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L. CHIERCHIA
E. ZEHNDER

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L. CHIERCHIA - E. ZEHNDER

1. - Introduction

We first describe the existence problem of quasiperiodic solutions in a general setting and consider a Lagrangian function $F = F(t, x, p)$,

$$F(t, x, p) \text{ defined on } T^{n+1} \times \mathbb{R}^n, \tag{1.1}$$

i.e. periodic in $(t, x) \in \mathbb{R}^{n+1}$ with periodic 1, $T^{n+1} = \mathbb{R}^{n+1}/\mathbb{Z}^{n+1}$. The aim is to find special solutions of the associated Euler-equations

$$\frac{\mathrm{d}}{\mathrm{d}t} F_p(t, x(t), \dot{x}(t)) = F_x(t, x(t), \dot{x}(t)). \tag{1.2}$$

We shall call, in the following, a solution $x(t)$ quasiperiodic with frequencies ω, if it is of the form

$$x(t) = U(t, \omega t), \tag{1.3}$$

where $\omega \in \mathbb{R}^n$ is a given vector with rationally independent components, and where

$$U(t, \vartheta) - \vartheta =: u(t, \vartheta) \quad \text{is defined on } T^{n+1} \tag{1.4}$$

i.e. is periodic in (t, ϑ). Inserting (1.3) into (1.2), one obtains the nonlinear partial differential equation for U:

$$\mathrm{D} F_p(t, U, \mathrm{D}U) = F_x(t, U, \mathrm{D}U), \tag{1.5}$$

where

$$\mathrm{D} = \mathrm{D}(\omega) = \sum_{j=1}^{n} \omega_j \frac{\partial}{\partial \vartheta_j} + \frac{\partial}{\partial t}. \tag{1.6}$$

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The differential operator D depends on the frequencies $\omega = (\omega_1, \cdots, \omega_n)$. It is the differentiation in the direction $(\omega, 1)$. Restricting our attention to functions of the special form

$$F(t, x, p) = \frac{1}{2}|p|^2 + f(t, x), \tag{1.7}$$

the equation to be solved becomes

$$\mathrm{D}^2 u = f_x(t, \vartheta + u) \tag{1.8}$$

for $u(t, \vartheta) = U(t, \vartheta) - \vartheta$ being a function on T^{n+1}. In order to solve (1.8) we shall assume f to be analytic and the frequencies ω to satisfy the diophantine conditions:

$$|\langle \omega, j \rangle + m| \geq \gamma(|j|)^{-\tau} \tag{1.9}$$

for two constants $\gamma > 0$ and $\tau \geq n$ and for all $(j, m) \in \mathbb{Z}^n \times \mathbb{Z} \setminus \{0\}$.

It is well know that under these conditions on f and ω the equation (1.8) has a solution, provided f is sufficiently small (in an appropriate sense). This is a consequence of the KAM theory, and we refer to [CC], [SZ] and [M1]. However, if f is not small, then (1.8) may not admit any solutions for frequencies contained in a compact region of $\mathbb{R}^n$, see [Ma]. We shall not impose any smallness conditions on f in the following. Instead we shall construct quasiperiodic solutions having sufficiently large frequencies. We point out, that the system under consideration, decribed by a Lagrangian function in the special form of (1.7), can be viewed as beeing "close to an integrable system" in the region in which $|p|$ is large. Introducing

$$\omega(\alpha) = \frac{1}{\alpha}\omega, \tag{1.10}$$

we look for quasiperiodic solutions having frequencies $\omega(\alpha)$ for sufficiently small $\alpha \in R$ with $\alpha \neq 0$. We shall abbreviate

$$E(u) =: \mathrm{D}^2 u - f_x(t, \vartheta + u), \tag{1.11}$$

with $\mathrm{D} = \mathrm{D}(\omega(\alpha))$.

In the second section we shall prove that there is unique formal powerseries expansion in α:

$$\tilde{u} \sim \sum_{j=2}^{\infty} \alpha^j u_j(\vartheta, t), \tag{1.12}$$

with analytic functions u_j on T^{n+1}, which solves the equation $E(\tilde{u}) = 0$ formally, and satisfies

$$\int_{T^{n+1}} u_j \mathrm{d}t \mathrm{d}\vartheta = 0, \text{ for all } j.$$

However, in general, the series diverges as it is well known, and our aim is to show that the formal series can be interpreted as an asymptotic expansion for the true quasiperiodic solutions u_α, as α tends to zero. For this purpose α is required to belong to the subset

$$A(\omega) = \left\{\alpha \in \mathbb{R} : \left|\frac{1}{\alpha}\langle \omega, j\rangle + m\right| \geq \gamma |j|^{-\tau}, \text{ for all } (j,m) \in \mathbb{Z}^n \times \mathbb{Z}\backslash 0\right\}. \tag{1.13}$$

If γ is sufficiently small and $\tau > n+1$, we will see that the set $\{\alpha \in A(\omega) : |\alpha| \leq \epsilon\}$ has positive Lebesgue measure for every $1 > \epsilon > 0$. Setting now for every $N \geq 2$

$$\tilde{u}_N := \sum_{j=2}^{N} \alpha^j u_j(t, \vartheta), \tag{1.14}$$

one concludes that, in proper norms,

$$|E(\tilde{u}_N)| \leq C_N |\alpha|^{N-1} \tag{1.15}$$

for all $|\alpha| \leq 1$, with a constant C_N independent of α. Consequently, $\tilde{u}$ can be interpreted as an approximate solution of $E(u) = 0$, if only α is small. Moreover, $\tilde{u}_N$ is stable in the sense that the matrixfunction on T^{n+1},

$$V_\vartheta^T F_{pp}(t, V, \mathrm{D}V) V_\vartheta, \tag{1.16}$$

with $V =: \vartheta + \tilde{u}_N(\vartheta, t)$, is close to the identity matrix. Thus the assumptions of the KAM theory are met and one concludes that there is an $\alpha^* = \alpha^*(N)$, such that for $\alpha \in A(\omega)$ satisfying $|\alpha| < \alpha^*$ there is a unique analytic solution u_α of (1.8) having frequencies $\omega(\alpha)$, hence solving

$$E(u_\alpha) = 0, \tag{1.17}$$

moreover

$$\int_{T^{n+1}} u_\alpha = 0.$$

In addition, one has an estimate of the form

$$|u_\alpha - \tilde{u}_N| \leq C_N |E(u_N)|. \tag{1.18}$$

This establishes the existence of uncountably many quasiperiodic solutions for every analytic f. We point out again, that f is not assumed to be small. Moreover, on account of (1.18) and (1.15) one concludes that for every $N \geq 2$ there are constants $C_N > 0$ and $\alpha^* = \alpha^*(N)$ such that

$$\left| u_\alpha - \sum_{j=2}^{N} \alpha^k u_j \right|_\infty \leq C_N |\alpha|^{N+1} \tag{1.19}$$

for all $\alpha \in A(\omega)$ satisfying $|\alpha| \leq \alpha^*$. This shows that indeed the formal series (1.12) serves as an asymptotic expansion for the solutions having large frequencies $\omega(\alpha)$. The precise statement and the details of this argument are given in section 3. For simplicity we shall only treat the case in which f is analytic. We point out that the asymptotic expansion holds true also for $f \in C^\infty(T^{n+1})$, in which case also the solutions u_α belong to $C^\infty(T^{n+1})$.

It should be mentioned that in the special case $n = 1$ the existence of quasiperiodic solutions having large frequencies can be used in order to prove that all solutions of

$$\ddot{x} - f_x(t, x) = 0, \quad (t, x) \in T^2 \tag{1.20}$$

are bounded, i.e.

$$\sup_t |\dot{x}(t)| < \infty.$$

This has already been pointed out in [M1] and we shall recall the argument. We shall write (1.20) as a system

$$\dot{x} = y, \;\; \dot{y} = f_x(t, x), \;\; \dot{t} = 1, \tag{1.21}$$

which is considered as a vectorfield on the phase space $T^2 \times \mathbb{R}$. Assume now that U is a solutions of

$$\begin{aligned} &\mathrm{D}^2 U = f_x(t, U) \\ &U(t, \vartheta) - \vartheta = u(t, \vartheta) \quad \text{on } T^2, \\ &\mathrm{D} = \frac{\omega}{\alpha}\frac{\partial}{\partial\vartheta} + \frac{\partial}{\partial t}. \end{aligned} \tag{1.22}$$

Then the map $\psi : T^2 \to T^2 \times \mathbb{R}$, defined by $(t, \vartheta) \to (t, x = U(t, \vartheta), y = \mathrm{D}U(t, \vartheta))$, describes an embedding of the torus T^2 into the phase space. In view of (1.22), the vectorfield (1.21) is tangential to $\psi(T^2) \subset T^2 \times \mathbb{R}$ so that its flow leaves this embedded torus invariant. If now $a_1 = \min \mathrm{D}U \leq \mathrm{D}U \leq a_2 = \max \mathrm{D}U$, then $\psi(T^2) \subset T^2 \times [a_1, a_2]$, and since $\psi(T^2)$ is invariant under the flow we conclude, for every solution $(t, x(t), y(t))$ satisfying $y(t^*) < a_1$ for some $t^* \in \mathbb{R}$, that $y(t) < a_2$ for all $t \in \mathbb{R}$. Since $\mathrm{D}U = \frac{\omega}{\alpha} + O(\alpha)$, we can construct for every $C > 0$ a quasiperiodic solution U satisfying $\mathrm{D}U > C$ by choosing α sufficiently small. This proves the claim, that all solutions are bounded. One can show that the analyticity of f is not necessary for the argument. It is sufficient to assume f to be sufficiently smooth, e.g. $f \in C^6(T^2)$, for the smooth case we refer to [M2]. Similar arguments allow to prove the boundedness of solutions of other equations, for example for the Euler equation associated to

$$F(t, x, p) = \frac{1}{2}p^2 + \sqrt{1 + p^2}\, f(t, x)$$

on $T^2 \times \mathbb{R}$. The above argument was used also in the more subtle proof in [DZ] of the boundedness of solutions for a nonlinear Duffing equation on $\mathbb{R}^2 \times \mathbb{R}$.

Observe that this note deals only with systems of very restricted nature and it is desirable to have asymptotic expansion for a more general class of Euler equations associated to

$$F(t,x,p) = g(p) + f(t,x,p)$$

on $T^{n+1} \times \mathbb{R}^n$, with

$$\frac{|f(t,x,p)|}{|g(p)|} \to 0, \text{ as } |p| \to \infty.$$

2. - The formal expansion

In order to solve $E(u) = 0$ we set formally

$$u =: \sum_{j=0}^{\infty} \alpha^j u_j(\vartheta, t), \tag{2.1}$$

and recall that

$$E(u) := D^2 u - f_x(t, \vartheta + u) \tag{2.2}$$

contains the parameter α also in the differential operator D. Introducing the operator

$$\partial =: \sum_{j=1}^{n} \omega_j \frac{\partial}{\partial \vartheta_j}$$

we can write

$$\mathrm{D}^2 = \frac{1}{\alpha^2}\partial^2 + \frac{2}{\alpha}\partial \mathrm{D}_t + \mathrm{D}_t^2, \tag{2.3}$$

where D_t denotes partial derivative with respect to t. Expanding $\alpha^2 E(u) = 0$ into powers of α we find the following equations to be solved for the functions u_j:

$$\begin{aligned} &\partial^2 u_0 = 0 \\ &\partial^2 u_1 + 2\partial \mathrm{D}_t u_0 = 0 \\ &\partial^2 u_j + 2\partial \mathrm{D}_t u_{j-1} + \mathrm{D}_t^2 u_{j-2} = \varphi_{j-2}, \end{aligned} \tag{2.4}$$

for $j \geq 2$, where

$$\varphi_j = \varphi_j(u_0, \cdots, u_j) = \frac{1}{j!}\left(\frac{\mathrm{d}}{\mathrm{d}\alpha}\right)^j f_x\left(t, \vartheta + \sum_{s=0}^{j} \alpha^s u_s\right)\Bigg|_{\alpha=0} \tag{2.5}$$

is a polynomial in $u_1, \cdots, u_j$.

We shall show that there are unique analytic solutions u_j defined on T^{n+1}, if we normalize

$$\int_{T^{n+1}} u_j \mathrm{d}\vartheta \mathrm{d}t = 0.$$

We first observe that the linear equation $\partial u = g$ on T^{n+1} admits a unique analytic solution u with meanvalue zero, provided g is analytic and has vanishing meanvalue. Since we will need it we formulate this well know result in quantitative terms. Denote by H_σ the space of holomorphic functions $g(t, x)$ defined in the complex strip $\sum_\sigma = \{(x, t) \in C^{n+1} : |\mathrm{Im}\ x_i| < \sigma, |\mathrm{Im}\ t| < \sigma\}$ and periodic in all its variables, and abbreviate

$$|g|_\sigma =: \sup_{\Sigma_\sigma} |g|.$$

LEMMA 1. *Let ω satisfy the diophantine conditions* (1.9). *Assume $g \in H_\sigma$ satisfies $|g|_\sigma < \infty$ and $\int g \mathrm{d}x = 0$. Then there is a unique analytic and periodic solution u satisfying*

$$\partial u = g \text{ on } \sum\nolimits_\sigma, \text{ and } \int_{T^n} u \mathrm{d}x = 0. \tag{2.6}$$

Moreover, there is a constant $C = C(n, \tau)$ such that

$$|u|_{\sigma-\delta} \leq \frac{1}{\gamma} \delta^{-\tau} C |f|_\sigma, \text{ for all } 0 < \delta \leq \sigma. \tag{2.7}$$

For a proof we refer e.g. to [R]. We notice that here the variable t is only a parameter. To construct the solutions one proceeds inductively.

a) First we show that $u_0 = u_1 = 0$. Indeed from the first two equations in (2.4) we conclude, in view of Lemma 1, that $u_0 = u_0(t)$ and $u_1 = u_1(t)$ are independent of the ϑ-variable. Integration of

$$\partial^2 u_2 + 2\partial \mathrm{D}_t u_1 + \mathrm{D}_t u_1 + \mathrm{D}_t^2 u_0 = f_x(t, \vartheta + u_0) \tag{2.8}$$

in the ϑ-variable gives $\mathrm{D}_t^2 u_0(t) = 0$ and hence $u_0 = 0$, if the meanvalue should vanish. Integrating now

$$\partial^2 u_3 + 2\partial \mathrm{D}_t u_2 + \mathrm{D}_t^2 u_1 = f_{xx}(t, \vartheta) u_1(t)$$

in the ϑ-variable over T^n we find $\mathrm{D}_t^2 u_1(t) = 0$ and hence $u_1(t) = 0$.

b) Next we proceed by induction and assume that

$$\partial^2 u_j + 2\partial \mathrm{D}_t u_{j-1} + \mathrm{D}_t^2 u_{j-1} = \varphi_{j-2}, \tag{2.9}$$
$$\int_{T^n} (\mathrm{D}_t^2 u_j - \varphi_j)\mathrm{d}\vartheta = 0$$

hold true for $0 \leq j \leq n$, where quantities with negative subscripts are defined to be zero. In order to prove the statement for $j = n+1$ we first solve

$$\partial^2 u_{n+1} = \varphi_{n-1} - 2\partial \mathrm{D}_t u_n - \mathrm{D}_t^2 u_{n-1}. \tag{2.10}$$

On account of the induction assumption the meanvalue over T^n of the right hand side vanishes, and by Lemma 1 there is a solution

$$u_{n+1} = a + b, \tag{2.11}$$

where $a = a(\vartheta, t)$ is uniquely determined, if we set

$$\int_{T^n} a(\vartheta, t)\mathrm{d}\vartheta = 0, \tag{2.12}$$

$b = b(t)$ is arbitrary. It will be determined by the condition

$$\int_{T^n} (\mathrm{D}_t^2 u_{n+1} - \varphi_{n+1})\mathrm{d}\vartheta = 0$$

or

$$\mathrm{D}_t^2 b = \int_{T^n} \left(\varphi_{n+1} - \mathrm{D}_t^2 a\right) \frac{\mathrm{d}\vartheta}{(2\pi)^n} = \int_{T^n} \varphi_{n+1} \frac{\mathrm{d}\vartheta}{(2\pi)^n}. \tag{2.13}$$

Observe that the average over T^n of φ_{n+1} does *not* depend on b. Indeed φ_{n+1} is, in view of (2.5), of the form

$$\varphi_{n+1} = f_{xx}(t, \vartheta)u_{n+1} + \tilde{\varphi},$$

where $\tilde{\varphi}$ depends on $u_n, u_{n-1}, \cdots, u_1$ only. Therefore, since the meanvalue of $f_{xx}(t, \vartheta)b(t)$ is zero, the meanvalue of φ_{n+1} is independent of b.

Now the necessary and sufficient condition for a solution of (2.13) is the vanishing of the meanvalue in the t variable:

$$\int_{T^{n+1}} \varphi_{n+1}\mathrm{d}\vartheta\mathrm{d}t = 0. \tag{2.14}$$

Assuming (2.14) to hold true there is a unique solution b of (2.13) having meanvalue zero and the induction is completed. It remains to prove (2.14).

c) For the proof of (2.14) we need

LEMMA 2. *For every* $u \in C^2(T^{n+1})$.

$$\int_{T^{n+1}} (1+u_\vartheta)^T E(u) \mathrm{d}\vartheta \mathrm{d}t = 0, \tag{2.15}$$

where u_ϑ *is the Jacobian matrix in the* ϑ*-variable.*

PROOF. Set $1 + u_\vartheta = V$, by integration:

$$\begin{aligned}\int_{T^{n+1}} V^T E(u) \mathrm{d}\vartheta \mathrm{d}t &= \int_{T^{n+1}} \left(V^T \mathrm{D} F_p - V^T F_x\right) \mathrm{d}\vartheta \mathrm{d}t \\ &= - \int_{T^{n+1}} \left(\mathrm{D} V^T F_p + V^T F_x\right) \mathrm{d}\vartheta \mathrm{d}t \\ &- \int_{T^{n+1}} \frac{\partial}{\partial \vartheta} F(t, \vartheta + u, \mathrm{D}(\vartheta + u)) \mathrm{d}\vartheta \mathrm{d}t = 0.\end{aligned}$$

Inserting the expansion for $\alpha^2 E(u)$ into (2.15) one finds the identities

$$\int_{T^{n+1}} \varphi_j = \int_{T^{n+1}} \sum_{s+\ell=j+2} u^T_{s,\vartheta} \Phi_\ell \mathrm{d}\vartheta \mathrm{d}t,$$

where
$$\Phi_\ell =: \partial^2 u_\ell + 2\partial \mathrm{D}_t u_{\ell-1} + \mathrm{D}^2_T u_{\ell-2} - \varphi_{\ell-2},$$

for all $j \geq 0$, and for every formal series u. The claim (2.14) follows immediately if we set $j = n+1$, since the integrand on the right hand side vanishes: indeed if $s = 0$ and $s = 1$, then $u_0 = u_1 = 0$. If $s \geq 2$, then by the induction assumption and by (2.10), $\Phi_\ell = 0$ for all $\ell \leq n+1$. This finishes the proof of the unique formal power series.

3. - Existence and asymptotic character

In this section we give the necessary details in order to prove (1.17)-(1.19). First we observe that the set

$$A := \left\{\alpha > 0 : \left|\frac{1}{\alpha}\langle \omega, j\rangle + m\right| \geq \gamma |j|^{-\tau}_{\cdot} \text{ for all } j, m \in \mathbb{Z}^{n+1}, j \neq 0\right\}$$

has positive Lebesgue measure μ provided the constant γ is sufficiently small. Here ω is a fixed vector with rationally independent components and τ is a constant satysfying $\tau > n+1$. More precisely:

LEMMA 3. *Fix* $0 < \lambda < 1$. *Then there is a constant* $\gamma^* = \gamma^*(\lambda)$ *such that for* $0 < \gamma \leq \gamma^*$

$$\mu\{\alpha \in A | 0 < \alpha \leq \epsilon\} \geq \epsilon(1-\lambda), \tag{3.1}$$

for every $0 < \epsilon \leq 1$.

PROOF. Assume $\gamma \leq \frac{1}{2}$, we prove that $\mu(B_\epsilon) \leq \epsilon\lambda$ if γ is sufficiently small, where $B_\epsilon = (0,\epsilon)\backslash A$ is the complement. We have

$$\mu(B_\epsilon) \leq \mu\left(\bigcup_{(j,m)\neq 0} A_{jm}\right)$$

where

$$A_{jm} = \left\{0 < \alpha \leq \epsilon : \left|\frac{1}{\alpha} - \frac{m}{\langle\omega, j\rangle}\right| < \frac{\gamma}{|\langle\omega, j\rangle||j|^\tau}\right\}.$$

In view of $\gamma \leq \frac{1}{2}$ one verifies readily that

$$\sum_{(j,m)} \mu(A_{jm}) \leq 4\gamma \sum_{j\neq 0} \frac{|\langle\omega, j\rangle|}{|j|^\tau} \sum_{|m| \geq \frac{1}{\epsilon}|\langle\omega,j\rangle|} \frac{1}{m^2}.$$

Since the sum over m is dominated by $\frac{2\epsilon}{|\langle\omega;j\rangle|}$ we conclude that

$$\mu(B_\epsilon) \leq 8\gamma\epsilon \sum_{j\neq 0} \frac{1}{|j|^\tau}.$$

In view of $\tau > n+1$, the right hand side is equal to $8\gamma\epsilon C$. Therefore, defining $\gamma^*(\lambda) = \min\left\{\frac{1}{2}, \frac{\lambda}{8C}\right\}$, one concludes that $\mu(B_\epsilon) \leq \lambda\epsilon$ as claimed.

Now, we can state our main result.

THEOREM. *Assume* $\gamma < \gamma^*$. *Assume* f *is real analytic in the (closure of the) complex strip* $\sum_\sigma$ *for some* $1 \geq \sigma > 0$. *For every* $N \geq 2$, *there exist positive constants* $\alpha^* = \alpha^*(N)$ *and* C_N *with the following properties:*

For $\alpha \in A(\omega)$ *satisfying* $|\alpha| < \alpha^*$ *there is a unique* u_α *real-analytic in, say,* $\sum_{\sigma/8}$ *and of mean value* 0 *such that*

$$E(u_\alpha) = 0 \quad in \ \sum\nolimits_{\sigma/8} \tag{3.2}$$

and

$$\left| u_\alpha - \sum_{j\geq 2}^{N} \alpha^j u_j \right|_{\sigma/8} \leq C_N |\alpha|^{N+1}. \tag{3.3}$$

The proof rests on the discussion in section 2 and on the following KAM result, for which we refer to [SZ] (Theorem 1) and [CC] (Lemma 6).

LEMMA 4. *Let f be as in the above Theorem. Let ω satisfy* (1.9) *and let $v \in H_\sigma$ with $|v|_\sigma \leq \sigma$, $|v_\vartheta|_\sigma \leq \frac{1}{2}$. There exists a costant $C = C(n, f, \sigma, \gamma, \tau)$ such that if*

$$C|E(v)|_\sigma \leq 1, \tag{3.4}$$

then there is a unique real analytic $u \in U_{\sigma/2}$ satisfying

$$E(u) = 0, \qquad \int (u - v) = 0, \qquad |u - v|_{\sigma/2} < C|E(v)|_\sigma. \tag{3.5}$$

PROOF OF THE THEOREM. Applying iteratively Lemma 1 and the Cauchy estimates (to control derivatives in terms of functions) to the u_i's constructed in section 2, one finds estimates of the form

$$|u_i|_{\sigma/2} \leq K_i, 2 \leq i \leq N,$$

with constants K_i depending on n, f and γ, τ. Thus one can find an α_0^* so small, that for $|\alpha| < \alpha_0^*$ one has

$$|\tilde{u}_N|_{\sigma/4} < \frac{\sigma}{4}, |\tilde{u}_{N,\vartheta}|_{\sigma/4} \leq \frac{1}{2}, \tag{3.6}$$

where, as above, $\tilde{u}_N =: \sum_{j=2}^{N} \alpha^j u_j$. Moreover, Taylor's formula leads to the bound

$$|E(\tilde{u}_N)|_{\sigma/4} \leq K_N^* |\alpha|^{N-1}. \tag{3.7}$$

Now, if we set

$$\alpha^* = \min\left\{\alpha_0^*, (CK_N^*)^{\frac{1}{1-N}}\right\},$$

the Theorem follows from Lemma 4 simply replacing ω by $\frac{\omega}{\alpha}$ $(\alpha \epsilon A(\omega))$, σ by $\frac{\sigma}{4}$ and v by $\tilde{u}_N$. In this case (3.3) holds with $C_N =: CK_N^*$.

This theorem gives a precise meaning to the asymptotic character of the series $\sum \alpha^i u_i$ which, as mentioned in the introduction, is in general divergent. It would, therefore, also be desirable to have good estimates for the functions

u_j. In the special case in which $n = 1$ the operator ∂ is simply the differential operator $\omega\frac{\partial}{\partial\vartheta}$. We may therefore assume $\omega = 1$ and find the following estimates:

PROPOSITION. *Assume $n = 1 = \omega$, and assume that f is analytic and bounded on the strip $\sum_\sigma$ with $0 < \sigma \leq 1$. Then the unique formal power series in section 2 satisfies*

$$|u_{j+2}|_{\sigma/2} \leq B^{j+2}j^{2j} \text{ for all } j \geq 0.$$

Here $B = \left(\frac{30M}{\sigma}\right)^2$ *with* $M = \max\left\{|f|_\sigma, |f_x|_\sigma, 1\right\}$.

We shall use the following

LEMMA 4. *For all $j \geq 1$:*

$$\sum_{k_1+2k_2+\cdots+jk_j=j} \prod_{s=1}^{j} \frac{1}{k_s!} < e^4.$$

PROOF. Using the generating functions, the left hand side of the inequality is equal to

$$\frac{1}{j!}\left(\frac{\mathrm{d}}{\mathrm{d}\alpha}\right)^j \exp\left(\sum_1^\infty \alpha^s\right)\Bigg|_{\alpha=0}$$

$$= \frac{1}{j!}\left(\frac{\mathrm{d}}{\mathrm{d}\alpha}\right)^j \exp\left(\frac{\alpha}{1-\alpha}\right)\Bigg|_{\alpha=0} = \frac{e^{-1}}{j!}\sum_{n=1}^{\infty}\frac{(n+j-1)(n+j-2)\cdots n}{n!},$$

so that the claim follows from

$$\frac{(n+j-1)(n+j-2)\cdots n}{j!} < 4^n \text{ for all } n, j \geq 1.$$

PROOF OF THE PROPOSITION. Recall that $u_0 = u_1 = 0$, and

$$u_j = a_j + b_j, \quad j \geq 2, \tag{3.8}$$

is determined by

$$\int a_j(\vartheta, t)\mathrm{d}\vartheta = 0, \int b_j(t)\mathrm{d}t = 0, \tag{3.9}$$

$$\partial_\vartheta^2 a_{j+2} = -2\partial_\vartheta\partial_t a_{j+1} - \partial^2 u_j - \varphi_j \tag{3.10}$$

$$\partial_t^2 b_{j+2} = \int \varphi_{j+2}\mathrm{d}\vartheta, \tag{3.11}$$

where $\varphi_0 = f_x(\vartheta, t), \varphi_1 = 0$ and where, for $j \geq 2$

$$\varphi_j = \sum_{k \in P_j} \left(\partial_x^{|k|} f_x\right) \prod_{s=2}^{j} \frac{u_s^{k_s}}{k_s!} = \frac{1}{j!} \left(\frac{\mathrm{d}}{\mathrm{d}\alpha}\right)^j f_x \left(t, \vartheta + \sum_{n \geq 2} \alpha^n u_n\right) \Bigg|_{\alpha=0} . \tag{3.12}$$

here

$$P_j = \{k_2, \cdots, k_j | 2k_2 + \cdots + jk_j = j\} .$$

and $|k| = k_2 + k_3 + \cdots + k_j$. Setting

$$P^*_{j+2} = \{k_2, \cdots, k_j | 2k_2 + \cdots + jk_j = j + 2\}$$

we can rewrite equation (3.11) as

$$\partial_t^2 b_{j+2} = \int \varphi_{j+2} \mathrm{d}\vartheta = \int \left\{ f_{xx} a_{j+2} + \sum_{P^*_{j+2}} \left(\partial_x^{|k|} f_x\right) \prod_{s=2}^{j+2} \frac{u_s^{k_s}}{k_s!} \right\} .$$

Integrating the first term by parts and inserting the equation (3.10) for a_{j+2} gives

$$\partial_t^2 b_{j+2} = - \int f \left(2 \partial_\vartheta \partial_t a_{j+1} + \partial_t^2 u_j - \varphi_j\right) + \int \Psi_j \mathrm{d}\vartheta, \tag{3.13}$$

where

$$\Psi_j = \Psi_j(u_j, u_{j-1}, \cdots, u_2) = \sum_{P^*_{j+2}} \left(\partial_x^{|k|} f_x\right) \prod_{s=2}^{j} \frac{u_s^{k^s}}{k_s!} . \tag{3.14}$$

We proof first the Lemma for $j = 0$. From

$$\partial_t^2 b_2 = \int f f_x \mathrm{d}\vartheta = 0$$

we conclude that $b_2 = 0$ so that $u_2 = a_2$. Since the meanvalue of a_2 vanishes we conclude that

$$|u_2|_\sigma \leq |\partial_\vartheta^2 u_2|_\sigma = |f_x|_\sigma \leq M,$$

which proves the Lemma for $j = 0$.

Assume now $j \geq 1$. We shall show that

$$|u_{i+2}|_{\sigma_i} \leq B^{i+1} j^{2i} \text{ for all } 0 \leq i \leq j, \tag{3.15}$$

where

$$\sigma_i = \sigma\left(1 - \frac{i}{2j}\right). \tag{3.16}$$

The Lemma then follows by setting $i = j$. The estimate (3.15) will be proved by induction in i. In the case $i = 0$, (3.15) is already proved above for $\sigma_0 = \sigma$ and we shall assume now that

$$|u_{s+2}|_{\sigma_s} \leq B^{s+1} j^{2s}, 0 \leq s \leq i-1, \tag{3.17}$$

where, of course, $1 \leq i \leq j$. From (3.9), (3.10) and (3.13) we conclude

$$\begin{aligned} |u_{i+2}|_{\sigma_i} &\leq |a_{i+2}|_{\sigma_i} + |b_{i+2}|_{\sigma_i} \\ &\leq |\partial_\vartheta^2 a_{i+2}|_{\sigma_i} + |\partial_t^2 b_{i+2}|_{\sigma_i} \\ &\leq 4M|\partial_\vartheta \partial_t u_{i+1}|_{\sigma_i} + 2M|\partial_t^2 u_i|_{\sigma_i} + 2M|\varphi_i|_{\sigma_i} + |\Psi_i|_{\sigma_i}. \end{aligned} \tag{3.18}$$

We estimate each term separately. Using the Cauchy estimates and the induction hypothesis (3.17) one finds

$$\begin{aligned} |\partial_\vartheta \partial_t u_{i+1}|_{\sigma_i} &\leq \frac{1}{(\sigma_{i-1} - \sigma_i)^2} |u_{i+1}|_{\sigma_{i-1}} = \left(\frac{2j}{\sigma}\right)^2 |u_{i+1}|_{\sigma_i} \\ &\leq \frac{4}{\sigma^2} B^i j^{2i}, \end{aligned} \tag{3.19}$$

similarly

$$|\partial_t^2 u_i|_{\sigma_i} \leq \left(\frac{j}{\sigma}\right)^2 |u_i|_{\sigma_{i-2}} \leq \frac{1}{\sigma^2} B^{i-1} j^{2(i-1)}. \tag{3.20}$$

Observe now that $\sigma_{s-2} \geq \sigma_s \geq \sigma_i$, and $B \geq \frac{2}{\sigma}, i \geq 1$ and that $|k| \geq 1$ for $k \in P_i$, then

$$\begin{aligned} |\varphi_i|_{\sigma_i} &\leq M \sum_{P_i} \frac{1}{(\sigma - \sigma_i)^{|k|}} \prod_{s=2}^{i} \frac{|u_s|_{\sigma_{s-2}}^{k_s}}{k_s!} \\ &\leq M \sum_{P_i} \left(\frac{2j}{i\sigma}\right)^{|k|} \frac{B^i j^{2i}}{B^{|k|} j^{4|k|}} \prod_{s=2}^{i} \frac{1}{k_s!} \\ &\leq \frac{2M}{\sigma} B^{i-1} j^{2i-3} \sum_{P_i} \prod_{s=1}^{i} \frac{1}{k_s!}, \end{aligned}$$

so that, by Lemma 4,

$$|\varphi_i|_{\sigma_i} \leq \frac{1}{\sigma} 2e^4 M B^{i-1} j^{2i-3}. \tag{3.21}$$

Observing that, if $k \in P^*_{i+2}$, then $2k_2 + \cdots + jk_j = j + 2$ and $|k| \geq 2$, one concludes similarly

$$|\Psi_i|_{\sigma_i} \leq 4e^4 M B^i j^{2(i-1)}. \tag{3.22}$$

Adding up we find from (3.18)-(3.22) that

$$\begin{aligned} |u_{i+2}|_{\sigma_i} &\leq \frac{1}{\sigma^2}(16 + 2 + 8e^4) M^2 B^i j^{2i}, \\ &< \frac{900 M^2}{\sigma^2} B^i j^{2i} = B^{i+1} j^{2i}, \end{aligned}$$

where we have used the definition of the constant B. This finishes the proof of the proposition.

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Dipartimento di Matematica
II Università di Roma
Via Orazio Raimondo
00173 Roma

Mathematik
ETH-Zentrum
8092 Zürich
Switzerland