegative but possibly infinite limit.

In Section 6 we have already pointed out that an expansion like (7.14) may or may not be of interest, depending on the behavior of the remainder term. We should stress that, even if the expansion (7.14) is useless, (7.13) still establishes the asymptotic equivalence of $D_N(s)$ and $d_N(\frac{1}{2}, s)$.

We conclude our discussion with one example of Theorem 7.3. For estimating normal location, the deficiency of either one of the Hodges–Lehmann estimators associated with the normal scores test and with van der Waerden's test with respect to $\bar{X}$ is asymptotic to $\frac{1}{2}\log\log N$. The deficiency of one of these Hodges–Lehmann estimators with respect to the other tends to zero for $N \to \infty$.

APPENDIX

1. Expansions for the contiguous case. Our purpose in this appendix will be the justification of the passage from (2.41) to (3.8) under the assumptions stated in Section 3. Thus we shall suppose throughout that f is positive and symmetric about 0 and that $g(x) = f(x - \theta)$.

Begin by defining a function $\xi(x, t)$ for $x \geqq 0$, $t \geqq 0$, by

$$F(\xi(x, t) - t) + F(\xi(x, t) + t) = 2F(x) . \tag{A1.1}$$

Introduce also two other functions of two variables, p and $\bar{p}$, by

$$p(x, t) = \frac{f(x - t)}{f(x - t) + f(x + t)} , \tag{A1.2}$$

$$\bar{p}(x, t) = p(\xi(x, t), t) . \tag{A1.3}$$

The basic property of the function ξ is, of course, that the joint distribution of $(\xi(Z_1, \theta), \cdots, \xi(Z_N, \theta))$ under F is the same as the joint distribution of $(Z_1, \cdots, Z_N)$ under G. It follows that the joint distribution of $(\bar{p}(Z_1, \theta), \cdots, \bar{p}(Z_N, \theta))$ under F is the same as the joint distribution of $(P_1, \cdots, P_N)$ under G. It is evident therefore that our task is essentially that of expanding $\bar{p}(x, t)$ around 0 as a function of t and giving suitable estimates of the remainder terms. We begin by differentiating formally. For convenience we shall, for any function of two variables $q(x, t)$, write

$$q_{i,j}(x, t) = \frac{\partial^{i+j} q(x, t)}{\partial x^i \, \partial t^j} .$$

Differentiating (A1.1) with respect to t we get

$$\xi_{0,1} = 2\bar{p} - 1 . \tag{A1.4}$$

It is now easy though tedious to obtain $\bar{p}_{0,j}(x, t)$ in terms of the $p_{i,k}(\xi(x, t), t)$ by replacing $\xi_{0,1}$ by $2\bar{p} - 1$ after each differentiation. Thus, for example,

$$\bar{p}_{0,1}(x, t) = [p_{0,1} + p_{1,0}(2p - 1)](\xi(x, t), t) , \tag{A1.5}$$

$$\begin{aligned} \bar{p}_{0,2}(x, t) = [p_{0,2} &+ 2p_{1,1}(2p - 1) + p_{2,0}(2p - 1)^2 + 2p_{1,0}p_{0,1} \\ &+ 2p_{1,0}^2(2p - 1)](\xi(x, t), t) . \end{aligned} \tag{A1.6}$$

Calculation of the $p_{i,j}$ is also tedious. Again we list the first few. Define

$$_1\psi_k(x, t) = \psi_k(x - t) , \qquad _2\psi_k(x, t) = \psi_k(x + t) , \tag{A1.7}$$

where $\psi_k = f^{(k)}/f$ as defined in (3.1), and let

$$_1\tilde{\psi}_k(x, t) = \psi_k(\xi(x, t) - t) , \qquad _2\tilde{\psi}_k(x, t) = \psi_k(\xi(x, t) + t) . \tag{A1.8}$$

Then

$$\begin{aligned} p_{0,1} &= -p(1 - p)[_1\psi_1 + {}_2\psi_1] , \qquad p_{1,0} = p(1 - p)[_1\psi_1 - {}_2\psi_1] , \\ p_{0,2} &= p(1 - p)[_1\psi_2 - {}_2\psi_2 - 2p \cdot {}_1\psi_1^2 + 2(1 - p)_2\psi_1^2 \\ &\qquad + 2(1 - 2p)_1\psi_1 \cdot {}_2\psi_1] , \end{aligned} \tag{A1.9}$$

$$\begin{aligned} p_{1,1} &= p(1 - p)[-_1\psi_2 - {}_2\psi_2 + 2p \cdot {}_1\psi_1^2 + 2(1 - p)_2\psi_1^2] , \\ p_{2,0} &= p(1 - p)[_1\psi_2 - {}_2\psi_2 - 2p \cdot {}_1\psi_1^2 + 2(1 - p)_2\psi_1^2 \\ &\qquad - 2(1 - 2p)_1\psi_1 \cdot {}_2\psi_1] . \end{aligned} \tag{A1.10}$$

Substituting (A1.9) and (A1.10) into (A1.5) and (A1.6) at $t = 0$ and employing similar manipulations with the third order derivatives we obtain

$$\text{(A1.11)} \qquad \tilde{p}(x, 0) = \tfrac{1}{2}, \qquad \tilde{p}_{0,1}(x, 0) = -\tfrac{1}{2}\psi_1(x), \qquad \tilde{p}_{0,2}(x, 0) = 0,$$
$$\tilde{p}_{0,3}(x, 0) = -\tfrac{1}{2}\psi_3(x) + 3\psi_1(x)\psi_2(x) - \tfrac{3}{2}\psi_1^3(x).$$

Moreover, from (A1.9), (A1.10) and the boundedness of p it is easy to see that constants b_1 and b_2 exist such that

$$\text{(A1.12)} \qquad |\tilde{p}_{0,1}| \leqq b_1 \textstyle\sum_{i=1}^{2} |{}_i\tilde{\psi}_1|, \qquad |\tilde{p}_{0,2}| \leqq b_2 \textstyle\sum_{i=1}^{2} \{|{}_i\tilde{\psi}_2| + {}_i\tilde{\psi}_1^2\}.$$

Similarly bounding first the $p_{i,k}$ and expressing $\tilde{p}_{0,j}$ appropriately, and invoking the inequality $|ab| \leqq r^{-1}|a|^r + s^{-1}|b|^s$, $r^{-1} + s^{-1} = 1$, we obtain for suitable b_3 and b_4

$$\text{(A1.13)} \qquad |\tilde{p}_{0,3}| \leqq b_3 \textstyle\sum_{i=1}^{2} \{|{}_i\tilde{\psi}_3| + |{}_i\tilde{\psi}_2|^{\frac{3}{2}} + |{}_i\tilde{\psi}_1|^3\},$$
$$|\tilde{p}_{0,4}| \leqq b_4 \textstyle\sum_{i=1}^{2} \{|{}_i\tilde{\psi}_4| + |{}_i\tilde{\psi}_3|^{\frac{4}{3}} + {}_i\tilde{\psi}_2^2 + {}_i\tilde{\psi}_1^4\}.$$

We need the following application of Taylor's formula with Cauchy's form of the remainder.

LEMMA A1.1. *Let $q(x, t)$ be a function of two variables possessing derivatives of order $\leqq k + 1$ in t in a neighborhood of 0. Then if S is any* rv *and $m \geqq 1$,*

$$\text{(A1.14)} \qquad E\left|q(S, t) - \textstyle\sum_{j=0}^{k} q_{0,j}(S, 0)\dfrac{t^j}{j!}\right|^m$$
$$\leqq \left[\frac{|t|^{k+1}}{(k + 1)!}\right]^m \sup\{E|q_{0,k+1}(S, \nu t)|^m : 0 \leqq \nu \leqq 1\}.$$

Suppose moreover that for $j = 0, \cdots, k$, $Eq_{0,j}(S, 0)$ exists and is finite. Then

$$\text{(A1.15)} \qquad E\left|\{q(S, t) - Eq(S, t)\} - \textstyle\sum_{j=0}^{k} \{q_{0,j}(S, 0) - Eq_{0,j}(S, 0)\}\dfrac{t^j}{j!}\right|^m$$
$$\leqq 2^m\left[\frac{|t|^{k+1}}{(k + 1)!}\right]^m \sup\{E|q_{0,k+1}(S, \nu t)|^m : 0 \leqq \nu \leqq 1\}.$$

PROOF. We have (cf. Dieudonné (1960), page 186, Titchmarsh (1939), page 368)

$$\text{(A1.16)} \qquad q(S, t) = \textstyle\sum_{j=0}^{k} q_{0,j}(S, 0)\dfrac{t^j}{j!}$$
$$+ \frac{t^{k+1}}{(k + 1)!} \int_0^1 (k + 1)(1 - \nu)^k q_{0,k+1}(S, \nu t)\, d\nu$$

provided that the integral converges. Hence the left-hand side of (A1.14) is bounded by

$$\left[\frac{|t|^{k+1}}{(k + 1)!}\right]^m E\left|\int_0^1 (k + 1)(1 - \nu)^k q_{0,k+1}(S, \nu t)\, d\nu\right|^m.$$

This obviously remains true even if the integral diverges for some values of S. An application of Ljapunov's inequality and Fubini's theorem complete the proof of (A1.14) and a similar argument disposes of (A1.15). □

Note that by using the same device one can show that the left-hand side of (A1.14) and (A1.15) is $o(|t|^{mk})$ for $t \to 0$ if q is k times continuously differentiable and

$$\text{(A1.17)} \qquad \lim_{t \to 0} E|q_{0,k}(S, t)|^m = E|q_{0,k}(S, 0)|^m .$$

Of course (A1.17) holds if $q_{0,k}(S, \cdot)$ is continuous at 0 and

$$\text{(A1.18)} \qquad \sup \{E|q_{0,k}(S, t)|^{m+\delta} : |t| \leqq \delta\} < \infty$$

for some $\delta > 0$.

We introduce two final pieces of notation. If $d_1, \cdots, d_N$ is a sequence of numbers we write

$$\text{(A1.19)} \qquad ||d|| = \frac{1}{N} \sum_{j=1}^N |d_j| .$$

If χ is a function of one variable and $\varepsilon > 0$ is fixed we define

$$\text{(A1.20)} \qquad ||\chi|| = \sup \{\textstyle\int_{-\infty}^{\infty} |\chi(x + y)| f(x)\, dx : |y| \leqq \varepsilon\} .$$

THEOREM A1.1. *Suppose that f is four times differentiable, that $E_0\psi_3(|X_1|)$, $E_0\psi_1(|X_1|)\psi_2(|X_1|)$ and $E_0\psi_1^3(|X_1|)$ exist and are finite and that $0 \leqq 2\theta \leqq \varepsilon$. Then if $r \geqq 1$, $r^{-1} + s^{-1} = 1$, there exists a constant B such that*

$$\text{(A1.21)} \qquad \begin{aligned} \sum_{j=1}^N a_j(2\pi_j - 1) = -\theta \sum_{j=1}^N a_j E_0\psi_1(Z_j) - \frac{\theta^3}{6} \sum_{j=1}^N a_j E_0[\psi_3(Z_j) \\ - 6\psi_1(Z_j)\psi_2(Z_j) + 3\psi_1^3(Z_j)] + M_1 , \\ |M_1| \leqq BN\theta^4 ||a^r||^{1/r}[||\psi_4^s|| + ||\psi_3^{4s/3}|| + ||\psi_2^{2s}|| + ||\psi_1^{4s}||]^{1/s} ; \end{aligned}$$

$$\text{(A1.22)} \qquad \begin{aligned} \sum_{j=1}^N a_j^3(2\pi_j - 1) = -\theta \sum_{j=1}^N a_j^3 E_0\psi_1(Z_j) + M_2 , \\ |M_2| \leqq BN\theta^3 ||a^{3r}||^{1/r}[||\psi_3^s|| + ||\psi_2^{3s/2}|| + ||\psi_1^{3s}||]^{1/s} ; \end{aligned}$$

$$\text{(A1.23)} \qquad \begin{aligned} \sum_{j=1}^N a_j^2 E_\theta(2P_j - 1)^2 = \theta^2 \sum_{j=1}^N a_j^2 E_0\psi_1^2(Z_j) + M_3 , \\ |M_3| \leqq BN\theta^3 ||a^{2r}||^{1/r}[||\psi_3^s|| + ||\psi_2^{3s/2}|| + ||\psi_1^{3s}||]^{1/s} ; \end{aligned}$$

$$\text{(A1.24)} \qquad \begin{aligned} \sigma_\theta^2\Big(\sum_{j=1}^N a_j P_j\Big) = \frac{\theta^2}{4} \sigma_0^2\Big(\sum_{j=1}^N a_j \psi_1(Z_j)\Big) + M_4 , \\ |M_4| \leqq BN^2\theta^{24/8} ||a^2||[||\psi_3^{4/3}|| + ||\psi_2^3|| + ||\psi_1^6||] + BN\theta^{19/8} ||a^3||^{1/3} \\ \times [||\psi^{4/3}|| + ||\psi_2^3|| + ||\psi_1^6||]^{2/3}[E_0|\sum a_j(\psi_1(Z_j) - E_0\psi_1(Z_j))|^3]^{1/3} . \end{aligned}$$

Moreover, for $m \geqq 1$ and $\rho > 0$ there exist B' and B'' depending only on m and on m and ρ respectively, and such that

$$\text{(A1.25)} \qquad \sum_{j=1}^N E_\theta|2P_j - 1|^m \leqq B'N\theta^m ||\psi_1^m|| ;$$

$$\text{(A1.26)} \qquad \begin{aligned} [\sum_{j=1}^N \{E_\theta|P_j - \pi_j|^m\}^\rho]^{1/\rho} \\ \leqq \theta^m[\sum \{E_0|\psi_1(Z_j) - E_0\psi_1(Z_j)|^m\}^\rho]^{1/\rho} \\ + B''N^{1/\rho}\theta^{2m}[||\psi_2^{m(\rho \vee 1)}|| + ||\psi_1^{2m(\rho \vee 1)}|| + 1]^{1/\rho} , \end{aligned}$$

where $\rho \vee 1$ denotes the larger of ρ and 1.

PROOF. In (A1.14) we take $E = E_0$, $q(Z, \theta) = \sum a_j(2\bar{p}(Z_j, \theta) - 1)$, $k = 3$,

$m = 1$, and find

$$|M_1| \leqq \frac{\theta^4}{4!} \sup \{E_0|2 \sum a_j \tilde{p}_{0,4}(Z_j, \nu\theta)| : 0 \leqq \nu \leqq 1\}$$

$$\leqq \frac{N\theta^4}{12} ||a^r||^{1/r} \sup \left\{ \left[\frac{1}{N} \sum E_0 |\tilde{p}_{0,4}(Z_j, \nu\theta)|^s \right]^{1/s} : 0 \leqq \nu \leqq 1 \right\},$$

by Hölder's and Ljapunov's inequalities. Since $\sum |\tilde{p}_{0,4}(Z_j, \nu\theta)|^s$ is symmetric in $Z_1, \cdots, Z_N$, we have

$$\frac{1}{N} \sum E_0 |\tilde{p}_{0,4}(Z_j, \nu\theta)|^s = E_0 |\tilde{p}_{0,4}(|X_1|, \nu\theta)|^s .$$

Now we apply (A1.13) and use the fact that the distribution of ${}_i\tilde{\psi}_j(|X_1|, \nu\theta)$ under $F(x)$ is the same as that of ${}_i\psi_j(|X_1|, \nu\theta)$ under $F(x - \nu\theta)$ to obtain

$$E_0|\tilde{p}_{0,4}(|X_1|, \nu\theta)|^s \leqq b_4^s E_{\nu\theta}[\textstyle\sum_{i=1}^2 \{|{}_i\psi_4(|X_1|, \nu\theta)| + |{}_i\psi_3(|X_1|, \nu\theta)|^{\frac{4}{3}}$$
$$+ {}_i\psi_2^2(|X_1|, \nu\theta) + {}_i\psi_1^4(|X_1|, \nu\theta)\}]^s .$$

Because $s \geqq 1$ and $0 \leqq 2\nu\theta \leqq \varepsilon$ for $0 \leqq \nu \leqq 1$, this implies that

$$E_0|\tilde{p}_{0,4}(|X_1|, \nu\theta)|^s \leqq 8^{s-1} b_4^s [||\psi_4^s|| + ||\psi_3^{4s/3}|| + ||\psi_2^{2s}|| + ||\psi_1^{4s}||] ,$$

which proves (A1.21).

The proof of (A1.22), (A1.23) and (A1.25) is similar. In each case we can apply (A1.14), taking $q(Z, \theta) = \sum a_j^3(2\tilde{p}(Z_j, \theta) - 1)$, $k = 2$, $m = 1$ to prove (A1.22), and $q(Z, \theta) = \sum a_j^2(2\tilde{p}(Z_j, \theta) - 1)^2$, $k = 2$, $m = 1$ to prove (A1.23). In (A1.25) the symmetry in $Z_1, \cdots, Z_N$ is already present from the start, so here we use (A1.14) with $q(|X_1|, \theta) = 2\tilde{p}(|X_1|, \theta) - 1$, $k = 0$ and the value of m as in (A1.25).

A rather delicate argument is needed to deal with (A1.24). Because $\tilde{p}_{0,2}(x, 0) = 0$,

$$\left(\tilde{p}(x, t) - \tfrac{1}{2} + \frac{t}{2} \psi_1(x)\right)^2$$
$$= \left| \frac{t^2}{2} \int_0^1 2(1 - \nu)\tilde{p}_{0,2}(x, \nu t)\, d\nu \right|^{\frac{6}{5}} \left| \frac{t^3}{6} \int_0^1 3(1 - \nu)^2 \tilde{p}_{0,3}(x, \nu t)\, d\nu \right|^{\frac{4}{5}}$$
$$\leqq |t|^{\frac{24}{5}} \{ |\tfrac{1}{2} \int_0^1 2(1 - \nu)\tilde{p}_{0,2}(x, \nu t)\, d\nu|^3 + |\tfrac{1}{6} \int_0^1 3(1 - \nu)^2 \tilde{p}_{0,3}(x, \nu t)\, d\nu|^{\frac{4}{3}} \}$$
$$\leqq |t|^{\frac{24}{5}} \int_0^1 \{ |\tilde{p}_{0,2}(x, \nu t)|^3 + |\tilde{p}_{0,3}(x, \nu t)|^{\frac{4}{3}} \}\, d\nu ,$$

and similarly,

$$\left| \tilde{p}(x, t) - \tfrac{1}{2} + \frac{t}{2} \psi_1(x) \right|^{\frac{3}{2}} \leqq |t|^{\frac{21}{5}} \int_0^1 \{ |\tilde{p}_{0,2}(x, \nu t)|^3 + |\tilde{p}_{0,3}(x, \nu t)|^{\frac{4}{3}} \}\, d\nu .$$

By now familiar manipulations yield

$$\left| \sigma_\theta^2(\textstyle\sum a_j P_j) - \dfrac{\theta^2}{4} \sigma_0^2(\sum a_j \psi_1(Z_j)) \right|$$
$$\leqq \sigma_0^2\left(\textstyle\sum a_j \left\{ \tilde{p}(Z_j, \theta) + \dfrac{\theta}{2} \psi_1(Z_j) \right\} \right)$$
$$+ \theta \left| \mathrm{Cov}_0 \left(\textstyle\sum a_j \left\{ \tilde{p}(Z_j, \theta) + \dfrac{\theta}{2} \psi_1(Z_j) \right\} , \sum a_j \psi_1(Z_j) \right) \right|$$

$$\leqq N^2\|a^2\|E_0\left\{\tilde{p}(|X_1|,\theta)-\tfrac{1}{2}+\frac{\theta}{2}\psi_1(|X_1|)\right\}^2+N\theta\|a^3\|^{\frac{1}{3}}\left[E_0\left|\tilde{p}(|X_1|,\theta)-\tfrac{1}{2}\right.\right.$$

$$\left.\left.+\frac{\theta}{2}\psi_1(|X_1|)\right|^{\frac{3}{2}}\right]^{\frac{2}{3}}[E_0|\sum a_j(\psi_1(Z_j)-E_0\psi_1(Z_j))|^3]^{\frac{1}{3}}$$

$$\leqq BN^2\theta^{\frac{24}{8}}\|a^2\|[\|\psi_3^{\frac{4}{3}}\|+\|\psi_2^3\|+\|\psi_1^6\|]+BN\theta^{\frac{19}{8}}\|a^3\|^{\frac{1}{3}}$$

$$\times[\|\psi_3^{\frac{4}{3}}\|+\|\psi_2^3\|+\|\psi_1^6\|]^{\frac{2}{3}}[E_0|\sum a_j(\psi_1(Z_j)-E_0\psi_1(Z_j))|^3]^{\frac{1}{3}}\,.$$

It remains to consider (A1.26). Since

$$\tilde{p}(Z_j,\theta)-E_0\tilde{p}(Z_j,\theta)=\theta[\tilde{p}_{0,1}(Z_j,0)-E_0\tilde{p}_{0,1}(Z_j,0)]$$
$$+\frac{\theta^2}{2}\textstyle\int_0^1[|\tilde{p}_{0,2}(Z_j,\nu\theta)|+E_0|\tilde{p}_{0,2}(Z_j,\nu\theta)|]2(1-\nu)\,d\nu\,,$$

and $m\geqq 1$, we have

$$E_\theta|P_j-\pi_j|^m\leqq 2^{m-1}\theta^m E_0|\tilde{p}_{0,1}(Z_j,0)-E_0\tilde{p}_{0,1}(Z_j,0)|^m$$
$$+\frac{\theta^{2m}}{2}E_0\textstyle\int_0^1\{|\tilde{p}_{0,2}(Z_j,\nu\theta)|+E_0|\tilde{p}_{0,2}(Z_j,\nu\theta)|^m\}2(1-\nu)\,d\nu$$
$$\leqq\frac{\theta^m}{2}E_0|\psi_1(Z_j)-E_0\psi_1(Z_j)|^m$$
$$+2^{m-1}\theta^{2m}\textstyle\int_0^1 E_0|\tilde{p}_{0,2}(Z_j,\nu\theta)|^m 2(1-\nu)\,d\nu\,.$$

Hence

$$\sum\{E_\theta|P_j-\pi_j|^m\}^\rho\leqq\theta^{m\rho}\sum\{E_0|\psi_1(Z_j)-E_0\psi_1(Z_j)|^m\}^\rho$$
$$+2^{m\rho}N\theta^{2m\rho}[1+\sup\{E_0|\tilde{p}_{0,2}(|X_1|,\nu\theta)|^{m(\rho\vee 1)}:0\leqq\nu\leqq 1\}]\,.$$

Proceeding as before we prove (A1.26) and the theorem. □

COROLLARY A1.1. *Suppose that positive numbers c, C and ε exist such that* (2.35), (3.2) *and* (3.3) *are satisfied. Let $\tilde{K}$, K_θ and η be defined by* (2.40), (3.4) *and* (3.5). *Then there exists $A>0$ depending on N, a, F and θ only through c, C and ε, and such that*

(A1.27)
$$\sup_x\left|\tilde{K}\left(x-\frac{\sum a_j(2\pi_j-1)}{(\sum a_j^2)^{\frac{1}{2}}}\right)-K_\theta(x-\eta)\right|$$
$$\leqq A\{N^{-\frac{5}{4}}+\theta^{\frac{19}{8}}[E_0|\sum a_j(\psi_1(Z_j)-E_0\psi_1(Z_j))|^3]^{\frac{1}{3}}$$
$$+N^{-\frac{1}{2}}\theta^5\sigma_0^2(\sum a_j\,\psi_1(Z_j))\}\,,$$

(A1.28) $$|\sum a_j^m E_0\psi_1(Z_j)|\leqq AN \qquad \textit{for}\ \ m=1,3\,,$$

(A1.29) $$|\sum a_j^2E_0\psi_1^2(Z_j)|\leqq AN\,,$$

(A1.30) $$|\sum a_jE_0[\psi_3(Z_j)-6\psi_1(Z_j)\psi_2(Z_j)+3\psi_1^3(Z_j)]|\leqq AN\,,$$

(A1.31) $$\sum E_\theta|2P_j-1|^m\leqq AN^{1-m/2} \qquad \textit{for}\ \ 1\leqq m\leqq 6\,,$$

(A1.32) $$[\sum\{E_\theta|P_j-\pi_j|^3\}^{\frac{4}{3}}]^{\frac{3}{4}}\leqq\theta^3[\sum\{E_0|\psi_1(Z_j)-E_0\psi_1(Z_j)|^3\}^{\frac{4}{3}}]^{\frac{3}{4}}+AN^{-\frac{3}{4}}\,.$$

PROOF. Since the corollary is trivially true for $N\leqq(2C/\varepsilon)^2$, we may assume that $2\theta\leqq 2CN^{-\frac{1}{2}}\leqq\varepsilon$ and use the results in Theorem A1.1. We note that (2.35) implies that $\|a^r\|\leqq[C^r\max(1,N^{r-4})]^{\frac{1}{4}}$. In the notation of this appendix (3.2) asserts that $\|\psi_i^{m_i}\|\leqq C$ for $m_1=6$, $m_2=3$, $m_3=\frac{4}{3}$ and $m_4=1$. All order symbols in this proof are uniform for fixed c, C and ε.

(A1.28)—(A1.30) follow from (2.35) and (3.2) by Hölder's and Ljapunov's inequalities, e.g.

$$|\sum a_j{}^3 E_0 \psi_1(Z_j)| = O(N \, |a| \; \|\psi_1{}^4\|^{\frac{1}{4}}) = O(N) \, .$$

(A1.31) and (A1.32) are immediate consequences of (A1.25) and (A1.26).

Taking $r = 4$, $s = \frac{4}{3}$ in (A1.22)—(A1.24) we find

$$\text{(A1.33)} \qquad M_2 = O(1) \, , \qquad M_3 = O(N^{-\frac{1}{4}}) \, ,$$
$$M_4 = O(N^{-\frac{2}{5}} + N\theta^{\frac{19}{8}}[E_0|\sum a_j(\psi_1(Z_j) - E_0\psi_1(Z_j))|^3]^{\frac{1}{3}}) \, .$$

Hence, uniformly in x,

$$\text{(A1.34)} \qquad \begin{aligned} \tilde{K}(x) = \Phi(x) + \phi(x)\Big\{ & \frac{\sum a_j{}^4}{12(\sum a_j{}^2)^2}(x^3 - 3x) - \theta \frac{\sum a_j{}^3 E_0 \psi_1(Z_j)}{3(\sum a_j{}^2)^{\frac{3}{2}}}(x^2 - 1) \\ & + \frac{\theta^2}{2\sum a_j{}^2}[\sum a_j{}^2 E_0 \psi_1{}^2(Z_j) - \sigma_0{}^2(\sum a_j \psi_1(Z_j))]x \Big\} \\ & + O(N^{-\frac{5}{4}} + \theta^{\frac{19}{8}}[E_0|\sum a_j(\psi_1(Z_j) - E_0\psi_1(Z_j))|^3]^{\frac{1}{3}}) \, . \end{aligned}$$

Taking $r = \infty$, $s = 1$ in (A1.21) we have

$$\text{(A1.35)} \qquad \begin{aligned} \frac{\sum a_j(2\pi_j - 1)}{(\sum a_j{}^2)^{\frac{1}{2}}} = \eta - \frac{\theta^3}{6(\sum a_j{}^2)^{\frac{1}{2}}} & \sum a_j E_0[\psi_3(Z_j) \\ & - 6\psi_1(Z_j)\psi_2(Z_j) + 3\psi_1{}^3(Z_j)] + O(N^{\frac{3}{4}}\theta^4) \, , \end{aligned}$$

where the second term on the right is $O(N^{\frac{1}{2}}\theta^3)$ by (A1.30). Now we substitute $x - (\sum a_j{}^2)^{-\frac{1}{2}} \sum a_j(2\pi_j - 1)$ for x in (A1.34) and expand the right-hand side around $x - \eta$. It follows from (A1.35), (A1.28) for $m = 3$ and (A1.29) that in this way we obtain (A1.27).

2. Asymptotic behavior of moments of functions of order statistics. Our aim in this appendix is twofold. In the first place we provide a proof of Theorem 3.2 where the order of the remainder in expansion (3.8) is evaluated. Secondly, we obtain asymptotic expressions for the leading terms in the expansion for the case where exact or approximate scores are used, thus in effect proving Theorems 4.1 and 4.2.

Let $U_{1:N} < U_{2:N} < \cdots < U_{N:N}$ be order statistics of a sample of size N from the uniform distribution on (0, 1).

LEMMA A2.1. *If $\lambda = j/(N+1)$ then for all $N = 1, 2, \cdots, j = 1, \cdots, N$ and $t \geqq 0$,*

$$P\left(|U_{j:N} - \lambda| \left(\frac{N}{\lambda(1-\lambda)}\right)^{\frac{1}{2}} \geqq t\right) \leqq 2\exp\left\{-\frac{3t^2}{6t+8}\right\} .$$

PROOF. The probability on the left is equal to

$$\text{(A2.1)} \qquad B\left(j, N, \lambda - t\left(\frac{\lambda(1-\lambda)}{N}\right)^{\frac{1}{2}}\right) + B\left(N - j + 1, N, 1 - \lambda - t\left(\frac{\lambda(1-\lambda)}{N}\right)^{\frac{1}{2}}\right)$$

where

$$B(j, N, p) = \sum_{k=j}^N \binom{N}{k} p^k (1-p)^{N-k} \, .$$

For $j > Np$ Bernstein's inequality (cf. Hoeffding (1963) page 17) yields

$$B(j, N, p) \leqq \exp\left\{-\frac{j - Np}{1-p}\, h\left(\frac{j - Np}{Np}\right)\right\}$$

with $h(s) = 3s(2s + 6)^{-1}$. Application of this result gives after some algebra

$$B\left(j, N, \lambda - t\left(\frac{\lambda(1-\lambda)}{N}\right)^{\frac{1}{2}}\right)$$
$$\leqq \exp\left\{-\frac{3}{2}\,\frac{[t + (\lambda/N(1-\lambda))^{\frac{1}{2}}]^2}{(3 + N^{-1}) + t(N\lambda(1-\lambda))^{-\frac{1}{2}}[\lambda(5 + N^{-1}) - 2] - 2N^{-1}t^2}\right\}.$$

Noting that $\lambda \leqq N(N+1)^{-1}$ and $(N\lambda(1-\lambda))^{-\frac{1}{2}} \leqq 1 + N^{-1}$, we see that $\exp\{-3t^2(6t+8)^{-1}\}$ is an upper bound for the first term in (A2.1). By interchanging j and $(N - j + 1)$ we find that the same is true for the second term in (A2.1) which proves the lemma. □

LEMMA A2.2. *If $\lambda = j/(N+1)$, k is a positive real number, ν_k is the kth absolute moment of the standard normal distribution and $I_{(a,b)}$ is the indicator of (a, b), then uniformly for $j = 1, \cdots, N$ and $\eta \geqq \frac{1}{2}\lambda(1-\lambda)$ we have for $N \to \infty$,*

$$\left(\frac{N}{\lambda(1-\lambda)}\right)^{\frac{1}{2}k} E(\lambda - U_{j:N})^k I_{(\lambda-\eta,\lambda)}(U_{j:N}) = \tfrac{1}{2}\nu_k + O((N\lambda(1-\lambda))^{-\frac{1}{2}}),$$
$$\left(\frac{N}{\lambda(1-\lambda)}\right)^{\frac{1}{2}k} E(U_{j:N} - \lambda)^k I_{(\lambda,\lambda+\eta)}(U_{j:N}) = \tfrac{1}{2}\nu_k + O((N\lambda(1-\lambda))^{-\frac{1}{2}}).$$

PROOF. Let f be the density of $Z = (N/\lambda(1-\lambda))^{\frac{1}{2}}(U_{j:N} - \lambda)$. Application of Stirling's formula in the form $\log n! = (n + \frac{1}{2})\log(n+1) - (n+1) + \frac{1}{2}\log 2\pi + O(n^{-1})$ followed by expansion of logarithms yields

$$\log f(z) = -\tfrac{1}{2}\log 2\pi + \frac{2\lambda - 1}{(N\lambda(1-\lambda))^{\frac{1}{2}}}\, z - \frac{1}{2}\left[1 - \frac{\lambda^3 + (1-\lambda)^3}{N\lambda(1-\lambda)}\right] z^2$$
$$+ O\left(\frac{|z|^3}{(N\lambda(1-\lambda))^{\frac{1}{2}}} + \frac{1}{N\lambda(1-\lambda)}\right)$$

for $z^2 < N\min(\lambda/(1-\lambda), (1-\lambda)/\lambda)$. Hence, for $|z| \leqq (N\lambda(1-\lambda))^{\frac{1}{6}} < [N\min(\lambda/(1-\lambda), (1-\lambda)/\lambda)]^{\frac{1}{2}}$,

$$\text{(A2.2)} \qquad f(z) = \frac{1}{(2\pi)^{\frac{1}{2}}} e^{-\frac{1}{2}z^2}\left[1 + O\left(\frac{|z| + |z|^3}{(N\lambda(1-\lambda))^{\frac{1}{2}}} + \frac{1}{N\lambda(1-\lambda)}\right)\right]$$

uniformly in j. Since $\eta(N/\lambda(1-\lambda))^{\frac{1}{2}} \geqq \frac{1}{2}(N\lambda(1-\lambda))^{\frac{1}{6}}$, (A2.2) and Lemma A2.1 imply that

$$EZ^k I_{(\lambda,\lambda+\eta)}(U_{j:N}) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_0^{\frac{1}{2}(N\lambda(1-\lambda))^{\frac{1}{6}}} z^k e^{-\frac{1}{2}z^2}\left[1 + O\left(\frac{1 + |z| + |z|^3}{(N\lambda(1-\lambda))^{\frac{1}{2}}}\right)\right] dz$$
$$+ O\left(\int_{\frac{1}{2}(N\lambda(1-\lambda))^{\frac{1}{6}}}^{\infty} z^k e^{-\frac{1}{2}z}\, dz\right) = \tfrac{1}{2}\nu_k + O((N\lambda(1-\lambda))^{-\frac{1}{2}}),$$

which proves the second part of the lemma. The first part now follows by noting that $U_{j:N}$ and $1 - U_{N-j+1:N}$ have the same distribution. □

REMARK. One easily verifies that Lemma A2.2 continues to hold when η is taken as small as $[c(\lambda(1-\lambda)/N)|\log N\lambda(1-\lambda)|]^{\frac{1}{2}}$ for any $c > 1$. It should also be noted that when j or $(N-j+1)$ remains bounded as $N \to \infty$, Lemma A2.2 merely states that $E|U_{j:N} - \lambda|^k = O(N^{-k})$.

Condition R_r. For real $r > 0$, a function h on $(0, 1)$ is said to satisfy condition R_r if h is twice continuously differentiable on $(0, 1)$ and

$$\limsup_{t \to 0,1} t(1-t)\left|\frac{h''(t)}{h'(t)}\right| < 1 + \frac{1}{r}.$$

LEMMA A2.3. *Let $r_1, \cdots, r_m, k_1, \cdots, k_m$ be positive real numbers, $j = 1, \cdots, N$, $\lambda = j/(N+1)$ and ν_k the* kth *absolute moment of the standard normal distribution. Suppose that $h_1, \cdots, h_m$ satisfy conditions $R_{r_1}, \cdots, R_{r_m}$ respectively and that $\sum k_i/r_i \leqq 1$. Define*

$$M = \left(\frac{\lambda(1-\lambda)}{N}\right)^{\frac{1}{2}\sum k_i} \left\{\left(\frac{\lambda(1-\lambda)}{N}\right)^{\frac{1}{2}} + (N\lambda(1-\lambda))^{-\frac{1}{2}} \prod_{i=1}^m |h_i'(\lambda)|^{k_i}\right\}.$$

Then, uniformly in j, we have for $N \to \infty$

$$E \prod_{i=1}^m |h_i(U_{j:N}) - h_i(\lambda)|^{k_i} = \left(\frac{\lambda(1-\lambda)}{N}\right)^{\frac{1}{2}\sum k_i} \nu_{\sum k_i} \prod_{i=1}^m |h_i'(\lambda)|^{k_i} + O(M)$$

and for integer $k_1, \cdots, k_m$

$$\begin{aligned} E \prod_{i=1}^m (h_i(U_{j:N}) - h_i(\lambda))^{k_i} &= O(M) \qquad \textit{if } \sum k_i \textit{ is odd,} \\ &= \left(\frac{\lambda(1-\lambda)}{N}\right)^{\frac{1}{2}\sum k_i} \nu_{\sum k_i} \prod_{i=1}^m (h_i'(\lambda))^{k_i} + O(M) \qquad \textit{if } \sum k_i \textit{ is even.} \end{aligned}$$

PROOF. For reasons of symmetry it is sufficient to consider only $j \leqq (N+1)/2$, i.e. $\lambda \leqq \frac{1}{2}$. Since h_i satisfies condition R_{r_i}, there exists $0 < \varepsilon < \frac{1}{6}$, $\tau > 1$ and $C > 0$ such that for $i = 1, \cdots, m$

$$\left|\frac{h_i''(t)}{h_i'(t)}\right| \leqq \left(1 + \frac{1}{r_i\tau}\right) t^{-1} \qquad \text{for} \quad 0 < t \leqq 3\varepsilon, \tag{A2.3}$$

$$|h_i''(t)| \leqq C \qquad \text{for} \quad \varepsilon \leqq t \leqq 1-\varepsilon, \tag{A2.4}$$

$$\left|\frac{h_i''(t)}{h_i'(t)}\right| \leqq \left(1 + \frac{1}{r_i\tau}\right)(1-t)^{-1} \qquad \text{for} \quad 1-3\varepsilon \leqq t < 1. \tag{A2.5}$$

Suppose first that $\lambda \leqq 2\varepsilon$. Integration of (A2.3) shows that for $0 < t \leqq \lambda$ and $i = 1, \cdots, m$,

$$\left(\frac{t}{\lambda}\right)^{1+1/r_i\tau} \leqq \frac{h_i'(t)}{h_i'(\lambda)} \leqq \left(\frac{\lambda}{t}\right)^{1+1/r_i\tau},$$

$$\frac{r_i\tau}{2r_i\tau+1}\lambda\left[1 - \left(\frac{t}{\lambda}\right)^{2+1/r_i\tau}\right] \leqq \frac{h_i(\lambda) - h_i(t)}{h_i'(\lambda)} \leqq r_i\tau\lambda\left[\left(\frac{\lambda}{t}\right)^{1/r_i\tau} - 1\right].$$

It follows that

$$\frac{h_i(\lambda) - h_i(t)}{h_i'(\lambda)} = (\lambda - t) + O\left(\frac{(\lambda-t)^2}{\lambda}\right) \qquad \text{for} \quad \tfrac{1}{2}\lambda \leqq t \leqq \lambda, \tag{A2.6}$$

$$\left|\frac{h_i(\lambda) - h_i(t)}{h_i'(\lambda)}\right| \leqq r_i\tau\lambda\left(\frac{\lambda}{t}\right)^{1/r_i\tau} \qquad \text{for} \quad 0 < t \leqq \tfrac{1}{2}\lambda.$$

Application of Lemma A2.2 with $\eta = \frac{1}{2}\lambda$ yields

$$E \prod_{i=1}^{m} \left(\frac{h_i(\lambda) - h_i(U_{j:N})}{h_i'(\lambda)}\right)^{k_i} I_{(0,\lambda)}(U_{j:N})$$

$$(\text{A2.7}) \qquad = \frac{1}{2}\left(\frac{\lambda(1-\lambda)}{N}\right)^{\frac{1}{2}\Sigma k_i} \nu_{\Sigma k_i}[1 + O((N\lambda(1-\lambda))^{-\frac{1}{2}})]$$

$$+ O\left(\lambda^{\Sigma k_i} E\left(\frac{\lambda}{U_{j:N}}\right)^{1/\tau} I_{(0,\frac{1}{2}\lambda)}(U_{j:N})\right),$$

where we have made use of $\sum k_i/r_i \leqq 1$. For $2 \leqq j \leqq \frac{1}{2}(N+1)$,

$$\lambda^{\Sigma k_i} E\left(\frac{\lambda}{U_{j:N}}\right)^{1/\tau} I_{(0,\frac{1}{2}\lambda)}(U_{j:N}) = \lambda^{\Sigma k_i + 1/\tau} \frac{N}{j-1} E U_{j-1:N-1}^{1-1/\tau} I_{(0,\frac{1}{2}\lambda)}(U_{j-1:N-1})$$

$$(\text{A2.8}) \qquad \leqq 2\lambda^{\Sigma k_i} P(U_{j-1:N-1} < \tfrac{1}{2}\lambda)$$

$$= O\left(\left(\frac{\lambda(1-\lambda)}{N}\right)^{\frac{1}{2}\Sigma k_i} (N\lambda(1-\lambda))^{-\frac{1}{2}}\right)$$

by Lemma A2.1. For $j = 1$ we have

$$\lambda^{\Sigma k_i} E\left(\frac{\lambda}{U_{j:N}}\right)^{1/\tau} I_{(0,\frac{1}{2}\lambda)}(U_{j:N})$$

$$(\text{A2.9}) \qquad = (N+1)^{-\Sigma k_i - 1/\tau} N \int_0^{1/2(N+1)} u^{-1/\tau}(1-u)^{N-1}\, du$$

$$= O(N^{-\Sigma k_i}) = O\left(\left(\frac{\lambda(1-\lambda)}{N}\right)^{\frac{1}{2}\Sigma k_i} (N\lambda(1-\lambda))^{-\frac{1}{2}}\right).$$

Together, (A2.8) and (A2.9) ensure that the second remainder term in (A2.7) may be omitted.

A similar analysis based on (A2.3)—(A2.5) shows that for $\lambda \leqq 2\varepsilon$ but $t \geqq \lambda$, (A2.6) holds for $\lambda \leqq t \leqq 3\lambda/2$ and

$$\left|\frac{h_i(t) - h_i(\lambda)}{h_i'(\lambda)}\right| \leqq r_i \tau \lambda \left(\frac{t}{\lambda}\right)^{2+1/r_i\tau} \qquad \text{for} \quad \frac{3\lambda}{2} \leqq t \leqq 3\varepsilon\,,$$

$$= O(\lambda^{-1-1/r_i\tau}(1-t)^{-1/r_i\tau}) \qquad \text{for} \quad 3\varepsilon \leqq t < 1\,.$$

Hence by Lemmas A2.2 and A2.1 and a change from $U_{j:N}$ to $U_{j:N-1}$ as in (A2.8),

$$E \prod_{i=1}^{m} \left(\frac{h_i(U_{j:N}) - h_i(\lambda)}{h_i'(\lambda)}\right)^{k_i} I_{(\lambda,1)}(U_{j:N})$$

$$= \frac{1}{2}\left(\frac{\lambda(1-\lambda)}{N}\right)^{\frac{1}{2}\Sigma k_i} \nu_{\Sigma k_i}[1 + O((N\lambda(1-\lambda))^{-\frac{1}{2}})]$$

$$(\text{A2.10}) \qquad + O(\lambda^{\Sigma k_i} \exp\{-\tfrac{1}{4}(N\lambda)^{\frac{1}{2}}\}$$

$$+ \lambda^{-\Sigma k_i - 1/\tau} E(1 - U_{j:N-1})^{1-1/\tau} I_{(3\varepsilon,1)}(U_{j:N-1}))$$

$$= \frac{1}{2}\left(\frac{\lambda(1-\lambda)}{N}\right)^{\frac{1}{2}\Sigma k_i} \nu_{\Sigma k_i}[1 + O((N\lambda(1-\lambda))^{-\frac{1}{2}})]\,.$$

Combining (A2.7)—(A2.10) and noting that (A2.7) and (A2.10) remain valid when absolute values are taken inside the expectation signs, we see that the lemma is proved for $\lambda \leqq 2\varepsilon$.

If $2\varepsilon < \lambda \leqq \frac{1}{2}$, (A2.3)—(A2.5) imply that

$$h_i(t) - h_i(\lambda) = h_i'(\lambda)(t-\lambda) + O((t-\lambda)^2) \qquad \text{for} \quad \varepsilon \leqq t \leqq 1-\varepsilon\,,$$
$$|h_i(t) - h_i(\lambda)| = O((t(1-t))^{-1/r_i\tau}) \qquad \text{for} \quad t < \varepsilon \quad \text{or} \quad t > 1-\varepsilon\,,$$

and the proof of the lemma for $2\varepsilon < \lambda \leqq \frac{1}{2}$ follows by noting that $h_i'(\lambda)$ is bounded and arguing as e.g. in (A2.10). □

REMARK. Although the remainder M in Lemma A2.3 consists of two terms, only one of these plays a role for any particular value of λ. For $2\varepsilon < \lambda < 1 - 2\varepsilon$, $h_i'(\lambda)$ and $(\lambda(1-\lambda))^{-1}$ are bounded and we need only retain the first term of M. It follows from (A2.7)—(A2.10) that for $\lambda \leqq 2\varepsilon$ or $\lambda \geqq 1 - 2\varepsilon$ only the second term of M is needed.

LEMMA A2.4. *Lemma* A2.3 *continues to hold for central moments,* i.e. *if* $h_i(\lambda)$ *is replaced by* $Eh_i(U_{j:N})$ *for* $i = 1, \cdots, m$, *provided only that* $r_i \geqq 1$ *for* $i = 1, \cdots, m$.

PROOF. As $r_i \geqq 1$, Lemma A2.3 contains as a special case

$$|Eh_i(U_{j:N}) - h_i(\lambda)| = O\left(\frac{\lambda(1-\lambda) + |h_i'(\lambda)|}{N}\right). \tag{A2.11}$$

The lemma is proved by expanding the central moments in terms of moments centered at the $h_i(\lambda)$ and applying (A2.11), Lemma A2.3 and the remark following it. □

We also note the following extension of a result of Hoeffding (1953).

LEMMA A2.5. *Let* $h_1, \cdots, h_m$ *be continuous functions on* $(0, 1)$, q *a continuous function on* R^m *and* Q *a convex function on* R^m *such that* $|q| \leqq Q$. *Suppose that* $\int_0^1 |h_i(t)|\, dt < \infty$ *for* $i = 1, \cdots, m$ *and that* $\int_0^1 Q(h_1(t), \cdots, h_m(t))\, dt < \infty$. *Then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^N q(Eh_1(U_{j:N}), \cdots, Eh_m(U_{j:N})) = \int_0^1 q(h_1(t), \cdots, h_m(t))\, dt\,.$$

PROOF. Because h_i is continuous and summable, Lemma 2.2 of Bickel (1967) implies that for any $\varepsilon > 0$, $Eh_i(U_{j_N:N}) - h_i(j_N(N+1)^{-1}) \to 0$ uniformly for $\varepsilon \leqq j_N(N+1)^{-1} \leqq 1 - \varepsilon$ as $N \to \infty$. Since q is continuous and $q(h_1, \cdots, h_m)$ is summable, the lemma is proved if we show that

$$\lim_{\varepsilon\downarrow 0} \limsup_N \frac{1}{N} \left(\sum_{j=1}^{[\varepsilon(N+1)]} + \sum_{j=[(1-\varepsilon)(N+1)]}^{N}{}'\right)|q(Eh_1(U_{j:N}), \cdots, Eh_m(U_{j:N}))| = 0\,.$$

It is obviously sufficient to prove this for Q instead of q, but as Q has the same properties as q and is moreover nonnegative, this is equivalent to showing that

$$\limsup_N \frac{1}{N} \sum_{j=1}^N Q(Eh_1(U_{j:N}), \cdots, Eh_m(U_{j:N})) \leqq \int_0^1 Q(h_1(t), \cdots, h_m(t))\, dt\,.$$

As Q is convex this follows from Jensen's inequality. □

LEMMA A2.6. *Let* $k_1, \cdots, k_m$ *be positive integers and* $r_1, \cdots, r_m$ *positive real numbers such that* $\sum k_i/r_i \leqq 1$. *Suppose that* $h_1, \cdots, h_m$ *are continuous functions* *on* $(0, 1)$ *for which* $\int_0^1 |h_i(t)|^{r_i}\, dt < \infty$ *for* $i = 1, \cdots, m$. *Then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^N \prod_{i=1}^m (Eh_i(U_{j:N}))^{k_i} = \int_0^1 \prod_{i=1}^m (h_i(t))^{k_i}\, dt\,.$$

If, in addition, h_1 *is monotone in neighborhoods of* 0 *and* 1, *then also*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^N \left(h_1\left(\frac{j}{N+1}\right)\right)^{k_1} \prod_{i=2}^m (Eh_i(U_{j:N}))^{k_i} = \int_0^1 \prod_{i=1}^m (h_i(t))^{k_i}\, dt\,.$$

PROOF. The first part of the lemma is a special case of Lemma A2.5, obtained by taking $q(x_1, \cdots, x_m) = \prod x_i^{k_i}$ and $Q(x_1, \cdots, x_m) = 1 + \sum |x_i|^{r_i}$. To establish the second part we follow the proof of Lemma A2.5 for these choices of q and Q but with $Eh_1(U_{j:N})$ replaced by $h_1(j(N+1)^{-1})$, until we arrive at the point where it suffices to show that

$$\limsup_N \frac{1}{N} \sum_{j=1}^N \left[\left|h_1\left(\frac{j}{N+1}\right)\right|^{r_1} + \sum_{i=2}^m |Eh_i(U_{j:N})|^{r_i}\right] \leqq \int_0^1 \sum_{i=1}^m |h_i(t)|^{r_i}\, dt\,.$$

As $|h_1|^{r_1}$ is continuous and summable, its monotonicity near 0 and 1 amply guarantees that $N^{-1} \sum |h_1(j(N+1)^{-1})|^{r_1} \to \int_0^1 |h_1(t)|^{r_1}\, dt$. Application of Jensen's inequality to the remaining terms completes the proof. □

We now state the results needed to prove Theorems 3.2, 4.1 and 4.2 in the form of two corollaries.

COROLLARY A2.1. *Suppose that positive numbers* C *and* δ *exist such that* $|h'(t)| \leqq C(t(1-t))^{-\frac{4}{3}+\delta}$ *for all* $0 < t < 1$. *Then there exists* $A > 0$ *depending on* N *and* h *only through* C *and* δ *and such that*

$$\sum_{j=1}^N \{E|h(U_{j:N}) - Eh(U_{j:N})|^3\}^{\frac{4}{9}} \leqq AN^{\frac{4}{9}}\,.$$

The above condition is fulfilled if h *satisfies condition* R_1 *and* $\int_0^1 h^6(t)\, dt < \infty$.

PROOF. Define $\lambda = j/(N+1)$. For all $0 < t < 1$, $|h(t) - h(\lambda)|$ is maximized by taking $h'(t) \equiv C(t(1-t))^{-\frac{4}{3}+\delta}$ and for this particular choice of h' the function h satisfies condition R_3. Hence, by Lemma A2.3, we have in general

$$E|h(U_{j:N}) - h(\lambda)|^k = O\left(\left(\frac{\lambda(1-\lambda)}{N}\right)^{\frac{1}{2}k} (\lambda(1-\lambda))^{-k(\frac{4}{3}-\delta)}\right)$$

for $0 < k \leqq 3$. It follows that

$$\begin{aligned}\sum_{j=1}^N \{E|h(U_{j:N}) - Eh(U_{j:N})|^3\}^{\frac{4}{9}} &= O\left(\sum_{j=1}^N \{N^{-\frac{3}{2}}(\lambda(1-\lambda))^{-\frac{5}{2}}\}^{\frac{4}{9}}\right)\\ &= O\left(N^{\frac{1}{3}} \int_{1/N}^{1-1/N} (t(1-t))^{-\frac{10}{9}}\, dt\right) = O(N^{\frac{4}{9}})\,.\end{aligned}$$

Condition R_1 ensures that for ε as in (A2.3) and $0 < t < \frac{1}{2}u < \varepsilon$, $|h(t) - h(2\varepsilon)| \geqq \frac{1}{4}u|h'(u)|$ and hence for $u \to 0$,

$$u^7(h'(u))^6 \leqq 2^{13} \int_0^{\frac{1}{2}u} (h(t) - h(2\varepsilon))^6\, dt \to 0\,.$$

In the same way one shows that $|h'(u)| = o((1-u)^{-\frac{7}{6}})$ for $u \to 1$, which completes the proof. □

For $i = 1, 2, 3$, let $\psi_i = f^{(i)}/f$ and $\Psi_i(t) = \psi_i(F^{-1}((1+t)/2))$ as in (3.1) and (3.15). Let J be a function on (0, 1).

COROLLARY A2.2. *Suppose that* (3.2) *holds, that* $0 < \int_0^1 J^4(t)\,dt < \infty$ *and that both* J *and* Ψ_1 *satisfy condition* R_2. *Let either* $a_j = a_{j,N} = EJ(U_{j:N})$ *for* $j = 1, \cdots, N$ *or* $a_j = a_{j,N} = J(j/(N+1))$ *for* $j = 1, \cdots, N$. *Then, as* $N \to \infty$,

$$\frac{1}{N}\sum_{j=1}^N a_j^2 = \int_0^1 J^2(t)\,dt + o(1)\,, \tag{A2.12}$$

$$\frac{1}{N}\sum_{j=1}^N a_j^k E\Psi_1^{4-k}(U_{j:N}) = \int_0^1 J^k(t)\Psi_1^{4-k}(t)\,dt + o(1)\,, \qquad k = 1, \cdots, 4\,, \tag{A2.13}$$

$$\frac{1}{N}\sum_{j=1}^N a_j E\Psi_1(U_{j:N})\Psi_2(U_{j:N}) = \int_0^1 J(t)\Psi_1(t)\Psi_2(t)\,dt + o(1)\,, \tag{A2.14}$$

$$\frac{1}{N}\sum_{j=1}^N a_j E\Psi_3(U_{j:N}) = \int_0^1 J(t)\Psi_3(t)\,dt + o(1)\,, \tag{A2.15}$$

$$\frac{1}{N}\sigma^2\left(\sum_{j=1}^N a_j \Psi_1(U_{j:N})\right) = \int_0^1\int_0^1 J(s)J(t)\Psi_1'(s)\Psi_1'(t)[s \wedge t - st]\,ds\,dt + o(1)\,. \tag{A2.16}$$

If $a_j = EJ(U_{j:N})$ *for* $j = 1, \cdots, N$, *then also*

$$\begin{aligned} N^{-\frac12}\frac{\sum_{j=1}^N a_j E\Psi_1(U_{j:N})}{(\sum_{j=1}^N a_j^2)^{\frac12}} &= \frac{\int_0^1 J(t)\Psi_1(t)\,dt}{(\int_0^1 J^2(t)\,dt)^{\frac12}} - \frac{1}{N}\frac{\sum_{j=1}^N \operatorname{Cov}(J(U_{j:N}), \Psi_1(U_{j:N}))}{(\int_0^1 J^2(t)\,dt)^{\frac12}} \\ &\quad + \frac{1}{2N}\frac{\int_0^1 J(t)\Psi_1(t)\,dt}{(\int_0^1 J^2(t)\,dt)^{\frac32}}\sum_{j=1}^N \sigma^2(J(U_{j:N})) + o(N^{-1}) \\ &= \frac{\int_0^1 J(t)\Psi_1(t)\,dt}{(\int_0^1 J^2(t)\,dt)^{\frac12}} - \frac{1}{N}\frac{\int_{1/N}^{1-1/N} J'(t)\Psi_1'(t)t(1-t)\,dt}{(\int_0^1 J^2(t)\,dt)^{\frac12}} \\ &\quad + \frac{1}{2N}\frac{\int_0^1 J(t)\Psi_1(t)\,dt}{(\int_0^1 J^2(t)\,dt)^{\frac32}}\int_{1/N}^{1-1/N}(J'(t))^2 t(1-t)\,dt + o(N^{-1}) \\ &\quad + O\left(N^{-\frac32}\int_{1/N}^{1-1/N}|J'(t)|(|J'(t)| + |\Psi_1'(t)|)(t(1-t))^{\frac12}\,dt\right). \end{aligned} \tag{A2.17}$$

If $J = -\Psi_1$ *and either* $a_j = -E\Psi_1(U_{j:N})$ *for* $j = 1, \cdots, N$ *or* $a_j = -\Psi_1(j/(N+1))$ *for* $j = 1, \cdots, N$, *then*

$$\begin{aligned} N^{-\frac12}\frac{\sum_{j=1}^N a_j E\Psi_1(U_{j:N})}{(\sum_{j=1}^N a_j^2)^{\frac12}} &= -\left(\int_0^1 \Psi_1^2(t)\,dt\right)^{\frac12} + \frac{\int_{1/N}^{1-1/N}(\Psi_1'(t))^2 t(1-t)\,dt}{2N(\int_0^1 \Psi_1^2(t)\,dt)^{\frac12}} \\ &\quad + o(N^{-1}) + O\left(N^{-\frac32}\int_{1/N}^{1-1/N}(\Psi_1'(t))^2(t(1-t))^{\frac12}\,dt\right). \end{aligned} \tag{A2.18}$$

PROOF. The assumptions imply that Ψ_1, Ψ_2, Ψ_3 and J are continuous, that Ψ_1^6, Ψ_2^3, $|\Psi_3|^{\frac{4}{3}}$ and J^4 are summable and that J is monotone near 0 and 1. Hence (A2.12)—(A2.15) follow from Lemma A2.6.

For $a_j = J(j/(N+1))$ a proof of (A2.16) is essentially contained in Stigler (1969). Our condition R_2 for Ψ_1 ensures that Ψ_1' will satisfy Stigler's condition T at 0 and 1. As in the proof of Corollary A2.1, one can argue that near 0 and 1

$$\Psi_1'(t) = o((t(1-t))^{-\frac{7}{6}}) , \qquad J'(t) = o((t(1-t))^{-\frac{5}{4}}) . \tag{A2.19}$$

Inspection of Stigler's conditions for (A2.16) shows that in our case the only missing ingredient is that Ψ_1 is not necessarily increasing on (0, 1). However, Ψ_1 is monotone where it matters, that is in a neighborhood of 0 and 1.

To prove that (A2.16) remains valid for $a_j = EJ(U_{j:N})$ we note that by Lemma A2.4 and (A2.19)

$$\begin{aligned}\sigma^2\Big(\textstyle\sum_{j=1}^N \Big(EJ(U_{j:N}) - J\Big(\frac{j}{N+1}\Big)\Big)\,\Psi_1(U_{j:N})\Big) &\\ \leqq \Big[\textstyle\sum_{j=1}^N \Big|EJ(U_{j:N}) - J\Big(\frac{j}{N+1}\Big)\Big|\,\sigma(\Psi_1(U_{j:N}))\Big]^2 &\\ = o(N^{-1}[\textstyle\int_{1/N}^{1-1/N} (t(1-t))^{-\frac{23}{12}}\, dt]^2) = o(N^{\frac{5}{6}}) .&\end{aligned}$$

For $a_j = EJ(U_{j:N})$ we have

$$\frac{1}{N}\sum_{j=1}^N a_j^2 = \int_0^1 J^2(t)\,dt - \frac{1}{N}\sum_{j=1}^N \sigma^2(J(U_{j:N})) , \tag{A2.20}$$

$$\begin{aligned}\frac{1}{N}\sum_{j=1}^N a_j E\Psi_1(U_{j:N}) &\\ = \int_0^1 J(t)\Psi_1(t)\,dt - \frac{1}{N}\sum_{j=1}^N \operatorname{Cov}(J(U_{j:N}), \Psi_1(U_{j:N})) .&\end{aligned} \tag{A2.21}$$

By Lemma A2.4, condition R_2 for J, and (A2.19)

$$\begin{aligned}\frac{1}{N}\sum_{j=1}^N \sigma^2(J(U_{j:N})) &\\ = \frac{1}{N}\int_{1/N}^{1-1/N} (J'(t))^2 t(1-t)\,dt + O(N^{-2}\int_{1/N}^{1-1/N} (J'(t))^2\,dt + N^{-\frac{3}{2}} &\\ + N^{-\frac{3}{2}}\int_{1/N}^{1-1/N} (J'(t))^2 (t(1-t))^{\frac{1}{2}}\,dt) &\\ = \frac{1}{N}\int_{1/N}^{1-1/N} (J'(t))^2 t(1-t)\,dt &\\ + O(N^{-\frac{3}{2}} + N^{-\frac{3}{2}}\int_{1/N}^{1-1/N} (J'(t))^2 (t(1-t))^{\frac{1}{2}}\,dt) = o(N^{-\frac{1}{2}}) .&\end{aligned} \tag{A2.22}$$

Similarly

$$\begin{aligned}\frac{1}{N}\sum_{j=1}^N \operatorname{Cov}(J(U_{j:N}), \Psi_1(U_{j:N})) &\\ = \frac{1}{N}\int_{1/N}^{1-1/N} J'(t)\Psi_1'(t)t(1-t)\,dt &\\ + O(N^{-\frac{3}{2}} + N^{-\frac{3}{2}}\int_{1/N}^{1-1/N} |J'(t)\Psi_1'(t)|(t(1-t))^{\frac{1}{2}}\,dt) = o(N^{-\frac{7}{12}}) .&\end{aligned} \tag{A2.23}$$

Together (A2.20)—(A2.23) are sufficient to prove (A2.17).

If $J = -\Psi_1$ and $a_j = -E\Psi_1(U_{j:N})$, then (A2.17) reduces to (A2.18). To prove that (A2.18) also holds if $a_j = -\Psi_1(j/(N+1))$, it suffices to show that

$$\sum_{j=1}^N \Psi_1\left(\frac{j}{N+1}\right) E\Psi_1(U_{j:N}) - \left[\sum_{j=1}^N \Psi_1^2\left(\frac{j}{N+1}\right) \sum_{j=1}^N (E\Psi_1(U_{j:N}))^2\right]^{\frac{1}{2}} = o(1) + O(N^{-\frac{1}{2}} \textstyle\int_{1/N}^{1-1/N} (\Psi_1'(t))^2(t(1-t))^{\frac{1}{2}}\, dt)\,. \tag{A2.24}$$

It follows from Lemma A2.3 and condition R_2 for Ψ_1 that

$$\sum_{j=1}^N \left\{E\Psi_1(U_{j:N}) - \Psi_1\left(\frac{j}{N+1}\right)\right\}^2 = O(N^{-1} + N^{-1} \textstyle\int_{1/N}^{1-1/N} (\Psi_1'(t)^2\, dt)$$

$$= O(N^{-1} + N^{-\frac{1}{2}} \textstyle\int_{1/N}^{1-1/N} (\Psi_1'(t))^2(t(1-t))^{\frac{1}{2}}\, dt)\,,$$

which suffices to establish (A2.24) and complete the proof. □

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