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Asymptotic expansions of the mean values of Dirichlet L-functions IV

Masanori KATSURADA (桂田昌紀)
Department of Mathematics, Faculty of Science
Kagoshima University, Kagoshima 890, Japan

1 Introduction

Let χ be a Dirichlet character mod q $(q \geq 2)$ and let $L(s,\chi)$ with a complex variable $s = \sigma + it$ denotes the Dirichlet L-function attached to χ. Let $\varphi(n)$ denote Euler's function. The aim of this article is to consider the asymptotical property of the mean square

$$\sum_{\chi(\mathrm{mod}\, q)} |L(s,\chi)|^2, \tag{1.1}$$

where the summation is taken over all the characters mod q.

Let $Q \geq 2$ be a real number. In 1971, P. D. T. A. Elliott [3] proved the asymptotic formula

$$\sum_{p \leq Q} \sum_{\substack{\chi(\mathrm{mod}\, p) \\ \chi \neq \chi_0}} |L(s,\chi)|^2 = \frac{Q^2}{2\log Q}\zeta(2\sigma) + O\left\{\frac{Q^2}{(\log Q)^2}\right\} \qquad (Q \to +\infty)$$

for $\Re s = \sigma > \frac{1}{2}$, where p runs through all prime numbers not exceeding Q, and χ_0 denotes the principal character with its respective modulus. Let $\zeta(s,\alpha)$ denote the Hurwitz zeta-function defined by

$$\zeta(s,\alpha) = \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s}, \qquad \alpha > 0,\ \sigma > 1.$$

Using the relation

$$\sum_{\chi(\mathrm{mod}\, q)} |L(s,\chi)|^2 = \frac{\varphi(q)}{q^{2\sigma}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left|\zeta\left(s, \frac{a}{q}\right)\right|^2, \tag{1.2}$$

P. X. Gallagher [4], in 1975, proved the asymptotic bound

$$\sum_{\chi(\mathrm{mod}\, q)} |L(\tfrac{1}{2}+it,\chi)|^2 = O\left\{(q+|t|)\log q(|t|+2)\right\}$$

for arbitrary $q \geq 2$ and real t. This was improved by R. Balasubramanian [2], in 1980, to

$$\sum_{\chi(\bmod q)} |L(\tfrac{1}{2}+it,\chi)|^2 = \frac{\varphi^2(q)}{q}\log qt + O\left\{q(\log\log q)^2\right\} + O\left(te^{10\sqrt{\log q}}\right) + O\left(q^{\frac{1}{2}}t^{\frac{2}{3}}e^{10\sqrt{\log q}}\right)$$

uniformly for all $q \geq 2$ and $t \geq 3$. In the special case $s = \frac{1}{2}$, the existence of a more explicit asymptotic formula was shown by D. R. Heath-Brown [5]. In 1981, he proved

$$\sum_{\chi(\bmod q)} |L(\tfrac{1}{2},\chi)|^2 = \frac{\varphi(q)}{q}\sum_{k|q}\mu(\tfrac{q}{k})T(k),$$

where k runs over all positive divisors of q and $T(k)$ can be expressed by the asymptotic form

$$T(k) = k\Big(\log\frac{k}{8\pi}+\gamma\Big) + 2\zeta^2(\tfrac{1}{2})k^{\frac{1}{2}} + \sum_{n=0}^{2N-1} c_n k^{-\frac{n}{2}} + O(k^{-N})$$

for any integer $N \geq 1$, with some numerical constants c_n and Euler's constant γ. In particular, when $q = p$ is a prime, his formula yields an asymptotic series with respect to $p^{-\frac{1}{2}}$, because the term corresponding to $k = 1$ can be caluculated explicitly. For the proof, he investigated the function $\sum_{\chi(\bmod q)} L(s,\chi)L(1-s,\bar{\chi})$, instead of using (1.2).

During 1989–1991, on the same lines as Gallagher and Balasubramanian, Zhang Wenpeng [13]-[17] obtained more precise asymptotic results for the following various mean values:

$$\sum_{q\leq Q}\frac{q}{\varphi(q)}\sum_{\chi(\bmod q)}|L(\tfrac{1}{2}+it,\chi)|^2, \qquad \sum_{\chi(\bmod q)}|L^{(h)}(\tfrac{1}{2}+it,\chi)|^2 \quad (h=0,1),$$

$$\sum_{\substack{\chi(\bmod q)\\ \chi\neq\chi_0}}|L(1,\chi)|^2, \qquad \sum_{\chi(\bmod q)}^{*} L'(\sigma+it,\chi)L'(1-\sigma-it,\bar{\chi}) \quad (0<\sigma<1),$$

where $*$ means that the summation is restricted to the primitive characters $(\bmod q)$. For example, he proved

$$\sum_{\chi(\bmod q)}|L(\tfrac{1}{2}+it,\chi)|^2 = \frac{\varphi^2(q)}{q}\left\{\log\left(\frac{qt}{2\pi}\right)+2\gamma+\sum_{p|q}\frac{\log p}{p-1}\right\} + O(qt^{-\frac{1}{12}}) + O\left[(t^{\frac{5}{6}}+q^{\frac{1}{2}}t^{\frac{5}{12}})\exp\left\{\frac{2\log(qt)}{\log\log(qt)}\right\}\right]$$

and

$$\sum_{\substack{\chi(\bmod q)\\ \chi\neq\chi_0}}|L(1,\chi)|^2 = \frac{\pi^2}{6}\varphi(q)\prod_{p|q}\left(1-\frac{1}{p^2}\right) - \frac{\varphi^2(q)}{q^2}\left(\log q+\sum_{p|q}\frac{\log p}{p-1}\right)^2 + O(\log\log q)$$

for all $q \geq 3$ and $t \geq 3$, where p runs through all prime divisors of q.

On the other hand, F. V. Atkinson [1] developed a new method which enabled him to treat $\zeta(u)\zeta(v)$ as a function of two independent variables to deduce the explicit formula for the error term

$$E(T) = \int_0^T |\zeta(\tfrac{1}{2}+it)|^2 dt - T\log(T/2\pi) - (2\gamma - 1)T.$$

In spite of its importance and applicability, Atkinson's formula had long been neglected. In 1985, Y. Motohashi [11], inspired by this Atkinson's work and investigated the function

$$Q(u,v;q) = \varphi(q)^{-1} \sum_{\chi(\mathrm{mod}\, q)} L(u,\chi)L(v,\bar{\chi}).$$

In his article, Atkinson's method was enlightened from a viewpoint of the theory of complex functions and the following "decomposition" of $Q(u,v;q)$ was proved:

$$\begin{aligned} Q(u,v;q) &= L(u+v,\chi_0) + \varphi(q)q^{-u-v}\Gamma(u+v-1)\zeta(u+v-1) \\ &\quad \cdot \left\{\frac{\Gamma(1-u)}{\Gamma(v)} + \frac{\Gamma(1-v)}{\Gamma(u)}\right\} + g(u,v;q) + g(v,u;q), \end{aligned}$$

where $g(u,v;p)$ can be expressed by certain infinite series which involves the confluent hypergeometric functions. From this formula in case $q = p$ is a prime, he obtained the asymptotic expansion

$$\begin{aligned} &(p-1)^{-1} \sum_{\chi(\mathrm{mod}\, p)} |L(\tfrac{1}{2}+it,\chi)|^2 \\ &= \log\frac{p}{2\pi} + 2\gamma + \Re\frac{\Gamma'}{\Gamma}(\tfrac{1}{2}+it) + 2p^{-\frac{1}{2}}|\zeta(\tfrac{1}{2}+it)|^2\cos(t\log p) \\ &\quad - p^{-1}|\zeta(\tfrac{1}{2}+it)|^2 + O(p^{-\frac{3}{2}}) \end{aligned}$$

for arbitrary fixed $t \in \boldsymbol{R}$.

More detailed utilization of the method of Atkinson and Motohashi improve the above asymptotic formula. In what follows we state this improvement in a more general form. Let

$$\binom{s}{n} = \frac{s(s-1)\cdots(s-n+1)}{n!} \qquad (n = 0,1,2,\ldots)$$

as usual and set

$$F(w;q) = q^{1-w}\Gamma(w-1)\zeta(w-1), \qquad G(u,v) = \frac{\Gamma(1-u)}{\Gamma(v)},$$

$$S_N(u,v;k) = \sum_{n=0}^{N-1}\binom{-v}{n}\zeta(u-n)\zeta(v+n)k^{u-n} \quad (N \geq 1), \quad P(w;q) = \prod_{p|q}(1-p^{-w}).$$

Our main theorem is stated as follows:

Theorem ([8, Theorem] and [9, Theorem 1 and 2]) *Let $\boldsymbol{Z}_{\leq 1}$ denote the set of all integers not greater than 1 and define*

$$E = \{\sigma + it;\ 2\sigma - 1 \in \boldsymbol{Z}_{\leq 1} \text{ or } \sigma + it \in \boldsymbol{Z}\},$$

then for any integer $N \geq 1$, in the region

$$\{\sigma + it;\ -N+1 < \sigma < N,\ t \in \boldsymbol{R}\} \tag{1.3}$$

with the exception of the points of E, we have

$$\begin{aligned}
&\varphi(q)^{-1} \sum_{\chi(\bmod q)} |L^{(h)}(\sigma + it, \chi)|^2 \\
&= \frac{d^{2h}}{dw^{2h}} \zeta(w) P(w;q)\Big|_{w=2\sigma} \\
&\quad + 2P(1;q) \sum_{\mu,\nu=0}^{h} \binom{h}{\mu}\binom{h}{\nu} F^{(2h-\mu-\nu)}(2\sigma;q) \Re\left\{\frac{\partial^{\mu+\nu} G}{\partial u^\mu \partial v^\nu}(\sigma+it, \sigma-it)\right\} \\
&\quad + 2q^{-2\sigma} \sum_{\mu,\nu=0}^{h} \binom{h}{\mu}\binom{h}{\nu} (-\log q)^{2h-\mu-\nu} \sum_{k|q} \mu(\tfrac{q}{k}) T^{(\mu,\nu)}(\sigma+it;k),
\end{aligned}$$

where $T^{(\mu,\nu)}(\sigma+it;k)$ has the asymptotic expression

$$T^{(\mu,\nu)}(\sigma+it;k) = \Re\left\{\frac{\partial^{\mu+\nu} S_N}{\partial u^\mu \partial v^\nu}(\sigma+it, \sigma-it;k) + E_N^{(\mu,\nu)}(\sigma+it;k)\right\}.$$

Here $E_N^{(\mu,\nu)}(\sigma+it;k)$ is the error term satisfying the estimate

$$E_N^{(\mu,\nu)}(\sigma+it;k) = O\left[k^{\sigma-N}(|t|+1)^{2N+\frac{1}{2}-\sigma}\log^{\mu+\nu}\{2k(|t|+1)\}\right] \tag{1.4}$$

in the region (1.3), with the O-constant depending only on σ, N and h. In particular, when $q = p$ is a prime, we have the asymptotic expansion

$$\begin{aligned}
&(p-1)^{-1} \sum_{\chi(\bmod p)} |L^{(h)}(\sigma+it,\chi)|^2 \\
&= \zeta^{(2h)}(2\sigma) - \frac{\partial^{2h}}{\partial u^h \partial v^h}\{p^{-u-v}\zeta(u)\zeta(v)\}\Big|_{(u,v)=(\sigma+it,\sigma-it)} \\
&\quad + 2\sum_{\mu,\nu=0}^{h} \binom{h}{\mu}\binom{h}{\nu} F^{(2h-\mu-\nu)}(2\sigma;p)\Re\left\{\frac{\partial^{\mu+\nu} G}{\partial u^\mu \partial v^\nu}(\sigma+it,\sigma-it)\right\} \\
&\quad + 2p^{-2\sigma} \sum_{\mu,\nu=0}^{h} \binom{h}{\mu}\binom{h}{\nu}(-\log p)^{2h-\mu-\nu} T^{(\mu,\nu)}(\sigma+it;p).
\end{aligned}$$

From Stirling's formula and the functional equation of $\zeta(s)$, we have

$$\binom{-\sigma+it}{n}\zeta(\sigma+it-n)\zeta(\sigma-it+n)k^{\sigma+it-n} = O\{k^{\sigma-n}(|t|+1)^{2n+\frac{1}{2}-\sigma}\},$$

for $-n+1<\sigma<n$ $(n\geq 1)$, and this estimate is best-possible because

$$\zeta(\sigma+it)=\Omega(1)$$

for $\sigma>1$ as $|t|\to+\infty$. Hence, when $h=0$, the upper bound in (1.4) cannot be replaced by a smaller one.

Moreover, the asymptotic expressions for (1.1), where $\sigma+it$ lies in the exceptional set E, can be deduced as the limiting cases of our Theorem. For example, we have the following corollaries:

Corollary 1 ([9, Theorem 1 and 2])

$$\begin{aligned}&\varphi(q)^{-1}\sum_{\chi(\bmod q)}|L(\tfrac12+it,\chi)|^2\\ &=\frac{\varphi(q)}{q}\left\{\log\frac{q}{2\pi}+2\gamma+\sum_{p|q}\frac{\log p}{p-1}+\Re\frac{\Gamma'}{\Gamma}(\tfrac12+it)\right\}\\ &\quad+2q^{-1}\sum_{k|q}\mu(\tfrac{q}{k})T^{(0,0)}(\tfrac12+it;k).\end{aligned}$$

In particular, when $q=p$ is a prime, we have the asymptotic expansion

$$\begin{aligned}&(p-1)^{-1}\sum_{\chi(\bmod q)}|L(\tfrac12+it,\chi)|^2\\ &=\log\frac{p}{2\pi}+2\gamma+\Re\frac{\Gamma'}{\Gamma}(\tfrac12+it)-p^{-1}|\zeta(\tfrac12+it)|^2+2p^{-1}T^{(0,0)}(\tfrac12+it;p).\end{aligned}$$

Corollary 2 ([8, Corollary 1]) *Let $\psi(s)=\frac{\Gamma'}{\Gamma}(s)$ be the digamma-function and put*

$$A_0(q)=\log\frac{q}{2\pi}+\gamma_0,\quad A_1(q)=\frac12\log^2\frac{q}{2\pi}+\gamma_0\log\frac{q}{2\pi}+\gamma_1+\frac{\pi^2}{8},$$

$$A_2(q)=\frac16\log^3\frac{q}{2\pi}+\frac{\gamma_0}{2}\log^2\frac{q}{2\pi}+(\gamma_1+\frac{\pi^2}{8})\log\frac{q}{2\pi}+\frac{\pi^2}{8}\gamma_0+\gamma_2,$$

where $\gamma_0(=\gamma)$, γ_1 and γ_2 are the coefficients of the Laurent expansion of $\zeta(s)$ at $s=1$ defined by

$$\zeta(s)=\frac{1}{s-1}+\gamma_0+\gamma_1(s-1)+\gamma_2(s-1)^2+\gamma_3(s-1)^3+\cdots.$$

Then we have

$$\begin{aligned}&\varphi(q)^{-1}\sum_{\chi(\bmod q)}|L'(\tfrac12+it,\chi)|^2\\ &=P(1;q)\Big\{2\gamma_2+2\gamma_1\frac{P'}{P}(1;q)+\gamma_0\frac{P''}{P}(1;q)+\frac13\frac{P'''}{P}(1;q)\\ &\quad-\frac16\Re\psi''(\tfrac12+it)+\frac13\Re\psi^3(\tfrac12+it)+A_0(q)\Re\psi^2(\tfrac12+it)\\ &\quad+2A_1(q)\Re\psi(\tfrac12+it)+2A_2(q)\Big\}\\ &\quad+2q^{-1}\sum_{\mu,\nu=0}^{1}(-\log q)^{2-\mu-\nu}\sum_{k|q}\mu(\tfrac{q}{k})T^{(\mu,\nu)}(\tfrac12+it;k).\end{aligned}$$

If $q = p$ is a prime, then

$$
\begin{aligned}
&(p-1)^{-1} \sum_{\chi(\bmod p)} |L'(\tfrac{1}{2}+it,\chi)|^2 \\
&= 2\gamma_2 - \frac{1}{6}\Re\psi''(\tfrac{1}{2}+it) + \frac{1}{3}\Re\psi^3(\tfrac{1}{2}+it) + A_0(p)\Re\psi^2(\tfrac{1}{2}+it) \\
&\quad +2A_1(p)\Re\psi(\tfrac{1}{2}+it) + 2A_2(p) - \frac{\partial^2}{\partial u \partial v}\{p^{-u-v}\zeta(u)\zeta(v)\}\Big|_{(u,v)=(\frac{1}{2}+it,\frac{1}{2}-it)} \\
&\quad +2p^{-1}\sum_{\mu,\nu=0}^{1}(-\log p)^{2-\mu-\nu}T^{(\mu,\nu)}(\tfrac{1}{2}+it;p).
\end{aligned}
$$

We note here

$$
\frac{P'}{P}(1;q) = \sum_{p|q}\frac{\log p}{p-1}, \quad \frac{P''}{P}(1;q) = \left(\sum_{p|q}\frac{\log p}{p-1}\right)^2 - \sum_{p|q}\frac{p\log^2 p}{(p-1)^2},
$$

$$
\frac{P'''}{P}(1;q) = \left(\sum_{p|q}\frac{\log p}{p-1}\right)^3 + \sum_{p|q}\frac{p(p+1)\log^3 p}{(p-1)^3} - 3\left(\sum_{p|q}\frac{\log p}{p-1}\right)\left(\sum_{p|q}\frac{p\log^2 p}{(p-1)^2}\right).
$$

Corollary 3 ([10, Theorem 1]) *Let χ_0 be the principal character mod q and define*

$$
\tilde{S}_N(u,v;k) = S_N(u,v;k) - \zeta(u)\zeta(v)k^u,
$$

then we have

$$
\begin{aligned}
&\varphi(q)^{-1}\sum_{\substack{\chi(\bmod q)\\ \chi\neq\chi_0}}|L(1,\chi)|^2 \\
&= \zeta(2)\prod_{p|q}(1-p^{-2}) \\
&\quad + q^{-2}\varphi(q)\Big\{\gamma_0^2 - 2\gamma_1 - 2\zeta(2) - \Big(\log q + \sum_{p|q}\frac{\log p}{p-1}\Big)^2\Big\} \\
&\quad + 2q^{-2}\sum_{k|q}\mu(\tfrac{q}{k})\Re\big\{\tilde{S}_N(1,1;k) + O(k^{1-N})\big\}.
\end{aligned}
$$

In particular, if $q = p$ is a prime, then we have the asymptotic expansion

$$
\begin{aligned}
&(p-1)^{-1}\sum_{\substack{\chi(\bmod p)\\ \chi\neq\chi_0}}|L(1,\chi)|^2 \\
&= \zeta(2) - \frac{\log^2 p}{p-1} + p^{-1}\big\{\gamma_0^2 - 2\gamma_1 - 2\zeta(2)\big\} \\
&\quad + 2p^{-2}\Re\big\{\tilde{S}_N(1,1;p) + O(p^{1-N})\big\},
\end{aligned}
$$

Corollary 3 can be applied to deduce upper estimates for class numbers of cyclotomic fields. Furthermore we have

Corollary 4 ([8, Corollary 2])

$$\varphi(q)^{-1}\sum_{\substack{\chi(\mathrm{mod}\, q)\\ \chi\neq\chi_0}}|L'(1,\chi)|^2$$

$$= q^{-1}P(1;q)\Big[-\frac{1}{4}\log^4 q+\gamma_0\log^3 q$$

$$+\Big\{\gamma_0\frac{P'}{P}(1;q)+\frac{1}{2}\frac{P''}{P}(1;q)-2\gamma_1-2\zeta(2)\Big\}\log^2 q$$

$$+\Big\{-\gamma_0\frac{P''}{P}(1;q)-2\gamma_0\gamma_1+6\gamma_2+4\zeta(2)\gamma_0-2\zeta(3)\Big\}\log q$$

$$-\gamma_0\frac{P'}{P}(1;q)\frac{P''}{P}(1;q)-\frac{1}{4}\Big(\frac{P''}{P}(1;q)\Big)^2-\zeta(4)-\zeta^2(2)+2\gamma_0\zeta(3)$$

$$+2\zeta(2)(\gamma_0^2-2\gamma_1)+(\gamma_1^2-6\gamma_3)\Big]+\frac{d^2}{dw^2}\zeta(w)P(w;q)\Big|_{w=2}$$

$$+2q^{-2}\sum_{\mu,\nu=0}^{1}(-\log q)^{2-\mu-\nu}\sum_{k|q}\mu(\tfrac{q}{k})\tilde{T}^{(\mu,\nu)}(1;k).$$

In particular, if $q=p$ is a prime, then

$$(p-1)^{-1}\sum_{\substack{\chi(\mathrm{mod}\, p)\\ \chi\neq\chi_0}}|L'(1,\chi)|^2$$

$$=\zeta''(2)-\frac{1}{4}\frac{\log^4 p}{p-1}+\gamma_0\frac{\log^3 p}{p-1}-\gamma_0^2\frac{\log^2 p}{p-1}$$

$$+p^{-1}\big\{(\gamma_0^2-2\gamma_1-2\zeta(2))\log^2 p+2(2\zeta(2)\gamma_0-\zeta(3)-\gamma_0\gamma_1+3\gamma_2)\log p$$

$$-\zeta(4)-\zeta^2(2)+2\gamma_0\zeta(3)+2\zeta(2)(\gamma_0^2-2\gamma_1)+(\gamma_1^2-6\gamma_3)\big\}$$

$$+2p^{-2}\sum_{\mu,\nu=0}^{1}(-\log p)^{2-\mu-\nu}\tilde{T}^{(\mu,\nu)}(1;p).$$

where $\tilde{T}^{(\mu,\nu)}(1;k)$ has the asymptotic expression

$$\tilde{T}^{(\mu,\nu)}(1;k)=\Re\Big\{\frac{\partial^{\mu+\nu}\tilde{S}_N}{\partial u^\mu\partial v^\nu}(1,1;k)+E_N^{(\mu,\nu)}(1;k)\Big\}$$

for any integer $N\geq 1$, with the error estimate for $E_N^{(\mu,\nu)}(1;k)$ in (1.4).

2 Outline of the proof of Theorem

We define the contour $\mathcal{C}$ which starts from infinity, proceeds along the real axis to δ $(0<\delta<\pi)$, rounds the origin counter-clockwise and returns to infinity. Let $h^{(N)}(z)$ denote the N-th derivative of the function

$$h(z)=\frac{e^z}{e^z-1}-\frac{1}{z},$$

and define for $N \geq 1$

$$\begin{aligned} R_N(u,v;k) &= \frac{1}{\Gamma(u)\Gamma(v)(e^{2\pi iu}-1)(e^{2\pi iv}-1)} \\ &\cdot \int_0^1 \frac{(1-\tau)^{N-1}}{(N-1)!} \int_C \frac{y^{v+N-1}}{e^y-1} \int_C h^{(N)}\Big(x+\frac{\tau y}{k}\Big) x^{u-1} dx dy d\tau, \end{aligned}$$

where $\Im x$ and $\Im y$ vary from 0 to 2π round $\mathcal{C}$. Here the contour integrals are absolutely convergent for $\Re u < N+1$ and any $v \in \boldsymbol{C}$, since the inequality

$$h^{(N)}\Big(x+\frac{\tau y}{k}\Big) = O\Big((1+|x|)^{-N-1}\Big)$$

holds uniformly for all $x, y \in \mathcal{C} \cup [0,+\infty[$ and $\tau \in [0,1]$ (cf. [9, Lemma 1]). Then we have

Lemma 2.1 ([7, Lemma 1] or [8, Lemma 2.1])

$$\begin{aligned} Q(u,v;q) &= \zeta(u+v)P(u+v;q) + P(1;q)F(u+v;q)\{G(u,v)+G(v,u)\} \\ &+ q^{-u-v}\sum_{k|q}\mu(\tfrac{q}{k})\{S(u,v;k)+S(v,u;k)\}, \end{aligned}$$

where

$$S(u,v;k) = S_N(u,v;k) + k^{u-N}R_N(u,v;k)$$

for any integer $N \geq 1$. In particular, if $q = p$ is a prime, then

$$\begin{aligned} Q(u,v;p) &= \zeta(u+v) - p^{-u-v}\zeta(u)\zeta(v) + F(u+v;p)\{G(u,v)+G(v,u)\} \\ &+ p^{-u-v}\{S(u,v;p)+S(v,u;p)\}. \end{aligned}$$

The assertions of Lemma 2.1 are proved by the procedure of Motohashi [11] and by integrating by parts N-times of the contour integral expression of $g(u,v;q)$ in [7, (2.2)]. By applying certain residue calculus for $R_N(u,v;k)$, and then by using the transformation formula of the confluent hypergeometric functions, we can show the following alternative expressions which are useful for the deduction of the estimate (1.4):

Lemma 2.2 ([8, Lemma 2.2]) *Let $\sigma_a(n)$ denote the sum of the a-th powers of the positive divisors of n. Then $R_N(u,v;k)$ is expressed by the following absolutely convergent infinite series: For $\Re u < N$, $\Re v > -N+1$ with $\Re(u+v) < 2$, we have*

$$\begin{aligned} R_N(u,v;k) &= (-1)^N(2\pi)^{u+v-1}\frac{\Gamma(N+1-u)}{\Gamma(v)}\int_0^1\frac{(1-\tau)^{N-1}}{(N-1)!}\sum_{l=1}^{\infty}\sigma_{u+v-1}(l) \\ &\cdot\Big\{e^{\frac{\pi i}{2}(u+v-1)}J_-(\tau,l;k) + e^{-\frac{\pi i}{2}(u+v-1)}J_+(\tau,l;k)\Big\}d\tau, \end{aligned}$$

where

$$J_\pm(\tau,l;k) = \int_0^\infty y^{v+N-1}\Big(1+\frac{\tau y}{k}\Big)^{u-N-1}e^{\pm 2\pi i l y}dy.$$

On the other hand, if $\Re u < N$, $\Re v > -N+1$ *with* $\Re(u+v) > 0$, *then*

$$\begin{aligned} R_N(u,v;k) &= (-1)^N \frac{\Gamma(v+N)}{\Gamma(v)} \int_0^1 \frac{(1-\tau)^{N-1}}{(N-1)!} \sum_{l=1}^{\infty} \sigma_{1-u-v}(l) \\ &\quad \cdot \left\{ \tilde{J}_-(\tau,l;k) + \tilde{J}_+(\tau,l;k) \right\} d\tau, \end{aligned}$$

where

$$\tilde{J}_\pm(\tau,l;k) = \int_0^\infty y^{-u+N} \left(1+\frac{\tau y}{k}\right)^{-v-N} e^{\pm 2\pi i l y} dy.$$

Successively differentiating both sides of the formulas in Lemma 2.1, we get

$$\begin{aligned} &\frac{\partial^{2h} Q}{\partial u^h \partial v^h}(u,v;q) \\ &= \frac{d^{2h}}{dw^{2h}} \zeta(w) P(w;q) \Big|_{w=u+v} \\ &\quad + P(1;q) \sum_{\mu,\nu=0}^{h} \binom{h}{\mu}\binom{h}{\nu} F^{(2h-\mu-\nu)}(u+v;q) \frac{\partial^{\mu+\nu} G^*}{\partial u^\mu \partial v^\nu}(u,v) \\ &\quad + q^{-u-v} \sum_{\mu,\nu=0}^{h} \binom{h}{\mu}\binom{h}{\nu} (-\log q)^{2h-\mu-\nu} \sum_{k|q} \mu(\tfrac{q}{k}) \frac{\partial^{\mu+\nu} S^*}{\partial u^\mu \partial v^\nu}(u,v;k) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^{2h} Q}{\partial u^h \partial v^h}(u,v;p) &= \zeta^{(2h)}(u+v) - \frac{\partial^{2h}}{\partial u^h \partial v^h} p^{-u-v} \zeta(u)\zeta(v) \\ &\quad + \sum_{\mu,\nu=0}^{h} \binom{h}{\mu}\binom{h}{\nu} F^{(2h-\mu-\nu)}(u+v;p) \frac{\partial^{\mu+\nu} G^*}{\partial u^\mu \partial v^\nu}(u,v) \\ &\quad + p^{-u-v} \sum_{\mu,\nu=0}^{h} \binom{h}{\mu}\binom{h}{\nu} (-\log p)^{2h-\mu-\nu} \frac{\partial^{\mu+\nu} S^*}{\partial u^\mu \partial v^\nu}(u,v;p), \end{aligned}$$

where we write $G^*(u,v) = G(u,v) + G(v,u)$ and $S^*(u,v;k) = S(u,v;k) + S(v,u;k)$ for brevity. If we specialize $u = \sigma + it$ and $v = \sigma - it$ in these formulas and write

$$E_N^{(\mu,\nu)}(\sigma+it;k) = \frac{\partial^{\mu+\nu}}{\partial u^\mu \partial v^\nu} k^{u-N} R_N(u,v;k) \Big|_{(u,v)=(\sigma+it,\sigma-it)},$$

then we obtain the right-hand expressions for $\varphi(q)^{-1} \sum_{\chi(\mathrm{mod}\, q)} |L^{(h)}(s,\chi)|^2$ in our Theorem by noting that

$$\overline{\frac{\partial^{\mu+\nu} S}{\partial u^\mu \partial v^\nu}(u,v;k) \Big|_{(u,v)=(\sigma+it,\sigma-it)}} = \frac{\partial^{\mu+\nu} S}{\partial u^\nu \partial v^\mu}(v,u;k) \Big|_{(u,v)=(\sigma+it,\sigma-it)},$$

which is a consequence of the reflection principle.

The following lemma is essential in proving the estimate for $E_N^{(\mu,\nu)}(\sigma+it;k)$:

Lemma 2.3 ([8, Lemma 2.3]) *Let γ be a non negative integer, and let α, β, δ, κ, a, b, t be real numbers such that $\alpha > -1$, $\delta \geq 0$, $\kappa \geq 1$, $0 < a < \min(\frac{1}{2}, \frac{t}{8\pi\kappa})$ and $1 \leq t \leq b$. Then*

$$\begin{aligned}
&\int_a^b x^\alpha (1+x)^\beta \log^\gamma x \log^\delta (1+x) \exp i \left\{ t \log\left(\frac{1+x}{x}\right) + 2\pi\kappa x \right\} dx \\
&= \left(U - \frac{1}{2}\right)^\alpha \left(U + \frac{1}{2}\right)^\beta \log^\gamma \left(U - \frac{1}{2}\right) \log^\delta \left(U + \frac{1}{2}\right) \frac{1}{2\kappa} \sqrt{\frac{t}{\pi}} U^{-\frac{1}{2}} \\
&\quad \cdot \exp i \left\{ tV + 2\pi\kappa \left(U - \frac{1}{2}\right) + \frac{\pi}{4} \right\} \\
&\quad + O(t^{-1} a^{\alpha+\delta+1} |\log a|^\gamma) + O(\kappa^{-1} b^{\alpha+\beta} \log^{\gamma+\delta}(2b)) + R(t,\kappa),
\end{aligned}$$

where

$$R(t,\kappa) \ll \begin{cases} \kappa^{-\frac{1}{2}(\alpha+\beta)-\frac{5}{4}} t^{\frac{1}{2}(\alpha+\beta)-\frac{1}{4}} \log^{\gamma+\delta}\left(\frac{2t}{\kappa}\right), & (1 \leq \kappa \leq t), \\ \kappa^{-\alpha-\delta-1} t^{\alpha+\delta-\frac{1}{2}} \log^\gamma \left(\frac{2\kappa}{t}\right), & (\kappa \geq t), \end{cases}$$

and

$$U = \sqrt{\frac{1}{4} + \frac{t}{2\pi\kappa}}, \qquad V = 2 \operatorname{Arcsinh} \sqrt{\frac{\pi\kappa}{2t}}.$$

Here the constants implied in the O- and Vinogradov's $\ll$ symbols depend at most on α, β, γ and δ. A similar result holds for the corresponding integral with $-\kappa$ in place of κ, except that in this case the explicit term on the right-hand side is to be ommited.

This lemma is proved by the saddle-point method.

Since the first infinite series for $R_N(u,v;k)$ in Lemma 2.2 is compact uniformly convergent in the region $\Re u < N$, $\Re v > -N+1$ with $\Re(u+v) < 2$, the term-by-term differentiation is permissible, and this gives for $N+1 < \sigma < 1$

$$\begin{aligned}
(2.1) \quad & E_N^{(\mu,\nu)}(\sigma+it;k) \\
&= (-1)^N \int_0^1 \frac{(1-\tau)^{N-1}}{(N-1)!} \Bigg[\sum_{\substack{\mu_0+\cdots+\mu_5=\mu \\ \nu_1+\cdots+\nu_5=\nu}} \frac{\mu!}{\mu_0! \cdots \mu_5!} \frac{\nu!}{\nu_1! \cdots \nu_5!} k^{\sigma+it-N} \log^{\mu_0} k \\
&\quad \cdot (2\pi)^{2\sigma-1} \log^{\mu_1+\nu_1}(2\pi) \left.\frac{d^{\mu_2}}{du^{\mu_2}} \Gamma(N+1-u)\right|_{u=\sigma+it} \left.\frac{d^{\nu_2}}{dv^{\nu_2}} \frac{1}{\Gamma(v)}\right|_{v=\sigma-it} \\
&\quad \cdot \sum_{l=1}^\infty \sigma_{2\sigma-1}^{(\mu_3+\nu_3)}(l) \Bigg\{ e^{\frac{\pi i}{2}(2\sigma-1)} \left(\frac{\pi i}{2}\right)^{\mu_4+\nu_4} J_-^{(\mu_5,\nu_5)}(\tau,l;k) \\
&\quad + e^{-\frac{\pi i}{2}(2\sigma-1)} \left(-\frac{\pi i}{2}\right)^{\mu_4+\nu_4} J_+^{(\mu_5,\nu_5)}(\tau,l;k) \Bigg\} \Bigg] d\tau,
\end{aligned}$$

where we write

$$\sigma_a^{(n)}(l) = \frac{d^n}{da^n} \sigma_a(l) = \sum_{d|l} d^a \log^n d$$

and

$$J_\pm^{(\mu_5,\nu_5)}(\tau,l;k) = \left.\frac{\partial^{\mu_5+\nu_5} J_\pm(\tau,l;k)}{\partial u^{\mu_5} \partial v^{\nu_5}}\right|_{(u,v)=(\sigma+it,\sigma-it)}.$$

Making use of Lemma 2.3, we can show

$$J_{\pm}^{(\mu_5,\nu_5)}(\tau,l;k) \ll \begin{cases} \left(\frac{k}{\tau}\right)^{N+\frac{1}{4}} l^{-\sigma+\frac{1}{4}} t^{\sigma-\frac{3}{4}} \log^{\mu_5+\nu_5}(2t) & \text{for } l \le k^{-1}t\tau, \\ l^{-\sigma-N} t^{\sigma+N-\frac{1}{2}} \left\{ \log^{\nu_5}\left(\frac{2k}{\tau}\right) + \log^{\nu_5} l \right\} & \text{for } l \ge k^{-1}t\tau. \end{cases}$$

If we substitute these bounds into (2.1) and estimate term-by-term, then we consequently obtain the estimate (1.4) in case $-N+1<\sigma<1$. The deduction of (1.4) in case $0<\sigma<N$ is the same as above except for the use of the second infinite series of $R_N(u,v;k)$ in Lemma 2.2. □.

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