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Asymptotic Expansions in Limits of Large Momenta and Masses

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Abstract. Asymptotic expansions of renormalized Feynman amplitudes in limits of large momenta and/or masses are proved. The corresponding asymptotic operator expansions for the S-matrix, composite operators and their time-ordered products are presented. Coefficient functions of these expansions are homogeneous within a regularization of dimensional or analytic type. Furthermore, they are explicitly expressed in terms of renormalized Feynman amplitudes (at the diagrammatic level) and certain Green functions (at the operator level).

1. Introduction

Thirty years ago Weinberg described the leading large momentum behavior of Feynman amplitudes [44]. Logarithmic corrections were characterized by Fink [20], and Slavnov [32] proved that the large momentum asymptotic expansions are always performed in powers and logarithms of the expansion parameter. Recently Hurd applied the three expansion renormalization method (see [24] and references therein) and analyzed the large momentum asymptotic behavior in the coordinate-space language in terms of short-distance expansion of Feynman amplitudes $G_\Gamma(x_1, x_2, \dots)$ at $x_1 \to x_2$.

In papers [4–6, 18, 25, 30] asymptotic expansions in various large momentum limits were obtained. A typical result is the expansion of a Feynman integral

$$F_\Gamma(\underline{Q}/\varrho, \underline{q}) \overset{\varrho\to 0}{\sim} \sum_{k,l} C_{k,l}\varrho^k \ln^l \varrho\,, \tag{1.1}$$

where $\underline{Q}(\underline{q})$ are large (small) momenta. However the coefficient functions $C_{k,l}$ in these papers are rather cumbersome. For instance, they are expressed in terms of numerous parametric integrals [4] or in terms of Mellin integrals [5, 6, 18, 25, 30]. At least, they are not naturally associated with renormalized and/or regularized Feynman amplitudes.

Some time ago, the situation for operator asymptotic expansions was reversed in its character. For example, coefficient functions of Wilson expansion [45]

$$T J_1(x+\varrho\xi_1) J_2(x+\varrho\xi_2) \overset{\varrho\to 0}{\sim} \sum_i C_i(\varrho\xi_1,\varrho\xi_2)\mathcal{O}_i(x) \tag{1.2}$$

were naturally expressed by Zimmermann [50], and by Anikin and Zavialov [2, 3, 46], in terms of Green functions of composite operators J_1 and J_2 (here $\{\mathcal{O}_i\}$ is a basis of composite operators). However this expansion is not in powers and logarithms of the expansion parameter ϱ because the coefficient functions $C_i(\varrho\xi_1,\varrho\xi_2)$ depend on it nontrivially. This is the reason why Zimmermann's method was not really applied at the diagrammatic level. On the other hand, the above mentioned results on the expansion (1.1) were never generalized to the operator level since the corresponding coefficient functions are not naturally characterized as Green functions of composite operators.

In a number of papers [9, 11, 12, 21–23, 26, 29, 36, 37] the asymptotic expansions with two desired properties were derived both at the diagrammatic and the operator level. In particular, at the diagrammatic level, the coefficients $C_{k,l}$ are renormalized (or R^*-normalized – see below) Feynman integrals. In turn, the corresponding operator expansions, e.g. (1.2), are in powers and logarithms of the expansion parameter and the coefficient functions are Green functions of composite operators. However these results have not been analytically justified. For example, in papers [21, 29] the authors implicitly assume the validity of some asymptotic expansion which serves as a starting point of combinatorial manipulations. Furthermore, in asymptotic expansions of Refs. [21–23, 26, 29] there is no explicit infrared (IR) finiteness. An alternative approach – the method of glueing [9, 11, 12] – is based on the existence of some asymptotic expansion. Note also that in the textbooks by Collins and Muta [17, 29] the proof of the expansions is substituted by lower order arguments.

The main purpose of this paper is to prove asymptotic expansions in limits of large momenta and/or masses with both above mentioned properties: the expansions are in powers and logarithms, and coefficient functions are written as renormalized Feynman amplitudes (at the diagrammatic level) or Green functions of composite operators (at the operator level). We shall apply a straightforward generalization of the method of Zimmermann, Anikin and Zavialov [2, 3, 46, 50] that is based on pre-subtractions in certain subgraphs. This procedure removes the ultraviolet (UV) divergences which are generated in the limit under consideration. However we define the pre-subtracting operator that provides the desired properties of the expansions. It is this technique that was applied in Refs. [36, 37] to derive the expansion of the effective action of low-energy theory in inverse powers of the heavy mass and the Wilson expansion for a product of several composite operators.

To derive the asymptotic expansions in limits of large momenta and/or masses it is natural to use the minimal subtraction scheme [7, 41] within the dimensional [42, 7] or analytic [40] regularization. It is also possible to derive "simple" asymptotic expansions in other renormalization schemes. If one wants, however, to provide the two basic properties of the expansions, then it is natural to turn to expansions with composite operators renormalized in the minimal subtraction scheme or its analytic generalization.

We shall derive asymptotic expansions in two forms. In the first form, some terms in the expansion may be divergent. To show that the divergences are cancelled we shall use the second form of the expansion where the coefficient functions are explicitly finite due to the R^*-operation [13–15] which is a generalization of the dimensional renormalization when both ultraviolet and infrared divergences are involved. In fact, the R^*-operation removes here the IR divergences that are induced by "naive" Taylor expansions of Feynman integrals around zero values of small momenta and masses.

In the next section we introduce definitions and notations relevant to Feynman graphs, amplitudes, integrals, and to the R-operation. We use the analytic renormalization in the style of the dimensional one: instead of Speer's evaluator which performs the renormalization and, at the same time, removes the regularization [40], we imply the recursive insertion of counterterms and do not remove the regularization to the very end.

In Sect. 3 diagrammatic expansions are derived, using Zimmermann identities [48] which connect the initial renormalization and an appropriate pre-subtracting procedure. In Sect. 4 these asymptotic expansions are written in the explicitly finite form. Section 5 contains the main result of the present work – a proof of the asymptotic estimate of the remainder. This proof is based on the α-representation technique. The corresponding asymptotic expansions at the operator level are derived in Sect. 6. The key method for this purpose turns out to be the counterterm technique of Anikin and Zavialov [1, 46] generalized to the case of Lagrangians and composite operators without normal ordering [39]. It is this technique in which operator Zimmermann identities are naturally written. Finally, in Appendix, integrands of α-representations are described and corresponding factorization formulae are listed.

2. Renormalized Feynman Amplitudes

Let Γ be a connected graph with L lines and V vertices. The corresponding dimensionally regularized Feynman amplitude is written as

$$G_\Gamma(\underline{q},\underline{m};\varepsilon) = (2\pi)^d \delta^{(d)}\left(\sum_i q_i\right) F_\Gamma(\underline{q},\underline{m};\varepsilon)\,, \tag{2.1}$$

where $F_\Gamma(\underline{q},\underline{m};\varepsilon)$ is a dimensionally regularized Feynman integral depending on masses $\underline{m} = (m_1,\ldots,m_L)$ and external momenta $\underline{q} = (q_1,\ldots,q_{N-1})$, and $d = 4-2\varepsilon$ is the space-time dimension. The Feynman integral may be formally represented as an integral over loop momenta. It is unambiguously defined in terms of the α-representation [7, 46]

$$F_\Gamma(\underline{q},\underline{m};\varepsilon) = \int_0^\infty d\underline{\alpha} I_\Gamma(\underline{q},\underline{m},\underline{\alpha};\varepsilon)\,, \qquad I_\Gamma = I_\Gamma^{(0)} \exp\left(-\frac{i}{2}\sum_l m_l^2\alpha_l\right),$$

$$I_\Gamma^{(0)} = (4\pi^2 i)^{-\mathfrak{N}d/4} 2^{-L} D_\Gamma^{\varepsilon-2}(\underline{\alpha}) \left\{\prod_l Z_l(-i\partial/\partial u_l) \exp\left(\frac{i}{2} W(\underline{q},\underline{u},\underline{\alpha})\right)\right\}\Bigg|_{\underline{u}=0}, \tag{2.2}$$

$$W = D_\Gamma^{-1}(A + 2B - K)\,.$$

Here $Z_l(p_l)$ is the numerator of the propagator of the l^{th} line (it is implied that vertex factors are included in Z_l which are supposed to be homogeneous polynomials with degrees n_l); $\mathfrak{N} = L - V + 1$ is the number of independent circuits (loops), and A, B, K are standard polylinear forms (see the Appendix).

To define the dimensional regularization we regard the parameter d in (2.2) as a complex number. Moreover, monomials in q_i and u_l which appear after the action of operators $Z_l(-i\partial/\partial u_l)$, as well as the metric tensor $g^{\mu\nu}$, are regarded as elements of the algebra of covariants where, in particular, one has $(\partial/\partial u_l^\mu)u_{l'}^\nu = g_\mu^\nu \delta_{ll'}$, $g_\mu^\mu = d$. Furthermore, this algebra includes, if necessary, the γ-matrices, the tensor $\varepsilon_{\kappa\lambda\mu\nu}$ etc. The algebra is characterized by the basis in which any element is supposed to be uniquely expanded. In other words, an element is transformed into the "normal form" – see detailed definitions in [7]. However, in the present work an explicit construction of such a basis is not essential.

Thus, any dimensionally regularized Feynman integral is defined by Eq. (2.2). It is a sum of tensor monomials which are built from external momenta multiplied by functions of scalar products q_iq_j. If all masses m_l are non-zero then, at sufficiently large $\mathrm{Re}\,\varepsilon$, the α-integral (2.2) is convergent. At other values of ε it is understood in the sense of analytic continuation. In case of zero masses one cannot always find a domain of complex values of ε where the α-integral would be convergent, if both UV and IR divergences are involved. Then, to define dimensionally regularized Feynman integrals it is convenient to introduce the analytic regularization by inserting the factor $\prod_l \alpha^{\lambda_l}$ into the integrand of the α-representation (such analytic regularization coincides with the "standard" one [40] up to a trivial product of Γ-functions). If there is a massless detachable subgraph (i.e. with zero external momenta) the Feynman integral is equal to zero. For a graph without such subgraphs, there is a domain of parameters $(\varepsilon, \underline{\lambda})$ where analytically and dimensionally regularized α-integral is absolutely convergent. It turns out to be a meromorphic function of $(\varepsilon, \underline{\lambda})$, and the dimensionally regularized Feynman integral is defined through analytic continuation to the point $(\varepsilon, \underline{0})$ [38]. An alternative definition of dimensionally regularized Feynman integrals is based on the use of Mellin transformation [19].

In the framework of the dimensional regularization one may regard Feynman integrals as tempered distributions when the regularization is not "completely" removed, i.e. when the momenta are considered as four-dimensional objects and the regularization parameter d is not yet equal to four. However this procedure seems unnatural. For instance, it is sensible to consider, at $d \neq 4$, the "formal" d-dimensional Fourier transform rather than the "true" Fourier transform that is always uniquely defined for any tempered distribution. Thus, in what follows, we shall imply that dimensionally regularized Feynman integrals are regarded in the domain of non-exceptional Euclidean momenta where $(q_{i_1} + q_{i_2} + \dots)^2 < 0$ for any subset of indices $i_1, i_2, \dots$.

We shall define an analytically regularized Feynman integral as a Feynman integral with one analytic regularization parameter: it is obtained by the analytic continuation to the point $\underline{\lambda} = (\lambda_1, \dots, \lambda_L)$ with $\lambda_l = \lambda$ for $l = 1, \dots, L$. As in the case of the dimensional regularization, a massless Feynman integral with zero external momenta is equal to zero. An analytically regularized Feynman amplitude is also naturally understood as a tempered distribution. It can be defined in terms of the α-representation

$$G_\Gamma(\underline{q},\underline{m};\lambda) = (2\pi)^4\delta\left(\sum_i q_i\right) \int d\underline{\alpha} \prod_l \alpha_l^{\lambda} I_\Gamma(\underline{q},\underline{m},\underline{\alpha}) . \tag{2.3}$$

When proving the estimate of the remainder of the asymptotic expansion we shall use the α-representation (2.3) and various "mixed" representations which are obtained from it by the Fourier transformation with respect to a part of momenta. This representation looks like [33, 35]

$$G_\Gamma(\underline{x},\underline{q},\underline{m};\lambda) = \int d\underline{\alpha} \prod_l \alpha_l^{\lambda} I_\Gamma^x(\underline{x},\underline{q},\underline{m}),$$

$$I_\Gamma^x = I_\Gamma^{(0)x} \exp\left(-\frac{i}{2}\sum_l m_l^2\alpha_l\right), \tag{2.4}$$

$$I_\Gamma^{(0)x} = (4\pi^2 i)^{-\Re(\hat{\Gamma}^x)} 2^{-L} \left\{ \prod_l Z_l(-i\partial/\partial u_l) \exp\left(\frac{i}{2} W^x(\underline{x},\underline{q},\underline{u},\underline{\alpha})\right)\right\}\Bigg|_{\underline{u}=0} .$$

Here $\hat{\Gamma}^x$ is the graph obtained from Γ by adding an extra vertex $\hat{v}^x$ which is connected by extra lines (they form the set $\hat{\mathscr{L}}^x$) with vertices considered in coordinate space. Explicit formulae for a part of polylinear forms in (2.4) are listed in the Appendix. Note that a Fourier transform in a part of the variables is denoted, for brevity, by the same letter withouth tilde.

As it was proved in [34, 35], both in dimensional and analytic regularization it is possible to perform explicitly the above mentioned analytic continuation procedure, respectively, to the points $(\varepsilon,\underline{\lambda})$ with $\underline{\lambda} = 0$ and $\underline{\lambda}$ with $\lambda_l = \lambda$, $l = 1,\dots,L$. To do this, it is necessary to insert into the α-integrand an operator that has the structure of the R^*-operation.

In theories with normally non-ordered Lagrangians the R-operation is based on subtractions in all divergent one-particle-irreducible (1PI) subgraphs:

$$R_\Gamma = \sum_{\mathscr{S}} \Delta(\mathscr{S}) \equiv {}'R_\Gamma + \Delta(\Gamma) . \tag{2.5}$$

Here the sum is over spinneys of Γ consisting of divergent 1PI subgraphs. A spinney [8] is a set of pairwise disjoint subgraphs. Counterterm operations are defined by the following recursive relations:

$$\Delta(\mathscr{S}) = \begin{cases} 1, & \text{if } \mathscr{S} = \emptyset \\ \prod_i \Delta(\gamma^i), & \text{if } \mathscr{S} = \{\gamma^1,\dots,\gamma^i,\dots\}, \end{cases} \tag{2.6}$$

where

$$\Delta(\gamma) = \begin{cases} -M_\gamma {}'R_\gamma, & \text{if } \gamma \text{ is 1PI} \\ 0, & \text{otherwise} \end{cases} \tag{2.7}$$

are counterterm operations for subgraphs, and M_γ is some subtraction operator that specifies a renormalization scheme. A resolution of recursive relations (2.5)–(2.7) is given by the forest formula [47, 48]

$$R_\Gamma = \sum_F \prod_{\gamma\in F} (-M_\gamma) , \tag{2.8}$$

the sum taken over forests (sets of non-overlapping subgraphs) consisting of 1PI subgraphs.

The minimal subtraction scheme [41, 7] is defined by the subtraction operator which picks out the singular part of the Laurent expansion at $\varepsilon = 0$. The action of the counterterm operation $\Delta(\mathscr{S}) \equiv \Delta(\gamma^1)\ldots\Delta(\gamma^k)$ on a Feynman integral F_Γ reduces, in this case, to inserting polynomials $\mathscr{P}_{\gamma^i}$, with degrees $\omega(\gamma^i)$ in masses of subgraphs γ^i and their external momenta, into the Feynman integral $F_{\Gamma/\gamma}$ for the reduced graph $\Gamma \big/ \bigcup_i \gamma^i$ [16]. Here $\omega(\gamma) = 4\mathfrak{N}(\gamma) - 2L(\gamma) + n(\gamma)$ is the UV-degree of divergence where $n(\gamma) = \sum_{l\in\gamma} n_l$ is the total degree of propagators' numerators of a subgraph γ, $\mathfrak{N}(\gamma)$ is the loop number, and $L(\gamma)$ is the number of lines. Coefficients of monomials in $\mathscr{P}_\gamma$ are represented in the form $\sum_{-\mathfrak{N}(\gamma)\leq j<-1} a_j \varepsilon^j$.

Let us analogously define the analytic renormalization using the subtraction operator that picks out the pole part of the Laurent expansion at $\lambda = 0$. In this case, analytic counterterms $\mathscr{P}_\gamma^{\mathrm{AR}}$ are linear combinations of poles in λ. As in the case of the dimensional renormalization, let us call a Feynman amplitude analytically renormalized if the regularization is not yet removed, i.e. for $\lambda \neq 0$.

3. Diagrammatic Asymptotic Expansions

At a fixed energy scale some masses and momenta of a Feynman amplitude are considered to be large. Generally, the momentum flowing into an external vertex of a Feynman graph is written as the sum $Q_i + q_i$ of its large part Q_i and small part q_i. Similarly, the masses are subdivided into heavy (large) masses $\underline{M} = \{M_l \mid l \in \mathscr{L}_M\}$ and light (small) masses $\underline{m} = \{m_l \mid l \in \mathscr{L}_m\}$, and the set $\mathscr{L}$ of lines of the graph is represented as the union of subsets of heavy and light lines $\mathscr{L}_M$ and $\mathscr{L}_m$.

A limit of large momenta and/or masses of a Feynman amplitude is characterized by a specific subdivision of masses and external momenta into large and small ones. The large and small momenta satisfy the momentum conservation law $\sum_i Q_i = 0, \sum_i q_i = 0$. In general, a limit may be defined with exceptional momenta, i.e. certain conditions of the type $\sum_{i\in\mathscr{C}} Q_i = 0$ may be imposed for some sets $\mathscr{C}$. The momenta may be also "essentially exceptional" when these sums are equal to zero for overlapping sets $\mathscr{C}$. The simplest example of such a limit is given by the equalities $Q_1 = Q_3 = -Q_2 = -Q_4$ when $Q_1 + Q_2 = Q_1 + Q_4 = Q_2 + Q_3 = Q_3 + Q_4 = 0$.

We define the asymptotic behavior of Feynman amplitudes $G_\Gamma(\underline{Q}, \underline{q}, \underline{M}, \underline{m})$ in the limit of large momenta and masses in terms of the asymptotic behavior of Feynman integrals $F_\Gamma(\underline{Q}/\varrho, \underline{q}, \underline{M}/\varrho, \underline{m})$ for $\varrho \to 0$. We shall consider two cases. First, the Feynman amplitude may be regarded as a tempered distribution in $\underline{Q}$. Second, the corresponding Feynman integral may be regarded as a function in the domain of Euclidean non-exceptional momenta. In particular, in the framework of dimensional regularization we shall always imply the latter case. As to the small momenta $\underline{q}$, they are always supposed to be Euclidean and non-exceptional. Note that for the Fourier transform of a tempered distribution $F \in \mathscr{S}'(\mathbb{R}^n)$ one has $F(\tilde{\varrho x}) = \varrho^{-n}\tilde{F}(Q/\varrho)$. Thus, the asymptotic behavior to be considered is uniquely connected with the small distance behavior which is understood in terms of the limit $F(\varrho x)$ at $\varrho \to 0$. Note that if auxiliary exceptional restrictions are imposed then one may also consider the Feynman amplitude as a tempered distribution or as a function.

To derive the asymptotic expansion of a Feynman amplitude in the given limit let us apply the method [2, 3, 46, 50] that reduces to the construction of the remainder of the expansion. This remainder is nothing but the initial Feynman amplitude renormalized in a special subtraction procedure. Such "auxiliary" renormalization is constructed not only to remove UV divergences but also to provide a zero of sufficiently large order in the expansion parameter. Consequently, the terms of the expansion result from the difference between initial and auxiliary renormalizations. To transform this difference into a sum of terms of asymptotic expansion one may use the Zimmermann identity [48–50]

$$R_\Gamma = \mathscr{R}_\Gamma + \sum_{\mathscr{S} \neq \emptyset} \hat{\mathscr{R}}_{\Gamma/\mathscr{S}} \prod_{\gamma \in \mathscr{S}} (\mathscr{M}_\gamma - M_\gamma)' R_\gamma , \tag{3.1}$$

the sum running over non-empty spinneys of Γ. Here R_Γ and $\mathscr{R}_\Gamma$ are the R-operations based, respectively, on subtraction operators M and $\mathscr{M}$, $\hat{\mathscr{R}}_{\Gamma/\mathscr{S}} = \sum_{F \gtrsim \mathscr{S}} \prod_{\gamma \in F} (-\mathscr{M}_\gamma)$, and the symbol $F \gtrsim \mathscr{S}$ means that $\forall \gamma \in F$, $\gamma' \in \mathscr{S}$ one has $\gamma \supsetneqq \gamma'$ or $\phi \cap \gamma' = \emptyset$. Note that in practically all important cases the action of subtraction operators reduces to the contraction of subgraphs to points and to the insertion of polynomials into reduced vertices of the graph $\Gamma/\mathscr{S}$. After such action the operation $\hat{\mathscr{R}}_{\Gamma/\mathscr{S}}$ is equal to $R_{\Gamma/\mathscr{S}}$ – the R-operation for the reduced graph.

For a given limit, the identity (3.1) is applied as follows. Let us insert the initial subtraction operator as M_γ and, instead $\mathscr{M}_\gamma$, let us use some auxiliary "pre-subtracting" operator which is certainly fixed by the considered limit. In particular, the limit characterizes the set of subgraphs on which this operator is defined. First, pre-subtraction operator $\mathscr{M}_\gamma$ coincides with M_γ on some subset of 1PI subgraphs. Second, it is defined on the set of subgraphs which will be called *asymptotically irreducible* (AI). Note that such sets may be different when the Feynman amplitude is defined as a tempered distribution or a function.

To be specific, let us consider the limit of large non-exceptional momenta which is characterized by the following external momenta of the Feynman graph: Q_i for $i = 1, \ldots, n-1$; q_i for $i = n, n+1, \ldots, V-1$ and $-\sum Q_{i'} - \sum q_{i'}$ for $i = 0$. Let us suppose that the Feynman integral $F_\Gamma(\underline{Q}, \underline{q}, \underline{m}; \lambda)$ is analytically renormalized and that it is considered in the domain of non-exceptional momenta. Remember that $\lambda \neq 0$ and λ is in a vicinity of the origin. In this limit, a pre-subtraction procedure is constructed as follows. Let $\mathscr{V}^A$ be the set of external vertices with large momenta, and let $\mathscr{V}_\gamma$ denote the set of subgraph's vertices. If $\mathscr{V}^A \backslash \mathscr{V}_\gamma \neq \emptyset$ let the pre-subtraction operator coincide with the given subtraction operator defined on 1PI subgraphs. If $\mathscr{V}^A \subset \mathscr{V}_\gamma$ let us define the pre-subtraction operator on the subgraphs which are connected and 1PI after contraction of vertices $\mathscr{V}^A$. These are the subgraphs which will be called AI in the considered limit. As a pre-subtraction operator $\mathscr{M}_\gamma$, let us use the operator $\mathscr{M}_\gamma^{a_\gamma} = \mathscr{T}^{-a_\gamma}_{\underline{q}^\gamma, \underline{m}^\gamma}$: it performs Taylor expansion of order a_γ in subgraph's masses $\underline{m}^\gamma$ and all its external momenta except large momenta $\underline{Q}^\gamma = \underline{Q}$. (We denote by $\mathscr{T}^a_{\ldots}$ the Taylor expansion operator of degree a in the corresponding set of variables.) This Taylor expansion is implied either in integrands of Feynman integrals over loop momenta, or in integrands of α-representations. Using the latter prescription and applying the well-known representation of the subtraction operator in the BPHZ renormalization [2, 3, 46] we obtain the definition

$$\mathscr{M}_{\gamma}^{a_{\gamma}} F_{\Gamma}(\underline{Q}, \underline{q}, \underline{m}; \lambda) = \int d\underline{\alpha} \prod_{l} \alpha_l^{\lambda} \mathscr{T}_{\kappa}^{a_{\gamma}} \left\{ \kappa^{4\mathfrak{N}(\gamma)+n(\gamma)} I_{\Gamma}(\underline{Q}/\kappa, \underline{q}, \underline{m}, \underline{\alpha}(\kappa)) \right\}\Big|_{\kappa=1} \tag{3.2}$$

with $\alpha_l(\kappa) = \kappa^2 \alpha_l$, $l \in \mathscr{L}_{\gamma}$; $\alpha_l(\kappa) = \alpha_l$, $l \notin \mathscr{L}_{\gamma}$.

Note that polynomial dependence of counterterms on momenta and masses [16] provides the equations $\mathscr{M}_{\gamma}(-M_{\gamma}' R_{\gamma}) = \mathscr{M}_{\gamma} \Delta_{\gamma}^{M} = \Delta_{\gamma}^{M}$, so that $(\mathscr{M}_{\gamma} - M_{\gamma})' R_{\gamma}^{M} = \mathscr{M}_{\gamma} R_{\gamma}^{M}$. Futhermore, spinneys of AI subgraphs for the considered limit consist of exactly one element. Therefore the Zimmermann identity (3.1) takes the form

$$R_{\Gamma} = \sum_{\gamma} \hat{\mathscr{R}}_{\Gamma/\gamma} \mathscr{M}_{\gamma}^{a_{\gamma}} R_{\gamma} + \mathscr{R}_{\Gamma}, \tag{3.3}$$

the sum taken over AI subgraphs.

Observe now that after the action of the operator $\mathscr{M}_{\gamma}^{a_{\gamma}}$ the Feynman integral $R_{\gamma} F_{\Gamma}$ is transformed into the Feynman integral $F_{\Gamma/\gamma}$ for the graph Γ/γ with the factor $\mathscr{M}_{\gamma}^{a_{\gamma}} R_{\gamma} F_{\gamma}$ inserted into the reduced vertex. (An explicit proof of this proposition is done by a slight modification of the corresponding proof for the operator which performs Taylor expansion in all momenta – see [7, Lemma 5].) The Feynman integral $F_{\Gamma/\gamma}$ is no longer dependent on large momenta. Hence the subtraction operators $\mathscr{M}_{\gamma'}^{a_{\gamma'}}$ in $\hat{\mathscr{R}}_{\Gamma/\gamma}$ perform Taylor expansions in all external momenta $\underline{q}^{\gamma'}$ of subgraphs $\gamma' \subset \Gamma/\gamma$. Thus, the action of such operators produces Feynman integrals for massless subgraphs with zero external momenta which, in the analytic and dimensional regularizations, are nullified. Hence, $\hat{\mathscr{R}}_{\Gamma/\gamma}$ acts as $R_{\Gamma/\gamma}$, and in $R_{\Gamma/\gamma}$ there remains only the contribution of subtracting operators for subgraphs γ' without the vertex v_{γ} into which the subgraph γ was reduced. As a result, we arrive at the following equation:

$$\begin{aligned} RF_{\Gamma}(\underline{Q}, \underline{q}, \underline{m}; \lambda) = & \sum_{\gamma} R^{\mathrm{un}} F_{\Gamma/\gamma}(\underline{q}, \underline{m}; \lambda) \circ \mathscr{T}_{\underline{q}, \underline{m}}^{a_{\gamma}} RF_{\gamma}(\underline{Q}, \underline{q}, \underline{m}; \lambda) \\ & + \mathscr{R} F_{\Gamma}(\underline{Q}, \underline{q}, \underline{m}; \lambda), \end{aligned} \tag{3.4}$$

where $R^{\mathrm{un}} = \sum_{v_{\gamma} \notin \mathscr{V}(\mathscr{S})} \Delta(\mathscr{S})$, and the symbol $\circ$ shows that the latter factor is inserted into the reduced vertex of the former factor. Let the subtraction degrees a_{γ} be sufficiently large with $a_{\gamma} = \omega(\hat{\gamma}) + \bar{a}$. Here $\hat{\gamma}$ is the graph obtained from γ by contracting the vertices $\mathscr{V}^{A}$. In Sect. 5 a proof of the asymptotic estimate for Feynman amplitudes as tempered distributions is described. This result shows that the function $Y_{\Gamma}^{\bar{a}} = \mathscr{R} F_{\Gamma}(\underline{Q}, \underline{q}, \underline{m}; \lambda)$ behaves like $Y_{\Gamma}^{\bar{a}}(\underline{Q}/\varrho, \underline{q}, \underline{m}; \lambda) \sim \varrho^{\bar{a}+1}$ for $\varrho \to 0$ up to powers $\varrho^{k\lambda}$, $k = 1, \ldots, L$. Therefore it is natural to refer to this function as the remainder, and to treat Eq. (3.4) as the asymptotic expansion in the considered limit of large non-exceptional momenta. Tending $\bar{a}$ to infinity yields the asymptotic expansion

$$RF_{\Gamma}(\underline{Q}, \underline{q}, \underline{m}; \lambda) \overset{Q \to \infty}{\sim} \sum_{\gamma} R^{\mathrm{un}} F_{\Gamma/\gamma}(\underline{q}, \underline{m}; \lambda) \circ \mathscr{T}_{\underline{q}, \underline{m}} RF_{\gamma}(\underline{Q}, \underline{q}, \underline{m}; \lambda) \tag{3.5}$$

with $\mathscr{T}_{\ldots} = \mathscr{T}_{\ldots}^{\infty} = \sum_{k=0} \mathscr{T}^{(k)}$.

It should be noted that this asymptotic expansion is over homogeneous functions because

$$\mathscr{T}^{(k)}_{\underline{q},\underline{m}} F_\gamma(\underline{Q}/\varrho, \underline{q}, \underline{m}; \lambda) = \varrho^{-\omega(\gamma)+k+2L_\gamma\lambda} \mathscr{T}^{(k)}_{\underline{q},\underline{m}} F_\gamma(\underline{Q}, \underline{q}, \underline{m}; \lambda)$$

and, for various contributions to the R-operation R, similar properties are valid. In particular, this fact trivially provides the asymptotic behaviors of Feynman integrals which are claimed by the Weinberg theorem [44, 17, 27].

Observe that the presence of a regularization is a rather essential condition since the terms in the right-hand side of (3.4) and (3.5), are, in general, divergent: in $\mathscr{T}_{\underline{q},\underline{m}} RF_\gamma$, there appear IR divergences because of Taylor expansions at zero momenta and masses, and in $R^{\text{un}} F_{\Gamma/\gamma}$, there is lack of counterterms to remove *all* UV divergences. However, after summation, the expansion is finite, i.e. all the λ- (or, ε-) poles are cancelled. To prove this important property we shall write, in the next section, the asymptotic expansion in an explicitly finite form.

Let us now regard the Feynman amplitude

$$G_\Gamma(q_0, \underline{Q}, \underline{q}, \underline{m}; \lambda) = (2\pi)^4 \delta(q_0 + \sum Q_i + \sum q_i) F_\Gamma(\underline{Q}, \underline{q}, \underline{m}; \lambda)$$

as a tempered distribution. In coordinate-space language, the considered limit is described as $G_\Gamma(x + \varrho\underline{\xi}, \underline{q}, \underline{m}; \lambda)$ for $\varrho \to 0$, i.e. it is the short-distance limit $\Big($the variables x and $\underline{\xi}$ are defined as $x = \sum_i \lambda_i x_i$, $\xi_i = x_i - x$ with $\sum_i \lambda_i = 1\Big)$. In this case, one should call a subgraph asymptotically irreducible if $\mathscr{V}_\gamma \supset \mathscr{V}^A$, and if it becomes 1PI after contraction of vertices $\mathscr{V}^A$. For instance, the disconnected subgraph Γ_A consisting of isolated vertices $\mathscr{V}^A$ is AI. Let us use, as a pre-subtraction operator, the operator (3.2) which, in coordinate space, looks like

$$\begin{aligned}&\mathscr{M}^{a_\gamma}_\gamma G_\Gamma(x + \underline{\xi}, \underline{q}, \underline{m}; \lambda)\\ &\quad = \int d\underline{\alpha} \prod_l \alpha_l^\lambda \mathscr{T}^{a_\gamma}_\kappa \left\{\kappa^{4\mathfrak{N}(\hat{\gamma})+n(\gamma)} I_\Gamma(x + \kappa\underline{\xi}, \underline{q}, \underline{m}, \underline{\alpha}(\kappa))\right\}\Big|_{\kappa=1}. \qquad (3.6)\end{aligned}$$

For disconnected subgraphs, the action of the pre-subtraction operator is also graphically described by the contraction procedure, but in $\hat{\Gamma}$ rather than in the initial graph Γ – this property may be proved by means of a generalization of the corresponding proof of Ref. [7]. Thus, the asymptotic expansion of the Feynman amplitude as a tempered distribution takes the form

$$RG_\Gamma(x + \underline{\xi}, \underline{q}, \underline{m}; \lambda) \overset{\xi \to 0}{\sim} \sum_\gamma R^{\text{un}} G_{\hat{\Gamma}/\hat{\gamma}}(x, \underline{q}, \underline{m}; \lambda) \circ \mathscr{T}_{\underline{q}, \underline{m}, \underline{\xi}'} RF_\gamma(\underline{\xi}, \underline{q}, \underline{m}; \lambda). \qquad (3.7)$$

Here $\underline{\xi}'$ is the set of variables which are difference of coordinates for distinct components of disconnected subgraphs γ. In this expansion, there appears a series of terms which are local in $\underline{\xi}$ and correspond to disconnected AI subgraphs (e.g. Γ_A).

It is the expansion (3.7) for which the basic asymptotic estimate will be proved. Let the subtraction degrees be $a_\gamma = \omega(\hat{\gamma}) + \bar{a}$, let $\mathscr{R}^{\bar{a}}$ be the corresponding pre-subtracting operation, and let $Y^{\bar{a}}_\Gamma = \mathscr{R}^{\bar{a}} G_\Gamma(x + \underline{\xi}, \underline{q}, \underline{m}; \lambda)$ be the remainder. Then the following proposition is valid.

Theorem. *The remainder $Y^{\bar{a}}_\Gamma$ regarded as a tempered distribution in $\underline{\xi}$ and as a function at non-exceptional Euclidean momenta $\underline{q}$ behaves like $Y^{\bar{a}}_\Gamma(x, \varrho\underline{\xi}, \underline{q}, \underline{m}; \lambda) = o(\varrho^{\bar{a}+1-2(L+1)|\lambda|})$ for $\varrho \to 0$, $\lambda \neq 0$, and λ in a vicinity of the origin.*

A proof of this theorem will be performed in two steps. In Sect. 4 we shall prove that the asymptotic expansion is finite for $\lambda = 0$ whence the finiteness of the remainder follows immediately. In Sect. 5 the second step is done: the remainder is decomposed into various terms that are meromorphic in the regularization parameter, and for each term, the necessary asymptotic estimate is established. Note that using this proof one may easily maintain the corresponding estimate for Feynman integrals considered at non-exceptional momenta. One may also straightforwardly obtain generalizations of this proof for other limits of large momenta and masses: the only crucial point is to use an appropriate notion of AI subgraphs that is fixed by the considered limit (see below).

Let us now discuss generalizations of asymptotic expansions to other limits. Consider first the large mass limit for which masses $\underline{M}$ are essentially large than the light masses $\underline{m}$ and all momenta $\underline{q}$ which are regarded as non-exceptional and Euclidean. In this case one should consider to be AI the subgraphs with $\mathscr{L}(\gamma) \supset \mathscr{L}_M$, every connectivity component γ^i being heavy (i.e. $\mathscr{L}_M \cap \mathscr{L}(\gamma^i) \neq \emptyset$) and 1PI in respect to light lines. Furthermore, in this case it is sensible to choose the pre-subtraction operator $\mathscr{T}^{a_\gamma}_{\underline{q}^\gamma, \underline{m}^\gamma}$ that performs Taylor expansion in all external momenta and light masses of a subgraph. If a subgraph is disconnected the operator expands the product of Feynman integrals corresponding to its connectivity components. Thus the large mass expansion takes the form

$$RF_\Gamma(\underline{q}, \underline{M}, \underline{m}; \lambda) \overset{M\to\infty}{\sim} \sum_{\gamma^1,\ldots,\gamma^k} R^{\mathrm{un}} F_{\Gamma/\bigcup_i \gamma^i}(\underline{q}, \underline{m}; \lambda) \circ \mathscr{T}_{\underline{q},\underline{m}} RF_\gamma(\underline{q}, \underline{M}, \underline{m}; \lambda)\,. \tag{3.8}$$

Here the sum is over disjoint subgraphs $\{\gamma^i\}$ with AI union $\bigcup_i \gamma^i$.

Let us consider the limit of large non-exceptional momenta and large masses. In this case, a subgraph is called AI if $\mathscr{V}_\gamma \supset \mathscr{V}^A$ and the connectivity component containing the vertices $\mathscr{V}^A$ is 1PI after their identification while other components γ^i are heavy [i.e. $\mathscr{L}_M \cap \mathscr{L}(\gamma^i) \neq \emptyset$] and 1PI in respect to the light lines. We shall not write the corresponding pre-subtraction operator as well as the asymptotic expansion that turns out to be a hybrid of expansions (3.5) (or (3.7)) and (3.8). Some of the corresponding generalizations at the operator level will be described in Sect. 6.

As to various limits of exceptional momenta, let us briefly discuss the simplest case. Let the external momenta be $-q - Q_1 - \sum_{i\geq 4} q_i, Q_1, Q_2, q - Q_2, q_4, q_5, \ldots$ (two large momenta). In coordinate-space language, this limit is described as follows: $x_0 \to x_1$, $x_2 \to x_3$. In this case, one should consider a subgraph to be AI if after identifying each subset of vertices (0, 1) and (2, 3) it is 1PI or it consists of exactly two 1PI connectivity components. We shall not write the diagrammatic expansion. However the corresponding operator expansion will be described in Sect. 6 – see (6.13).

4. Explicitly Finite Expansions

To prove UV and IR finiteness of asymptotic expansions derived in Sect. 3 let us derive them in an explicitly finite form and show that these two forms of expansions are equivalent. As in Sect. 3, let us first consider the limit of large non-exceptional momenta. Let us apply the Zimmermann identity (3.3) with operator $\mathcal{M}_\gamma$ replaced now by operator $\mathfrak{X}_\gamma^{a_\gamma} = \tilde{R}_\gamma \mathcal{M}_\gamma^{a_\gamma} \equiv \tilde{R}_\gamma \mathcal{T}^{a_\gamma}_{\underline{q}^\gamma, \underline{m}^\gamma}$ rather than by $\mathcal{M}_\gamma^{a_\gamma}$. Here $\tilde{R}_\gamma$ is the $\tilde{R}$-operation i.e. the IR part of the R^*-operation [13–15]

$$R_\Gamma^* = \sum_{L(\mathcal{S}\cap\mathcal{S}')=0} \Delta(\mathcal{S})\tilde{\Delta}(\mathcal{S}') \equiv \Delta(\Gamma) + \tilde{\Delta}(\Gamma) + {}'R^*(\Gamma) . \tag{4.1}$$

The sum is over sets $\mathcal{S} = \{\gamma^i\}$ and $\mathcal{S}' = \{\gamma'^j\}$ such that subgraphs from $\mathcal{S}$ are 1PI and pairwise disjoint, and subgraphs from $\mathcal{S}'$ have not lines in common, any union of them being IR-reducible. (At least, if γ is IR-irreducible then $\tilde{\gamma} = \Gamma/(\Gamma\setminus\gamma)$ is one-vertex-irreducible – detailed definitions may be found in [13, 14].) IR-counterterm operations $\tilde{\Delta}(\mathcal{S}') = \prod \tilde{\Delta}(\gamma^j)$ are defined for IR-irreducible subgraphs. The existence of the R^*-operation was conjectured by Parisi [28]. It was described in [13, 15], and in [14], the basic theorem on the R^*-operation was proved: it states that any "R^*-normalized" Feynman integral $R^* F_\Gamma$ is both UV- and IR-finite.

The $\tilde{R}$-operation looks like

$$\tilde{R}_\gamma = \sum_{\mathcal{S}'\subset\gamma} \tilde{\Delta}(\mathcal{S}') . \tag{4.2}$$

One may represent R^* as the product $\tilde{R}R$ in the following sense: first, the R-operation transforms a Feynman integral into a sum of Feynman integrals with inserted UV-counterterms, and then each term is operated by $\tilde{R}$. Thus, the R^*-operation removes all divergences, and the $\tilde{R}$-operation removes only IR divergences. In particular, in the pre-subtracting operator $\mathfrak{X}_\gamma$ the $\tilde{R}$-operation cancels IR-divergences induced by nullification of masses and external momenta of a subgraph. Note that in all above formulae we use R^*- and $\tilde{R}$-operation based on dimensional or on analytic regularization.

When deriving asymptotic expansions we shall employ the following proposition proved in [36, 37].

Proposition 1. *Let external momenta of a subgraph be small, and let subtraction degrees satisfy $a_\gamma \geq \omega_\gamma$. Then the R-operation $R^{\mathfrak{X}}$ and* UV-*counterterm operations $\Delta^{\mathfrak{X}}$ based on the operator $\mathfrak{X}$ are, respectively, equal to the R-operation and* UV-*counterterm operations based on the initial minimal subtractions in respect to the regularization parameter.*

Repeating the arguments of Sect. 3 with operator $\mathcal{M}_\gamma$ substituted by $\mathfrak{X}_\gamma$ yields a relation similar to (3.3). Using Proposition 1 we have that, after preliminary action of operator $\mathfrak{X}$, the operation $\hat{R}^{\mathfrak{X}}_{\Gamma/\gamma}$ is transformed into the operation $R_{\Gamma/\gamma}$ based on the initial subtraction procedure. As a result, we obtain the expansion

$$RF_\Gamma(\underline{Q},\underline{q},\underline{m};\lambda) = \sum_\gamma RF_{\Gamma/\gamma}(\underline{q},\underline{m};\lambda) \circ R^* \mathcal{T}^{a_\gamma}_{\underline{q},\underline{m}} F_\gamma(\underline{Q},\underline{q},\underline{m};\lambda) + \mathcal{R}^{\mathfrak{X}} F_\Gamma(\underline{Q},\underline{q},\underline{m};\lambda) \tag{4.3}$$

and – in the asymptotic form –

$$RF_\Gamma(\underline{Q},\underline{q},\underline{m};\lambda) \overset{Q\to\infty}{=\sim} \sum_\gamma RF_{\Gamma/\gamma}(\underline{q},\underline{m};\lambda) \circ R^* \mathcal{T}_{\underline{q},\underline{m}} F_\gamma(\underline{Q},\underline{q},\underline{m};\lambda) . \tag{4.4}$$

Because of the basic property of the R^*-operation, the expansions (4.3) and (4.4) are explicitly finite so that one may remove the regularization by setting $\lambda = 0$.

It is possible, similarly, to derive explicitly finite expansions in other limits. They may be formally obtained from the corresponding expansions of Sect. 3 by substituting $R^{\mathrm{un}}_{\Gamma/\gamma}$ by $R_{\Gamma/\gamma}$, and R_γ by R^*_γ. Let us now prove that the expansions (3.4) and (4.3) are equivalent and, therefore, the remainders coincide, i.e. $\mathscr{R}F_\Gamma = \mathscr{R}^{\ddagger}F_\Gamma$. Here we present a brief version of the proof of the equivalence which was developed by Chetyrkin [10]. For this purpose, let us apply the formula [10] that expresses IR-counterterm operation $\tilde{\Delta}$ in terms of UV one. Let F_Γ be a massless Feynman integral with zero external momenta. Then

$$\tilde{\Delta}(\Gamma) = \sum_{F \ni \Gamma} (-1)^{|F|} \Delta(F) . \tag{4.5}$$

The sum runs over forests of Γ consisting of 1PI subgraphs, and the product of counterterm operations in $\Delta(F) = \prod_{\gamma \in F} \Delta(\gamma)$ is such that $\Delta(\gamma')\Delta(\gamma) = \Delta(\gamma'/\gamma)\Delta(\gamma)$ for $\gamma' \supsetneqq \gamma$.

Furthermore, it is necessary to use the equality

$$\sum_{\gamma \ni v_0} \sum_{F \le \gamma', F \ni \gamma} (-1)^{|F|} R(\Gamma/\gamma)\Delta(F) = R^{\mathrm{un}}(\Gamma) , \tag{4.6}$$

where v_0 is a fixed vertex, $F \le \gamma$ is the set of forests of γ, and $R^{\mathrm{un}}(\Gamma) = \sum_{v_0 \notin \mathscr{S}} \Delta(\mathscr{S})$. To prove this equality one should represent $R(\Gamma/\gamma)$ according to the counterterm formula (2.5), use the equation $\Delta(\mathscr{S})\Delta(F) = \Delta(\gamma^0 \cup F)\Delta(\mathscr{S}')$ with $\mathscr{S} = \mathscr{S}' \cup \{\gamma^0\}$ (here γ^0 is the element of $\mathscr{S}$ with the vertex v_0), and sum over F and $F' = \gamma^0 \cup F$.

Let us first take the asymptotic expansion in the form (4.4) and let us there insert $R^*_\gamma = \sum_{\mathscr{S}} R_{\gamma\backslash\mathscr{S}} \tilde{\Delta}(\mathscr{S})$, the sum running over sets of IR-disjoint subgraphs in γ. According to the definition of IR-counterterm operation [13, 14], the action of $\tilde{\Delta}(\mathscr{S})$ on a Feynman integral $\mathscr{T}_{\dots}F_\gamma$ equals $(\tilde{\Delta}(\gamma'')F_{\gamma''}) \circ (\mathscr{T}_{\dots}R_{\gamma\backslash\mathscr{S}}F_{\gamma\backslash\mathscr{S}})$ with $\gamma'' = (\gamma' \cup \mathscr{S})/\gamma' = \gamma/(\gamma\backslash\mathscr{S})$, and $\gamma' = \gamma\backslash\mathscr{S}$. Furthermore, from summation over γ and $\mathscr{S}$ with $\mathscr{S} \subsetneqq \gamma$, it is necessary to turn to summation over γ' and $\mathscr{S}$ with $\mathscr{S} \cap \gamma' = \emptyset$. Let then apply (4.6) with $\Gamma \to \Gamma/\gamma'$, $\Gamma/\gamma \to \Gamma/(\gamma' \cup \mathscr{S})$, $\gamma \to \gamma''$ and, as v_0, take the vertex into which the subgraph γ' was reduced. Finally the expansion takes the form (3.5). The equivalence of (3.7) and the corresponding explicitly finite expansion is demonstrated in the same way.

5. Proof of Asymptotic Estimate

Let us represent the operation $\mathscr{R} = \mathscr{R}^{\bar{a}}$ as

$$\mathscr{R} = \sum_{\mathscr{S}} \mathscr{R}_{(\mathscr{S})} , \qquad \mathscr{R}_{(\mathscr{S})} = \left(\sum_{\mathscr{N} \gtrsim \mathscr{S}} \prod_{\gamma \in \mathscr{N}} (-\mathscr{M}^{a_\gamma}_\gamma) \right) \prod_{\gamma \in \mathscr{S}} \Delta(\gamma) , \tag{5.1}$$

where the first sum is over spinneys $\mathscr{S}$ such that $\mathscr{V}^A \backslash \mathscr{V}_\gamma \neq \emptyset \ \forall \gamma \in \mathscr{S}$, and the second sum is over nests (a set $\mathscr{N}$ is a nest if for any $\gamma, \gamma' \in \mathscr{N}$ either $\gamma \subsetneqq \gamma'$ or $\gamma' \subsetneqq \gamma$) of AI subgraphs. As before the symbol $\mathscr{N} \gtrsim \mathscr{S}$ means that $\gamma \supsetneqq \gamma'$ or $\gamma \cap \gamma' = \emptyset \ \forall \gamma \in \mathscr{N}, \gamma' \in \mathscr{S}$. To prove the asymptotic estimate of $\mathscr{R}G_\Gamma$ it suffices to prove it for any term $\mathscr{R}_{(\mathscr{S})}G_\Gamma$. We shall consider only the case $\mathscr{S} = \emptyset$ because the corresponding generalization for $\mathscr{S} \neq \emptyset$ is rather straightforward (see comments in the end of this section). Note that $\mathscr{R}_{(\emptyset)}G_\Gamma$, as well as other contributions with $\mathscr{S} \neq \emptyset$, may contain UV poles in the regularization parameter

λ. Thus, the regularization will not be removed to the very end. We shall prove that $\mathscr{R}_{(\emptyset)} G_\Gamma$ is represented as the finite sum

$$\sum_{i<0} G_i(x+\underline{\xi},\underline{q},\underline{m})\lambda^i + G_0(x+\underline{\xi},\underline{q},\underline{m};\lambda), \tag{5.2}$$

where G_0 is analytic in λ and any G_i has necessary asymptotic behavior. In accordance with the results of Sect. 4, the poles in λ should be cancelled after summation in (5.1). Therefore the asymptotic property of G_i will provide the desired asymptotic estimate of $\mathscr{R}G_\Gamma$.

To be specific let us consider the short distance limit (see Sect. 3) with $\lambda_0 = 1$; $\lambda_i = 0$, $i \geq 1$, i.e. $x_i = x_0 + \xi_i$. Using the translation invariance we may fix $x_0 = 0$. Let us analyze the behavior of $\mathscr{R}_{(\emptyset)} G_\Gamma(\varrho\underline{\xi},\underline{q},\underline{m};\lambda)$ at $\varrho \to 0$, with $\mathscr{R}_{(\emptyset)} = \sum_{\mathscr{N}} \prod_{\gamma\in\mathscr{N}} (-\mathscr{M}_\gamma^{a_\gamma})$, the sum running over the nests of AI subgraphs. Using the scaling of variables $\alpha_l \to \alpha_l' = \varrho^2\alpha_l$ in the α-representation we have

$$\mathscr{R}_{(\emptyset)} G_\Gamma(\varrho\underline{\xi},\underline{q},\underline{m};\lambda) = \varrho^{-\omega(\hat{\Gamma})+2L\lambda}\mathscr{R}_{(\emptyset)} G_\Gamma(\underline{\xi},\varrho\underline{q},\varrho\underline{m};\lambda). \tag{5.3}$$

Let us pick out the overall subtraction in $\mathscr{R}_{(\emptyset)}$:

$$\begin{aligned}\mathscr{R}_{(\emptyset)} G_\Gamma(\underline{\xi},\varrho\underline{q},\varrho\underline{m};\lambda) &= (1-\mathscr{M}_\Gamma)\,{}'\mathscr{R}_{(\emptyset)} G_\Gamma(\underline{\xi},\varrho\underline{q},\varrho\underline{m};\lambda)\\ &= (1-\mathscr{T}_\varrho^{a_\Gamma})\,{}'\mathscr{R}_{(\emptyset)} G_\Gamma(\underline{\xi},\varrho\underline{q},\varrho\underline{m};\lambda).\end{aligned} \tag{5.4}$$

To prove the necessary asymptotic estimate we shall demonstrate that the function ${}'\mathscr{R}_{(\emptyset)}$
$G_\Gamma(\underline{\xi},\varrho\underline{q},\varrho\underline{m};\lambda)$ has a_Γ continuous derivatives in ϱ and the $(a_\Gamma+1)$-st derivative behaves as a linear combination of powers of the type $\varrho^{k\lambda}$.

Let us now use the fact that due to the homogeneity of the propagators' numerators Z_l the Feynman integral for any graph γ satisfies $\mathscr{T}_\varrho^k F_\gamma(\varrho\underline{q},\varrho\underline{m};\lambda) = 0$ for $k < \omega_\gamma$ and λ in a sufficiently small neighbourhood of the origin. Thus we may include in the operation $\mathscr{R}_{(\emptyset)}$ additional (trivial) subtractions of the degree $a_\gamma = \omega_\gamma - 1$ in masses and momenta in all 1PI subgraphs with $\mathscr{V}_\gamma \cap \mathscr{V}^A = \emptyset$. Futhermore, let us include in $\mathscr{R}_{(\emptyset)}$ trivial subtractions (with $a_\gamma = -1$) in the single lines and let us extend the summation in the forest formula to UV-forests [35]. By definition, a set F of 1PI subgraphs and/or single lines is called UV-forest if a) $\forall\gamma,\gamma' \in F$ either $\gamma \subsetneqq \gamma'$, or $\gamma' \subsetneqq \gamma$ or $L(\gamma\cap\gamma') = 0$; b) for any subset $\gamma^1,\dots,\gamma^k$ of pairwise disjoint elements of F the subgraph $\bigcup_i \gamma^i$ is disconnected or one-particle-reducible. Let us denote the obtained operation by R_0. It is represented as $R_0 = \sum_F \prod_{\gamma\in F} (-\mathscr{M}_\gamma^{a_\gamma})$, where the sum is over UV-forests which consist of 1PI subgraphs and single lines with $\mathscr{V}^A \cap \mathscr{V}(\gamma) = \emptyset$ (respectively, with $a_\gamma = \omega_\gamma - 1$ and $a_\gamma = -1$) and AI subgraphs.

Let us use the α-representation (2.3) or (2.4) for the Feynman amplitude $G_\Gamma(\underline{\xi},\underline{q},\underline{m};\lambda)$ and let us decompose the integration region over the sectors [33, 35]

$$\Delta_{\rho,\bar{l}} = \{\underline{\alpha}\,|\,\alpha_{\rho(1)} \leq \dots \leq \alpha_{\rho(\bar{l})} \leq 1 \leq \alpha_{p(\bar{l}+1)} \leq \dots \leq \alpha_{\rho(L)}\}$$

which are characterized by permutations p of the numbers $1,\dots,L$ and by the number $\bar{l}$. Without loss of generality let us consider only the contribution G^Δ of the sector with $\rho(l) = l$. For $l \geq \bar{l}+1$ let us introduce the variables $\beta_l = 1/\alpha_l$, and let us turn to the (sector) variables

$$t_l = \alpha_l/\alpha_{l+1}\,, \quad l = 1,\ldots,\bar{l}-1; \quad t_{\bar{l}} = \alpha_{\bar{l}};$$

$$\tau_l = \beta_l/\beta_{l-1}\,, \quad l = \bar{l}+2,\ldots,L; \quad \tau_{\bar{l}+1} = \alpha_{\bar{l}+1}\,. \tag{5.5}$$

The corresponding Jakobian is $\prod_l t_l^{l-1} \prod_{l'} \tau_{l'}^{L-1'}$.

To analyze the asymptotic behavior of the sector contribution $R_0 G^\Delta(\underline{\xi}, \varrho \underline{q}, \varrho \underline{m}; \lambda)$ at $\varrho \to 0$ it is convenient to rearrange the terms in R_0 according to some equivalence relation and to prove the necessary estimate for the contribution of any equivalence class. To define an appropriate equivalence relation let us introduce the operation $F \to \bar{F}$ that transforms any given UV-forest F into the corresponding maximal UV-forest $\bar{F} = \mathscr{F}$ as follows. First we include in F the elements Γ_A and Γ (if they were not in F from the very beginning). We may do this since Γ does not overlap with any other subgraph, and Γ_A overlaps only with subgraphs γ with $\mathscr{V}^A \backslash \mathscr{V}_\gamma \neq \emptyset$ and $\mathscr{V}_\gamma \cap \mathscr{V}^A \neq \emptyset$. However in the considered situation such subgraphs cannot contribute to R_0. Let now $\gamma \in F$, and let γ_- be the union of elements of F which are inside γ. Let us consider the family $\{\gamma' \,|\, \gamma' = \gamma \cap \gamma_l \cup \gamma_-, \, l = 1,\ldots,L\}$ where by γ_l we denote the subgraph consisting of the lines $\{1,\ldots,l\}$. There are $L(\gamma) - L(\gamma_-)$ distinct elements in this family. Let us enumerate them in the natural order: $\gamma^1 \subsetneqq \gamma^2 \subsetneqq \ldots$ If $\mathscr{V}^A \cap \mathscr{V}(\gamma^i) = \emptyset$ let us include, for any $i = 1,\ldots,L(\gamma) - L(\gamma_-) - 1$, in the UV-forest F the subgraph which is the bridge (i.e. a cut-line) or 1PI-component of subgraph γ^i containing the line $\mathscr{L}(\gamma^i \backslash \gamma^{i-1})$. If $\mathscr{V}_\gamma \supset \mathscr{V}^A$ then we include in F the AI-component of the subgraph $\gamma^i \cup \mathscr{V}^A$ with the line $\mathscr{L}(\gamma^i \backslash \gamma^{i-1})$.

As a result, for any given UV-forest it is possible to construct the maximal UV-forest $\bar{F}$ which consists of $L+1$ elements. Let us now call two UV-forests *equivalent* if $\bar{F}_1 = \bar{F}_2$. The set of all UV-forests is naturally decomposed over classes in respect to this equivalence relation, and the operation R_0 is represented as the sum over maximal UV-forests (with $\bar{F} = F$)

$$R_0 = \sum_{F:\bar{F}=F} R_0^F\,, \quad R_0^F = \sum_{F':\bar{F}'=F} \prod_{\gamma \in F'} (-\mathscr{M}_\gamma^{a_\gamma})\,. \tag{5.6}$$

(Observe that the notion of UV-forest is closely related with that of labelled forest [7] and other related notions.) Let us represent a maximal UV-forest F as $\mathscr{F} \cup \{\Gamma_A\}$ and introduce, for the contribution $R_0^F G^\Delta$ of an equivalence class of UV-forests, the auxiliary sector variables $\underline{t}' = \{t_\gamma \,|\, \gamma \in \mathscr{F}\}$ (they will not be used as integration variables)

$$t_\gamma = \begin{cases} \alpha_{\sigma(\gamma)}/\alpha_{\sigma(\gamma_+)}\,, & \gamma \neq \Gamma\,; \\ \alpha_{\sigma(\Gamma)}\,, & \gamma = \Gamma \end{cases} \tag{5.7}$$

Here γ_+ is the minimal element of $\mathscr{F}$ that includes γ, and $\sigma : \mathscr{F} \to \mathscr{L}$ is the mapping such that $\sigma(\gamma) \in \mathscr{L}(\gamma)$ and $\sigma(\gamma) \notin \mathscr{L}(\gamma')$ $\forall \gamma' \subsetneqq \gamma$. We shall shortly use a factorization of integrands in α-representations in these variables. Now, let us write the formulae which connect the auxiliary variables and the real sector variables:

$$t'_\gamma = \begin{cases} t_{\sigma(\gamma)} \dots t_{\sigma(\gamma_+)-1}, & \sigma(\gamma) < \sigma(\gamma_+) \le \bar{l} \\ (t_{\sigma(\gamma_+)} \dots t_{\sigma(\gamma)-1})^{-1}, & \sigma(\gamma_+) \le \sigma(\gamma) \le \bar{l} \\ t_{\sigma(\gamma)} \dots t_{\bar{l}} \tau_{\bar{l}+1} \dots \tau_{\sigma(\gamma_+)}, & \sigma(\gamma) \le \bar{l} < \sigma(\gamma_+) \\ (t_{\sigma(\gamma_+)} \dots t_{\bar{l}} \tau_{\bar{l}+1} \dots \tau_{\sigma(\gamma)})^{-1}, & \sigma(\gamma_+) \le \bar{l} < \sigma(\gamma) \\ \tau_{\sigma(\gamma)+1} \dots \tau_{\sigma(\gamma_+)}, & \bar{l} < \sigma(\gamma) < \sigma(\gamma_+) \\ (\tau_{\sigma(\gamma_+)+1} \dots \tau_{\sigma(\gamma)})^{-1}, & \bar{l} < \sigma(\gamma_+) < \sigma(\gamma) \end{cases} \tag{5.8}$$

if $\gamma \neq \Gamma$, and

$$t'_\Gamma = \begin{cases} t_{\sigma(\Gamma)} \dots t_{\bar{l}}, & \sigma(\Gamma) \le \bar{l} \\ (\tau_{\bar{l}+1} \dots \tau_{\sigma(\Gamma)})^{-1}, & \sigma(\Gamma) > \bar{l}. \end{cases} \tag{5.9}$$

Let F be a maximal UV-forest. Let us decompose $\mathscr{F} = F \backslash \Gamma_A$ as the union $\mathscr{F}^+ \cup \mathscr{F}^-$ with $\mathscr{F}^+ \cap \mathscr{F}^- = \emptyset$ as follows. By definition, $\gamma \neq \Gamma$ belongs to $\mathscr{F}^+$ if $\sigma(\gamma) < \sigma(\gamma_+)$, and $\Gamma \in \mathscr{F}^+$ if $\sigma(\Gamma) \le \bar{l}$. In all other cases $\gamma \in \mathscr{F}^-$. It is clear that the UV-forest $\mathscr{F} = \bar{F} \backslash \Gamma_A$ is built from F by elements of the subset $\mathscr{F}^+$ (and by the element $\Gamma \in \mathscr{F}^-$, if $\sigma(\Gamma) > \bar{l}$). Therefore, an element of an equivalence class of UV-forests is characterized by the set $\mathscr{F}^-$ and by some subset of the set $\mathscr{F}^+$. Thus, the contribution R_0^F of the equivalence class corresponding to the maximal UV-forest $F = \mathscr{F} \cup \{\Gamma_A\}$ is represented as

$$R_0^F = \prod_{\gamma \in \mathscr{F}^- \backslash \Gamma} (-\mathscr{M}_\gamma^{a_\gamma}) \prod_{\gamma \in \mathscr{F}^+ \cup \Gamma_A \cup \Gamma} (1 - \mathscr{M}_\gamma^{a_\gamma}) = (1 - \mathscr{M}_\Gamma^{a_\Gamma})\, ' R_0^F . \tag{5.10}$$

Let us use the mixed representation (2.4):

$$' R_0^F G^\Delta(\underline{\xi}, \underline{Q}, \underline{q}, \underline{m}; \lambda) = \int_\Delta d\underline{\alpha} d\underline{\beta} \prod_l \alpha_l^\lambda \, ' R_0^F I_\Gamma^x(\underline{\xi}, \underline{Q}, \underline{q}, \underline{m}, \underline{\alpha}, \underline{\beta}),$$

and rewrite the integrand as

$$I_\Gamma^x(\dots, \underline{\alpha}, \underline{\beta}, \dots) = \prod_{l=1}^{\bar{l}} t_l^{-2l} \prod_{\gamma \in \mathscr{F}} (t'_\gamma)^{2L(\gamma)} I_\Gamma^x(\dots, \underline{t}', \dots).$$

This relation follows from the formulae that connect polylinear forms defined at different choices of variables $(\underline{\alpha}', \underline{\beta}')$ and $(\underline{\alpha}, \underline{\beta})$ [35] (in the case under consideration there are no auxiliary IR variables). As a result we have

$$' R_0^F G^\Delta(\underline{\xi}, \underline{Q}, \underline{q}, \underline{m}; \lambda) = \int_0^1 d\underline{t} d\underline{\tau} \prod_l t_l^{l\lambda - l - 1} \prod_{l'} \tau_{l'}^{-(L-l'+1)\lambda + L - l'} J_\Gamma^x;$$

$$J_\Gamma^x(\underline{\xi}, \underline{Q}, \underline{q}, \underline{m}, \underline{t}, \underline{\tau}) = \prod_{\gamma \in \mathscr{F}} (t'_\gamma)^{2L(\gamma)} \prod_{\gamma \in \mathscr{F}^+ \cup \Gamma_A \backslash \Gamma} (1 - \mathscr{T}_{\kappa_\gamma}^{a_\gamma}) \prod_{\gamma \in \mathscr{F}^- \backslash \Gamma} (-\mathscr{T}_{\kappa_\gamma}^{a_\gamma})$$
$$\times \prod_{\gamma \in F \backslash \Gamma} (\kappa_\gamma)^{4\mathfrak{R}(\hat{\gamma}^x) + n(\gamma)} I_\Gamma^x(\underline{\xi}', \underline{Q}', \underline{q}, \underline{m}, \underline{\alpha}(\underline{\kappa}; \underline{t}, \underline{\tau})) \Big|_{\kappa_\gamma = 1 \, \forall \gamma}, \tag{5.11}$$

where $\underline{\xi}' = \prod_{\gamma \in \mathscr{N}} \kappa_\gamma \underline{\xi}$, $\underline{Q}' = \prod_{\gamma \in \mathscr{N}} \kappa_\gamma^{-1} \underline{Q}$, $\mathscr{N} = \{\gamma \in \mathscr{F} \mid \mathscr{V}_\gamma \supset \mathscr{V}^A\}$, and $\hat{\gamma}^x$ is the subgraph of $\hat{\Gamma}^x$ composed of lines $\mathscr{L}(\gamma)$ and $\hat{\mathscr{L}}^x$.

Let us enumerate the elements of the nest $\mathcal{N} \subset \mathcal{F}$ of AI subgraphs as $\gamma^1 \subsetneqq \gamma^2 \subsetneqq \dots$. Let us first analyze the case (A) when $\sigma(\gamma^1) > \bar{l}$, $\sigma(\Gamma) > \bar{l}$. In this situation it is convenient to consider in coordinate space all the variables corresponding to the vertices $\mathcal{V}^A$. (Remember that the variables corresponding to the vertices which do not belong to $\mathcal{V}^A$ are always considered in momentum space.) This trick provides a smooth dependence of the exponent of the parametric representation on the auxiliary parameters κ_γ. Furthermore, we have a possibility to regard the given contribution to the Feynman amplitude as a tempered distribution of the variables $\underline{\xi}$ because, after the action of the operators in (5.11), the terms of the exponent depending on $\underline{\xi}$ turn to be infinitely differentiable in $(\underline{t}, \underline{\tau})$. To see that both above conditions are satisfied it suffices to apply the factorization (A.1).

If the auxiliary parameters κ_γ are involved then the polylinear forms in the integrand of the α-representation are expressed in terms of the sums of products

$$\prod_T (\underline{\alpha}, \underline{\kappa}) = \prod_{l \notin T} \alpha_l \prod_{\gamma \in F} (\kappa_\gamma)^{2L(\gamma \backslash T)} = \prod_{\gamma \in F} (\kappa_\gamma^2 t'_\gamma)^{L(\gamma \backslash T)}$$

over various sets of pseudo-, 1- and 2-trees of the graph $\hat{\Gamma}^x$ (in the considered case $\hat{\Gamma}^x = \hat{\Gamma}$). We shall obtain the factorization of the integrand in respect to variables $(\underline{t}, \underline{\tau})$ from the factorization (A.4). To do this let us remember that, after the action of subtraction operators for $\gamma \in \mathcal{F}^-$, parameters κ_γ (which are coefficients at t'_γ) are nullified only if $\gamma \neq \Gamma$. Therefore, at $\gamma = \Gamma$, it is necessary to factorize maximal, rather than minimal power of the variable t'_Γ. If we properly change the arguments in proving factorization (A.4) [35] then we obtain the power of t'_Γ which equals $-[(\omega(\hat{\Gamma}^x) + 1)/2] - L$. (The square brackets are used to denote the integer part of a number.) Using factorizations (A.1)–(A.4) we come to the representation

$$I_\Gamma^x(\underline{\xi}', \underline{q}, \underline{m}, \underline{t}', \underline{\kappa}) = (t'_\Gamma)^{-[(\omega(\hat{\Gamma}^x)+1)/2]-L} \prod_{\gamma \in F \backslash \Gamma} (\kappa_\gamma^2 t'_\gamma)^{-[\omega(\hat{\gamma}^x)/2]-L(\gamma)} f e^{ih},$$

$$\begin{aligned} h(\underline{\xi}, \underline{q}, \underline{m}, \underline{t}', \underline{\kappa}) = &\sum_{i,j \in \mathcal{V}^x} \prod_{\gamma \in \mathcal{N}: i \overset{\gamma}{\nsim} j} \kappa_\gamma^2 \prod_{\gamma \in \mathcal{N}: i \overset{\gamma}{\sim} j} (t'_\gamma)^{-1} u_{ij}(\underline{t}', \underline{\kappa})(\xi_i - \xi_j)^2 \\ &+ \sum_{i \in \mathcal{V}^x, j \notin \mathcal{V}^A} \prod_{\gamma \in \mathcal{N}} \kappa_\gamma u_{ij}(\underline{t}', \underline{\kappa}) \xi_i q_j \\ &+ \sum_{i,j \notin \mathcal{V}^A} u_{ij}(\underline{t}', \underline{\kappa}) q_i q_j - \sum_l m_l^2 \alpha_l(\underline{t}'). \end{aligned} \tag{5.12}$$

Here the symbol $i \overset{\gamma}{\sim} j$ ($i \overset{\gamma}{\nsim} j$) means that vertices i and j are (not) connected in γ; the functions $f(\underline{\xi}, \underline{q}, \underline{t}', \underline{\kappa})$ and $u_{ij}(\underline{t}', \underline{\kappa})$ depend (smoothly) on $\underline{t}'$ and $\underline{\kappa}$ only through combinations $\kappa_\gamma^2 t'_\gamma$ (for $\gamma \neq \Gamma$). Furthermore, u_{ij} do not depend on t'_Γ, and f polynomially depends on $(t'_\Gamma)^{-1}$.

The formulae (5.8) and (5.9) show that t'_γ contains negative powers of the variables $(\underline{t}, \underline{\tau})$ if, and only if $\gamma \in \mathcal{F}^-$. However, due to the action of corresponding operators $\mathcal{T}_\gamma^{a_\gamma}$, parameters κ_γ (for $\gamma \neq \Gamma$) are nullified. Hence, the representation (5.12) produces a factorization of the integrand in (5.11) in variables t_l and τ_l as well. Since $\kappa_\Gamma \neq 0$, the non-analytic dependence in (5.12) enters through the exponent h due to the terms quadratic in $(\underline{q}, \underline{m})$, where negative powers of τ_l appear because of the variable t'_Γ [see (5.9)] which enters trivially: as $(t'_\Gamma)^k$ for

$k = 0, \pm 1$. By means of the standard factorization technique [35, 46] the sum of terms $\sum A^x_{i,i'} q_i q_{i'} / A^x$ quadratic in $\underline{q}$ takes the form $(\tau_{\bar{l}+1} \dots \tau_{l_0})^{-1} P(\underline{q}, \underline{t}, \underline{\tau})$, where $l_0 \le \sigma(\Gamma)$ and P is a polynomial that is quadratic and positive, at Euclidean $\underline{q}$, whose coefficients are infinitely differentiable in $(\underline{t}, \underline{\tau})$. The massive terms are similarly transformed. Thus,

$$'R_0^F G^\Delta(\underline{\xi}, \underline{q}, \underline{m}; \lambda) = \int_0^1 d\underline{t} d\underline{\tau} \prod_{l \le \bar{l}} t_l^{N_l + l\lambda} \prod_{l > \bar{l}} \tau_l^{\tilde{N}_l - (L-l+1)\lambda} g(\underline{\xi}, \underline{q}, \underline{m}, \underline{t}, \underline{\tau}), \tag{5.13}$$

where

$$g = \sum_k g_k(\underline{\xi}, \underline{q}, \underline{m}, \underline{t}, \underline{\tau}) \exp\left\{ -\frac{i}{2} (\tau_{\bar{l}+1} \dots \tau_{l_0(k)})^{-1} P_k(\underline{q}, \underline{t}, \underline{\tau}) \right.$$
$$\left. - \frac{1}{2} \sum'_{\bar{l} < l \le l_1(k)} m_l^2 (\tau_{\bar{l}+1} \dots \tau_l)^{-1} \right\}; \qquad l_{0,1}(k) \le \sigma(\Gamma), \qquad g_k \in C^\infty,$$

and the prime over the sum implies that it is over $l \notin \mathscr{L}(\gamma)\ \forall \gamma \in \mathscr{F}^- \backslash \Gamma$.

The differentiability properties of the integral (5.13) in $\underline{q}$ and $\underline{m}$ are governed by the values of the degrees $\tilde{N}_l$ for $\bar{l} < l \le \sigma(\Gamma)$, because the following proposition is valid.

Proposition 2. *The integral*

$$f(\varrho) = \int_0^1 \prod_l d\tau_l \tau_l^{\tilde{N}_l - r_l \lambda} \exp(-i\varrho/\tau_1 \dots \tau_L) \phi(\underline{\tau}) \tag{5.14}$$

with $\phi \in C^\infty$, integer $\tilde{N}_l$, and $r_1 > r_2 > \dots$, has the asymptotic expansion

$$f(\varrho) \overset{\varrho \to 0}{\sim} \sum_{n = \tilde{N}+1} \sum_{i=0}^{r_1} f_{n,i} \varrho^{n - i\lambda} + \sum_{n=0}^{\infty} f_n^{(0)} \varrho^n, \tag{5.15}$$

where $\tilde{N} = \min_l \{\tilde{N}_l\}$.

Thus, this propostion shows that $f \in C^{\tilde{N}}$, and the $(\tilde{N}+1)$-st derivative of f possesses singularities of the type $\varrho^{-k_i \lambda}$. A proof of the proposition may be reduced to the asymptotic estimate of integrals (5.14) with $\phi \equiv 1$. Using the change of variables $\tau_l = \beta_l / \beta_{l-1}$, $l = 2, \dots, L$; $\tau_1 = \beta_1$ with $d\underline{\tau} = \prod_l \beta_l^{-1} d\underline{\beta}$, one gets

$$\int_0^1 \prod_l d\tau_l \tau^{\tilde{N}_l - r_l \lambda} \exp(-i\varrho/\tau_1 \dots \tau_L) = \int_0^1 d\beta_L \beta_L^{\tilde{N}_L - 1} e^{-i\varrho/\beta_L}$$
$$\times \int_{\beta_L}^1 d\beta_{L-1} \beta_{L-1}^{\tilde{N}_{L-1} - \tilde{N}_L - 1} \dots \int_{\beta_2}^1 d\beta_1 \beta_1^{\tilde{N}_1 - \tilde{N}_2 - 1}$$

with $\bar{N}_l = \tilde{N}_l - r_l\lambda$. Taking explicitly $L-1$ integrals over $\beta_1, \ldots, \beta_{L-1}$ we obtain the integral $\int_0^1 d\beta_L \, e^{-i\varrho/\beta_L} H(\beta_L)$, the function H being a sum of powers $\beta_L^{\bar{N}_l}$, $l = 1, \ldots, L$. Using the asymptotic expansion

$$\int_0^1 d\beta \, \beta^{N-r\lambda} e^{-i\varrho/\beta} \overset{\varrho\to 0}{\sim} \varrho^{-r\lambda} \sum_{n=N+1}^{\infty} c_n \varrho^n + \sum_{n=0}^{\infty} c_n^{(0)} \varrho^n ,$$

for $\lambda \neq 0$, $\lambda \sim 0$, we obtain (5.14) and (5.15).

Note that the unregularized version (for $\lambda = 0$) of the asymptotic expansion (5.15) looks like [46]

$$f(\varrho)|_{\lambda=0} \overset{\varrho\to 0}{\sim} \sum_{n=\tilde{N}+1} \sum_i g_{n,i} \varrho^n \log^i \varrho + \sum_{n=0}^{\infty} g_n^{(0)} \varrho^n .$$

Therefore, $f \in C^{\tilde{N}}$, and $f^{(\tilde{N}+1)}$ behaves as $\log^k \varrho$.

Thus, after a suitable change of variables the asymptotic analysis of the integral (5.13) for $\underline{q}, \underline{m} \to \varrho\underline{q}, \varrho\underline{m}$ and $\varrho \to 0$ reduces to estimates of integrals of the type (5.14) and, consequently, to the power counting for $\tilde{N}_l$. As for the powers of variables τ_l at $\bar{l} < l \leq \sigma(\Gamma)$, which may present in a negative power in the exponent, we shall prove that they are sufficiently large. For other variables τ_l, as well as for UV variables t_l, we shall not perform such power counting. When the corresponding powers are negative, there appear poles in λ: if the integration with $t_l^{N_l+l\lambda}$ is considered as a result of the action of a distribution of the type x_+^λ then this fact follows from its well-known meromorphic properties in λ. However the results of Sect. 4 show that after summation over sectors, maximal forests, and spinneys in (5.1), these poles are to be cancelled. Thus, we shall prove that the integral (5.13) takes the form (5.2) where any term has necessary asymptotic behavior.

We perform the power counting of variables τ_l for $\bar{l} < l \leq \sigma(\Gamma)$ by means of the technique of [35, 46]. Let us denote by N_γ^+ minimal powers of variables t'_γ, for $\gamma \in \mathscr{F}^+$, in the function J_Γ^x in (5.11), and let N_γ^- be maximal powers of variables t'_γ for $\gamma \in \mathscr{F}^-$. These powers are represented as sums of contributions of factors in the representation (5.12), of the factor $\prod (t'_\gamma)^{2L(\gamma)}$, and of contributions that appear after the action of operators in (5.11). Using representations (5.8) and (5.9) we have

$$\tilde{N}_l \geq (L - l + 1) - 1 + \sum_{\gamma \in \mathscr{F}_l^+} N_\gamma^+ - \sum_{\gamma \in \mathscr{F}_l^-} N_\gamma^- + N^\xi(l) , \qquad (5.16)$$

where $\mathscr{F}_l^+ = \{\gamma \in \mathscr{N} \mid \sigma(\gamma) < l \leq \sigma(\gamma_+)\}$, $\mathscr{F}_l^- = \{\gamma \in \mathscr{N} \mid \sigma(\gamma_+) < l \leq \sigma(\gamma)\}$ (for convenience we set $\sigma(\Gamma_+) = 0$), and $N^\xi(l)$ is the contribution of the operator $1 - \mathscr{M}_{\Gamma_A}$.

Note that when differentiating in κ_γ one also obtains powers of other auxiliary variables $t'_{\gamma'}$, e.g. for $\gamma' \in \mathscr{F}^-$. However, for $\gamma' \neq \Gamma$, the corresponding parameters $\kappa_{\gamma'}$ are nullified. Therefore it is sufficient to take into account the variable t'_Γ. But the representation (5.13) shows that powers of τ_l^{-1} are accompanied by powers of masses $\underline{m}$ and small momenta $\underline{q}$. These powers explicitly improve the differentiability in $(\underline{q}, \underline{m})$ of the considered contribution $'R_0^F G^\Delta$. This makes it possible, according to Proposition 2, a weaker estimate of powers $\tilde{N}_l$. Thus, for simplicitly, it suffices to imply that derivatives $\partial/\partial\kappa_\gamma$ in (5.11) do not act on the terms which are quadratic in $\underline{q}$ and $\underline{m}$.

In the case $\gamma \in \mathscr{F}^-$ the relations (5.11), (5.12) and the described properties of functions in (5.11), in auxiliary parameters and variables $\underline{t}'$, give the following estimates:

$$N_\gamma^- \leq [a^\gamma/2] + L(\gamma) \qquad \gamma \in \mathscr{N};$$
$$N_\gamma^- \leq L(\gamma) - 1 \qquad \gamma \in \mathscr{F} \backslash \mathscr{N}. \tag{5.17}$$

For $\gamma \in \mathscr{F}^+$, let us use the formula $(1 - \mathscr{T}_\kappa^n) g(\kappa)|_{\kappa=1} = \frac{1}{n!} \int\limits_0^1 d\kappa (1-\kappa)^n g^{(n+1)}(\kappa)$. Thus, as far as the power counting is concerned, the action of operators $(1 - \mathscr{T}_\kappa^n)$ is equivalent to the action of operations $(\partial/\partial\kappa)^{n+1}$ which results in appearing auxiliary (convergent) integrals in κ. Let us now suppose that derivatives $\partial/\partial\kappa_\gamma$ do not operate on the terms of the exponent in (5.12) which depend on $\underline{\xi}$. Then the relations (5.11) and (5.12) lead to the inequalities

$$N_\gamma^+ \geq [a^\gamma/2] + L(\gamma) + 1 \qquad \gamma \in \mathscr{N};$$
$$N_\gamma^+ \geq L(\gamma) \qquad \gamma \in \mathscr{F} \backslash \mathscr{N}. \tag{5.18}$$

Let us apply the estimates (5.17), (5.18) and the relations

$$\sum_{\sigma(\gamma_+) < l \leq \sigma(\gamma)} L(\gamma) = \sum_{\sigma(\gamma) \geq l} L(\gamma) - \sum_{\sigma(\gamma), \sigma(\gamma_+) \geq l} L(\gamma),$$
$$\sum_{\sigma(\gamma) < l \leq \sigma(\gamma_+)} L(\gamma) = \sum_{\gamma : \sigma(\gamma_+) \geq l} L(\gamma) - \sum_{\sigma(\gamma), \sigma(\gamma_+) \geq l} L(\gamma), \tag{5.19}$$
$$\sum_{\gamma : \sigma(\gamma_+) \geq l} L(\gamma) = \sum_{\sigma(\gamma) \geq l} \sum_{\gamma : \gamma'_+ = \gamma} L_{\gamma'}.$$

By adding and subtracting the absent contribution from $\mathscr{M}_\Gamma$ and using conditions $a^\gamma = a_\gamma - \omega(\hat{\gamma}) = \bar{a} \ \forall \gamma \in \mathscr{N}$, we come, from (5.16), to the inequalities

$$\tilde{N}_l \geq [(\omega(\hat{\Gamma}) + 1)/2] + [\bar{a}/2] - (N^-(l) - N^+(l)) [\bar{a}/2]$$
$$+ N^+(l) + N^\xi(l) - 1 \tag{5.20}$$

for $\bar{l} < l \leq \sigma(\Gamma)$. Here $N^\pm(l) = |\mathscr{F}_l^\pm|$ are the numbers of elements in the corresponding sets. The difference $N^-(l) - N^+(l)$ takes the form $\sum\limits_{\gamma \in \mathscr{N}} (-1)^{\theta_l(\gamma)}$ with $\theta_l(\gamma) = (\varepsilon_l(\gamma) + 1)/2$, and $\varepsilon_l(\gamma) = \pm 1$ for $\gamma \in \mathscr{F}_l^\pm$. Hence the sign in this sum is alternating and the last term equals $+1$. Therefore, $N^-(l) - N^+(l) = 1$ for $\sigma(\gamma^1) > l$, and $= 0$, otherwise.

Let us now analyze the action of the operator $1 - \mathscr{M}_{\Gamma_A} = 1 - \mathscr{T}_{\kappa_0}^{\bar{a}}$ (here κ_0 is the parameter on which $\underline{\xi}$ is multiplied). It reduces to differentiating $(\partial/\partial\kappa_0)^{\bar{a}+1}$. Let this derivative act on the term of the exponent with $(\xi_i - \xi_j)^2$. The representation (5.12) shows that the contribution $N^\xi(l)$ equals $([\bar{a}/2] + 1) N_0^\xi(l)$, where $N_0^\xi(l)$ is due to the product $\prod\limits_{\gamma \in \mathscr{N} : i^\gamma \nsim j} \kappa_\gamma^2 \prod\limits_{\gamma \in \mathscr{N} : i^\gamma \sim j} (t'_\gamma)^{-1}$. The latter of these factors increases the power of τ_l by $N^-(l;i^\gamma \sim j) - N^+(l;i^\gamma \sim j)$; here $N^\pm(l;i^\gamma \sim j)$ is the contribution to $N^\pm(l)$ of elements γ with $i^\gamma \sim j$. As to the former factor, it decreases the degrees of all operators $\mathscr{M}_\gamma$ with $i^\gamma \nsim j$: this is equivalent to the contribution $N^-(l;i^\gamma \nsim j) - N^+(l;i^\gamma \nsim j)$. Thus, the considered factor leads to the total contribution $N^-(l) - N^+(l)$ that does not depend on i and j. Let now the derivative act in (5.12) on the linear terms in respect to $\underline{\xi}$. Then, because of the factor $\sum_{\gamma\in\mathscr{N}} \kappa_\gamma$, this action effectively reduces degrees of all operators $\mathscr{M}_\gamma$ for $\gamma \in \mathscr{N}$ whence the same estimate follows. Therefore, we have the inequality $N^\xi(l) \geq ([\bar{a}/2] + 1)(N^-(l) - N^+(l))$. Now, by virtue of condition $N^-(l) - 1 \geq 0$, we obtain

$$\tilde{N}_l \geq [(\omega(\hat{\Gamma}) + 1)/2] + [\bar{a}/2]\,. \tag{5.21}$$

Let now the derivative $\partial/\partial\kappa_\gamma$ (for disconnected γ) act on the terms of the exponent depending on $\underline{\xi}$. If $\gamma \in \mathscr{F}_l^-$ then differentiating the terms with $(\xi_i - \xi_j)^2$ does not at all reduce the powers of τ_l. If $\gamma \in \mathscr{F}_l^+$ this procedure gives the factor $\prod_{\gamma'\supsetneq\gamma:i^{\gamma'}\nsim j} \kappa_{\gamma'}^2 \prod_{\gamma'\supsetneq\gamma:i^{\gamma'}\sim j} (t'_{\gamma'})^{-1}$. Using the same arguments as for $N^\xi(l)$ we come to the estimates obtained in the previous case.

Let us now turn to case (B): $\sigma(\gamma^1) \leq \bar{l}$, $\sigma(\Gamma) > \bar{l}$. Let γ^k be the first of the elements of $\mathscr{N} \cap \mathscr{F}^-$, and suppose that

$$\sigma(\gamma^1) < \ldots < \sigma(\gamma^r) \leq \bar{l} < \sigma(\gamma^{r+1}) < \ldots < \sigma(\gamma^k) > \sigma(\gamma^{k+1})\,.$$

Let us represent the product $\prod_{0\leq i\leq r} (1 - \mathscr{M}_{\gamma^i})$ with $\gamma^0 = \Gamma_A$ as $-\sum_{0\leq i\leq r} \mathscr{M}_{\gamma^i} \prod_{i<j\leq r} (1 - \mathscr{M}_{\gamma^j}) + 1$. As a result, ${}'R_0^F G^\Delta$ equals $\sum_{0\leq i\leq r+1} G^{(i)}$, the $(r+1)$-st term corresponding to the unity. For the i^{th} term, with $i \leq r$, let us choose the mixed representation as follows. Let us include in the set $\mathscr{V}^x$ exactly one vertex of the set $\mathscr{V}^A$ from any connectivity component of the subgraph γ^i, and let us consider in the momentum space all the variables corresponding to other external vertices. For the $(r+1)$-st term in the sum over i, let us similarly choose the mixed representation: it is necessary to include into the set $\mathscr{V}^x$ exactly one vertex from any connectivity component of γ^k.

In case (B) it is also necessary to consider the situation $\sigma(\gamma^1) < \ldots < \sigma(\gamma^k) > \sigma(\gamma^{k+1})$; $\sigma(\gamma^k) \geq \bar{l}$. Then one should use the equality

$$\prod_{0\leq i<k} (1 - \mathscr{M}_{\gamma^i}) = \sum_{0\leq i<k} \mathscr{M}_{\gamma^i} \prod_{i<j<k} (1 - \mathscr{M}_{\gamma^j}) + 1$$

and represent ${}'R_0^F G^\Delta$ as the sum $\sum_{0\leq i\leq k} G^{(i)}$, where the k^{th} term corresponds to unity. The choice of the mixed representation is similarly performed. For the i^{th} term it is determined by the subgraph γ^i.

As in case (A) the described choice of the mixed representation leads to the absence of negative powers of auxiliary variables κ_γ in the corresponding exponent. Furthermore, factorizations (A.1) and (A.2) show that there are no negative powers of the variables $(\underline{t}, \underline{\tau})$ in the terms of the exponent depending on $\underline{\xi}$; this makes trivial the definition of the considered contribution $G^{(i)}$ as a tempered distribution in $(\underline{\xi}, \underline{Q})$. The factorization of the integrand in respect to the variables $\underline{t}'$ and $\underline{\kappa}$ is obtained with minimal modifications of arguments: in

(5.12) it is necessary to take into account the dependence on $\underline{Q}$ (for vertices $\mathscr{V}^A \backslash \mathscr{V}^x$) and to add to the function $h(\underline{\xi}, \underline{Q}, \underline{m}, \underline{t}', \underline{\kappa})$ the terms

$$\sum_{i,j \in \mathscr{V}^A \backslash \mathscr{V}^x} \prod_{\gamma \in \mathscr{N}} t'_\gamma u_{ij}(\underline{t}', \underline{\kappa}) Q_i Q_j + \sum_{i \in \mathscr{V}^A \backslash \mathscr{V}^x, j \notin \mathscr{V}^A} \prod_{\gamma \in \mathscr{N}} \kappa_\gamma t'_\gamma u_{ij}(\underline{t}', \underline{\kappa}) Q_i q_j .$$

Then one should use the same arguments as in case (A) which result in power counting. It is also necessary to use the equalities $\omega(\hat{\Gamma}^x) - \omega(\hat{\gamma}^x) = \omega(\hat{\Gamma}) - \omega(\hat{\gamma})$, for $\gamma \in \mathscr{N}$, which hold for the described choice of the mixed representation. The estimate on the power $\tilde{N}_l$ of the variable τ_l for $\bar{l} < l \leq \sigma(\Gamma)$ is described by (5.16) where, in this case, one should drop the term $N^\xi(l)$. Then this inequality leads, similarly, to the estimate (5.20) (with $N^\xi(l) = 0$). Since, however, $\sigma(\gamma^1) \leq \bar{l}$, in case (B) for any l with $\bar{l} < l \leq \sigma(\Gamma)$ we have $N^-(l) = N^+(l)$ and $N^+(l) \geq 1$. Hence, the estimate (5.21) is again valid.

Now it suffices to consider case (C) when $\sigma(\Gamma) \leq \bar{l}$. However, the described factorizations show that in this case the exponent of the properly chosen mixed representation does not involve negative powers of the variables $\underline{\tau}$. Therefore the infinite differentiability of $'R_0^F G^\Delta(\underline{\xi}, \varrho \underline{q}, \varrho \underline{m}; \lambda)$ in ϱ is provided. Now, applying Proposition 2 (with ϱ replaced by ϱ^2) and summing over sectors and maximal forests, we see that $'\mathscr{R}_{(\emptyset)} G_\Gamma(\underline{\xi}, \varrho \underline{q}, \varrho \underline{m}; \lambda)$ behaves like $\varrho^{\omega(\hat{\Gamma})+\bar{a}+1}$ up to powers $\varrho^{-2k\lambda}$ with $1 \leq k \leq L$. Hence, the operator $(1 - \mathscr{T}_\varrho^{a_\Gamma})$ in (5.4) gives the asymptotic behavior described by this power. Taking into account the power $\varrho^{-\omega(\hat{\Gamma})+2L\lambda}$ in (5.3) we obtain the desired behavior $\varrho^{\bar{a}+1}$ for $\mathscr{R}_{(\emptyset)} G_\Gamma(\underline{\xi}, \varrho \underline{q}, \varrho \underline{m}; \lambda)$ (up to corrections $\varrho^{2k\lambda}$ with $1 \leq k \leq L$).

Finally we observe that the proof of the asymptotic estimate on the other terms in the sum over $\mathscr{S}$ in (5.1) follows straightforwardly from the proof described above. In fact, the action of counterterm operation $\Delta(\mathscr{S})$ in (5.1) on G_Γ produces the Feynman amplitude $G_{\Gamma/\mathscr{S}} \circ \prod_i \Delta(\gamma^i) F_{\gamma^i}$ for the graph $\Gamma/\mathscr{S}$ where counterterms $\Delta(\gamma^i) F_{\gamma^i}$, polynomially dependent on masses of γ^i and its external momenta, are inserted into reduced vertices. As it was shown in this section, the asymptotic estimate is governed by differences $a_\gamma - \omega(\hat{\gamma})$ for AI subgraphs. Let γ' be an AI subgraph of $\Gamma/\mathscr{S}$ obtained by reduction of some $\gamma \subset \Gamma$: $\gamma' = \gamma/(\gamma \cap \mathscr{S})$. Let us consider the contribution to counterterms for $\gamma \cap \mathscr{S}$ of a monomial of degree $\omega(\gamma \cap \mathscr{S}) = \omega_1(\gamma \cap \mathscr{S}) + \omega_2(\gamma \cap \mathscr{S})$. Here the first term is the power of internal (in respect to γ') momenta, and the second term is the power of masses and internal momenta. The first term effectively increases the degree of divergence of subgraph $\hat{\gamma}'$ to $\omega(\hat{\gamma}') + \omega_1(\hat{\gamma} \cap \mathscr{S})$, and the second term implies that the subtracting operator $\mathscr{M}_\gamma$ is transformed into the subtracting operator for γ', with degree $a_{\gamma'} = a_\gamma - \omega_2(\hat{\gamma} \cap \mathscr{S})$. Since $\omega(\hat{\gamma}') = \omega(\hat{\gamma}) - \omega(\hat{\gamma} \cap \mathscr{S})$, the difference $a_{\gamma'} - \omega(\hat{\gamma}')$ is again equal to $\bar{a}$. Thus, we obtain the same estimate as in the case $\mathscr{S} \neq \emptyset$.

6. Operator Asymptotic Expansions

To derive operator asymptotic expansions one may also use pre-subtraction operators and Zimmermann identities. At the operator level, these identities are naturally written with the help of the counterterm technique of Anikin and Zavialov [1, 46]. In this section we shall describe various asymptotoc expansions of the S-matrix, composite operators and their products. Now, let us list definitions of these operators as well as formulae of counterterm technique generalized for Lagrangians and composite operators without normal ordering.

The renormalized S-matrix, composite operators $\check{J}_i$ and their time-ordered products $T\check{J}_1(x_1)\ldots\check{J}_n(x_n)$ are represented in perturbation theory by normal symbols off the mass shell [46, 49],

$$F(\underline{x}) \equiv F(x_1,\ldots,x_n) = \sum_{[k]} \frac{1}{[k]} \int F_{[k]}(\underline{x}\,|\,\underline{y}) : j_{[k]}(\underline{y}) : d\underline{y}\,, \tag{6.1}$$

which are regarded as functionals in classical fields. The sum in (6.1) is over normal products of asymptotic fields $: j_{[k]}(\underline{y}) : = : \phi_1(y_1)\ldots\phi_k(y_k) :$. A monomial $j_{[k]}$ involves k_i fields of the i^{th} type, and $\sum k_i = k$, $[k]! = \prod k_i!$. The index i may include, if necessary, all Lorentz and internal indices. The dimension $d_{[k]}$ of $j_{[k]}$ equals $N_k^b + \frac{3}{2}N_k^f$, where $N_k^{b(f)}$ is the number of boson (fermion) fields in $j_{[k]}$.

In the framework of counterterm technique [1, 46] without normal ordering [39] the renormalized S-matrix, composite operators and their time-ordered products are represented as

$$S = \mathrm{R}e^s = e^{s_r}\,, \tag{6.2}$$

$$\check{J}_i(x) = S^+ \otimes J_i(x)\,, \qquad J_i(x) = \mathrm{R}e^s j_i(x) = E_0(s_r)\,{}'J_i(x)\,,$$

$${}'J_i(x) = \frac{1}{1+XE_1(s_r)} : j_i(x) :\,, \tag{6.3}$$

$$\check{J}_1(x_1)\ldots\check{J}_1(x_n) = S^+ \otimes (J_1(x_1)\ldots J_n(x_n))\,,$$

$$(J_1(x_1)\ldots J_n(x_n)) = \mathrm{R}e^s j_1(x_1)\ldots j_n(x_n) = E_0(s_r)\,\mathbb{R}\!: \prod_i {}'J_i(x_i) :. \tag{6.4}$$

Here $s = i\int \mathscr{L}_{\text{int}}(x)dx$; $s_r = i\int \mathscr{L}_r(x)dx$ is the counterterm and interaction part of the action multiplied by i. The symbol $\otimes$ denotes the ordinary product of functionals while the symbol T of time-ordering is everywhere omitted for brevity. The asymptotic *in*-currents $j_i(x)$ are monomials in fields and their derivatives. Furthermore, $E_0(s_r) = e^{s_r}\mathscr{W} \equiv E_1(s_r) + 1$, where $\mathscr{W}$ is the operation [39] which removes normal ordering when applied to functionals (6.1), i.e. $\mathscr{W}\!: j_{[k]} : = j_{[k]}$. Finally,

$$\mathbb{R} = \sum_F \prod_{v\in F}(-P_v) \equiv (1-P)\,{}'\mathbb{R}\,, \tag{6.5}$$

the sum taken over non-overlapping subsets F of indices $\mathscr{J} = \{1,\ldots,n\}$ with $|v| > 1$. The operation P_v acts according to the rule

$$P_v : \prod_i {}'J_i : = : \prod_{i\notin v} {}'J_i \frac{1}{1+XE_1(s_r)} XE_0(s_r) \prod_{i\in v} {}'J_i :$$

with $P = P_{\mathscr{J}}$. Here and in (6.3) X is the subtraction operator which, in our case, specifies the analytic or dimensional minimal subtraction scheme. Its action on a functional (6.1) reduces to actions on diagrammatic contributions of Feynman amplitudes to coefficient functions $F_{[k]}$.

To derive the expansion of the product $\left(\prod_k J_k(x+\xi_k)\right)$ at short distances, i.e. at $\xi_k \to 0$, let us use the identity

$$1 = \mathscr{P}^a + (1 - \mathscr{P}^a), \qquad \mathscr{P}^a = \frac{1}{1 + \mathscr{M}^a E_1(s_r)} \mathscr{M}^a E_0(s_r). \tag{6.6}$$

Here $\mathscr{M}^a$ is the "functional" version of the diagrammatic pre-subtraction operator $\mathscr{M}_\gamma^{a_\gamma}$. In strictly renormalizable theories, its action on an arbitrary functional (6.1) reduces to actions of operators $\mathscr{M}_\gamma^{a-d_{[k]}}$ on the graphs' contributions to coefficient functions $F_{[k]}$. To obtain an explicit representation of the operator $\mathscr{M}^a$ it suffices to use the corresponding formula for the operator $\mathfrak{M}_x^a$ applied for the Wilson expansion in the BPHZ renormalization [2, 3, 46] and to include Taylor expansion in masses:

$$\mathscr{M}^a F(x+\underline{\xi}) = \sum_{[l],\{\lambda\}:d_{\{\lambda\}}\le a} c_{\{\lambda\}} \mathscr{T}_{\underline{m}}^{a-d_{\{\lambda\}}} \langle F(\underline{\xi}) \tilde{j}^{\{\lambda\}}(\underline{0})\rangle^{\mathrm{AI}} : j_{\{\lambda\}}(x) :, \tag{6.7}$$

where

$$j_{\{\lambda\}}(x) = \phi_1^{(\lambda_1)}(x) \dots \phi_l^{(\lambda_l)}(x) \qquad \phi_i^{(\lambda_i)} = \left(\frac{\partial}{\partial x_{\lambda_{i1}}}\right) \dots \left(\frac{\partial}{\partial x_{\lambda_{ir_i}}}\right) \phi_i;$$

$$\{\lambda\} = \{(\lambda_1), \dots, (\lambda_l)\} \qquad (\lambda_i) = (\lambda_{i1}, \dots, \lambda_{ir_i})$$

is a collection of Lorentz indices: $d_{\{\lambda\}} = \sum_i (r_i + \dim\phi_i)$ is the dimension of $j_{\{\lambda\}}$;

$$\tilde{j}^{\{\lambda\}}(\underline{0}) = \left(\frac{\partial}{\partial q_1}\right)^{(\lambda_1)} \dots \left(\frac{\partial}{\partial q_l}\right)^{(\lambda_l)} \tilde{\phi}_1(q_1) \dots \tilde{\phi}_l(q_l)\Big|_{\underline{q}=\underline{0}},$$

$c_{\{\lambda\}} = (-i)^{\sum r_i}/[l]! \prod r_i!$. The symbol AI denotes the contribution of AI graphs (in which the external lines corresponding to fields ϕ_i are amputated).

Inserting the identity (6.6) into (6.4) and using the relation $(1-\mathscr{P}^a)(1-P) = 1-\mathscr{P}^a$ yields the Zimmermann operator identity

$$\left(\prod_k J_k(x+\xi_k)\right) = E_0(s_r) \frac{1}{1+\mathscr{M}^a E_1(s_r)} \mathscr{M}^a \left(\prod_k J_k(x+\xi_k)\right) + Y^a(x,\underline{\xi}), \tag{6.8}$$

where

$$Y^a(x,\underline{\xi}) = E_0(s_r)(1-\mathscr{P}^a)'\mathrm{R} : \prod_k {}'J_k(x_k) : \tag{6.9}$$

is the remainder. Note that the action of the operator $\mathscr{M}^a$ after $E_1(s_r)$ produces massless Feynman integrals with zero external momenta which are nullified in the considered regularizations. Hence,

$$E_0(s_r) \frac{1}{1+\mathscr{M}^a E_1(s_r)} : j_{\{\lambda\}}(x) : = E_0(s_r) : j_{\{\lambda\}}(x) : = J^{\mathrm{B}}_{\{\lambda\}}(x)$$

is nonrenormalized composite operator (in its expansion in Feynman amplitudes, all UV divergences, except those connected with the vertex x, are removed). Thus we obtain the Wilson expansion

$$\left(\prod_k J_k(x+\xi_k)\right) \overset{\xi\to 0}{\sim} \sum_{[l],\{\lambda\}} c_{\{\lambda\}} R\mathscr{T}_{\underline{m}} \langle e^s \prod_k j_k(\xi_k) \tilde{j}^{\{\lambda\}}(\underline{0})\rangle A^{\mathrm{AI}} J^{\mathrm{B}}_{\{\lambda\}}(x). \tag{6.10}$$

The explicitly finite expansion is derived by the same arguments with $\mathcal{M}^a$ replaced by the IR-finite operator $\mathfrak{X}^a = \tilde{R}\mathcal{M}^a$. It takes the form (6.10) with $J^{\mathrm{B}} \to J$ and $R \to R^*$ [37].

The validity of the asymptotic expansion (6.10) follows immediately from the theorem proved in Sects. 4 and 5. The fact is that any diagrammatic contribution to the remainder (6.9) turns out to be a remainder of expansion of some Feynman amplitude. The analogous assertion holds for asymptotic operator expansions that will be described below.

The operator expansion (6.10) is written with the use of the monomial basis of fields $\{j_{\{\lambda\}}\}$. In gauge theories this basis becomes rather non-convenient since one should choose gauge-invariant combinations of fields and their derivatives. If some basis $\{\mathcal{O}_i\}$, with $\mathcal{O}_i = \mathrm{Re}^s o_i$, is fixed one may define the set of operators Π_i (projectors) which satisfy the conditions $\Pi_i \, o_{i'} = \delta_{ii'}$. For example, for the monomial basis, we have

$$\Pi_{\{\lambda\}} F(\underline{x}) = c_{\{\lambda\}} \langle F(\underline{x}) \tilde{j}^{\{\lambda\}}(\underline{0}) \rangle^{1\mathrm{PI}} j_{\{\lambda\}} .$$

Furthermore, in schemes with polynomial dependence of counterterms in masses it is natural to include powers of masses into composite operators: therefore, coefficient functions will not at all depend on masses. In this case, the elements of an arbitrary basis are equal to linear combinations of composite operators $J_{\{\lambda\}}$ with coefficients which are polynomial in masses. Using the equation $\mathcal{O}_i = \sum_i Z_{ii'} \mathcal{O}^{\mathrm{B}}_{i'}$ where $Z_{ii'}$ is the renormalization matrix of composite operators, we obtain the Wilson expansion in an arbitrary basis [23]:

$$\left(\prod_k J_k(x + \xi_k) \right) \overset{\xi \to 0}{\sim} \sum_{i,i'} \Pi_{i'} \, (J_1 \dots J_n)^{\mathrm{AI}} Z^{-1}_{ii'} \mathcal{O}_i(x) , \tag{6.11}$$

and – in the explicitly finite form [9, 11, 37] –

$$\left(\prod_k J_k(x + \xi_k) \right) \overset{\xi \to 0}{\sim} \sum_i \tilde{R} \, \Pi_i \, (J_1 \dots J_n)^{\mathrm{AI}}_{+} \mathcal{O}_i(x) . \tag{6.12}$$

The limits of large exceptional momenta, with non-intersecting sets $\mathscr{C}$ (see Sect. 3), are also connected with short distances. In these limits, coordinates are subdivided into groups, and coordinates of any group tend to each other. Let us write, in the monomial basis, the asymptotic expansion of the product

$$F(\underline{X}, \underline{\xi}) = \left(\prod_{k \in \mathscr{V}^A_1} J_k(X_1 + \xi_k) \prod_{k \in \mathscr{V}^A_2} J_k(X_2 + \xi_k) \right)$$

in the case of two groups (it is also derived with the use of an appropriate pre-subtracting operator):

$$\begin{aligned} F(\underline{X}, \underline{\xi}) \overset{\xi \to 0}{\sim} & \sum_{[l], \{\lambda\}, \{\mu\}} c^{\{\mu\}}_{\{\lambda\}} \tilde{R} \mathscr{T}_{\underline{m}} \langle \tilde{F}^{\{\mu\}}(\underline{0}, \underline{\xi}) \tilde{j}^{\{\lambda\}}(\underline{0}) \rangle^{\mathrm{AI}} \\ & \times \int dx \delta^{\{\mu\}}(x - \underline{X}) J_{\{\lambda\}}(x) + \sum_{[l_r], \{\lambda_r\}} c_{\{\lambda_1\}} c_{\{\lambda_2\}} \\ & \times \tilde{R} \mathscr{T}_{\underline{m}} \langle F(\underline{0}, \underline{\xi}) \tilde{j}^{\{\lambda_1\}}(\underline{0}) \tilde{j}^{\{\lambda_2\}}(\underline{0}) \rangle^{\mathrm{AI}} (J_{\{\lambda_1\}}(X_1) J_{\{\lambda_2\}}(X_2)) , \end{aligned} \tag{6.13}$$

where $X_r = \sum_{k\in\mathscr{V}_r^A} \lambda_k x_k$, with $\sum_{k\in\mathscr{V}_r^A} \lambda_k = 1$, are the "centers" of the groups $\mathscr{V}_r^A$;

$$\{\mu\} = \{(\mu_1), (\mu_2)\}, \qquad (\mu_r) = (\mu_{r1}, \ldots, \mu_{rt_r}), \qquad c_{\{\lambda\}}^{\{\mu\}} = i^{\Sigma t_r} c_{\{\lambda\}} / \prod t_r!;$$

$$\langle \tilde{F}^{\{\mu\}}(\underline{0}, \underline{\xi}) \tilde{j}^{\{\lambda\}}(\underline{0}) \rangle^{\mathrm{AI}} = \left(\frac{\partial}{\partial \underline{P}}\right)^{\{\mu\}} \left(\frac{\partial}{\partial \underline{q}}\right)^{\{\lambda\}} \langle \tilde{F}(\underline{P}, \underline{\xi}) \tilde{j}_{[l]}(\underline{q}) \rangle^{\mathrm{AI}} \Big|_{\underline{P}=\underline{q}=\underline{0}};$$

$$(2\pi)^4 \delta\left(P_1 + P_2 + \sum_i q_i\right) \langle \tilde{F}(\underline{P}, \underline{\xi}) \tilde{j}_{[l]}(\underline{q}) \rangle^{\mathrm{AI}}$$
$$= \int d\underline{Z} d\underline{y} \exp\{i \sum q_i y_i + i \sum P_r Z_r\} \langle F(\underline{Z}, \underline{\xi}) \tilde{j}_{[l]}(\underline{y}) \rangle^{\mathrm{AI}}.$$

In the first sum in (6.13) the symbol AI denotes the contribution of graphs which are 1PI after identifying the vertices of each group, and in the second sum, it denotes the contribution of graphs with two connectivity components, each of them being 1PI after the corresponding set $\mathscr{V}_r^A$ is contracted.

In limits with essentially exceptional large momenta (when the sets $\mathscr{C}$ are overlapping – see Sect. 3) in the operator expansions there appear products of operators whose coordinates are linear dependent (in [29] it was noted that in these limits local and multilocal operators are not sufficient, and "paralocal" operators appear). For instance, the expansion in the limit described in Sect. 3 involves products like $J^1(X + X^1) J^2(X - X^2) J^3(X - X^1) J^4(X + X^2)$.

Let us now briefly describe operator expansions in limits of large masses. They may be also derived with the help of a pre-subtraction operator. The expansion of the contribution $(\mathrm{Re}^s)_L$ of the graphs with light external lines to the S-matrix looks like [36]:

$$(\mathrm{Re}^s)_L \overset{M\to\infty}{\sim} \mathrm{Re}^{s_{\mathrm{eff}}} \qquad s_{\mathrm{eff}} = \int \mathscr{L}_{\mathrm{eff}}(x) dx, \tag{6.14}$$

$$\mathscr{L}_{\mathrm{eff}}(x) = \mathscr{L}_L(x) + \frac{1}{i} \sum_i R^* \Pi_i ((e^s)^{\mathrm{AI}} - 1) o_i(x). \tag{6.15}$$

Here s_{eff} is multiplied by i effective action of the low-energy theory, $\mathscr{L}_{\mathrm{eff}}$ is the effective Lagrangian, $\mathscr{L}_L$ is the light part of the initial Lagrangian (it consists of light fields); as before the symbol AI specifies the contribution of AI graphs (see the definition for the large mass limit in the end of Sect. 3).

Generalizing the arguments of [36] one straightforwardly gets the expansion of the contribution of graphs with light external lines $(\mathrm{Re}^s j(x))_L$ to a composite operator $J(x)$:

$$(\mathrm{Re}^s j(x))_L \overset{M\to\infty}{\sim} \sum_i \tilde{R} \Pi_i (J^{\mathrm{AI}}) \mathscr{O}_i^{\mathrm{eff}}(x), \tag{6.16}$$

where $\mathscr{O}_i^{\mathrm{eff}}(x) = \mathrm{Re}^{s_{\mathrm{eff}}} o_i(x)$, and the effective action is defined in (6.14) and (6.15). One may similarly derive the expansions of products of composite operators in the large mass limits. Let us consider only the case $n = 2$ and write the expansion in momentum space:

$$(\tilde{J}_1(q) J_2(0))_L = \mathrm{Re}^s \tilde{j}_1(q) j_2(0) \overset{M\to\infty}{\sim} \sum_i \tilde{R} \mathscr{T}_{\underline{m},q} \Pi_i (\tilde{J}_1(q) J_2(0))^{\mathrm{AI}} \mathscr{O}_i^{\mathrm{eff}}(0)$$
$$+ \sum_{i,i'} \tilde{R} \mathscr{T}_{\underline{m}} \Pi_i (J_1(0))^{\mathrm{AI}} \tilde{R} \mathscr{T}_{\underline{m}} \Pi_{i'} (J_2(0))^{\mathrm{AI}} (\tilde{\mathscr{O}}_i(q) \mathscr{O}_{i'}(0))^{\mathrm{eff}}, \tag{6.17}$$

with $(\mathcal{O}_i(x)\mathcal{O}_{i'}(y))^{\text{eff}} = \text{Re}^{s_{\text{eff}}} o_i(x) o_{i'}(y)$ and s_{eff}, $\mathcal{O}_i^{\text{eff}}$ defined by (6.14)–(6.16).

Furthermore, let us describe the expansion of the contribution of graphs with light external lines to an operator product in the limit of short distances ($\underline{\xi} \to 0$) and large masses ($\underline{M} \to \infty$):

$$(\Pi_k J_k(x+\xi_k))_L = (\text{Re}^s \Pi_k j_k(x+\xi_k))_L \overset{\substack{\xi\to 0\\ \underline{M}\to\infty}}{\sim} \sum_i \tilde{R}\mathcal{T}_{\underline{m}} \Pi_i (J_1(\xi_1)\dots J_n(\xi_n))^{\text{AI}} \mathcal{O}_i(x).$$

The symbol AI denotes the contribution of graphs which are AI in the considered limit (for any such graph, one of connectivity components includes all the vertices $\mathcal{V}^A$ and, after their identifying, becomes 1PI in respect to the light lines, and other components are heavy and 1PI in respect to light lines).

Asymptotic expansions in a form similar to (6.11) are not so compact. Some of them were obtained in [22].

Finally, it should be noted that operator asymptotic expansions have a great number of applications which are beyond the scope of this paper. Applications for deep-inelastic scattering are well-known [17, 27]. Moreover, in the past decade, operator expansions in various limits of large momenta and/or masses turned out to be an essential ingredient of the QCD sum rules method – see, e.g., [31].

Appendix

Polylinear forms in (2.2) are defined as follows [7, 35, 46]:

$$A(\underline{q},\underline{\alpha}) = \sum_{i,i'=1}^{V} A_{ii'}(\underline{\alpha}) q_i q_{i'},$$

$$B(\underline{q},\underline{u},\underline{\alpha}) = \sum_{l\in\mathcal{L}} u_l \alpha_l^{-1} \sum_{i=1} q_i \sum_{i'=1} e_{i'l} A_{ii'}(\underline{\alpha}),$$

$$K(\underline{u},\underline{\alpha}) = \sum_{ll'} u_l u_{l'} \left\{ \delta_{ll'} \alpha_l^{-1} D(\underline{\alpha}) - \alpha_l^{-1} \alpha_{l'}^{-1} \sum_{i,i'=1}^{V} e_{il} e_{i'l'} A_{ii'}(\underline{\alpha}) \right\},$$

$$D(\underline{\alpha}) = \sum_{T\in T^{[1]}} \Pi_T(\underline{\alpha}), \qquad A_{ii'}(\underline{\alpha}) = \sum_{T\in T^{[2]}_{ii'}} \Pi_T(\underline{\alpha}),$$

$$\Pi_T(\underline{\alpha}) = \prod_{l\notin T} \alpha_l .$$

Here $q_i = \sum_{l\in\mathcal{E}} e_{il} q_l$, e_{il} is the incidence matrix, $T^{[1]}$ is the set of trees of Γ, and $T^{[2]}_{ii'}$ is the set of 2-trees including the i^{th} and the i'^{th} vertex in the same connectivity component and the vertex with $i = 0$ – in the other component.

The sum of terms quadratic in $\underline{x}$ and $\underline{q}$ in the function W^x in (2.4)) is

$$H^x(\underline{x},\underline{q},\underline{\alpha}) = (A^x)^{-1} \left\{ \frac{1}{2} \sum_{jj'} A^{x,jj'} (x_j - x_{j'})^2 - 2 \sum_{ij} A_i^{x,j} q_i x_j - \sum_{ii'} A^x_{ii'} q_i q_{i'} \right\},$$

where polylinear forms (implicitly depending on the choice of the set $\mathscr{V}^x$) are written as [33, 35]

$$A^x(\underline{\alpha}) = \sum_{T\in\hat{T}^{[1]}} \Pi_T(\underline{\alpha}) \qquad A_i^{x,j}(\underline{\alpha}) = \sum_{T\in\hat{T}_i^{[1]j}} \Pi_T(\underline{\alpha}),$$

$$A_{ii'}^x(\underline{\alpha}) = \sum_{T\in\hat{T}_{ii'}^{[2]}} \Pi_T(\underline{\alpha}) \qquad A^{x,jj'}(\underline{\alpha}) = \sum_{T\in\hat{T}^{[0]jj'}} \Pi_T(\underline{\alpha}).$$

Here the sums are over pseudo-, 1- and 2-trees of the graph $\hat{\Gamma}^x$ including all the set $\hat{\mathscr{L}}^x$: $\hat{T}^{[1]}$ is the set of all trees of $\hat{\Gamma}^x$; $\hat{T}_i^{[1]j}$ is the subset of trees in which the vertex i is connected with the vertex j by a path without other vetices from $\mathscr{V}^x$ on it; $\hat{T}_{ii'}^{[2]}$ is the set of 2-trees in which the vertices i and i' are in the connectivity component without the vertex $\hat{v}^x$; $\hat{T}^{[0]jj'}$ is the set of pseudo-trees with a circuit going through the vertices j, $\hat{v}^x$, and j'.

The functions in the exponent of the mixed representation are factorized as follows [34, 35]:

$$A^{x,jj'}(\underline{\alpha})/A^x(\underline{\alpha}) = \prod_{\gamma\in\mathscr{F}} (t'_\gamma)^{d^{x,jj'}(\gamma)-1} a^{x,jj'}(\underline{t}'), \tag{A.1}$$

$$A_i^{x,j}(\underline{\alpha})/A^x(\underline{\alpha}) = \prod_{\gamma\in\mathscr{F}} (t'_\gamma)^{d_i^{x,j}(\gamma)} a_i^{x,j}(\underline{t}'), \tag{A.2}$$

$$A_{ii'}^x(\underline{\alpha})/A^x(\underline{\alpha}) = \prod_{\gamma\in\mathscr{F}} (t'_\gamma)^{d_{ii'}^x(\gamma)} a_{ii'}^x(\underline{t}'). \tag{A.3}$$

Here $a_{\cdots}(\underline{t}') \in C^\infty$; $d^{x,jj'}(\gamma) = 0$ if j, $j' \in \mathscr{V}^x$ are connected in $\hat{\gamma}^x$ by a path which does not go through other vertices from $\mathscr{V}^x$, and $d^{x,jj'}(\gamma) = 1$ otherwise; $d_i^{x,j}(\gamma) = 1$ if the vertex i is connected with $j \in \mathscr{V}^x$ in $\hat{\gamma}^x$, and any such path goes through some other vertex from $\mathscr{V}^x$, and $d_i^{x,j}(\gamma) = 0$ otherwise; $d_{ii'}^x(\gamma) = 0$ if the vertices i and i' are not connected in $\hat{\gamma}^x$ with $\hat{v}^x$, and $d_{ii'}^x(\gamma) \geq 1$ otherwise.

The rest of the integrand in (2.4),

$$(A^x)^{-2}Z^x(\underline{x}, \underline{q}, \underline{\alpha}) = (A^x)^{-2} \prod_l Z_l(-i\partial/\partial u_l) \exp\left\{\frac{i}{2}(2B^x - K^x)/A^x\right\}_{\underline{u}=\underline{0}}$$

(here B^x, K^x are certain polylinear forms – see [35]), is factorized as

$$(A^x)^{-2}Z^x(\underline{x}, \underline{q}, \underline{\alpha}) = \prod_{\gamma\in\mathscr{F}:\mathfrak{N}_\gamma>0} (t'_\gamma)^{-[\omega(\hat{\gamma}^x)/2]-L(\gamma)} z(\underline{x}, \underline{q}, \underline{t}') \tag{A.4}$$

with $z \in C^\infty$.

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