# ASYMPTOTIC EXPANSIONS IN SCALAR LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS 

Dedicated to Professor Taro Yoshizawa on his sixtieth birthday

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0. Introduction. We consider a scalar linear functional differential equation

$$
\begin{equation*}
\dot{x}(t)=F\left(t, x_{t}\right) \tag{0.1}
\end{equation*}
$$

Hereafter the following notations are used: $\omega$ is a nonnegative number. $C$ denotes the space of all complex valued functions continuous on the interval $[-\omega, 0]$ with the norm $\|\phi\|=\sup \{|\phi(\theta)| ;-\omega \leqq \theta \leqq 0\}$ for any $\phi$ in $C$. If $x=x(t)$ is a complex valued function continuous in $t$ on the interval $[\sigma-\omega, \sigma+\gamma]$ for some $\gamma \geqq 0$, the symbol $x_{t}$ denotes the element in $C$ with $x_{t}(\theta)=x(t+\theta)$ for $-\omega \leqq \theta \leqq 0$ and $\sigma \leqq t \leqq \sigma+\gamma$. Moreover, the following hypotheses are imposed on the equation (0.1). $F(t, \phi)$ is a complex valued functional which is continuous in $t \geqq 0$ and $\phi$ in $C$, linear in $\phi$ and has the asymptotic expansion of the form

$$
\begin{equation*}
F(t, \phi) \sim \sum_{n=0}^{\infty} L_{n}(\phi) t^{-n} \quad \text { as } \quad t \rightarrow \infty \tag{0.2}
\end{equation*}
$$

where $L_{n}(n=0,1, \cdots)$ are complex valued bounded linear functionals on the space $C$ of the form

$$
\begin{equation*}
L_{n}(\phi)=\int_{-\omega}^{0} \phi(\theta) d \eta_{n}(\theta) \quad(n=0,1, \cdots) \tag{0.3}
\end{equation*}
$$

for any $\phi$ in $C$ and some functions $\eta_{n}(\theta) \quad(n=0,1, \cdots)$ of bounded variation on the interval $[-\omega, 0]$. The asymptotic expansion ( 0.2 ) means that for any nonnegative integer $N$ there exist constants $\gamma_{N} \geqq 0$ and $\sigma_{N} \geqq 0$ satisfying the relation

$$
\left|F(t, \phi)-\sum_{n=0}^{N} L_{n}(\phi) t^{-n}\right| \leqq \gamma_{N} t^{-(N+1)}\|\phi\| \text { for any } t \geqq \sigma_{N} \text { and any } \phi \text { in } C .
$$

The linear functional differential equation

$$
\begin{equation*}
\dot{u}(t)=L_{0}\left(u_{t}\right) \tag{0.4}
\end{equation*}
$$

is called the homogeneous equation corresponding to (0.1). The equation
in the variable $\lambda$

$$
\begin{equation*}
\Delta(\lambda)=\lambda-\int_{-\omega}^{0} e^{\lambda \theta} d \eta_{0}(\theta)=0 \tag{0.5}
\end{equation*}
$$

is called the characteristic equation of (0.4). The roots $\lambda$ of (0.5) are called the characteristic values of (0.4).

In the present paper we prove the following theorems:
THEOREM 1. If $\lambda$ is a simple characteristic value of the equation (0.4), then the equation (0.1) has a formal solution $x=x(t)$ of the type

$$
\begin{equation*}
e^{\lambda t} t^{r} \sum_{m=0}^{\infty} c_{m} t^{-m} \tag{0.6}
\end{equation*}
$$

where the coefficient $c_{0}$ may be chosen arbitrarily.
Theorem 2. Let $\lambda$ be a simple characteristic value of the equation (0.4). Suppose that any other characteristic value with its real part equal to $\operatorname{Re} \lambda$ is simple and that the equation (0.1) has a formal solution of the type (0.6). Then there exists a constant $\sigma \geqq 0$ such that the equation (0.1) has a solution $x=x(t)$ for $t$ on the interval $[\sigma-\omega, \infty)$ with the asymptotic expansion

$$
\begin{equation*}
x(t) \sim e^{i t} t^{r} \sum_{m=0}^{\infty} c_{m} t^{-m} \quad \text { as } \quad t \rightarrow \infty \tag{0.7}
\end{equation*}
$$

For a linear differential difference equation

$$
\begin{equation*}
\dot{x}(t)=a(t) x(t)+b(t) x(t-\omega) \tag{0.8}
\end{equation*}
$$

which is a special case of the equation (0.1), assume that the coefficients $a(t)$ and $b(t)$ have the asymptotic expansions

$$
a(t) \sim \sum_{n=0}^{\infty} a_{n} t^{-n} \quad \text { and } \quad b(t) \sim \sum_{n=0}^{\infty} b_{n} t^{-n} \quad \text { as } \quad t \rightarrow \infty
$$

The characteristic equation of $\dot{x}(t)=a_{0} x(t)+b_{0} x(t-\omega)$ is

$$
\begin{equation*}
\Delta(\lambda)=\lambda-\left(a_{0}+b_{0} e^{-\lambda \omega}\right)=0 \tag{0.9}
\end{equation*}
$$

and the roots of (0.9) are the characteristic values. Bellman [1] as well as Bellman and Cooke [2] [3] studied the equation (0.8) and proved the existence of a formal solution of (0.8) of the type (0.6) for any simple characteristic value $\lambda$ and for the constant $r=\left(a_{1}+b_{1} e^{-\lambda \omega}\right) /\left(1+b_{1} e^{-\lambda \omega}\right)$. Moreover, they proved the existence of an exact solution of ( 0.8 ) with the asymptotic expansion of the form (0.7) for any simple characteristic value $\lambda$ under some other conditions. Our main theorems are generalizations of these results to the case of linear functional differential equations.

For a system of linear ordinary differential equations whose coefficients have the asymptotic expansions. Hukuhara [6] proved the existence of a solution with its asymptotic expansion equal to the formal solution. The method in our proof of Theorem 2 is based on that by Hukuhara [6].

In Section 1 we give a proof of Theorem 1 by the formal power series expansion of a solution. In order to prove Theorem 2, we state, in Section 2, some facts due to Hale [4] [5] concerning linear functional differential equations. We then convert the problem of solving our equation (0.1) to that of solving an integral equation in Section 3. In Section 4 we prove an existence theorem and a uniqueness theorem for the integral equation derived in the previous section. In Section 5 we complete the proof of Theorem 2.

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1. Proof of Theorem 1. Let $\lambda$ be a simple characteristic value of (0.4). Thus we have (0.5) as well as

$$
\begin{equation*}
\Delta^{\prime}(\lambda)=1-\int_{-\omega}^{0} \theta e^{\lambda 0} d \eta_{0}(\theta) \neq 0 \tag{1.1}
\end{equation*}
$$

Substituting the series (0.6) into the equation (0.1) with the expansion (0.2), we obtain

$$
\begin{aligned}
& e^{\lambda t} t^{r}\left\{\lambda c_{0}+\sum_{m=0}^{\infty}\left[\lambda c_{m}+(r-m+1) c_{m-1}\right] t^{-m}\right\} \\
& \quad=e^{\lambda t} t^{r} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty}\binom{r-m}{k}\left[\int_{-\omega}^{0} e^{\lambda \theta} \theta^{k} d \eta_{n}(\theta)\right] c_{m} t^{-(m+n+k)},
\end{aligned}
$$

where

$$
\binom{r-m}{k}=(r-m)(r-m-1) \cdots(r-m-k+1) / k!
$$

Comparing the coefficients of $e^{\lambda t} t^{r}$ and $e^{\lambda t} t^{r-1}$, respectively, we have $\Delta(\lambda) c_{0}=$ 0 and $\Delta(\lambda) c_{1}+\left[\Delta^{\prime}(\lambda) r-\Delta_{1}(\lambda)\right] c_{0}=0$, where

$$
\Delta_{1}(\lambda)=\int_{-\omega}^{0} e^{\lambda \theta} d \eta_{1}(\theta)
$$

Then we choose $c_{0}$ arbitrarily and let $r=\Delta_{1}(\lambda) / \Delta^{\prime}(\lambda)$, which is justified by (1.1). Furthermore, comparing the coefficient of $e^{\lambda t} t^{r-m}$, we have (1.2) $\Delta(\lambda) c_{m}+\left\{\left[\Delta^{\prime}(\lambda) r-\Delta_{1}(\lambda)\right]-(m-1) \Delta^{\prime}(\lambda)\right\} c_{m-1}+H\left(c_{0}, \cdots, c_{m-2}\right)=0$
for $m \geqq 2$, where $H\left(c_{0}, \cdots, c_{m-2}\right)$ denotes the sum of the terms containing the coefficients $c_{0}, \cdots, c_{m-2}$ alone. It follows that the coefficients $c_{m}$,
$m \geqq 1$ can be determined recursively starting from an arbitrary $c_{0}$. Thus we are done.
2. Linear functional differential equations. We state some facts, due to Hale [4] [5], on linear functional differential equations, which we need for the proof of Theorem 2. If $\sigma \geqq 0$ is a given real number and $\phi$ is a given function defined on the interval [ $0-\omega, 0$ ], a solution of the equation (0.1) with initial value $\phi$ at $\sigma$ is defined to be any continuous extension of $\phi(\theta-\sigma)$ on $[\sigma-\omega, \sigma]$ to the right of $\sigma$ which satisfies the equation (0.1). It is well known that for any given $\phi$ in $C$ there exists a unique solution with initial value $\phi$ at $\sigma$ defined for $t \geqq \sigma$ and the solution is continuous and linear in $\phi$ under the hypotheses stated in Section 0. If $u(\phi)$ is the solution of the equation (0.4) with initial value $\phi$ at zero, we define the family of linear operators $U(t), t \geqq 0$ by $U(t) \dot{\phi}=u_{t}(\phi)$. Let $X_{0}$ be the function on $[-\omega, 0]$ defined by $X_{0}(\theta)=0$ for $-\omega \leqq \theta<0$ and $X_{0}(0)=1$. Then the solution $x=x(t)$ of the equation (0.1) with initial value $\phi$ at $\sigma$ has the integral representation

$$
x_{t}(\theta)=U(t-\sigma) \phi(\theta)+\int_{\sigma}^{t} U(t-\tau) X_{0}(\theta) F\left(\tau, x_{\tau}\right) d \tau
$$

for $-\omega \leqq \theta \leqq 0$ or, in a more compact form,

$$
\begin{equation*}
x_{t}=U(t-\sigma) \dot{\phi}+\int_{\sigma}^{t} U(t-\tau) X_{0} F\left(\tau, x_{z}\right) d \tau . \tag{2.1}
\end{equation*}
$$

For any characteristic value $\lambda$ with multiplicity $m(\lambda)$, there are exactly $m(\lambda)$ linearly independent solution of the equation (0.4) of the form $p_{j}(\lambda, t) e^{\lambda t}$ for $j=1, \cdots, m(\lambda)$ and $-\infty<t<\infty$, where $p_{j}(\lambda, t)$ are polynomials in $t$. We define the functions $\phi_{j}(\lambda)$ in $C$ by the relation $\phi_{j}(\lambda)(\theta)=p_{j}(\lambda, \theta) e^{\lambda \theta}$ for $j=1, \cdots, m(\lambda)$ and $-\omega \leqq \theta \leqq 0$. Let $\Phi_{\lambda}=$ ( $\left.\phi_{1}(\lambda), \cdots, \phi_{m}(\lambda)(\lambda)\right)$. Then there exists a square matrix $B_{\lambda}$ of order $m(\lambda)$ whose characteristic values are $\lambda$ alone such that

$$
\begin{equation*}
\Phi_{\lambda}(\theta)=\Phi_{\lambda}(0) \exp \left[B_{\lambda} \theta\right] \quad \text { for } \quad-\omega \leqq \theta \leqq 0 . \tag{2.2}
\end{equation*}
$$

Furthermore, if $\phi=\Phi_{\lambda} a$ for some constant vector $a$ and if $u$ is a solution of the equation (0.4) with initial value $\phi$ at zero, then $u_{t}=\Phi_{\lambda} \exp \left[B_{\lambda} t\right] a$.

The equation adjoint to (0.4) is defined to be

$$
\begin{equation*}
\dot{v}(\tau)=-\int_{-\omega}^{0} v(\tau-\theta) d \eta_{0}(\theta) . \tag{2.3}
\end{equation*}
$$

$C^{*}$ denotes the space of complex valued continuous functions defined on the interval $[0, \omega]$. For any $\psi$ in $C^{*}$ and $\phi$ in $C$ we define

$$
\begin{equation*}
(\psi, \phi)=\psi(0) \phi(0)-\int_{-\omega}^{0} \int_{0}^{\theta} \psi(\xi-\theta) \phi(\xi) d \xi d \eta_{0}(\theta) \tag{2.4}
\end{equation*}
$$

The characteristic equation for the adjoint equation (2.3) is also defined by (0.5). For any characteristic value $\lambda$ with multiplicity $m(\lambda)$, there exist also exactly $m(\lambda)$ linearly independent solutions of the equation (2.3) of the form $q_{j}(\lambda, \tau) e^{-\lambda \tau}$ for $j=1, \cdots, m(\lambda)$ and $-\infty<\tau<\infty$. We define functions $\psi_{j}(\lambda)$ in $C^{*}$ by $\psi_{j}(\lambda)(\theta)=q_{j}(\lambda, \theta) e^{-\lambda \theta}$ for $j=1, \cdots, m(\lambda)$ and $0 \leqq \theta \leqq \omega$. If $\Psi_{\lambda}=\operatorname{col}\left(\psi_{1}(\lambda), \cdots, \psi_{m(\lambda)}(\lambda)\right)$, then the matrix $\left(\Psi_{\lambda^{\prime}}, \Phi_{\lambda}\right)=$ $\left(\left(\psi_{j}(\lambda), \phi_{k}(\lambda)\right) ; j, k=1, \cdots, m(\lambda)\right)$ is nonsingular and hence, without any loss of generality, can be assumed to be the identity.

Suppose $\Lambda=\left\{\lambda_{1}, \cdots, \lambda_{k}\right\}$ is a finite set of characteristic values of (0.4). Let $\left\{\Phi_{\lambda_{1}}, \cdots, \Phi_{\lambda_{k}}\right\}$ and $\left\{\Psi_{\lambda_{1}}, \cdots, \Psi_{\lambda_{k}}\right\}$ be the corresponding sets of functions in $C$ and those in $C^{*}$, respectively, defined above. If we let $\Phi_{A}=$ $\left(\Phi_{\lambda_{1}}, \cdots, \Phi_{\lambda_{k}}\right)$ and $\Psi_{A}=\operatorname{col}\left(\Psi_{\lambda_{1}}, \cdots, \Psi_{\lambda_{k}}\right)$, then the matrix $\left(\Psi_{\Lambda}, \Phi_{A}\right)$ is nonsingular and may be assumed to be the identity. Thus the matrix $B=$ $\operatorname{diag}\left(B_{\lambda_{1}}, \cdots, B_{\lambda_{k}}\right)$, where $B_{\lambda_{1}}, \cdots, B_{\lambda_{k}}$ are as defined in (2.2), is such that $\Phi_{A}(\theta)=\Phi_{A}(0) \exp [B \theta]$ for $-\omega \leqq \theta \leqq 0$. If $\phi=\Phi_{1} a$ for some constant vector $a$ and if $u(\phi)$ is the solution of the equation (0.4) with the initial value $\phi$ at zero, then we have $u_{t}(\phi)=\Phi_{\Lambda} \exp [B t] a$ for $-\infty<t<\infty$.

The above facts allow us to conclude that any $\phi$ in $C$ has a unique decomposition of the form $\phi=\phi^{P}+\phi^{Q}$ with $\phi^{P}$ in $P$ and with $\phi^{Q}$ in $Q$, where $P=P(\Lambda)=\left\{\phi\right.$ in $C ; \phi=\Phi_{\Lambda} b$ for a constant vector $\left.b\right\}$ and $Q=Q(\Lambda)=$ $\left\{\phi\right.$ in $\left.C ;\left(\Psi_{\Lambda}, \phi\right)=0\right\}$. In fact, $\phi^{P}=\Phi_{A}\left(\Psi_{1}, \phi\right)$. If we make this decomposition on the integral equation (2.1), we have the equivalent equation

$$
\begin{gather*}
x_{t}=U(t-\sigma) \dot{\phi}^{P}+\int_{\sigma}^{t} U(t-\tau) X_{0}^{P} F\left(\tau, x_{\tau}\right) d \tau+U(t-\sigma) \phi^{Q}  \tag{2.5}\\
\\
+\int_{\sigma}^{t} U(t-\tau) X_{0}^{Q} F\left(\tau, x_{\tau}\right) d \tau
\end{gather*}
$$

where $X_{0}^{P}=\Phi_{A}\left(\Psi_{A}, X_{0}\right)=\Phi_{A} \Psi_{A}(0)$ and $X_{0}^{Q}=X_{0}-X_{0}^{P}$.
3. Conversion to integral equations. It is well known that for any formal power series of the form $\sum_{m=0}^{\infty} c_{m} t^{-m}$, there exists an analytic function $q(t)$ with the asymptotic expansion $q(t) \sim \sum_{m=0}^{\infty} c_{m} t^{-m}$ as $t \rightarrow \infty$. A proof of the fact is given, for example, in Wasow [7].

Suppose there exists a formal solution of the type (0.6) of the equation (0.1). Then we have an analytic function $h(t)$ in $t$ on an interval $\left[\sigma_{0}, \infty\right)$ for some $\sigma_{0}>\omega$, which has the asymptotic expansion

$$
\begin{equation*}
h(t) \sim e^{\lambda t} t^{r} \sum_{m=0}^{\infty} c_{m} t^{-m} \quad \text { as } \quad t \rightarrow \infty \tag{3.1}
\end{equation*}
$$

Changing the variable in the equation (0.1) by

$$
\begin{equation*}
x(t)=y(t)+h(t), \tag{3.2}
\end{equation*}
$$

we obtain

$$
\begin{equation*}
\dot{y}(t)=L_{0}\left(y_{t}\right)+G\left(t, y_{t}\right)+g(t) \tag{3.3}
\end{equation*}
$$

where

$$
\begin{equation*}
G(t, \phi)=F(t, \phi)-L_{0}(\phi) \quad \text { and } \quad g(t)=-\dot{h}(t)+F\left(t, h_{t}\right) \tag{3.4}
\end{equation*}
$$

for any $t \geqq 0$ and $\phi$ in $C$.
Let us convert the problem of solving the equation (3.3) to that of solving an integral equation by making use of the facts stated in Section 2. Choose any number $\sigma \geqq \sigma_{0}$. If we let $y(t)=0$ for $t \leqq \sigma$, we have by (2.1) the integral representation of a solution of the equation (3.3)

$$
\begin{equation*}
y_{t}=\int_{\sigma}^{t} U(t-\tau) X_{0}\left[G\left(\tau, y_{\tau}\right)+g(\tau)\right] d \tau \tag{3.5}
\end{equation*}
$$

Let $\lambda$ be a simple characteristic value of (0.4) and let $\operatorname{Re} \lambda=\mu$. Put $\Lambda=\{\nu ; \Delta(\nu)=0, \operatorname{Re} \nu \geqq \mu\}$, which is known to be finite, and denote by $P=P(\Lambda)$ and $Q=Q(\Lambda)$ the spaces in $C$ corresponding to $\Lambda$. Therefore we obtain the unique decomposition of $C$ by the subspaces $P$ and $Q$. Hence we have

$$
\begin{align*}
y_{t}= & \int_{\sigma}^{t} U(t-\tau) X_{0}^{P}\left[G\left(\tau, y_{\tau}\right)+g(\tau)\right] d \tau  \tag{3.6}\\
& +\int_{\sigma}^{t} U(t-\tau) X_{0}^{Q}\left[G\left(\tau, y_{\tau}\right)+g(\tau)\right] d \tau
\end{align*}
$$

Suppose that any other characteristic value with its real part equal to $\mu$ is simple. It can be shown that there exist constants $K \geqq 0$ and $\varepsilon>0$ such that

$$
\begin{equation*}
\left\|U(t) X_{0}^{P}\right\| \leqq K e^{\mu t} \quad \text { for } \quad t \leqq 0 \tag{3.7}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|U(t) X_{0}^{Q}\right\| \leqq \mathrm{Ke}^{(\mu-\varepsilon) t} \quad \text { for } \quad t \geqq 0 \tag{3.8}
\end{equation*}
$$

If the integral

$$
\begin{equation*}
-\int_{\sigma}^{\infty} U(t-\tau) X_{0}^{P}\left[G\left(\tau, y_{\tau}\right)+g(\tau)\right] d \tau \tag{3.9}
\end{equation*}
$$

is convergent, it is a solution of the equation (0.4). Adding the integral (3.9) and a continuous function $f_{t}(\theta)=f(t+\theta)$ for $t \geqq \sigma$ and $-\omega \leqq \theta \leqq 0$ to the right-hand side of the equation (3.6), we have the integral equation

$$
\begin{align*}
y_{t}= & f_{t}-\int_{t}^{\infty} U(t-\tau) X_{0}^{P}\left[G\left(\tau, y_{\tau}\right)+g(\tau)\right] d \tau  \tag{3.10}\\
& +\int_{\sigma}^{t} U(t-\tau) X_{0}^{Q}\left[G\left(\tau, y_{\tau}\right)+g(\tau)\right] d \tau .
\end{align*}
$$

A solution $y=y(t)$ of the integral equation (3.10) is also a solution of the functional differential equation (3.3) if $f_{t}=0$ for $t \geqq \sigma$ and if the integral (3.9) is convergent. Hence the function $x=x(t)$ in (3.2) is a solution of our fuctional differential equation (0.1).
4. Existence and uniqueness theorem. It follows from the hypothesis (0.2) and the relation (3.4) that $G(t, \phi) \sim \sum_{n=1}^{\infty} L_{n}(\phi) t^{-n}$ as $t \rightarrow \infty$ for any $\phi$ in $C$. Then there exist constants $\sigma_{1} \geqq \sigma_{0}>\omega$ and $A \geqq 0$ such that

$$
\begin{equation*}
|G(t, \phi)| \leqq A t^{-1}\|\phi\| \quad \text { for } \quad t \geqq \sigma_{1} \text { and } \phi \text { in } C . \tag{4.1}
\end{equation*}
$$

Moreover, for any nonnegative integer $N$ there exist constants $B_{N}$ and $\sigma_{N}$ satisfying $|g(t)| \leqq B_{N} e^{\mu t} t^{\rho-N}$ for $t \geqq \sigma_{N}$. Here $g(t)$ is the function defined in (3.4) and

$$
\begin{equation*}
\operatorname{Re} \lambda=\mu \quad \text { and } \quad \operatorname{Re} r=\rho \tag{4.2}
\end{equation*}
$$

Here is a theorem concerning the existence of solutions of the integral equation (3.10).

Theorem 3. Suppose that there exist constants $N>\rho+1, \sigma \geqq \sigma_{1}$ and $\alpha \geqq 0$ satisfying the relations

$$
\begin{gather*}
2 A K /(N-\rho-1)<1 / 2,  \tag{4.3}\\
\varepsilon \sigma>N-\rho-1,  \tag{4.4}\\
2 A K /(\varepsilon \sigma-N+\rho+1)<1 / 2,  \tag{4.5}\\
|g(t)| \leqq B_{N} e^{\mu t} t^{\rho+1-N} \quad \text { for } t \geqq \sigma,  \tag{4.6}\\
\left\|f_{t}\right\| \leqq \alpha e^{\mu t} t^{\rho+1-N} \quad \text { for } t \geqq \sigma \tag{4.7}
\end{gather*}
$$

and

$$
\begin{equation*}
\left(2 A \alpha+B_{N}\right) K[1 /(N-\rho-1)+1 /(\varepsilon \sigma-N+\rho+1)] \leqq \alpha \tag{4.8}
\end{equation*}
$$

Then the equation (3.10) has a solution $y=y(t)$ continuous in $t$ on the interval $[\sigma-\omega, \infty)$ satisfying the relation

$$
\begin{equation*}
\left\|y_{t}-f_{t}\right\| \leqq \alpha e^{\mu t} t^{\rho+1-N} \quad \text { for } \quad t \geqq \sigma \tag{4.9}
\end{equation*}
$$

Proof. Denote by $S$ the class of continuous functions $y=y(t)$ in $t$ on the interval $[\sigma-\omega, \infty)$ which satisfy the relation (4.9). On $S$ we define an operator $T$ by $w(t)=(T y)(t)$ for $t \geqq \sigma-\omega$, where

$$
\begin{align*}
w_{t}=f_{t} & -\int_{t}^{\infty} U(t-\tau) X_{0}^{P}\left[G\left(\tau, y_{\tau}\right)+g(\tau)\right] d \tau  \tag{4.10}\\
& +\int_{\sigma}^{t} U(t-\tau) X_{0}^{Q}\left[G\left(\tau, y_{\tau}\right)+g(\tau)\right] d \tau
\end{align*}
$$

$w=T y$ is well-defined for any $y$ in $S$ and is continuous on the interval $[\sigma-\omega, \infty)$. For any member $y=y(t)$ in $S$ we obtain

$$
\begin{equation*}
\left\|y_{t}\right\| \leqq 2 \alpha e^{\mu t} t^{\rho+1-N} \quad \text { for } \quad t \geqq \sigma \tag{4.11}
\end{equation*}
$$

by (4.7) and (4.9). Thus by (4.10) we have

$$
\left\|w_{t}-f_{t}\right\| \leqq\left(2 A \alpha+B_{N}\right) K \int_{t}^{\infty} \tau^{\rho-N} d \tau+\left(2 A \alpha+B_{N}\right) K e^{(\mu-\varepsilon) t} \int_{\sigma}^{t} e^{\varepsilon \tau} \tau^{\rho-N} d \tau
$$

using (3.7), (3.8), (4.1) and (4.6). On the other hand, we have the inequality $e^{s t} t^{\rho-N} \leqq(d / d t)\left(e^{\varepsilon t} t^{\rho+1-N}\right) /(\varepsilon \sigma-N+\rho+1)$ for $t \geqq \sigma$ by (4.4). Then we obtain

$$
\begin{array}{r}
\left\|w_{t}-f_{t}\right\| \leqq\left(2 A \alpha+B_{N}\right) K[1 /(N-\rho-1)+1 /(\varepsilon \sigma-N+\rho+1)] e^{\mu t} t^{\rho+1-N}  \tag{4.12}\\
\text { for } t \geqq \sigma
\end{array}
$$

Thus from (4.8) it follows that $T$ is a mapping from $S$ to $S$.
Moreover, we see that the mapping $T: S \rightarrow S$ is continuous with respect to the topology of uniform convergence on any compact subinterval of the interval $[\sigma-\omega, \infty)$ and that the class $S$ is closed with respect to the same topology. It can be also proved that the family $T(S)$ is uniformly bounded and equicontinuous on any compact subinterval of the interval $[\sigma-\omega, \infty)$. It is clear that the class $S$ is convex. Therefore we conclude that there exists a member $y=y(t)$ in $S$ which is invariant under our mapping $T$ by applying the following lemma proved by Hukuhara [6]. The function $y=y(t)$ is the desired solution of the integral equation (3.10). This proves Theorem 3.

Lemma. Let $S$ be a convex family of continuous functions in $t$ on an interval $I$. Suppose that a transformation $T$ from $S$ to $S$ is continuous with respect to the topology of uniform convergence on any compact subinterval of $I$ and that $S$ is closed with respect to the same topology. Moreover, suppose that the family $T(S)$ is uniformly bounded and equi-continuous on any compact subinterval of $I$. Then there exists at least one function which is invariant under the transformation $T$, that is, a function $x(t)$ in $S$ such that $T\{x(t)\}=x(t)$.

We have the following uniqueness theorem.
Theorem 4. Suppose that there exists a solution $y=y(t)$ of the
equation (3.10), continuous in $t$ on the interval $[\sigma-\omega, \infty)$, which satisfies the relation

$$
\begin{equation*}
\left\|y_{t}\right\| \leqq \beta e^{\mu t} t^{\rho+1-N} \quad \text { for } \quad t \geqq \sigma \tag{4.13}
\end{equation*}
$$

and for some constant $\beta \geqq 0$, where $N>\rho+1$ and $\sigma \geqq \max \left\{\sigma_{1}, 1\right\}$ satisfy

$$
\begin{equation*}
\varepsilon \sigma-N+\rho+1>0 \tag{4.14}
\end{equation*}
$$

and

$$
\begin{equation*}
A K[1 /(N-\rho-1)+1 /(\varepsilon \sigma-N+\rho+1)] \leqq 1 \tag{4.15}
\end{equation*}
$$

Then the solution $y=y(t)$ is unique.
Proof. Let $y=y(t)$ and $y^{\prime}=y^{\prime}(t)$ be continuous solutions in $t$, on the interval $[\sigma-\omega, \infty)$, of the equation (3.10) which satisfy, respectively, (4.13) and

$$
\begin{equation*}
\left\|y_{t}^{\prime}\right\| \leqq \beta^{\prime} e^{\mu t} t^{\rho+1-N^{\prime}} \quad \text { for } \quad t \geqq \sigma \tag{4.16}
\end{equation*}
$$

and for some constants $\beta \geqq 0$ and $\beta^{\prime} \geqq 0$, where $N>\rho+1, N^{\prime}>\rho+1$ and $\sigma \geqq \max \left\{\sigma_{1}, 1\right\}$ satisfy (4.14), (4.15),

$$
\begin{equation*}
\varepsilon \sigma-N^{\prime}+\rho+1>0 \tag{4.17}
\end{equation*}
$$

and

$$
\begin{equation*}
A K\left[1 /\left(N^{\prime}-\rho-1\right)+1 /\left(\varepsilon \sigma-N^{\prime}+\rho+1\right)\right]<1 \tag{4.18}
\end{equation*}
$$

The function $z=y-y^{\prime}$ is a solution of the integral equation

$$
\begin{equation*}
z_{t}=-\int_{t}^{\infty} U(t-\tau) X_{0}^{P} G\left(\tau, z_{\tau}\right) d \tau+\int_{\sigma}^{t} U(t-\tau) X_{0}^{Q} G\left(\tau, z_{\tau}\right) d \tau \tag{4.19}
\end{equation*}
$$

since the functional $G(t, \phi)$ is linear in $\dot{\phi}$. On the other hand, the solution $z=z(t)$ satisfies

$$
\begin{equation*}
\left\|z_{t}\right\| \leqq\left\|y_{t}\right\|+\left\|y_{t}^{\prime}\right\| \leqq \beta^{\prime \prime} e^{\mu t} t^{\rho+1-N^{\prime \prime}} \quad \text { for } \quad t \geqq \sigma \tag{4.20}
\end{equation*}
$$

where $\beta^{\prime \prime}=\beta+\beta^{\prime}$ and $N^{\prime \prime}=\min \left\{N, N^{\prime}\right\}$ by (4.13) and (4.16). Using the relations (3.7), (3.8), (4.1), (4.14), (4.15), (4.17) and (4.18) for the equation (4.19), we have
(4.21) $\quad\left\|z_{t}\right\| \leqq A K\left[1 /\left(N^{\prime \prime}-\rho-1\right)+1 /\left(\varepsilon \sigma-N^{\prime \prime}+\rho+1\right)\right] \beta^{\prime \prime} e^{r t} t^{\rho+1-N^{\prime \prime}}$

$$
\text { for } t \geqq \sigma
$$

by the same argument as in the proof of Theorem 3. Repeating the same argument, we have, for any positive integer $m$,

$$
\begin{array}{r}
\left\|z_{t}\right\| \leqq\left\{A K\left[1 /\left(N^{\prime \prime}-\rho-1\right)+1 /\left(\varepsilon \sigma-N^{\prime \prime}+\rho+1\right)\right]\right\}^{m} \beta^{\prime \prime} e^{\mu t} t^{\rho+1-N^{\prime \prime}} \\
\text { for } t \geqq \sigma .
\end{array}
$$

This implies that $z(t)=0$ for $t \geqq \sigma-\omega$ by (4.15) or (4.18). This proves Theorem 4.
5. Proof of Theorem 2. Now we are in a position to prove Theorem 2. Under the hypotheses stated in Section 0 for the equation (0.1) and the assumptions in Theorem 2 for a characteristic value $\lambda$ and a formal solution of the type ( 0.6 ) of the equation ( 0.1 ), we consider the integral equation (3.10) with $f_{t}=0$ for $t \geqq \sigma$, where $G(t, \phi)$ and $g(t)$ are as defined in (3.4) and (3.1). Note the relations (3.7), (3.8), (4.1) and (4.2). First we choose a nonnegative integer $N>\rho+1$ satisfying the relation (4.3), and next choose a constant $\sigma \geqq \max \left\{\sigma_{1}, 1\right\}$ satisfying the relations (4.4), (4.5) and (4.6). Finally we choose a constant $\alpha \geqq 0$ satisfying the relation (4.8). The assumption (4.7) is automatically satisfied. Then it follows from Theorem 3 that there exists a solution $y=y(t)$, continuous in $t$ on the interval $\left[\sigma-\omega, \infty\right.$ ), of the equation (3.10) with $f_{t}=0$ for $t \geqq \sigma$ satisfying the relation (4.9). Thus we have

$$
\begin{equation*}
\left\|y_{t}\right\| \leqq \alpha e^{\mu t} t^{\rho+1-N} \quad \text { for } \quad t \geqq \sigma \tag{5.1}
\end{equation*}
$$

Since the integral (3.9) is clearly convergent for the solution $y=y(t)$, it is also a solution of the functional differential equation (3.3), for which the function $x=x(t)$ defined in (3.2) is a solution of our equation (0.1) on the interval $[\sigma-\omega, \infty)$.

To investigate the properties of the solution $x=x(t)$, we choose any nonnegative integer $N^{\prime}>\rho+1$ satisfying $2 A K /\left(N^{\prime}-\rho-1\right)<1 / 2$. There exist constants $\sigma^{\prime} \geqq \sigma$ and $B_{N^{\prime}} \geqq 0$ satisfying the relations $\varepsilon \sigma^{\prime}-N^{\prime}+$ $\rho+1>0,2 A K /\left(\varepsilon \sigma^{\prime}-N^{\prime}+\rho+1\right)<1 / 2$,

$$
\begin{equation*}
|g(t)| \leqq B_{N}, e^{\mu t} t^{\rho-N^{\prime}} \quad \text { for } \quad t \geqq \sigma^{\prime} \tag{5.2}
\end{equation*}
$$

and

$$
\begin{equation*}
e^{-\mathrm{s} t} \leqq t^{\rho+1-N^{\prime}} \quad \text { for } \quad t \geqq \sigma^{\prime} \tag{5.3}
\end{equation*}
$$

We consider another integral equation of the form

$$
\begin{align*}
z_{t}=f_{t} & -\int_{t}^{\infty} U(t-\tau) X_{0}^{P}\left[G\left(\tau, z_{\tau}\right)+g(\tau)\right] d \tau  \tag{5.4}\\
& +\int_{\sigma^{\prime}}^{t} U(t-\tau) X_{0}^{Q}\left[G\left(\tau, z_{\tau}\right)+g(\tau)\right] d \tau
\end{align*}
$$

where

$$
\begin{equation*}
f_{t}=\int_{\sigma}^{o^{\prime}} U(t-\tau) X_{0}^{Q}\left[G\left(\tau, y_{\tau}\right)+g(\tau)\right] d \tau \quad \text { for } \quad t \geqq \sigma^{\prime} \tag{5.5}
\end{equation*}
$$

For the function (5.5) we have $\left\|f_{t}\right\| \leqq \beta e^{(\mu-s) t} \leqq \beta e^{\mu t} t^{\rho+1-v^{\prime}}$ for $t \geqq \sigma^{\prime}$ by
(3.8), (4.1), (4.6), (5.1) and (5.3). It is clear that $y=y(t)$ is a solution, continuous in $t$ on the interval $[\sigma-\omega, \infty$ ), of the equation (5.4) and satisfies (5.1) for $t \geqq \sigma^{\prime}$. On the other hand, since we can choose a constant $\alpha^{\prime} \geqq \beta$ so that $\left(2 A \alpha^{\prime}+B_{N^{\prime}}\right) K\left[1 /\left(N^{\prime}-\rho-1\right)+1 /\left(\varepsilon \sigma^{\prime}-N^{\prime}+\right.\right.$ $\rho+1)] \leqq \alpha^{\prime}$, it follows that the conditions (4.3)-(4.5) and (4.8) of Theorem 3 are fulfilled for the constants $N^{\prime}, \sigma^{\prime}$ and $\alpha^{\prime}$. Then there exists a solution $z=z(t)$, continuous in $t$ on the interval $\left[\sigma^{\prime}-\omega, \infty\right)$, satisfying $\left\|z_{t}-f_{t}\right\| \leqq \alpha^{\prime} e^{\mu t} t^{\rho+1-N^{\prime}}$ for $t \geqq \sigma^{\prime}$, which implies the relation $\left\|z_{t}\right\| \leqq$ $\beta^{\prime} e^{\mu t} t^{\rho+1-N^{\prime}}$ for $t \geqq \sigma^{\prime}$ and for some $\beta^{\prime} \geqq 0$ by Theorem 3. Moreover, we have $y(t)=z(t)$ for $t \geqq \sigma^{\prime}-\omega$ by Theorem 4.

Hence the solution $y=y(t)$ of the equation (3.10) with $f_{t}=0$ for $t \geqq \sigma$ satisfies the asymptotic property $y(t)=O\left(e^{\mu t} t^{\rho+1-N^{\prime}}\right)$ as $t \rightarrow \infty$ for any nonnegative integer $N^{\prime} \geqq N$. Then $e^{-\lambda t} t^{-r} y(t) \sim 0$ as $t \rightarrow \infty$. Thus it follows that the solution $x=x(t)$ of the equation (0.1), obtained in (3.2), has the same asymptotic expansion as that of the function $h(t)$. This implies the relation (0.7). This completes the proof of Theorem 2.

## References

[1] R. Bellman, Asymptotic series for the solutions of linear differential-difference equations, Rend. Circ. Mat. Palermo Ser. 2, 7 (1958), 261-269.
[2] R. Bellman and K. L. Cooke, Asymptotic Behavior of Solutions of Differential-Difference Equations, Memoirs Amer. Math. Soc. No. 35, 1959.
[3] R. Bellman and K. L. Cooke, Differential-Difference Equations, Academic Press, New York-London, 1963.
[4] J. K. Hale, Linear asymptotically autonomous functional differential equations, Rend. Circ. Mat. Palermo 5 (1966), 331-351.
[5] J. K. Hale, Theory of Functional Differential Equations, Springer-Verlag, New York-Heidelberg-Berlin, 1977.
[6] M. Hukuhara, Sur les points singuliers des équations différentielles linéaires: Domain réel, Jour. Facul. Sci. Hokkaido Imp. Univ. 2 (1934), 13-88.
[7] W. Wasow, Asymptotic Expansions for Ordinary Differential Equations, John Wiley and Sons, Inc., New York-London-Sydney, 1965.

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