Given any $\Sigma \in J'_V$, there is a global rotation $\sigma \in O'(V)$ with spinor norm 1 such that $\sigma\Sigma \in J^c_V$.

Here

$$J^c_V = \{\Sigma \in J_V\colon\ \Sigma_\mathfrak{p} \in \Omega(V_\mathfrak{p}) \text{ for all (finite \& infinite) spots } \mathfrak{p}\}.$$

For any spot $\mathfrak{p}$, $O'(V_\mathfrak{p})$ always equals to $\Omega(V_\mathfrak{p})$ except when $\mathfrak{p}$ is discrete, non-dyadic and $V_\mathfrak{p}$ is the unique four dimensional anisotropic space. See 61C, 95 : 1, [7]. Thus, we may suppose $\dim(V) = 4$. Let $\boldsymbol{T} = \{\mathfrak{p}\colon \Sigma_\mathfrak{p} \notin \Omega(V_\mathfrak{p})\}$ and let $\boldsymbol{X}$ be the remaining real and non-dyadic spots at which V is anisotropic. We may suppose $\boldsymbol{T}$ is not empty. By scaling, we may further assume V represents 1. Fix a unit $\Delta_\mathfrak{p}$ of quadratic defect $4\mathfrak{o}_{F_\mathfrak{p}}$ and a prime element $\pi_\mathfrak{p}$ at each $\mathfrak{p} \in \boldsymbol{T}$. The Weak Approximation Theorem gives us two elements α, β in the global field F such that: α is close to $\Delta_\mathfrak{p}$ at $\mathfrak{p} \in \boldsymbol{T}$, and close to 1 at $\mathfrak{p} \in \boldsymbol{X}$; β is close to $\pi_\mathfrak{p}$ at $\mathfrak{p} \in \boldsymbol{T}$, and close to 1 at $\mathfrak{p} \in \boldsymbol{X}$. Then, $\alpha\beta$ is close to $\Delta_\mathfrak{p}\pi_\mathfrak{p}$ at $\mathfrak{p} \in \boldsymbol{T}$ and to 1 at $\mathfrak{p} \in \boldsymbol{X}$. Locally, $V_\mathfrak{p}$ (being 4-dimensional) is universal for every non-real spot $\mathfrak{p}$ and of course also at all the real spots with $V_\mathfrak{p}$ isotropic. On the other hand, $V_\mathfrak{p}$ is positive definite for the remaining real spots. Thus, the Local-Global Representation Theorem gives us vectors $u, v, w \in V$ such that $Q(u) = \alpha$, $Q(v) = \beta$, and $Q(w) = \alpha\beta$. Choose $x \in V$ with $Q(x) = 1$, and put $\sigma = S_x S_u S_v S_w$. One sees that σ lies in $O'(V)$ and $\sigma\Sigma_\mathfrak{p} \in \Omega(V_\mathfrak{p})$ at every $\mathfrak{p}$, see 95 : 1a, [7]. Hence, $\sigma\Sigma \in J^c_V$.

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Asymptotic expansions of finite theta series

by

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Dedicated to Th. Schneider on his 65th birthday

1. Introduction. We are interested in the approximate evaluation of sums like

$$S_N(x) = \sum_{n=1}^{N} e^{\pi i n^2 x} \qquad (x \text{ real}).$$

For instance we shall prove (see Remark 2):

THEOREM 1.

$$S_N(x) = a(p, q) \int_0^N e^{\pi i t^2 \xi/q}\, dt + O(\sqrt{q}) \tag{1}$$

for

$$x = \frac{p}{q} + \frac{\xi}{q}, \quad |\xi| \leqslant \frac{1}{4N}, \quad 0 < q \leqslant 4N, \quad (p, q) = 1. \tag{2}$$

Here $a(p, q)$ is an arithmetical function of p and q, whose modulus is zero or $1/\sqrt{q}$ according as pq is odd or even. The exact order of magnitude of the integral in (1) is known (see (9)) to be

$$\asymp \frac{N\sqrt{q}}{\sqrt{q} + N\sqrt{|\xi|}}.$$

Hence the main term in (1) is zero for odd pq and dominates $\sqrt{q}$ for even pq in the permitted range. (The symbols $O(\ )$ and $\asymp$ are explained at the beginning of the next section.)

By Dirichlet's box principle one can find for any given pair x, N a triple p, q, ξ satisfying (2). Therefore (1) is applicable to every real x and evaluates $S_N(x)$ up to an error $O(\sqrt{N})$ at most. (Observe that $\sqrt{N}$ is the exact order of the L_2-norm of $S_N(x)$.)

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Sums of the above type were treated by Hardy and Littlewood ([4], [5]) by means of their approximate functional equation. It became clear that the rational approximations of x (or the continued fraction of x) play an important role (see also Behnke [1], [2], Koksma [6], Ostrowski [7] and Walfisz [11], [12]). Those investigations were concerned with upper and lower estimates of $|S_N|$. Particularly successful with regards to upper estimates is a combination of Weyl's and Vinogradov's method, but the result can still be improved upon (see Th. 6 and the discussion preceding it). It is remarkable that one can actually isolate a main term in $S_N(x)$ as given in (1). Of course in the immediate neighborhood of rational points $\frac{p}{q}$ (the so-called major arcs) this represents no problem, and the precise form of it is given e.g. in Vinogradov's book [9] (chapt. III, p. 57) with an error term $O(q)$. This form of the error term is only appropriate, if $q = O(\sqrt{N})$, leaving the so-called minor arcs open for further discussion. Vinogradov obtains his error $O(q)$ by means of Euler's summation formula and van der Corput's method. By turning Euler's sum formula into the Poisson sum formula (by inserting the Fourier expansion of the first Bernoulli-polynomial) Vinogradov could have improved the error term to $O(\sqrt{q}\log q)$, if he had postponed estimating terms which later combine (cf. Lemma 2). However, the error term requires further attention: an application of the mentioned Weyl–Vinogradov method brings the error term down to $O(\sqrt{q\log q})$. This still does not eliminate the minor arcs, i.e. intervals where the error might be above $\sqrt{N}$. It is remarkable that the well-known (see [4], [12], [8]) iteration of the approximate functional equation of Hardy–Littlewood can be used to estimate incomplete Gaussian sums by $O(\sqrt{q})$ (see Lemma 4); applying this, one can improve the estimate of the error term to $O(\sqrt{q})$. Consequently the minor arcs dissappear entirely. Thus we find that the asymptotic behavior of $S_N(x)$ is controlled by a generalized approximate functional equation (see Theorems 3, 4, 5) around $\frac{p}{q}$ instead of 0 with an error term of the mentioned quality.

As an application of the asymptotic expansion (1) we shall deduce in the last section of this paper the following

THEOREM 2. *Let* $\{g(n)\}_1^\infty$ *be a non-decreasing sequence of positive numbers. Then for almost all real* x *one has*

$$S_N(x) = O_x\big(\sqrt{N}\,g(N)\big) \qquad (N \geqslant 1;\ N \text{ integer})$$

or

$$S_N(x) \neq O_x\big(\sqrt{N}\,g(N)\big) \qquad (N \geqslant 1;\ N \text{ integer}),$$

according as the series

$$\sum_{n=1}^{\infty} \frac{1}{n g^4(n)} \tag{3}$$

converges or diverges. "Almost all" means "up to a set of Lebesgue measure zero".

In this notation the dependence of the O-symbols on the given sequence g is suppressed. Since there is no minimal (maximal) $g(n)$ for which (3) converges (diverges), O_x may be replaced by o_x in Theorem 2. If, in particular, $g^4(n)$ belongs to the logarithmic scale, then Theorem 2 is contained implicitely in results of Walfisz [11], p. 383–384, on infinite theta series.

2. Notations. $f(t) = O_{a,b,\ldots}\big(g(t)\big)$, $t \in D$, for two functions f, g means that there is a constant $c > 0$ depending only on the parameters $a, b, \ldots$ such that $|f(t)| \leqslant c|g(t)|$ for all t in the set D. If c does not depend on any parameter, i.e. if c is an absolute constant, we drop the subscripts of the O-symbol. If simultaneously $f(t) = O\big(g(t)\big)$ and $g(t) = O\big(f(t)\big)$ in D, we write $f(t) \asymp g(t)$, $t \in D$.

A prime on the sign of summation $\sum$ has the following meaning: If $a \leqslant b$ is a pair of reals and $\{c_n\}$ a sequence of complex numbers, then we put

$$\sum_{n=a}^{b}{}' c_n = \sum_{a \leqslant n \leqslant b} c_n - \frac{c_a + c_b}{2},$$

where c_a or c_b is defined to be zero, whenever a or b is no integer. Furthermore we use the definitions

$$\sum_{n} c_n = \lim_{M\to\infty} \sum_{n=-M}^{M} c_n,$$

and

$$\sum_{n\neq a} c_n = \lim_{M\to\infty} \sum_{\substack{n=-M \\ n\neq a}}^{M} c_n$$

in cases of existence.

In the sequel q always denotes a positive integer and p an integer relatively prime to q. For an integer N the number $\langle N\rangle_q$ is defined by the conditions $\langle N\rangle_q \equiv N \bmod q$, $0 \leqslant \langle N\rangle_q < q$. By $\sum_{n \bmod q}$ we mean summation over any complete residue system modulo q. We use the abbreviations

$$E(x) = e^{\pi i x}, \qquad \sqrt{i} = E(1/4),$$

and define $\sqrt{x}$ for real x to be $\geqslant 0$ or $i\sqrt{|x|}$ according as $x \geqslant 0$ or < 0.

3. A general approximation. For integers n we shall need the generalized Gaussian sums

$$g_n(p,q) = \sum_{h \bmod 2q} E\left(\frac{ph^2+nh}{q}\right) = g_{-n}(p,q). \tag{4}$$

The ordinary Gaussian sum $g_0(p,q)$ is explicitely known (see e.g. [10], p. 13, 16) as

$$\frac{g_0(p,q)}{2\sqrt{q}} = \begin{cases} 0 & \text{for} \quad 2\nmid pq, \\ \left(\frac{2p}{q}\right) E\left(\frac{(q-1)^2}{8}\right) & \text{for} \quad 2|p, \\ \left(\frac{q}{p}\right) E\left(\frac{p}{4}\right) & \text{for} \quad 2|q \end{cases}$$

in terms of the quadratic residue symbol $\left(\frac{a}{b}\right)$. For general n the sum $g_n(p,q)$ can be reduced to $g_0(p,q)$ by

Lemma 1. *One has $|g_n(p,q)| = 0$ or $2\sqrt{q}$ according as $pq+n$ is odd or even, and in the latter case more precisely*

$$g_n(p,q) = E\left(\frac{p^*n^2}{4q}\right) g_0(p+\delta q, q),$$

where $\delta = 1$ or 0 according as pq is odd or even, and where

$$p^* = -p(p')^2 \tag{5}$$

with any solution p' of the congruence $p'p \equiv 1 \bmod q$, subject to the additional condition $4|p'$ if q is odd.

Proof. (a) First let $pq+n$ be odd. Since $h+q$ runs through a complete residue system modulo $2q$ if h does so, and since

$$p(h+q)^2+n(h+q) \equiv ph^2+nh+(pq+n)q \bmod 2q,$$

we have

$$g_n(p,q) = \sum_{h \bmod 2q} E\left(\frac{p(h+q)^2+n(h+q)}{q}\right) = E(pq+n)g_n(p,q) = -g_n(p,q),$$

hence $g_n(p,q) = 0$.

(b) Secondly let $pq+n$ be even.

(b_1) If pq is odd, n is odd, and we have $n = pp'n+uq$ with some odd integer u, hence for all integers h

$$ph^2+nh = p\big(h+(p'n)/2\big)^2+uhq+(p^*n^2)/4$$

and

$$uh \equiv \big(h+(p'n)/2\big) \equiv \big(h+(p'n)/2\big)^2 \bmod 2,$$

consequently

$$g_n(p,q) = E\left(\frac{p^*n^2}{4q}\right) \sum_{h \bmod 2q} E\left(\frac{(p+q)\big(h+(p'n)/2\big)^2}{q}\right) = E\left(\frac{p^*n^2}{4q}\right) g_0(p+q,q).$$

(b_2) If pq is even, n is even, and we have $n \equiv p'pn \bmod 2q$, hence for all integers h:

$$ph^2+nh \equiv p\big(h+(p'n)/2\big)^2+(p^*n^2)/4 \bmod 2q,$$

therefore

$$g_n(p,q) = E\left(\frac{p^*n^2}{4q}\right) g_0(p,q). \ \blacksquare$$

Corollary of Lemma 1. *For integers n and $2a$ we have*

$$g_{2(n+a)}(p,q) = E\left(\frac{p^*}{q}(n^2+2an)\right) g_{2a}(p,q).$$

The main object of this paper will be the sum

$$S_N(x,\theta) = \sideset{}{'}\sum_{n=0}^{N} E(n^2x+2n\theta),$$

defined for real N, x, θ with $N \geqslant 0$. Emphasizing a rational approximation p/q of x, it will be also convenient to use the abbreviation

$$S_N(p,q,\xi,\theta) = S_N\left(\frac{p}{q}+\frac{\xi}{q}, \frac{\theta}{q}\right).$$

Lemma 2. *Let N, ξ, a be real, $N \geqslant 0$ and A such that $pq+2A$ is an even integer. Then*

$$S_N(p,q,\xi,A+\alpha) = \sum_n \frac{g_{2(n+A)}(p,q)}{2q} \int_0^N E\left(\frac{t^2\xi+2t(\alpha-n)}{q}\right) dt. \tag{6}$$

Proof. Put $A+\alpha = \theta$. We start with the identity

$$S_N(p,q,\xi,\theta) = \sum_{h \bmod 2q} E(ph^2/q) S_{N,h}, \tag{7}$$

where

$$S_{N,h} = \sideset{}{'}\sum_{\substack{n=0 \\ n\equiv h \bmod 2q}}^{N} E\left(\frac{n^2\xi+2n\theta}{q}\right) = \sideset{}{'}\sum_{m=-h/2q}^{(N-h)/2q} E\left((h+m2q)^2\frac{\xi}{q}+2(h+m2q)\frac{\theta}{q}\right).$$

To $S_{N,h}$ we apply Poisson's summation formula

$$\sideset{}{'}\sum_{m=a}^{b} f(m) = \sum_n \int_a^b f(t) E(-2nt)\, dt,$$

valid for a, b real, $a \leqslant b$, and complex-valued functions f with a continuous derivative in $[a, b]$ (see [14], vol. I, chap. II, § 13). We obtain

$$S_{N,h} = \sum_n \int_{-h/2q}^{(N-h)/2q} E\left((h+t2q)^2\frac{\xi}{q} + 2(h+t2q)\frac{\theta}{q} - 2nt\right)dt$$

$$= \sum_n \frac{E(nh/q)}{2q} \int_0^N E\left(\frac{t^2\xi + t(2\theta - n)}{q}\right)dt.$$

Inserting this in (7) yields

$$S_N(p, q, \xi, \theta) = \sum_n \frac{g_n(p, q)}{2q} \int_0^N E\left(\frac{t^2\xi + t(2\theta - n)}{q}\right)dt.$$

Now Lemma 1 implies (6). ■

For the special case $p = 0$, $q = 1$ Lemma 2 contains the identity which is the basis of Wilton's proof [13] of the approximate functional equation of $S_N(x, \theta)$ (see Theorem A).

For an analysis of the preceding formulae we introduce for complex y the normed Fresnel integral

$$F(y) = \frac{1}{\sqrt{i}} \int_0^y E(t^2)\,dt,$$

hence

$$F(-y) = -F(y), \quad F(0) = 0, \quad F'(0) = 1/\sqrt{i}.$$

We have

$$\text{(8)} \qquad \operatorname{Re}\big(\sqrt{i}F(y)\big) > 0 \quad \text{and} \quad \operatorname{Im}\big(\sqrt{i}F(y)\big) > 0 \quad \text{for } y > 0,$$
$$F(\infty) = 1/2, \quad F(-\infty) = -1/2,$$

where $F(\infty)$ and $F(-\infty)$ are defined as the limit of $F(y)$ for $y \to \infty$ along the positive or negative real axis respectively. From this we infer that

$$\text{(9)} \qquad F(y) \asymp \frac{y}{1+|y|} \quad \text{for real } y.$$

Furthermore we obtain by partial integration (along the real axis) of

$$\text{(10)} \qquad F(y) - F(\operatorname{sign} y \cdot \infty) = \frac{1}{\sqrt{i}} \int_{\operatorname{sign} y \cdot \infty}^{y} E(t^2)\,dt$$

that for real $y \neq 0$

$$\text{(11)} \qquad F(y) = \frac{\operatorname{sign} y}{2} + \frac{E(y^2)}{\sqrt{i}2\pi i y} + O\left(\frac{1}{y^3}\right).$$

Note that for real x and negative ξ

$$\text{(12)} \qquad F\left(\frac{x}{\sqrt{\xi}}\right) = \overline{F\left(\frac{-x}{\sqrt{|\xi|}}\right)},$$

the bar denoting the conjugate complex value.

Also the series

$$\text{(13)} \qquad \Phi_{p,q,r,a}(x) = \frac{1}{4\pi i} \sum_{n \neq 0} \frac{g_{2(n+a)}(p, q)E(-2rn/q)}{x - n},$$

defined for $x \in (-1, 1)$ and integers $r, 2a$, will play an important role. Since the numerators of its terms have period q, the series is convergent. Now it is convenient to begin with our most general approximation result.

THEOREM 3. *Let ξ, θ be real and N be a non-negative integer. Define p^* by (5) and let $0 < \varepsilon \leqslant 1/2$. Choose A and B such that $pq+2A$ and $pq+2B$ are even integers and that $\theta = A + a$, $N\xi + \theta = B + \beta$ and $(B-A)\xi \geqslant 0$ holds with $|a| \leqslant 1-\varepsilon$, $|\beta| \leqslant 1-\varepsilon$. Put $M = B - A$ and $r = \langle N \rangle_q$. Then*

$$\text{(a)} \qquad S_N(p, q, 0, \theta) = \frac{g_{2A}(p, q)}{2q} \int_0^N E\left(\frac{2ta}{q}\right)dt + E\left(\frac{2Na}{q}\right)\Phi_{p,q,r,A}(a) - \Phi_{p,q,0,A}(a),$$

(b) *for $\xi \neq 0$:*

$$S_N(p, q, \xi, \theta) = \frac{\omega_{2A}(p, q)}{\sqrt{\xi}} E\left(\frac{-a^2}{q\xi}\right) S_{|M|}\left(p^*, q, -\frac{1}{\xi}, \frac{\xi}{|\xi|}\left(Ap^* + \frac{a}{\xi}\right)\right) +$$
$$+ \frac{\omega_{2B}(p, q)}{\sqrt{\xi}} E\left(-\frac{(M-a)^2}{q\xi}\right) F\left(\frac{\beta}{\sqrt{q\xi}}\right) -$$
$$- \frac{\omega_{2A}(p, q)}{\sqrt{\xi}} E\left(\frac{-a^2}{q\xi}\right) F\left(\frac{a}{\sqrt{q\xi}}\right) +$$
$$+ E\left(\frac{-N^2\xi + 2N\beta}{q}\right)\Phi_{p,q,r,B}(\beta) - \Phi_{p,q,0,A}(a) + O_\varepsilon(\xi q\sqrt{q}),$$

where

$$\omega_n(p, q) = \frac{\sqrt{i}g_n(p, q)}{2\sqrt{q}} \quad (pq + n \text{ even})$$

is an $8q$-th root of unity (see Lemma 1).

Proof. For the case $\xi = 0$ the assertion follows immediately from Lemma 2 by evaluating the terms of the series for $n \neq 0$. In view of the formula

$$S_N(p, q, \xi, \theta) = \overline{S_N(-p, q, -\xi, -\theta)}$$

and (12) it suffices to prove the theorem for $\xi > 0$. We start with formula (6). There we distinguish several cases for n in the integral

$$(14)\qquad \int_0^N := \int_0^N E\left(\frac{t^2\xi+2t(a-n)}{q}\right)dt$$
$$= \sqrt{i}\sqrt{\frac{q}{\xi}}\,E\left(-\frac{(a-n)^2}{q\xi}\right)\left\{F\left(\frac{N\xi+a-n}{\sqrt{q\xi}}\right)-F\left(\frac{a-n}{\sqrt{q\xi}}\right)\right\}.$$

(a) Let $n<0$. Then $0<a-n\leqslant N\xi+a-n$, hence (11) applied to (14) yields

$$(15)\qquad \int_0^N = K_n-L_n+O\left(\frac{\xi q^2}{|a-n|^3}+\frac{\xi q^2}{|N\xi+a-n|^3}\right),$$

where

$$K_n = \frac{q}{2\pi i}\,\frac{E\big((N^2\xi+2N(a-n))/q\big)}{N\xi+a-n},\qquad L_n=\frac{q}{2\pi i}\,\frac{1}{a-n}.$$

(b) Let $n>B-A$. Then $a-n\leqslant N\xi+a-n<0$, then as in (a) we obtain also formula (15). The following cases are treated similarly.

(c) Let $0<n<B-A$. Then $a-n<0<N\xi+a-n$, hence

$$\int_0^N = \sqrt{i}\sqrt{\frac{q}{\xi}}\,E\left(-\frac{(a-n)^2}{q\xi}\right)+K_n-L_n+O\left(\frac{\xi q^2}{|a-n|^3}+\frac{\xi q^2}{|N\xi+a-n|^3}\right).$$

(d) Let $n=0<B-A$. Then $N\xi+a-n>0$, hence

$$\int_0^N = \sqrt{i}\sqrt{\frac{q}{\xi}}\,E\left(-\frac{a^2}{q\xi}\right)\left(\frac{1}{2}-F\left(\frac{a}{\sqrt{q\xi}}\right)\right)+K_n+O\left(\frac{\xi q^2}{|N\xi+a-n|^3}\right).$$

(e) Let $n=B-A>0$. Then $a-n<0$, hence

$$\int^N = \sqrt{i}\sqrt{\frac{q}{\xi}}\,E\left(-\frac{(a-B+A)^2}{q\xi}\right)\left(\frac{1}{2}+F\left(\frac{N\xi+a-B+A}{\sqrt{q\xi}}\right)\right)-$$
$$-L_n+O\left(\frac{\xi q^2}{|a-n|^3}\right).$$

(f) If $n=0=B-A$, we leave (14) in the given form.

Inserting the results of (a)–(f) into (6) and using the corollary of Lemma 1, we obtain Theorem 3.

Remark 1. By repeated partial integration of (10) one can refine the asymptotic relation (11) to the asymptotic expansion

$$F(y) = \frac{\operatorname{sign} y}{2}+\frac{E(y^2)}{\sqrt{i}}\sum_{\nu=1}^{k}\frac{1\cdot 3\cdots(2\nu-3)}{(2\pi i)^\nu y^{2\nu-1}}+O\left(\frac{1\cdot 3\cdots(2k-1)}{(2\pi)^k y^{2k+1}}\right)$$

for real $y\neq 0$ and arbitrary positive integers k. Using this, one can (as the preceding proof shows) replace the remainder term in Theorem 3 (b) by an asymptotic series which after being chopped off after its kth term leaves us with the remainder $O_{\varepsilon,k}(|\xi q|^k\sqrt{q})$.

4. The function Φ and incomplete Gaussian sums. Since $\Phi=\Phi_{p,q,r,a}(x)$ vanishes for odd $pq+2a$ (see Lemma 1 and (13)), we may assume $2\,|\,(pq+2a)$ in the sequel. Furthermore the case of an odd $2a$ can be reduced to the one of an even $2a$ by

LEMMA 3. *For odd $2a$ and odd pq we have*

$$\Phi_{p,q,r,a}(x) = E\left(\frac{p^*}{q}(a-\tfrac14)\right)\Phi_{p+q,q,r-p^*/2,a-1/2}(x),$$

where as usual $x\in(-1,1)$ and r integral.

Proof. $p\equiv p+q \bmod q$ implies $p^*\equiv(p+q)^* \bmod q$. Since p^* and $(p+q)^*$ are even by construction, the latter congruence holds also modulo $2q$. This in conjunction with Lemma 1 yields

$$g_{2(n+a)}(p,q) = E\left(\frac{p^*}{q}(n+a)^2-\frac{(p+q)^*}{q}(n+a-\tfrac12)^2\right)g_{2(n+a-1/2)}(p+q,q)$$
$$= E\left(\frac{p^*}{q}(n+a-\tfrac14)\right)g_{2(n+a-1/2)}(p+q,q)$$

for all integers n. Inserting this in (13), we obtain the assertion.

For the further investigation of Φ we start with a simple identity. Since $g_{2(n+a)}(p,q)$ has the period q with respect to the variable n, the well-known expansion

$$\pi\cot\pi z = \sum_n \frac{1}{z-n}\qquad (z\not\equiv 0 \bmod 1)$$

and (13) lead for $x\neq 0$ to the identity

$$(16)\qquad \Phi_{p,q,r,a}(x)$$
$$= \frac{1}{4qi}\sum_{n \bmod q}E\left(\frac{-2nr}{q}\right)g_{2(n+a)}(p,q)\cot\left(\frac{\pi}{q}(x-n)\right)-\frac{g_{2a}(p,q)}{2\pi x i},$$

in particular for integers r and a

$$(17)\qquad \Phi_{0,1,r,a}(x) = \frac{1}{2\pi i}\left(\pi\cot\pi x-\frac{1}{x}\right).$$

This and Theorem 3 imply the "approximate functional equation" of Hardy and Littlewood [4] which can be formulated as follows.

THEOREM A. *For all real N, x, θ with $N \geq 0$, $0 < x < 1$ one has*

$$S_N(x, \theta) = \frac{\sqrt{i}\,E(-\theta^2/x)}{\sqrt{x}} S_{Nx}\left(-\frac{1}{x}, \frac{\theta}{x}\right) + O\left(\frac{1+|\theta|}{\sqrt{x}}\right).$$

Proof. Since changing N into $N+O(1)$ yields only the error $O(1/\sqrt{x})$ on both sides, it suffices to prove the assertion for integral N. In Theorem 3 we choose $p = 0$, $q = 1$, $\varepsilon = 1/2$ and observe that $\Phi_{0,1,r,a}(t) = O(1)$ for $|t| \leq 1/2$ by (17). Thus we arrive immediately at the assertion. ■

Now we want to estimate Φ for $q > 1$. From (16) and Lemma 1 one infers that $\Phi = O_\varepsilon(\sqrt{q}\log(q+1))$ for $|x| \leq 1-\varepsilon$, $0 < \varepsilon < 1$. In this estimate one can get rid of the factor $\log(q+1)$ by the following considerations (see Lemma 6). We use the following lemma on incomplete Gaussian sums (which we did not find in the literature and which seems to be new).

LEMMA 4. *For all real N, θ with $N \geq 0$ the estimate*

$$S_N\left(\frac{p}{q}, \theta\right) = O\left(\frac{N}{\sqrt{q}} + \sqrt{q}\right)$$

holds.

Proof. Since the left hand side depends only on the residue classes of p modulo $2q$ and of θ modulo 1, and since $S_N(-p/q, \theta) = \overline{S_N(p/q, -\theta)}$, we may assume $0 < p < q$ and $|\theta| \leq 1/2$. As in [4] we employ the expansion of reals $x_0 \in (0, 1)$ into a simple continued fraction

$$x_0 = \frac{1}{a_1 + x_1}, \quad x_1 = \frac{1}{a_2 + x_2}, \quad \ldots, \quad x_{\nu-1} = \frac{1}{a_\nu + x_\nu}, \quad \ldots,$$

where the a_ν are positive integers and $0 \leq x_\nu < 1$. For the rational number $x_0 = p/q$ this process terminates, say after n steps, i.e. $x_n = 0$, $0 < x_\nu < 1$ ($\nu = 1, \ldots, n-1$). By Theorem A we obtain

$$S_N(x_0, \theta) = \frac{\omega_1}{\sqrt{x_0}} \overline{S_{Nx_0}(x_1, \theta_1)} + O\left(\frac{1}{\sqrt{x_0}}\right),$$

where ω_1 is the unimodular number $\sqrt{i}\,E(-\theta^2/x_0)$ and θ_1 is the real number defined by $\theta_1 \equiv -\theta/x_0 + a_1/2 \bmod 1$ and $-1/2 \leq \theta_1 < 1/2$. Expressing in the same way $S_{Nx_0}(x_1, \theta_1)$ by the sum $S_{Nx_0x_1}(x_2, \theta_2)$ for some $\theta_2 \in [-1/2, 1/2)$ and so forth, and observing that $x_{\nu-1}x_\nu < 1/2$ ($\nu = 1, \ldots, n$), one gets by this iteration as in [4], p. 212–213, the relation

(18) $$S_N(x_0, \theta) = \frac{\omega_n}{\sqrt{x_0 x_1 \ldots x_{n-1}}} S_{Nx_0 \ldots x_{n-1}}\left((-1)^n x_n, (-1)^n \theta_n\right) + O\left(\frac{1}{\sqrt{x_0 x_1 \ldots x_{n-1}}}\right),$$

where $|\omega_n| = 1$ and $|\theta_n| \leq 1/2$. Using the identity

(19) $$x_0 x_1 \ldots x_{\nu-1} = (q_\nu + x_\nu q_{\nu-1})^{-1} \quad (q_0 = 1)$$

with the denominators q_ν of the convergents

$$\frac{p_\nu}{q_\nu} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_\nu}}} \quad (\nu = 1, \ldots, n)$$

of x_0 and observing that $q = q_n$, we see that

$$S_N\left(\frac{p}{q}, \theta\right) = \omega_n \sqrt{q}\, S_{N/q}\left(0, (-1)^n \theta_n\right) + O(\sqrt{q}).$$

Estimating the first sum on right hand side by the number of its terms, we obtain the assertion. ■

A representation of Φ suitable for applications is given in

LEMMA 5. *For $x \in (-1, 1)$, integers r and a and even pq we have*

$$\Phi_{p,q,r,a}(x) = E\left(\frac{p^* a^2}{q}\right)\left\{S_v\left(\frac{p}{q}, 0\right) - \frac{v}{2q} g_0(p, q)\right\} + \int_0^x \varphi_{p,q,r,a}(t)\,dt,$$

where $v = \langle r - ap^* \rangle_q$ and

$$\varphi_{p,q,r,a}(x) = -\frac{g_0(p, q)}{4\pi i} \sum_{n \neq 0} \frac{E\left((p^*(n+a)^2 - 2nr)/q\right)}{(x-n)^2}.$$

Proof. In Theorem 3 (a) we choose $\theta = A = a = 0$ and note that $\Phi_{p,q,0,0}(0) = 0$ because of (13) and (4). We obtain

(20) $$\Phi_{p,q,N,0}(0) = S_N\left(\frac{p}{q}, 0\right) - \frac{N}{2q} g_0(p, q)$$

for all integers $N \geq 0$. Applying the corollary of Lemma 1 to (13) yields

(21) $$\Phi_{p,q,r,a}(x) = E\left(\frac{p^* a^2}{q}\right) \Phi_{p,q,r-ap^*,0}(x).$$

Since $\varphi_{p,q,r,a}(t)$ converges uniformly in t in every compact subset of $(-1, 1)$, it follows from (13) and Lemma 1 that

$$\Phi_{p,q,r,a}(x) = \Phi_{p,q,r,a}(0) + \int_0^x \varphi_{p,q,r,a}(t)\,dt.$$

This, (21) for $x = 0$ and (20) imply the assertion.

From Lemmas 3, 5, 4 and 1 we obtain

LEMMA 6. *For all integers r, $2a$ and real x with $|x| \leq 1-\varepsilon$, $0 < \varepsilon < 1$:*

$$\Phi_{p,q,r,a}(x) = O_\varepsilon(\sqrt{q}).$$

5. Special cases of Theorem 3. The behavior of elliptic theta functions at rational points can be discussed by means of well-known transformation formulas. There are corresponding approximate functional equations, generalizing the famous result of Hardy–Littlewood (see Theorem A), viz. by Theorem 3 and Lemma 6 with the choice of $\varepsilon = 1/2$ one obtains

THEOREM 4. *Let N, ξ, θ be real with $N \geq 0$, $\xi \neq 0$. Choose A such that $pq+2A$ is an even integer and $\theta = A+\alpha$ holds with $|\alpha| \leq 1/2$. Then*

$$S_N(p,q,\xi,\theta) = \frac{\omega_{2A}(p,q)}{\sqrt{\xi}} E\left(\frac{-\alpha^2}{q\xi}\right) S_{N|\xi|}\left(p^*, q, -\frac{1}{\xi}, \frac{\xi}{|\xi|}\left(Ap^*+\frac{\alpha}{\xi}\right)\right) + O\left(\frac{1}{\sqrt{|\xi|}} + |\xi| q\sqrt{q}\right).$$

This result is useful, if $|\xi|$ is approximately in the range $[1/N, 1/q]$. Estimating $S_{N|\xi|}$ trivially as $O(N|\xi|+1)$ yields the

COROLLARY OF THEOREM 4. *For real N, ξ, θ with $N \geq 0$, $\xi \neq 0$ we have*

$$S_N(p,q,\xi,\theta) = O\left(N\sqrt{|\xi|} + \frac{1}{\sqrt{|\xi|}} + |\xi| q\sqrt{q}\right).$$

Now we want to show how Theorem 3 can be used to obtain asymptotic expansions for $S_N(x,\theta)$. We discuss the special example $A = B$, $N\xi+\alpha = \beta$. To this we apply Lemma 6 and get immediately

THEOREM 5. *Let θ be real, N an integer > 0 and $0 < \varepsilon \leq 1/2$. Choose A such that $pq+2A$ is an even integer and that $\theta = A+\alpha$ holds with $|\alpha| \leq 1/2$. Then for real $\xi \neq 0$ with $|N\xi+\alpha| \leq 1-\varepsilon$ we have*

$$(22) \quad S_N(p,q,\xi,\theta) = \frac{g_{2A}(p,q)}{2q} \int_0^N E\left(\frac{t^2\xi+2t\alpha}{q}\right) dt + O_\varepsilon\left(\sqrt{q}(1+|\xi|q)\right)$$
$$= \frac{\omega_{2A}(p,q)}{\sqrt{\xi}} E\left(-\frac{\alpha^2}{q\xi}\right)\left\{F\left(\frac{N\xi+\alpha}{\sqrt{q\xi}}\right) - F\left(\frac{\alpha}{\sqrt{q\xi}}\right)\right\} + O_\varepsilon\left(\sqrt{q}(1+|\xi|q)\right).$$

If $\varepsilon < 1/2$ the permitted range of ξ is a neighborhood of zero, e.g. $N|\xi| \leq 1/4$ is permitted in case $\varepsilon = 1/4$. Furthermore it suffices to consider $0 < q \leq 4N$, if we want to approximate $S_N(x,t)$ for all pairs of reals x, t. Namely, by Dirichlet's box principle we can find p, q, ξ such that (2) holds. Then we choose A and α such that $t = (A+\alpha)/q$ with $pq+2A$ even and $|\alpha| \leq 1/2$ and apply Theorem 5 to $S_N(x,t) = S_N(p,q,\xi, A+\alpha)$ with $\varepsilon = 1/4$. (Of course the case $\xi = 0$ is obtained by letting $\xi \to 0$.) We see that $S_N(x,t)$ is evaluated up to an error $O(\sqrt{q}) = O(\sqrt{N})$. It is interesting that the order of the main term can be described in terms of simple functions. We have

LEMMA 7. *For real y_1, y_2 we have*

$$F(y_2)-F(y_1) \asymp \begin{cases} \left|\dfrac{1}{y_1} - \dfrac{1}{y_2}\right| + \dfrac{|\gamma|}{\sqrt{y_1 y_2}}, & \text{if } y_1, y_2 \geq 1 \text{ or } y_1, y_2 \leq -1, \\ \dfrac{|y_2-y_1|}{1+|y_2-y_1|}, & \text{else,} \end{cases}$$

where $y_2^2-y_1^2 = 2k+\gamma$ with an integer k and $|\gamma| \leq 1$.

Proof. We may assume $y_2 \geq y_1$ and, since $F(y)$ is an odd function, also that $y_1+y_2 \geq 0$. We start with the identity

$$(23) \quad |F(y_2)-F(y_1)| = \frac{1}{2}\left|\int_0^{y_2^2-y_1^2} \frac{E(t)}{\sqrt{t+y_1^2}}\, dt\right| \quad (y_1 \geq 0).$$

(a) Let $y_1 \geq 1$. Integrating the right hand side of (23) twice by parts we obtain

$$2|F(y_2)-F(y_1)| = \left|\frac{1}{\pi i}\left\{\frac{E(\gamma)}{y_2} - \frac{1}{y_1}\right\} - G\right|$$

with

$$G = \frac{1}{2\pi^2}\left\{\frac{E(\gamma)}{y_2^3} - \frac{1}{y_1^3}\right\} + \frac{3}{4\pi^2}\int_0^{y_2^2-y_1^2} \frac{E(t)}{(\sqrt{t+y_1^2})^5}\, dt.$$

Since

$$|G| \leq \frac{1}{\pi^2}\left|\frac{E(\gamma)}{y_2^3} - \frac{1}{y_1^3}\right| \leq \frac{3}{\pi^2}\left|\frac{E(\gamma)}{y_2} - \frac{1}{y_1}\right|,$$

we obtain

$$F(y_2)-F(y_1) \asymp \left|\frac{E(\gamma)}{y_2} - \frac{1}{y_1}\right| \asymp \left(\frac{1}{y_1} - \frac{1}{y_2}\right) + \frac{|\gamma|}{\sqrt{y_1 y_2}}.$$

(b) Let $0 \leq y_1 < 1$. Estimating the real or imaginary part of

$$\int_0^{y_2^2-y_1^2} \frac{E(t)}{\sqrt{t+y_1^2}}\, dt$$

according as $y_2^2-y_1^2 \leq 1/4$ or $> 1/4$ yields

$$\frac{y_2-y_1}{1+(y_2-y_1)} = O\left(F(y_2)-F(y_1)\right).$$

From this the assertion follows, since apparently

$$F(y_2)-F(y_1) = O\left(\frac{y_2-y_1}{1+(y_2-y_1)}\right).$$

(c) In the remaining case $0 \leqslant y_1+y_2$, $y_1 < 0$ (and therefore $y_2 > 0$), and this implies

$$F(y_2)-F(y_1) \asymp |F(y_2)|+|F(y_1)| \asymp \frac{y_2-y_1}{1+y_2-y_1}$$

in view of (8) and (9). ■

Now we are in a position to prove for

$$T = \frac{\omega_{24}(p,q)}{\sqrt{\xi}} E\left(\frac{-\alpha^2}{q\xi}\right)\left\{F\left(\frac{N\xi+\alpha}{\sqrt{q\xi}}\right)-F\left(\frac{\alpha}{\sqrt{q\xi}}\right)\right\}$$

(with the notation of Theorem 5) the

COROLLARY OF THEOREM 5. *Let* $0<|\xi|\leqslant 1/(4N)$, $0<q\leqslant 4N$, $\alpha' = \alpha \operatorname{sign}\xi$, $|\alpha'|\leqslant 1/2$, $(N^2|\xi|+2N\alpha')/q = 2k+\gamma$ *for some integer* k *and* $|\gamma|\leqslant 1$. *Then*

$$(24)\qquad T \asymp \begin{cases} \dfrac{N}{\sqrt{q}+N\sqrt{|\xi|}}, & \text{if}\quad -N|\xi|-\sqrt{q|\xi|}\leqslant \alpha'\leqslant\sqrt{q|\xi|},\\[2ex] \sqrt{q}\left(\dfrac{N|\xi|}{\alpha'(N|\xi|+\alpha')}+\dfrac{|\gamma|}{\sqrt{\alpha'(N|\xi|+\alpha')}}\right), & \text{otherwise}.\end{cases}$$

In particular: $T = O(\sqrt{q})$, *if* $|\alpha'|\geqslant 3\sqrt{N|\xi|}$ *and* $|\gamma| < \dfrac{|k|q}{N}$; *in all other cases*: $\sqrt{q} = O(T)$.

Proof. (24) is an immediate consequence of Lemma 7 and (12).

(a) Let $|\alpha'|\geqslant 3\sqrt{N|\xi|}$. Then $|N\xi/\alpha'|\leqslant 1/6$, hence $\alpha' \asymp N|\xi|+\alpha' \asymp N|\xi|+2\alpha' = (2k+\gamma)q/N$, hence

$$T \asymp \sqrt{q}\left(\vartheta+\frac{|\gamma|N}{|2k+\gamma|q}\right)\quad\text{with}\quad 0\leqslant\vartheta<1.$$

Therefore $T = O(\sqrt{q})$ if $|\gamma| < |k|q/N$, and $\sqrt{q} = O(T)$ otherwise.

(b) If $\sqrt{q|\xi|}<\alpha'<3\sqrt{N|\xi|}$ or $-3\sqrt{N|\xi|}<\alpha'<-N|\xi|-\sqrt{q|\xi|}$ then $N|\xi|/\big(\alpha'(N|\xi|+\alpha')\big)\geqslant 2/21$, hence $\sqrt{q} = O(T)$.

(c) In the remaining case $T \asymp \dfrac{N}{\sqrt{q}+N\sqrt{|\xi|}} \geqslant \dfrac{\sqrt{q}}{5}$, since $q\leqslant 4N$ and $|\xi|\leqslant 1/(4N)$.

Remark 2. From Theorem 5 (with $\varepsilon = 1/4$) and its corollary we deduce Theorem 1 by observing that we are in the case $\theta = 0$. Therefore we may choose $\alpha = A = 0$ for even pq and $\alpha = -A = \dfrac{\operatorname{sign}\xi}{2}$ for odd pq. In the latter case $\alpha'(N|\xi|+\alpha')\asymp 1$, and if $1/2 = \alpha'\leqslant\sqrt{q|\xi|}$, then $q\asymp N$.

Looking for an upper estimate of $S_N(x,\theta)$, one obtains by combining Weyl's and Vinogradov's method ([9], chapter I, lemma 8a), as is well known, that

$$S_N(x,\theta) = O\left(\frac{N}{\sqrt{q}}+(\sqrt{N}+\sqrt{q})\sqrt{\log q}\right)\quad\text{for}\quad |x-p/q|\leqslant 1/q^2.$$

That the factor $\sqrt{\log q}$ can be removed is shown by Theorem 6. Of course $\sqrt{N}$ may be dropped also, and the resulting estimate is optimal under these general conditions.

THEOREM 6. *Let* x *be real with* $\left|x-\dfrac{p}{q}\right|\leqslant\dfrac{1}{q^2}$. *Then for any real* N, θ *with* $N\geqslant 0$ *we have*(1)

$$S_N(x,\theta) = O\left(\frac{N}{\sqrt{q}}+\sqrt{q}\right).$$

Proof. We may assume N to be integral. Put $\xi = qx-p$. Note that then always $|\xi|\leqslant 1/q$. If $|N\xi|>1/4$, then the assertion is clear by the corollary of Theorem 4. If $|N\xi|\leqslant 1/4$, then Theorem 5 with $\varepsilon = 1/4$ implies the assertion, where g_{24} is estimated by Lemma 1 and $\int_0^N$ as $O(N)$. ■

Remark 3. There is another proof of Theorem 5 for $\theta = 0$, using Theorem 3 only in the special case $p = 0$, $q = 1$, $|Nx|<1/2$. We sketch such a proof for the case $2|pq$, $q\leqslant N$, $0<|\xi|<1/(2N)$. Put $x_0 = \dfrac{p}{q}+\dfrac{\xi}{q}$. Assume $0<x_0<1$. From the theory of continued fractions one knows that $p = p_n$, $q = q_n$ for some convergent p_n/q_n of x_0 (see notation for (18)) and that $x_n = (q+x_nq_{n-1})|\xi|$. This, (18) and (19) yield

$$S_N\left(\frac{p}{q}+\frac{\xi}{q}\right) = \omega_n\sqrt{q+x_nq_{n-1}}\,S_{N/(q+x_nq_{n-1})}\big((q+x_nq_{n-1})\xi,\ \sigma\theta_n\big)+O(\sqrt{q})$$

with $\sigma = \operatorname{sign}\xi$. Now applying Theorem 3 in the mentioned special case and computing ω_n and θ_n, we obtain Theorem 5. (Observe that θ_n, ω_n are independent of N and that, e.g., θ_n can only equal 0, $-1/2$, or $x_n/2$, where the alternatives do not depend upon ξ.)

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(1) Obviously sums of the kind

$$\sum_{n=L}^{N+L}{}' E(n^2x) = E(L^2x)S_N(x,Lx)\qquad (L\text{ integer})$$

are then also $O(N/\sqrt{q}+\sqrt{q})$.

6. An application. We want to prove Theorem 2. Apparently its first part is a special case of

LEMMA 8. *Let $\{g(n)\}_1^\infty$ be a non-decreasing sequence of positive numbers such that*

$$\sum_{n=1}^{\infty} \frac{1}{ng^4(n)}$$

converges. Then for all real θ and almost all real x

$$S_N(x, \theta) = O_x\big(\sqrt{N} g(N)\big) \quad (N \geqslant 1, N \text{ integer}).$$

Proof. We may restrict ourselves to irrational x in $(0, 1)$. Let p_ν/q_ν $(\nu = 1, 2, \ldots)$ be the convergents of x (see proof of Lemma 4). According to a well-known theorem of Khintchine (see [3], Theorem I, p. 120) for almost all x

(25) $$|q_\nu x - p_\nu| > q_\nu^{-1} g^{-4}(q_\nu) \quad \text{for all } \nu \geqslant \nu_0,$$

where ν_0 is a positive integer depending on x. Take any positive integer N. If $Ng^{-2}(N) < q_{\nu_0}$, we have $|S_N(x, \theta)| \leqslant N < \sqrt{q_{\nu_0}}\sqrt{N} g(N)$, hence the desired estimate. Now let $Ng^{-2}(N) \geqslant q_{\nu_0}$, and let ν be the index $\geqslant \nu_0$ with $q_\nu \leqslant Ng^{-2}(N) < q_{\nu+1}$. From the theory of continued fractions one knows that

(26) $$|q_\nu x - p_\nu| < q_{\nu+1}^{-1} < N^{-1} g^2(N).$$

From the Corollary of Theorem 4 for the case $p = p_\nu$, $q = q_\nu$ and from (25) and (26) it follows that (note that $g(n) \nearrow \infty$ for $n \to \infty$) for all sufficiently large N:

$$\begin{aligned} S_N(x, \theta) &= O\big(\sqrt{N} g(N) + \sqrt{q} g^2(q) + g^2(N) N^{-1} q \sqrt{q}\big) \\ &= O\big(\sqrt{N} g(N) + \sqrt{q} g^2(q)\big) \\ &= O\big(\sqrt{N} g(N) + \sqrt{N} g^{-1}(N) g^2(q)\big) = O\big(\sqrt{N} g(N)\big). \ \blacksquare \end{aligned}$$

For the proof of the second part of Theorem 2 we use

LEMMA 9. *Let $\{\psi(n)\}_1^\infty$ be a non-increasing sequence of positive numbers such that $\sum_1^\infty \psi(n)$ diverges. Then for almost all real x the inequality*

$$|qx - p| < \psi(q)$$

has infinitely many solutions p, q with $2|q$ (and $(p, q) = 1$ as usual).

Of course it suffices to show that for almost all irrational x the inequality $|rx - s| < \psi(r)$ has infinitely many solutions r, s with even $r > 0$ and odd s. Without the condition $2|q$ Lemma 9 is a well-known theorem of Khintchine (see [3], Theorem I, p. 120). Under the additional condition that $\psi(n)/\psi(2n)$ be bounded, Lemma 9 is essentially equivalent to a result of Walfisz [11]. A proof of Lemma 9 can be obtained by slight modifications of the proof given in [3] for the second part of Theorem I, chap. VII. First one observes that

$$\sum_{n=1}^{N} \frac{\varphi(2n)}{2n} \geqslant \frac{1}{2} \sum_{n=1}^{N} \frac{\varphi(n)}{n} \quad (N \geqslant 1),$$

where $\varphi(\ )$ denotes Euler's function. Then, as in [3] there exists a sequence $\tau(n) \to 0$ and a set E of positive measure such that for $y \in E$ there are infinitely many p, q satisfying

$$|qy - p| < \tau(q)\psi(q), \quad 2|q.$$

Next we observe, analogously to [3], Corollary, p. 126, that for almost all x there exist integers $T(x) > 0$, $z(x)$, and an element $y(x) \in E$ such that $x = Ty + z$, where T is odd. Then we have

$$|qx - (Tp + qz)| = T|qy - p| < T\tau(q)\psi(q) < \psi(q)$$

for infinitely many p, q with $2|q$. Since p must be odd, so is $s = Tp + qz$. ■

Proof of the second part of Theorem 2. Let $\{g(n)\}_1^\infty$ be any non-decreasing sequence of positive numbers such that $\sum_1^\infty n^{-1} g^{-4}(n) = \infty$. It suffices to prove the result in the case $g(n) \to \infty$ for $n \to \infty$. Take any sequence $\{h(n)\}_1^\infty$ of positive numbers such that $h(n)/g(n)$ is monotone and tends to ∞ for $n \to \infty$ and such that still $\sum_1^\infty n^{-1} h^{-4}(n) = \infty$. By Cauchy's condensation theorem we infer that $\sum_{k=1}^\infty h^{-4}(2^k) = \infty$, hence $\sum_{k=1}^\infty h^{-4}(2^{2k}) = \infty$, hence $\sum_1^\infty n^{-1} h^{-4}(n^2) = \infty$. Put $f(n) = h(n^2)$. By Lemma 9 for almost all real x the conditions

(27) $$|qx - p| < q^{-1} f^{-4}(q), \quad 2|q$$

have infinitely many solutions p, q. By (27) and by Khintchine's theorem, already used for Lemma 8, we may assume that these pairs p, q satisfy the additional condition

(28) $$1 \leqslant f^4(q) < q.$$

Apply Theorem 1 to these pairs with $\xi = qx - p$, $N = [qf^2(q) + 1]$, then for some absolute constants $c_1, c_2, c_3 > 0$ and all sufficiently large q

(29) $$|S_N(x)| > c_1 \frac{N}{\sqrt{q}} - c_2\sqrt{q} \geqslant c_1\sqrt{N} f(q) - c_2\sqrt{q} > c_3\sqrt{N} f(q).$$

Because of $q \leqslant N$ and (28) we have

$$q \geqslant (N-1)f^{-2}(q) > (N-1)q^{-1/2} \geqslant (N-1)f(q)/\sqrt{N} \geqslant [\sqrt{N}+1]$$

for all sufficiently large q, hence by (29) it follows that

$$|S_N(x)| > c_3\sqrt{N}f([\sqrt{N}+1]) = c_3\sqrt{N}h([\sqrt{N}+1]^2) \geqslant c_3\sqrt{N}h(N). \blacksquare$$

Remark 4. Another proof of Lemma 8 and Theorem 2 can be given by means of formula (18), Theorem 3 in the special case $p = 0$, $q = 1$, and by more extensive use of continued fractions (see Remark 3).

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Der Satz von Erdös und Fuchs in reell-quadratischen Zahlkörpern

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Herrn Prof. Dr. Th. Schneider zum 65. Geburtstag gewidmet

Es sei K ein reell-quadratischer Zahlkörper mit der Diskriminante $d > 0$. Es bezeichne A eine unendliche Folge von ganzen Zahlen $a_i \in K$ mit $a_i \succeq 0$, d.h., es sei sowohl $a_i \geqslant 0$ als auch die zu a_i konjugierte Zahl $a_i' \geqslant 0$; ferner sei $0 \in A$. Für ganze Zahlen $\xi \in K$ werde erklärt:

$$f(\xi) := \sum_{\substack{(i,j)\\ a_i+a_j=\xi,\, a_i,a_j\in A}} 1;$$

in $f(\xi)$ werden für $i \neq j$ demnach sowohl die Zerlegung $a_i + a_j$ als auch die Zerlegung $a_j + a_i$ gezählt. Für reelle Zahlen $x \geqslant 0$, $x' \geqslant 0$ setze man

$$F(x, x') := \sum_{\substack{0 \leqslant \xi \leqslant x\\ 0 \leqslant \xi' \leqslant x'}} f(\xi).$$

Über die Folge A werde vorausgesetzt:

(1) $$F(x, x') = a \cdot xx' + r(x, x'), \quad a > 0;$$

$r(x, x')$ hänge nur vom Produkt $x \cdot x'$ ab, und es gelte $r(x, x') = O(xx')$. Unter dieser Voraussetzung wird in der vorliegenden Arbeit gezeigt:

Satz. *Es gilt:*

$$\overline{\lim_{xx' \to \infty}} |r(x, x')|(xx')^{-1/4}\log(xx') > 0.$$

Eine Anwendung des Satzes liefert das Kreisproblem des Körpers K ([3], [4]). In dem Falle ist

$$A = \{a^2;\ a \in K \text{ ganz}\},$$

wobei also für $a \neq 0$ a^2 und $(-a)^2$ verschiedene Elemente der Folge A sind. Die Bedingung (1) ist erfüllt.

In [3] wird zwar das schärfere Resultat

$$r(x, x') \neq o\big((xx')^{1/4}\big)$$