*Asymptotic Factorization in Nondissipative Wiener–Hopf Problems**

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1. Introduction. In two earlier papers [1,2], James Radlow and the author have presented a general asymptotic method for solving problems of Wiener–Hopf type which involve singular perturbations. One such problem arises when studying the diffraction of an electromagnetic wave by a semi-infinite metallic sheet (*cf.* [3]). Several other examples are discussed in Noble's book ([4], pp. 93–94, 160–164).

The asymptotic method of [1,2] labored under an unfortunate limitation on its range of applicability: it required that all the analytic functions occurring in the (Fourier) transform equation for the problem possess a common strip of regularity of positive width. To show the existence of such a strip, one often had to distort the physical situation by introducing some artificial dissipative mechanism. In a diffraction problem, for instance, one had to assume that the waves were attenuated slightly by the medium (*cf.* example 3 in Section 5).

The purpose of this paper is to show that the limitation of our asymptotic method to problems with a common strip of regularity is unnecessary. In particular, we shall prove that the central procedure of the method, the asymptotic factorization of the transformed kernel into functions analytic in upper

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and lower half-planes, remains valid when the two half-planes have only a single line in common. A more precise formulation of this result, together with a summary of the contents of the paper, appears in the following section.

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2. Statement of the singular perturbation problem. Let $K(x)$ be a function of the real variable x which is continuous for all x, nowhere zero, and of *algebraic growth* near ∞:

$$(2.1) \qquad |\log K(x)| = O(\log |x|) \qquad \text{as} \quad x \to \pm\infty.$$

Definition 1. By a *factorization* of $K(x)$ we shall mean a pair of functions $P_+(z)$, $P_-(z)$ of the complex variable z which satisfy the following conditions:

A) $P_+(z)$ is nonzero and analytic in the upper half-plane $\operatorname{Im} z > 0$ and continuous in its closure $\operatorname{Im} z \geqq 0$, while $P_-(z)$ behaves similarly in the lower half-plane.

B) $P_\pm(z)$ are of *subexponential growth* in their respective closed half-planes:

$$(2.2) \qquad |\log P_\pm(z)| = O(|z|^\gamma), \qquad \gamma < 1, \quad \text{as} \quad |z| \to \infty.$$

C) For all real x,

$$(2.3) \qquad P_+(x)P_-(x) = K(x).$$

When such a factorization exists, we shall write it symbolically in the form $K = P_+P_-$ and shall call P_+ and P_- the *factors* of K.

Factorizations, when they exist, are *unique* up to multiplicative constants. In other words, if both $K = P_+P_-$ and $K = Q_+Q_-$, then there is a nonzero complex constant a such that

$$(2.4) \qquad P_+(z) = aQ_+(z), \qquad P_-(z) = a^{-1}Q_-(z).$$

To see this, we may symbolically "divide" the second factorization by the first, obtaining the factorization

$$(2.5) \qquad 1 = R_+R_- ,$$

where $R_\pm(z) = Q_\pm(z)/P_\pm(z)$ satisfies all the conditions of Definition 1. In particular, since $R_-(z)$ is nonzero in the lower half-plane, the function $1/R_-(z)$ is analytic there, while by (2.5) this function coincides with $R_+(z)$ on the real axis. Hence R_+ and $(R_-)^{-1}$ are analytic continuations of one another, and together they make up an entire function of subexponential growth which is nowhere zero. But the extended Liouville theorem asserts that such a function is necessarily constant. Hence $R_+(z) = a$, $R_-(z) = a^{-1}$ and (2.4) follows.

We may now state the singular perturbation problem. Suppose that one must find a factorization of the function

$$K^*(x) = K_0(x) + \epsilon K_1(x) \tag{2.6}$$

for small $\epsilon > 0$, where $K_0(x)$ has the known factorization $K_0 = P_+^{(0)} P_-^{(0)}$. When the perturbing term $K_1(x)$ is at least as well-behaved as $K_0(x)$ near $\pm\infty$, no special difficulty arises; a power series expansion in ϵ will do the trick (*cf.* [4], p. 163). We shall consider in this paper the more interesting "singular" case where $K_1(x)$ grows more rapidly (or decays more slowly) than $K_0(x)$. Specifically, we assume in this and the four following sections that

$$\frac{K_1(x)}{K_0(x)} = c\,|x|^\alpha\,[1 + g(x)], \tag{2.7}$$

where c and α are positive constants, and the real function $g(x)$ approaches zero at infinity at some definite rate:

$$g(x) = O(|x|^{-\beta}) \quad \text{as} \quad x \to \pm\infty \tag{2.8}$$

with $0 < \beta \leqq 1$. Hypotheses somewhat more general than (2.7) will be discussed in Section 7.

For functions K^* satisfying (2.7), we seek to do three things. First, we must find a factorization $K^* = P_+^* P_-^*$ whose factors differ but little from those of K_0 when ϵ is small. We shall do this in Section 3. Second, we want to write down asymptotically correct perturbation expansions of $P_\pm^*$ in terms of $P_\pm^{(0)}$ and ϵ. General formulas of this type will be derived in Section 4 and illustrated by a detailed discussion of some special examples in Section 5. A noteworthy feature of these formulas is that their leading perturbation terms depend only on the values of α and c and, to some extent, β, but not on the detailed behavior of $K_0(x)$ and $K_1(x)$ for finite values of x. This should make them especially valuable in practical situations, particularly those involving random fluctuations, where M and K may not be precisely known. Finally, we shall investigate in Section 6 the behavior of $P_\pm^*(z)$ for large $|z|$, where again we find asymptotic expansions whose leading terms depend essentially only on α, c and β. Our guiding principle throughout the paper will be to seek half-plane estimates and expansions which remain valid right up to, and even on, the real axis.

To simplify notation, we may divide K^* by K_0 and look for a factorization of the function

$$L(x) = \frac{K^*(x)}{K_0(x)} = 1 + \epsilon K_1/K_0 \tag{2.9}$$

in the form

$$L = Q_+ Q_- . \tag{2.10}$$

It is clear from Definition 1 that this formulation is perfectly equivalent to the original problem; we need only set

$$Q_\pm(z) = P_\pm^*(z)/P_\pm^{(0)}(z). \tag{2.11}$$

Indeed, this substitution also shows that Q_+ and Q_- must both be unity when $\epsilon = 0$, so that we shall be searching for asymptotic expansions of $Q_\pm$ which begin with the constant term 1.

3. The factorization procedure. As soon as we try to factorize $L(x)$, the nature of the difficulty caused by the singular perturbation becomes manifest. For example, the "standard" factorization formula of Wiener and Hopf ([4], pp. 15–16; [5], pp. 179–180) reads, for a factorization $K = P_+P_-$, as follows:

$$\log P_\pm(z) = \pm\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\log K(\xi)}{\xi - z}\,d\xi. \tag{3.1}$$

Now (3.1) makes sense (*i.e.*, converges) only when the function to be factored approaches one at infinity. Our function $L(x)$, however, does not approach one, but instead grows at the rate $\epsilon c\,|x|^\alpha$. This in itself would not be immediately fatal, since there is a standard procedure for reducing functions with arbitrary growth properties to functions which approach one (*cf.* [4], pp. 41–42). In our case, this procedure would consist of dividing $L(x)$ by $\epsilon c(1 + x^2)^{\alpha/2}$ and inserting the quotient as $K(x)$ in (3.1). But if we did this, the resulting integral would become infinite as $\epsilon \to 0$, so that no expansion for small ϵ could be obtained.

This same difficulty arose in the strip factorization problem [1]. There, it was circumvented by deriving in place of (3.1) a new factorization formula with an integrand which vanished at infinity quadratically in ξ. Such a formula would be valid for all functions $K(x)$ of algebraic growth. It would therefore accommodate the direct substitution of a function like our $L(x)$, and could afterwards be expanded in ϵ to give asymptotic approximations.

A quadratically convergent factorization formula was derived in [1] by deformation of contours within the strip. This device is not available here, however, as there is no room to push contours about. We shall rely instead on the fact that, for real $K(x)$, the real part of formula (3.1) itself converges quadratically. Thus we will be able to use the asymptotic methods and formulas of [1] for the real part of $\log Q_\pm(z)$—that is, for the moduli of the factors—and the imaginary part will have to be "pulled along behind" with a tow-chain constructed from various function-theoretic devices. An important link in the chain will be the Plemelj–Privalov theory (*cf.* [6], chapter 2) on the behavior of Cauchy integrals near the path of integration. For this reason it will be necessary to limit consideration to functions $K(x)$ which are locally Hölder-continuous. This restriction should be relatively harmless, since the functions to be factorized in most practical applications are piecewise smooth, with perhaps algebraic singularities at the junctions.

Definition 2. A function $f(z)$ of the real or complex variable z is called (strictly) *Hölder-continuous* in a domain D if there exist positive constants A, q such that $|f(z_1) - f(z_2)| \leqq A\,|z_1 - z_2|^q$ for all z_1 and z_2 in D.

Definition 3. A function defined in a domain D and Hölder-continuous in every compact subdomain of D is called *locally Hölder-continuous* in D.

In both definitions, we can without loss of generality take $q \leqq 1$. Furthermore, if $f(z)$ is locally Hölder-continuous and φ is a continuously differentiable function over the range of f, then the composite function $\varphi[f(z)]$ is also locally Hölder-continuous.

The technique of factorization which we shall adopt in this paper is set out in the two following theorems.

Theorem 1. Let $K(x)$ be a positive real function of algebraic growth which is locally Hölder-continuous on the real axis. Define

$$u(x, y) = \frac{y}{2\pi} \int_{-\infty}^{\infty} \frac{\log K(\xi)}{(\xi - x)^2 + y^2} \, d\xi. \tag{3.2}$$

Then $u(x, y)$ is a harmonic function in the half-plane $y > 0$, and has there a conjugate harmonic function $v(x, y)$. Moreover, both u and v are locally Hölder-continuous in the closed half-plane $y \geqq 0$.

Theorem 2. Let $K(x)$, $u(x, y)$ and $v(x, y)$ be as in Theorem 1, and suppose also that $\log K(x)$ *is (strictly) Hölder-continuous on the full real axis. For $z = x + iy$, define*

$$P_\pm(z) = \exp\,[u(x, \pm y) \pm iv(x, \pm y)]. \tag{3.3}$$

Then $K = P_+P_-$ is a factorization of $K(x)$ in the sense of Definition 1.

Proof of Theorem 1. We note first that, as a consequence of (2.1), the integral in (3.2) converges uniformly in x and y in any compact subdomain of the open half-plane $y > 0$, as do all its formal partial derivatives. Thus for $y > 0$ we may differentiate $u(x, y)$ under the integral sign. Since the kernal $y/[(\xi - x)^2 + y^2] = \operatorname{Im}\,(\xi - z)^{-1}$ is harmonic, u must be harmonic also.

We next let v denote an arbitrary one of the infinitely many harmonic functions conjugate to u in the upper half-plane. (The possible choices of v differ only by additive constants, which would become permissible multiplicative constants in the factors $P_\pm$.) We must show that u and v possess limiting values $u(x, 0)$, $v(x, 0)$ as $y \to 0$, and that the functions so extended are Hölder-continuous on any compact subdomain D of the closed half-plane.

To do this, let ξ_0 be a positive number larger in modulus than the real part of any point in D, and set

$$u_1(x, y) = \frac{y}{2\pi} \int_{-\xi_0}^{\xi_0} \frac{\log K(\xi)}{(\xi - x)^2 + y^2} \, d\xi. \tag{3.4}$$

Clearly u_1 is harmonic for $y > 0$, and a function v_1 conjugate to u_1 can be written down at once:

$$v_1(x, y) = \frac{1}{2\pi} \int_{-\xi_0}^{\xi_0} \frac{(x - \xi) \log K(\xi)}{(x - \xi)^2 + y^2} \, d\xi. \tag{3.5}$$

But now

$$w_1(z) = u_1 + iv_1 = \frac{1}{2\pi i}\int_{-\xi_0}^{\xi_0} \frac{\log K(\xi)}{\xi - z}\, d\xi \tag{3.6}$$

has the form of a Cauchy integral whose kernel $\log K(\xi)$ is Hölder-continuous. By the Plemelj–Privalov theory ([6], sections 17 and 22), $w_1(z)$ then has as $y \to 0^+$ a limiting value $w_1(x)$ defined for $|x| < \xi_0$ by the formula

$$w_1(x) = \tfrac{1}{2}\log K(x) + \frac{1}{2\pi i}\,\text{P.V.}\int_{-\xi_0}^{\xi_0} \frac{\log K(\xi)}{\xi - x}\, d\xi, \tag{3.7}$$

where P.V. denotes the Cauchy principal value of the integral at the singularity $\xi = x$. Moreover, the extended $w_1(z)$ is Hölder-continuous in any compact subset of $y \geqq 0$ which excludes the points $z = \pm\xi_0$. Thus w_1, and therefore also u_1 and v_1, is Hölder-continuous in D.

Now set

$$u_2(x, y) = u - u_1 = \frac{y}{2\pi}\int_{|\xi|>\xi_0} \frac{\log K(\xi)}{(\xi - x)^2 + y^2}\, d\xi. \tag{3.8}$$

This function is harmonic in the entire x, y-plane with the exception of the two rays $-\infty < x \leqq -\xi_0$ and $\xi_0 \leqq x < \infty$ on the line $y = 0$. But these rays lie outside of D, so u_2 is, in particular, harmonic at all interior and boundary points of D. Thus its conjugate function $v_2 = v - v_1$ is also harmonic in all of D. But a function harmonic in a closed region is certainly Hölder-continuous, so that u_2 and v_2 are Hölder-continuous in D. By addition, u and v are also. This proves Theorem 1.

Proof of Theorem 2. We verify, in turn, each of the three conditions of Definition 1.

A) Since u and v are conjugate harmonic functions in $y > 0$, $P_\pm(z)$ are analytic functions in $\operatorname{Im} z > 0$, $\operatorname{Im} z < 0$ respectively. By Theorem 1, both are continuous up to the real axis, and both are nonzero since they are defined as exponentials of continuous functions.

B) To verify the subexponential growth of $P_\pm$, we must show that $u(x, y)$ and $v(x, y)$ are each $O(|z|^\gamma)$ in $y \geqq 0$, with a fixed $\gamma < 1$. We shall prove the even stronger result that

$$u = O(\log |z|), \qquad v = O(\log^2 |z|). \tag{3.9}$$

We begin with u, writing (3.2) in the form

$$u(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\log K(x + y\tau)}{1 + \tau^2}\, d\tau. \tag{3.10}$$

From (2.1) and the non-vanishing of $K(\xi)$ we may deduce that for all x, y and some fixed C

$$\begin{aligned} |\log K(x + y\tau)| &\leqq C \log(2 + |x| + y\,|\tau|) \\ &\leqq C[\log(2 + |x|) + \log(1 + y\,|\tau|)] \\ &\leqq C[\log(2 + |x|) + \log(1 + y) + \log(1 + |\tau|)]. \end{aligned} \tag{3.11}$$

Substituting into (3.10), we find

$$|u(x, y)| \leqq \frac{C}{2}[\log(2 + |x|) + \log(1 + y)] + \frac{C}{\pi}\int_0^\infty \frac{\log(1 + \tau)}{1 + \tau^2}\, d\tau. \tag{3.12}$$

As this last integral converges, we can assert that $u(x, y)$ is bounded in any finite part of the closed half-plane $y \geqq 0$ and is $O(\log |x| + \log y)$ near infinity. But $\log |x| + \log y \leqq 2 \log |z|$, and the first part of (3.9) is established.

We note, in passing, that the same estimate, with the same proof, is also valid uniformly in ξ_0 for the functions $u_1(x, y)$ and $u_2(x, y)$ defined by (3.4) and (3.8).

To prove the second part of (3.9), we look at $v(x, y)$ in a closed semicircle of radius R centered at the origin. Setting $\xi_0 = 3R$, we write $v = v_1 + v_2$ as in the proof of Theorem 1. Transforming (3.5), we find

$$v_1(x, y) = -\frac{1}{2\pi}\int_a^b \frac{\tau \log K(x + y\tau)}{1 + \tau^2}\, d\tau \tag{3.13}$$

with $a = -(3R + x)/y$ and $b = (3R - x)/y$. But in our semicircle $|x| \leqq R$, so that $b \leqq B = 4R/y$ and $a \geqq -B$. Moreover $0 \leqq y \leqq R$, so that inequality (3.11) yields

$$\begin{aligned} |v_1(x, y)| &\leqq \frac{C}{2\pi}\int_{-B}^{B} \frac{|\tau|\,\{2\log(2 + R) + \log(1 + |\tau|)\}}{1 + \tau^2}\, d\tau \\ &\leqq \frac{C}{\pi}\{2\log(2 + R) + \log(1 + B)\}\int_0^B \frac{\tau}{1 + \tau^2}\, d\tau \\ &= \frac{C}{2\pi}\{2\log(2 + R) + \log(1 + B)\}\log(1 + B^2). \end{aligned} \tag{3.14}$$

Hence $v_1 = O(\log B + \log^2 B)$ as both $R, B \to \infty$. But $\log B = \log 4R - \log y$, so that for y greater than any fixed positive y_0 we can assert $\log B = O(\log R)$. Thus

$$v_1 = O(\log^2 R) \tag{3.15}$$

for $x^2 + y^2 \leqq R^2$ and $y \geqq y_0 > 0$.

To deal with the excluded strip $0 \leqq y \leqq y_0$, we require an estimate for the derivative

$$\begin{aligned} \frac{\partial v_1}{\partial y} &= -\frac{y}{\pi}\int_{-3R}^{3R} \frac{(x - \xi)\log K(\xi)}{[(x - \xi)^2 + y^2]^2}\, d\xi \\ &= \frac{1}{\pi y}\int_a^b \frac{\tau \log K(x + y\tau)}{(1 + \tau^2)^2}\, d\tau \\ &= \frac{\log K(x)}{\pi y}\int_a^b \frac{\tau}{(1 + \tau^2)^2}\, d\tau + \frac{1}{\pi y}\int_a^b \frac{\tau[\log K(x + y\tau) - \log K(x)]}{(1 + \tau^2)^2}\, d\tau \\ &= I_1 + I_2\,. \end{aligned} \tag{3.16}$$

But now $-B \leqq a \leqq -B/2 \leqq B/2 \leqq b \leqq B$ and the integrand in I_1 is even, so that

$$|I_1| \leqq \frac{|\log K(x)|}{\pi y} \int_{B/2}^{B} \frac{\tau}{(1+\tau^2)^2}\, d\tau < \frac{4\,|\log K(x)|}{\pi B^2 y} \tag{3.17}$$
$$= \frac{y\,|\log K(x)|}{4\pi R^2} \leqq \text{const.}\, y$$

uniformly in R. As for I_2, we use the Hölder condition

$$|\log K(x + y\tau) - \log K(x)| \leqq A\,|y\tau|^{\alpha} \tag{3.18}$$

to obtain

$$|I_2| \leqq \frac{A}{\pi y^{1-\alpha}} \int_{-\infty}^{\infty} \frac{|\tau|^{1+\alpha}}{(1+\tau^2)^2}\, d\tau. \tag{3.19}$$

Since the integral converges, $I_2 = O(1/y^{1-\alpha})$ and, combining with (3.17), we have

$$\left|\frac{\partial v_1}{\partial y}\right| \leqq \frac{A_1}{y^{1-\alpha}} \quad \text{for} \quad |x| \leqq R, \quad 0 \leqq y < y_0\,, \tag{3.20}$$

with A_1 independent of R. But in this strip

$$v_1(x, y) = v_1(x, y_0) - \int_y^{y_0} \left(\frac{\partial v_1}{\partial y}\right) dy. \tag{3.21}$$

Estimating $v_1(x, y_0)$ by (3.15) and the integral term from (3.20), we find that (3.15) holds in the entire semicircle.

We turn now to v_2, which is harmonic and conjugate to u_2 in a *full* circle $x^2 + y^2 \leqq 4R^2$ of radius $2R$. Thus the Poisson integral formula for the conjugate function (*cf.* [7]) reads

$$v_2(re^{i\theta}) = v_2(0) + \frac{2Rr}{\pi} \int_0^{2\pi} \frac{\sin(\theta - \phi)}{4R^2 - 4Rr\cos(\theta - \phi) + r^2}\, u_2(2Re^{i\phi})\, d\phi, \tag{3.22}$$

where we have written $v_2(x + iy)$ in place of $v_2(x, y)$ and similarly for u_2. But we have already seen that $u_2(z) = O(\log |z|)$, while for $r \leqq R$ the denominator in the integrand is at least R^2. Hence everywhere in the circle $|z| \leqq R$ we have

$$|v_2(z)| \leqq |v_2(0)| + O(\log R). \tag{3.23}$$

But from (3.7)

$$v_2(0) = v(0) - v_1(0) = v(0) + \frac{1}{2\pi}\,\text{P.V.} \int_{-3R}^{3R} \xi^{-1} \log K(\xi)\, d\xi. \tag{3.24}$$

Now $v(0)$ is independent of R, while the principal value is at worst proportional to $\log^2 R$. Thus

$$v_2(z) = O(\log^2 R). \tag{3.25}$$

Adding (3.15) yields the second part of (3.9) and completes the verification of condition B).

C) For $z = x$ real, we have from (3.3)

$$P_+(x)P_-(x) = e^{2u(x,0)}. \tag{3.26}$$

But for $|x| < \xi_0$ we have from (3.7) and (3.8)

$$u(x, 0) = u_1(x, 0) + u_2(x, 0) = \tfrac{1}{2} \log K(x) + 0 = \tfrac{1}{2} \log K(x). \tag{3.27}$$

Substituting into (3.26) yields (2.3) for $|x| < \xi_0$. Since ξ_0 is arbitrary, Theorem 2 is proved.

We remark that strict Hölder-continuity of $\log K(x)$ right up to $x = \pm\infty$ is needed in Theorem 2 only to establish the subexponential growth of the factors near the real axis. If one is willing to allow factors which may grow exponentially or even faster, then ordinary local Hölder-continuity of $K(x)$ will be enough to establish the existence of a factorization. Such factors, however, need no longer be uniquely determined up to multiplicative constants.

4. Asymptotic formulas for the factors. In this section, we shall apply the general factorization procedure described in Section 3 to the function $L(x) = 1 + \epsilon K_1/K_0$ defined in (2.9).

We must first make sure that $L(x)$ has the requisite growth and smoothness properties. If K_0 is nonzero and K_1 satisfies (2.7), then when ϵ is sufficiently small L is certainly positive for all real x and has algebraic growth near ∞. As for smoothness, we assume that K_0 and K_1 are locally Hölder-continuous on the real axis and that the function $g(x)$ appearing in (2.7) is Hölder-continuous near infinity (*i.e.*, Definition 2 is satisfied for some domain $|x| \geqq x_0$).

From the first of these assumptions, it follows immediately that L is locally Hölder-continuous with each local Hölder coefficient *proportional* to ϵ:

$$|L(x_1) - L(x_2)| \leqq \epsilon A_r |x_1 - x_2|^a \qquad \text{for} \quad |x_1|, |x_2| \leqq r, \tag{4.1}$$

where A_r is a local Hölder coefficient for K_1/K_0. But now take r so large that both $r > x_0$ and $|g(x)| < \frac{1}{2}$ for all $|x| > r$. Then for $|x_1|$ and $|x_2|$ both *greater* than r we have from (2.7)

$$\begin{aligned} \log L(x_1) - \log L(x_2) &= \log \left(\frac{1 + \epsilon c\, |x_1|^\alpha\, [1 + g(x_1)]}{1 + \epsilon c\, |x_2|^\alpha\, [1 + g(x_2)]} \right) \\ &= \log \left(\frac{1 + \epsilon c\, |x_1|^\alpha\, [1 + g(x_2) + O(|x_1 - x_2|^\gamma)]}{1 + \epsilon c\, |x_2|^\alpha\, [1 + g(x_2)]} \right) \\ &= \log\, [1 + O(|x_1 - x_2|^\gamma) + O(|x_1/x_2|^\alpha - 1)], \end{aligned} \tag{4.2}$$

where all O symbols are uniform in ϵ and in x_1, x_2. From this it follows easily that $\log L(x)$ is (strictly) Hölder-continuous for $|x| > r$ with a Hölder coefficient *independent* of ϵ. Moreover, the local Hölder-continuity and non-vanishing of

$L(x)$ imply that $\log L(x)$ is also Hölder-continuous for $|x| \leqq r$. Hence under our assumptions $\log L(x)$ is Hölder-continuous on the entire axis, indeed uniformly in ϵ, and we may apply Theorems 1 and 2 to $L(x)$.

These theorems tell us that $L(x)$ possesses factors $Q_\pm(z)$ of the form

$$Q_\pm(z) = \exp\,[u(x, \pm y; \epsilon) \pm iv(x, \pm y; \epsilon)], \tag{4.3}$$

where for each $\epsilon \geqq 0$ we have defined

$$u(x, y; \epsilon) = \frac{y}{2\pi} \int_{-\infty}^{\infty} \frac{\log\,[1 + \epsilon K_1(\xi)/K_0(\xi)]}{(\xi - x)^2 + y^2}\, d\xi \tag{4.4}$$

in the open half-plane $y > 0$, while $v(x, y; \epsilon)$ is a harmonic function conjugate to u. To fix the additive constant which enters into the determination of v, we impose the normalization requirement that

$$v(0, 0; \epsilon) = 0 \qquad \text{for all} \quad \epsilon. \tag{4.5}$$

We are entitled to impose a condition at the origin because Theorem 1 asserts that v is continuous up to the real axis.

We may now state and prove our main result, which is the analogue of Theorem 3 of [1].

Theorem 3. *Let the factorization* $L = Q_+Q_-$ *be determined by formulas* (4.3) *through* (4.5). *Then the factors* Q_+ *and* Q_- *possess at any point* z *in their respective* (*closed*) *half-planes of definition the following asymptotic expansions as* $\epsilon \to 0$:

$$Q_\pm(z) = \begin{cases} 1 + O(\epsilon) & \text{when} \quad 0 < \alpha < 1, \\ 1 \pm i\pi^{-1}cz\epsilon \log \epsilon + O(\epsilon) & \text{when} \quad \alpha = 1, \\ 1 \mp ihz\epsilon^{1/\alpha} + O(\epsilon^{\nu}) & \text{when} \quad \alpha > 1. \end{cases} \tag{4.6}$$

Here α *and* c *are the constants defined in* (2.7), *while the real constant* h *and exponent* ν *are given by*

$$h = \pi^{-1} \int_0^\infty t^{-2} \log\,(1 + ct^\alpha)\, dt, \tag{4.7}$$

$$\nu = \min\left(1, \frac{1 + \beta}{\alpha}\right). \tag{4.8}$$

Moreover, each expansion is uniform in z *in every compact subset of the closed half-plane.*

Proof. Let D be a compact subset of the closed upper half-plane $y \geqq 0$. Suppose all points of D have moduli less than R. We shall first find an expansion for $u(x, y; \epsilon)$ in D; from this we shall deduce an expansion for $v(x, y; \epsilon)$ and thus one for $Q_\pm(z)$. The O symbols used in this proof are meant to apply uniformly in D as $\epsilon \to 0$; in the initial expansions (4.9), (4.10) and (4.11) of u, they also apply uniformly in the entire semicircle $x^2 + y^2 \leqq R^2$, $y \geqq 0$.

We consider separately the three cases distinguished in (4.6). When $\alpha < 1$, we employ in (4.4) the inequality

$$|\log(1 + \epsilon K_1/K_0)| \leqq \text{const. } |\epsilon K_1/K_0|.$$

We then obtain from (2.7) the estimate

$$\begin{aligned} |u(x, y; \epsilon)| &\leqq \text{const.} \int_{-\infty}^{\infty} \frac{\epsilon\, |K_1(\xi)|}{K_0(\xi)} \cdot \frac{y}{(\xi - x)^2 + y^2}\, d\xi \\ &\leqq \text{const. } \epsilon \int_{-\infty}^{\infty} (1 + |x + \xi|^{\alpha}) \frac{y}{\xi^2 + y^2}\, d\xi = O(\epsilon), \end{aligned} \tag{4.9}$$

since the integral converges for $\alpha < 1$ and is uniformly bounded for $(x, y) \in D$.

When $\alpha = 1$, we have from (2.7) and (4.4)

$$\begin{aligned} u(x, y; \epsilon) &= \frac{y}{2\pi} \int_{-\infty}^{\infty} \frac{\log[1 + \epsilon c\, |\xi|\, (1 + g)]}{(\xi - x)^2 + y^2}\, d\xi \\ &= \frac{y}{2\pi} \int_{-\infty}^{\infty} \frac{\log(1 + \epsilon c\, |\xi|)}{(\xi - x)^2 + y^2}\, d\xi + O(\epsilon) \end{aligned}$$

by the same sort of estimation as that used for (4.9), with $1 - \beta$ in place of α. Substituting $\omega = \epsilon(\xi - x)$, we obtain

$$u(x, y; \epsilon) = \frac{y\epsilon}{2\pi} \int_{-\infty}^{\infty} \frac{\log(1 + c\, |\omega + \epsilon x|)}{\omega^2 + \epsilon^2 y^2}\, d\omega + O(\epsilon).$$

The contribution to the integral from values of $|\omega|$ greater than R is clearly $O(1)$, so that

$$u(x, y; \epsilon) = \frac{y\epsilon}{2\pi} \int_{-R}^{R} \frac{\log(1 + c\, |\omega + \epsilon x|)}{\omega^2 + \epsilon^2 y^2}\, d\omega + O(\epsilon).$$

But for $|\omega| \leqq R$ and $|x| \leqq R$ a finite Taylor expansion yields

$$\log(1 + c\, |\omega + \epsilon x|) = c\, |\omega + \epsilon x| + O[(\omega + \epsilon x)^2] = c\, |\omega| + O(\omega^2) + O(\epsilon),$$

while from explicit evaluations we can assert that

$$\int_{-R}^{R} \frac{\omega^2}{\omega^2 + \epsilon^2 y^2}\, d\omega = O(1)$$

and also

$$y \int_{-R}^{R} \frac{\epsilon}{\omega^2 + \epsilon^2 y^2}\, d\omega = O(1).$$

Therefore, finally,

$$\begin{aligned} u(x, y; \epsilon) &= \frac{cy\epsilon}{2\pi} \int_{-R}^{R} \frac{|\omega|}{\omega^2 + \epsilon^2 y^2}\, d\omega + O(\epsilon) \\ &= \frac{cy\epsilon}{2\pi} \log(1 + R^2/\epsilon^2 y^2) + O(\epsilon). \end{aligned}$$

But $\log (1 + R^2/\epsilon^2 y^2) = -2(\log \epsilon + \log y) + O(1)$, and $y \log y = O(1)$, so that we obtain the following asymptotic formula for u in the case $\alpha = 1$:

$$u(x, y; \epsilon) = -\pi^{-1} c y \epsilon \log \epsilon + O(\epsilon). \tag{4.10}$$

When $\alpha > 1$, we write the integral (4.4) for $u(x, y; \epsilon)$ as a sum of two integrals I_1 and I_2, with

$$I_1 = \frac{1}{2\pi} \int_{-2R}^{2R} \frac{y \log [1 + \epsilon K_1(\xi)/K_0(\xi)]}{(\xi - x)^2 + y^2} d\xi$$

and, after the substitution $t = \delta\xi$ with $\delta = \epsilon^{1/\alpha}$,

$$I_2 = \frac{y\ \delta}{2\pi} \int_{|t|>2\delta R} \frac{\log \{1 + c\,|t|^\alpha\,[1 + g(t/\delta)]\}}{(t - x\ \delta)^2 + y^2\ \delta^2} dt.$$

Now $I_1 = O(\epsilon)$, since the logarithm is $O(\epsilon)$ in a finite region, while the indefinite integral of the remaining factors is an arctangent and therefore uniformly bounded. In I_2, we have $g(t/\delta) = O(\delta^\beta)$ by (2.8). Hence the integrand in I_2 may be written as $t^{-2} \log (1 + c\,|t|^\alpha) + O(\delta^\beta)$. But for $\alpha > 1$ the integral of this quantity is convergent at $t = 0$, so that $I_2 = hy\delta + O(\delta^{1+\beta})$ with h as given by (4.7). (For additional details, see the similar evaluation in [1], pp. 250–252.) Adding our estimate for I_1, we find that

$$u(x, y; \epsilon) = hy\epsilon^{1/\alpha} + O(\epsilon^\nu) \tag{4.11}$$

with exponent ν defined by (4.8).

It would now be easy enough to obtain (4.6) in a formal way. One would simply take the formal harmonic conjugates of (4.9) through (4.11) as follows: the conjugate of y is $-x$, while an O term is its own conjugate. This would yield expansions for $v(x, y; \epsilon)$ which, when themselves substituted into (4.3), would give precisely (4.6). To turn this formal procedure into a rigorous proof, we shall justify it by means of the following lemma.

Lemma 1. *For each value of a parameter ϵ, let $\Phi(x, y; \epsilon)$ be a harmonic function of x, y in the semi-circle $x^2 + y^2 \leqq R^2$, $y > 0$, which is continuous up to the boundary $|x| \leqq R$, $y = 0$. Suppose that the Φ are uniformly bounded in the closed semi-circle:*

$$|\Phi(x, y; \epsilon)| \leqq B \tag{4.12}$$

and that the boundary-values $\Phi(x, 0; \epsilon)$ are Hölder-continuous uniformly in ϵ:

$$|\Phi(x_1, 0; \epsilon) - \Phi(x_2, 0; \epsilon)| \leqq A\ |x_1 - x_2|^a \tag{4.13}$$

for all $|x_1|$ and $|x_2|$ not greater than R, and all ϵ. Let $\Psi(x, y; \epsilon)$ be a harmonic function conjugate to Φ for $y > 0$, continuous up to $y = 0$ and vanishing at the origin. Then the $\Psi(x, y; \epsilon)$ are uniformly bounded in any closed semi-circle (centered at the origin) of radius smaller than R.

Proof of Lemma 1. Let C denote the closed semi-circle $x^2 + y^2 \leqq R^2$, $y \geqq 0$, and call its curvilinear and straight boundaries Γ and Δ respectively. The corner points $(\pm R, 0)$ are assigned to Γ, but not to Δ. Denote by $G(z; \zeta)$ the conjugate Green's function for Laplace's equation in C; that is, let G be the kernel such that

$$\int_{\Gamma+\Delta} G(z; \zeta)\Phi(\zeta; \epsilon)\, |d\zeta| = \Psi(z; \epsilon) \tag{4.14}$$

for z in the interior of C. The function G has the following well-known properties (*cf.* [8]):

1) G is independent of ϵ [because of the normalization $\Psi(0; \epsilon) = 0$].
2) G is continuous in both variables except at $z = \zeta$.
3) As z approaches a point $\zeta = \xi$ on Δ, G behaves asymptotically like the corresponding Green's function for a half-plane. Specifically,

$$G(z; \zeta) = G(x + iy; \xi) = \frac{\pi^{-1}(x - \xi)}{(x - \xi)^2 + y^2} + G_1(z; \zeta), \tag{4.15}$$

where G_1 is continuous at $z = \zeta$.

Now suppose z lies in the smaller semicircle $C' : x^2 + y^2 \leqq (R - \delta)^2, y \geqq 0$. Denote by Δ' the open segment $|x| < R - \delta/2$, $y = 0$, and set $\Gamma' = \Gamma + (\Delta - \Delta')$, as indicated in Figure 1. Then (4.14) becomes

$$\Psi(z; \epsilon) = \int_{\Gamma'+\Delta'} G(z; \zeta)\Phi(\zeta; \epsilon)\, |d\zeta|, \tag{4.16}$$

and we shall separately estimate the integrals over Γ' and Δ'. For ζ on Γ', $G(z; \zeta)$ is continuous, hence bounded, so that by (4.12) the integrals over Γ' are bounded uniformly in ϵ. When $\zeta = \xi$ is on Δ', we use (4.15), and observe that $G_1(z; \zeta)$ is continuous for z in C' and ζ on the closure of Δ'. Hence G_1 is bounded, and so the integrals involving G_1 are also uniformly bounded. There remains only the integral

$$\begin{aligned} &\frac{1}{\pi}\int_{\Delta'} \frac{x - \xi}{(x - \xi)^2 + y^2}\, \Phi(\xi, 0; \epsilon)\, d\xi \\ &\qquad = \frac{1}{\pi}\int_{\Delta'} \frac{x - \xi}{(x - \xi)^2 + y^2}\, [\Phi(\xi, 0; \epsilon) - \Phi(x, 0; \epsilon)]\, d\xi \\ &\qquad\quad + \pi^{-1}\Phi(x, 0; \epsilon)\int_{\Delta'} \frac{x - \xi}{(x - \xi)^2 + y^2}\, d\xi. \end{aligned} \tag{4.17}$$

Here we can estimate the first term using (4.13) and the coefficient of the second term from (4.12); the integral in the second term is

$$\tfrac{1}{2}\log\left[\frac{(R - \delta/2 + x)^2 + y^2}{(R - \delta/2 - x)^2 + y^2}\right]$$

and is bounded because $|x| \leqq R - \delta$. The final result is the uniform boundedness of (4.17), and hence of (4.16), for z in C'. This proves the lemma.

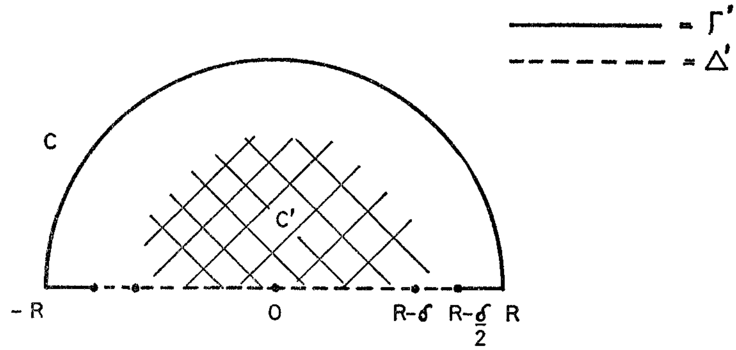

FIGURE 1

We illustrate the application of this lemma to the proof of Theorem 3 only in the case $\alpha > 1$; the other two cases proceed in exactly the same way. Observing (4.11), we take for small $\epsilon > 0$

$$\Phi(x, y; \epsilon) = \epsilon^{-\nu}[u(x, y; \epsilon) - hy\epsilon^{1/\alpha}] \tag{4.18}$$

and correspondingly

$$\Psi(x, y; \epsilon) = \epsilon^{-\nu}[v(x, y; \epsilon) + hx\epsilon^{1/\alpha}]. \tag{4.19}$$

From the results of Section 3, we know that u and v are conjugate harmonic functions in $y > 0$ and are both continuous up to $y = 0$. Hence for each fixed positive ϵ the functions Φ and Ψ also have these properties. Moreover, Ψ vanishes at the origin by (4.5), while the uniform boundedness of Φ in the semicircle $x^2 + y^2 \leqq R^2$, $y \geqq 0$ is precisely the content of (4.11). To verify the uniform Hölder condition (4.13), we note from (3.27) that

$$\Phi(x, 0; \epsilon) = \epsilon^{-\nu}u(x, 0; \epsilon) = \tfrac{1}{2}\epsilon^{-\nu} \log L(x). \tag{4.20}$$

Thus by the mean value theorem

$$\Phi(x_1, 0; \epsilon) - \Phi(x_2, 0; \epsilon) = \tfrac{1}{2}\epsilon^{-\nu}[L(\hat{x})]^{-1}[L(x_1) - L(x_2)].$$

But from (4.1) and the fact that the intermediate value $L(\hat{x})$ is uniformly bounded away from zero we can say that

$$|\Phi(x_1, 0; \epsilon) - \Phi(x_2, 0; \epsilon)| \leqq \text{const.}\ \epsilon^{1-\nu}\ |x_1 - x_2|^a.$$

As $\nu \leqq 1$ by (4.8), $\Phi(x, 0; \epsilon)$ is uniformly Hölder-continuous, and all the hypotheses of Lemma 1 are therefore satisfied.

We may conclude from the lemma that Ψ is bounded uniformly in ϵ in the compact set D defined at the beginning of the proof of Theorem 3. But this means precisely that

$$v(x, y; \epsilon) = -hx\epsilon^{1/\alpha} + O(\epsilon^{\nu}), \tag{4.21}$$

which is the relation formally conjugate to (4.11). Therefore the formal conjugation procedure is justified in the case $\alpha > 1$. As the same arguments apply in the other two cases, Theorem 3 is proved.

5. Examples. We present at this point a number of simple examples of exact factorizations for which we can verify the asymptotic formulas derived in Section 4.

It is worth emphasizing that we did not employ the procedure of Section 3 to find these exact factorizations. This would be much too complicated in practice. We should rather regard Sections 3 and 4 as a means of finding the behavior of factors for ϵ small without carrying out the exact factorization at all.

In any specific example, the problem arises of how to be sure that a factorization found by some independent means is identical to the factorization defined in Section 3 and for which Theorem 3 holds. As long as all factors are of subexponential growth, two factorizations of the same function can differ only by constant multipliers. Thus if the two factorizations agree at one point, they will agree everywhere. Should they not agree at this point, one of them can easily be normalized to make them agree.

For the factorization $L = Q_+Q_-$ of Section 4, the most convenient point to look at is the origin $z = 0$. Indeed, from (3.27) and (4.5) we may assert that

$$Q_\pm(0) = [L(0)]^{1/2}, \tag{5.1}$$

with the positive branch for the square root. Hence in all our examples we will normalize the factors so as to satisfy (5.1).

Example 1. $L(x) = 1 + \epsilon(x^2 - k^2)$, $k > 0$.
Here the factors

$$Q_\pm(z) = (1 - \epsilon k^2)^{1/2} \mp i\epsilon^{1/2}z \tag{5.2}$$

can be written down by inspection, and the expansion

$$Q_\pm = 1 \mp i\epsilon^{1/2}z + O(\epsilon) \tag{5.3}$$

is immediate. But this agrees exactly with (4.6), since in this example $\alpha = 2$, $c = 1$, $\beta = 1$, and therefore $\nu = 1$ and, integrating (4.7) by parts, also $h = 1$. In the limiting case $k = 0$, the $O(\epsilon)$ "error term" disappears, but otherwise things are not significantly different.

Example 2. $L(x) = 1 + \epsilon(x - a)^2$, $a \neq 0$ real.
Here the natural attempt at a factorization would be $L = P_+P_-$ with

$$P_\pm(z) = 1 \mp i\epsilon^{1/2}(z - a). \tag{5.4}$$

These factors, however, do not satisfy (5.1), and so must be normalized, yielding

$$Q_\pm(z) = \left(\frac{1 \mp i\epsilon^{1/2}a}{1 \pm i\epsilon^{1/2}a}\right)^{1/2} P_\pm(z). \tag{5.5}$$

To expand this, we note that $P_\pm$ has (5.4) itself as its natural expansion, while its coefficient has the binomial expansion $1 \mp i\epsilon^{1/2}a + O(\epsilon)$. Putting these together, we find that $Q_\pm$ has the same expansion (5.3) as in Example 1. But again this agrees exactly with (4.6), since α, c and β are just the same as before.

Example 3. $L(x) = 1 + \epsilon(x^2 - k^2)^{1/2}$, $k > 0$ real.
This example, in which $\alpha = 1$, figures prominently in the solution by Wiener–Hopf methods of the problem of diffraction of an electromagnetic wave by a half-plane made of an imperfectly conducting material. Here k represents the wave-number of the incident wave, and a real k corresponds to a medium in which there are no attenuation losses.

An exact factorization in this problem was first found by Senior [3] (*cf.* also [4], pp. 91–92), and another was obtained, by a different method, in [1]. Both of these treatments assumed that k had a small but nonzero imaginary part k_2, so that $L(x)$ could be extended as a regular function into a complex strip of positive width $2\,|k_2|$ surrounding the real axis. When k is real, $L(x)$ already has two branch points $x = \pm k$ on the axis itself, and no such strip can exist. The factors as found in [1], however, do possess limits as $k_2 \to 0$, and we shall prove that these limiting functions constitute a factorization of $L(x)$ for real k.

Our prospective factorization $L = Q_+Q_-$, as obtained from equation (4.10) of [1] by appropriate transformations and limit processes, is given by the formula

$$
\begin{aligned}
\log Q_\pm(z) &= \tfrac{1}{2}\log\epsilon + \tfrac{1}{2}\log\left[(\nu^2 + k^2)^{1/2} \mp z\right] \\
&\quad - \tfrac{1}{4}\log\left[\frac{1 - \epsilon(z^2 - k^2)^{1/2}}{1 + \epsilon(z^2 - k^2)^{1/2}}\right]\left\{1 \mp \frac{2i}{\pi}\log\left[\frac{(z^2 - k^2)^{1/2} - z}{ik}\right]\right\} \\
&\quad \mp \frac{iz}{\pi}\int_\nu^\infty \frac{t\log\{k^{-1}[t + (t^2 + k^2)^{1/2}]\}}{(t^2 + k^2 - z^2)(t^2 + k^2)^{1/2}}\,dt,
\end{aligned}
\tag{5.6}
$$

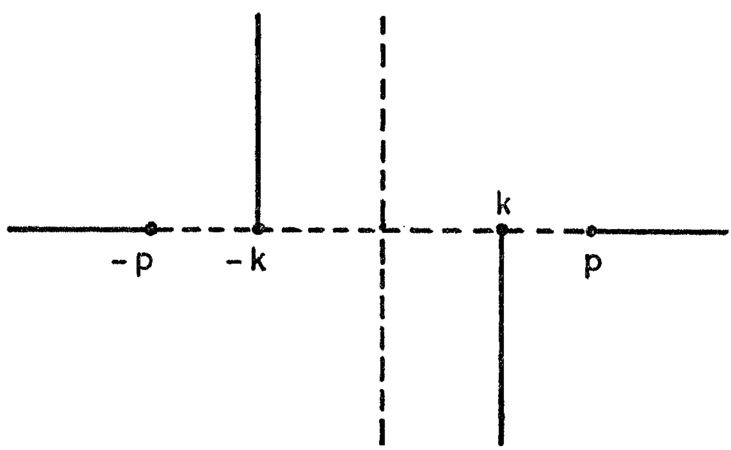

FIGURE 2

with $\nu = \epsilon^{-1}$. To make this precise, we must indicate the branch choices for the square roots and logarithms. Setting $p = (\nu^2 + k^2)^{1/2}$, we cut the complex z-plane as shown in Figure 2: horizontal slits along the real axis from p to $+\infty$ and from $-p$ to $-\infty$, vertical slits downward from k to $-i\infty$ and upward from $-k$ to $+i\infty$. All functions in (5.6) are to be continuous in this slit plane, and $(z^2 - k^2)^{1/2}$ is to have the value $+ik$ at $z = 0$. The logarithm of ϵ and the logarithm in the integral term are of course always real, while the three remaining logarithms are taken to be real at $z = 0$, $z = \pm k$, and $z = 0$ respectively.

To verify Definition 1, we first investigate the domains of analyticity of the functions $Q_\pm(z)$. Proceeding as in [1], we differentiate (5.6) with respect to ν and find

$$\begin{aligned} \frac{d}{d\nu} \log Q_\pm(z) &= -\frac{1}{2\nu} + \frac{\nu}{2p(p \mp z)} \\ &\quad - \frac{(z^2 - k^2)^{1/2}}{2(p^2 - z^2)} \left\{ 1 \mp \frac{2i}{\pi} \log \left[\frac{(z^2 - k^2)^{1/2} - z}{ik} \right] \right\} \\ &\quad \pm \frac{iz\nu}{\pi p(p^2 - z^2)} \log \left(\frac{\nu + p}{k} \right). \end{aligned} \tag{5.7}$$

Since (5.6) vanishes at $\nu = \infty$ (or $\epsilon = 0$), it must be equal to the definite integral of (5.7) from ∞ to ν. Therefore, (5.6) will be regular in z at any particular z and ν provided (5.7) is regular at that z for the same ν and all larger real ν. But an elementary residue calculation reveals that (5.7) is perfectly regular in the entire z-plane except for a single branch point of square-root type, located at $+k$ for Q_+ and at $-k$ for Q_-. (The second "branch point" at $\mp k$ and the two "poles" at $\pm p$ are only optical illusions.) Thus each of the factors given by (5.6) is analytic, not only in the slit plane, but also across three of its four cuts. Specifically, $\log Q_+(z)$ is analytic except on the slit running downward from k, while $\log Q_-(z)$ is analytic except on the slit going upward from $-k$. Thus $Q_+(z)$, for example, is certainly nonzero and analytic in the upper half-plane and continuous up to the axis except possibly at $z = k$. But for $z \to k$ we have directly from (5.6)

$$\begin{aligned} \lim_{z \to k} \log Q_+(z) &= \log Q_+(k) \\ &= \tfrac{1}{2} \log \epsilon + \tfrac{1}{2} \log (p - k) - \frac{ik}{\pi} \int_\nu^\infty \frac{\log \{k^{-1}[t + (t^2 + k^2)^{1/2}]\}}{t(t^2 + k^2)^{1/2}} \, dt; \end{aligned} \tag{5.8}$$

the potentially troublesome third term of (5.6) vanishes in the limit. Hence Q_+ (and similarly Q_-) satisfies condition A) of Definition 1.

To check the condition B) of subexponential growth, we observe that all terms of (5.6) except the last grow at worst logarithmically in $|z|$. In the last term, we can write the expression $t^2 + k^2 - z^2$ occurring in the denominator as the product of the benign factor $t + (z^2 - k^2)^{1/2}$ and the possibly difficult factor $t - (z^2 - k^2)^{1/2}$. The benign factor grows rapidly enough with $|z|$ to counteract the factor of z outside the integral sign, and what remains is just a Cauchy integral in t with a Hölder-continuous kernel which grows logarithmically at $t = \infty$. The Plemelj theory may then be invoked as in the proof of Theorem 2 to establish that the whole integral can grow only logarithmically. Thus $Q_\pm(z)$ is not only subexponential, but is even of algebraic growth. The details are omitted.

Finally, we must compute for real x the product $Q_+(x)Q_-(x)$. From (5.6), we compute that anywhere within the common domain of analyticity of Q_+ and Q_-

$$\begin{aligned}\log Q_+(z) &+ \log Q_-(z)\\ &= \log \epsilon + \tfrac{1}{2}\log(\nu^2 + k^2 - z^2) - \tfrac{1}{2}\log\left[\frac{1-\epsilon(z^2-k^2)^{1/2}}{1+\epsilon(z^2-k^2)^{1/2}}\right]\\ &= \tfrac{1}{2}\log[1+\epsilon(z^2-k^2)^{1/2}]^2\end{aligned} \tag{5.9}$$

and therefore

$$Q_+(z)Q_-(z) = \pm[1+\epsilon(z^2-k^2)^{1/2}], \tag{5.10}$$

with one or the other sign holding for all z. To determine which sign, we simply let $z \to k$ and observe that all three logarithms in the intermediate member of (5.9) then approach real values, so that $\log(Q_+Q_-)$ is real in the limit, and the sign in (5.10) should be the positive one. Specializing (5.10) to real $z = x$, we obtain (at first for $x \neq \pm k$, then by continuity for all x):

$$Q_+(x)Q_-(x) = L(x). \tag{5.11}$$

Thus (5.6) is indeed a factorization of $L(x)$ in the sense of Definition 1.

This factorization also satisfies our normalization condition (5.1), since all the expressions which appear in (5.6) preceded by an ambiguous sign vanish at $z = 0$. Hence Theorem 3 should apply to these factors, and, since $\alpha = \beta = c = 1$, formula (4.6) predicts the expansion

$$Q_\pm(z) = 1 \pm i\pi^{-1}z\epsilon\log\epsilon + O(\epsilon). \tag{5.12}$$

This expansion is easily verified directly from (5.6). Indeed, the first two terms of (5.6) taken together give

$$\tfrac{1}{2}\log[(1+\epsilon^2k^2)^{1/2} \mp \epsilon z] = O(\epsilon).$$

The first factor of the third term is $O(\epsilon)$ everywhere, even at the branch points. Finally, the integral term may be rewritten, after the change of variables $s = t^{-1}$, in the form

$$\begin{aligned}\mp\frac{iz}{\pi}\int_0^\epsilon &\frac{\log\{(ks)^{-1}[1+(1+k^2s^2)^{1/2}]\}}{[1+(k^2-z^2)s^2](1+k^2s^2)^{1/2}}\,ds\\ &= \pm\frac{iz}{\pi}\int_0^\epsilon \log(ks)\,ds + O(\epsilon^2\log\epsilon)\\ &= \pm i\pi^{-1}z\epsilon[\log(k\epsilon) - 1] + O(\epsilon^2\log\epsilon)\\ &= \pm i\pi^{-1}z\epsilon\log\epsilon + O(\epsilon).\end{aligned} \tag{5.13}$$

Adding and taking exponentials yields (5.12).

We note in passing that the branch cuts from $\pm p$ to ∞, which are necessary in defining the integral term, do not disturb this expansion. This is because $p \to \infty$ as $\epsilon \to 0$, so that these cuts recede beyond any given z-region if ϵ is made small enough.

Example 4. $L(x) = 1 + \epsilon |x|$.
This is, of course, the limiting case $k = 0$ in Example 3, but it is of independent interest, since this $L(x)$ cannot be made into a single analytic function by *any* choice of branch cuts in the complex plane. Hence one cannot simply set $k = 0$ in formula (5.6), but must proceed with some care. Skipping over the details of the procedure, we shall merely write down the result, and then verify that it is the correct one.

We work initially in a cut plane with three slits. Two of the slits are horizontal, from $\nu = \epsilon^{-1}$ to $+\infty$ and from $-\nu$ to $-\infty$ along the real axis. The third slit runs vertically from the origin, but is positioned differently for Q_+ and Q_-. Specifically, in defining Q_+ we slit the plane along the negative imaginary axis, while for Q_- we make a slit along the positive imaginary axis. We then define

$$\text{(5.14)} \qquad \log Q_\pm(z) = \tfrac{1}{2} \log (1 \mp \epsilon z) - \tfrac{1}{4} \log \left(\frac{1 + \epsilon z}{1 - \epsilon z}\right)\left[1 \mp \frac{2i}{\pi} \log 2iz\right] \mp \frac{iz}{\pi} \int_\nu^\infty \frac{\text{Log}\, 2t}{t^2 - z^2}\, dt,$$

with $\log (1 \pm \epsilon z)$ taken as real at $z = 0$ and

$$\log 2iz = \text{Log}\,(2\,|z|) - i\pi/2$$

on the negative real z-axis. Log with the capital letter denotes a real-valued logarithm.

To verify that (5.14) yields a true factorization, we proceed just as in Example 3. The derivative of (5.14) with respect to ν turns out to be regular everywhere except for a branch point at the origin. Each of the functions $Q_\pm(z)$ is therefore nonzero and analytic in a plane with only one slit, namely the single vertical cut from 0 to ∞ described above. Continuity at $z = 0$ is immediate from (5.14); in fact, $\log Q_\pm(0) = 0$, so that (5.1) is satisfied. The argument for subexponential growth is the same as before.

The product $Q_+(z)Q_-(z)$ is now not a single analytic function, since the two vertical cuts join up to form a solid barrier along the imaginary axis. Instead, this product has two distinct forms in the right and left halves of the z-plane. For $\text{Re}\, z < 0$, (5.14) yields

$$\log Q_+ + \log Q_- = \tfrac{1}{2} \log (1 - \epsilon z)^2$$

and therefore

$$\text{(5.15)} \qquad Q_+(z)Q_-(z) = 1 - \epsilon z \qquad \text{for} \quad \text{Re}\, z < 0.$$

When $\text{Re}\, z > 0$, however, the quantity $\log 2iz$ has different values in Q_+ and

Q_-, so that this logarithm no longer cancels out in forming the product, but makes instead a net contribution of $+2\pi i$. Hence in this half-plane

$$\log Q_+ + \log Q_- = \tfrac{1}{2} \log (1 - \epsilon z)^2 + \log \left(\frac{1 + \epsilon z}{1 - \epsilon z}\right) = \tfrac{1}{2} \log (1 + \epsilon z)^2$$

and

$$Q_+(z)Q_-(z) = 1 + \epsilon z \qquad \text{for} \quad \operatorname{Re} z > 0. \tag{5.16}$$

Specializing (5.15) and (5.16) to real $z = x$, we find

$$Q_+(x)Q_-(x) = 1 + \epsilon\, |x| = L(x) \tag{5.17}$$

and the factorization (5.14) is established.

By theorem 3, these factors should have the asymptotic representations (5.12) for small ϵ, and indeed they do, as the first two terms in (5.14) are $O(\epsilon)$, while the last term yields to the substitution $t = s^{-1}$ already employed in (5.13):

$$\pm\frac{iz}{\pi} \int_0^\epsilon \frac{\log (s/2)}{1 - s^2 z^2}\, ds = \pm i\pi^{-1} z\epsilon \log \epsilon + O(\epsilon).$$

6. Behavior of factors at infinity. In solving any problem of Wiener–Hopf type, it is of immense theoretical and practical value to know the precise behavior of the factors at infinity. In an integral equation problem, for example, the rate of growth or decay of the factors at infinity will determine both the number of linearly independent solutions and the behavior of those solutions near the endpoint of the integration path. The vicinity of this endpoint is usually the region of greatest interest in such a problem.

For the situation where a common strip of regularity exists, the behavior at infinity of the factors in a singular perturbation problem was discussed at length in [2]. Asymptotic formulas were found for large $|z|$ and fixed ϵ. These formulas contained a multiplicative complex constant V whose value could be determined in principle, but not in practice. In this section, we improve upon the results of [2] in two ways. First, we find asymptotic formulas valid for the present problem, where no common strip need exist. Second, while a multiplicative constant analogous to V enters into these formulas also, here we are at least able to determine explicitly the modulus of the constant, leaving undetermined only its relatively unimportant complex argument.

We begin with a lemma which is the analogue of Theorem 3.1 of [2], but which is harder to prove because its conclusion must be established right up to the path of integration.

Lemma 2. Let $K(x)$ be a positive real function which is (strictly) Hölder-continuous on the real axis and which approaches 1 *as $x \to \pm\infty$:*

$$K(x) = 1 + O(|x|^{-\sigma}), \qquad 0 < \sigma \leqq 1. \tag{6.1}$$

Let $P_\pm(z)$ be defined by formula (3.1). *Then $K = P_+P_-$, and each of the factors $P_\pm(z)$ satisfies*

$$P_\pm(z) = 1 + O(|z|^{-\sigma} \log |z|) \tag{6.2}$$

as $z \to \infty$ along any path in the closed half-plane $y \geqq 0$ ($y \leqq 0$).

Proof. It is immediate from (3.1) that $P_\pm(z)$ are nonzero and analytic in their respective half-planes. Continuity up to the real axis is a consequence of the Plemelj theory, which asserts that the limiting values are given by

$$\log P_\pm(x) = \tfrac{1}{2} \log K(x) \pm \frac{1}{2\pi i} \text{ P.V.} \int_{-\infty}^{\infty} \frac{\log K(\xi)}{\xi - x} \, d\xi. \tag{6.3}$$

The integral here converges at infinity because of (6.1). The product-relation $P_+(x)P_-(x) = K(x)$ is now immediate, and therefore $K = P_+P_-$ if the $P_\pm(z)$ have subexponential growth. But subexponential, and even algebraic, growth would follow from (6.2), so it only remains to prove (6.2).

For simplicity, we consider only $P_+(z)$, whose logarithm is given by

$$\log P_+(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log K(\xi)}{\xi - z} \, d\xi. \tag{6.4}$$

Let $z = x + iy$, and suppose first that $y \geqq y_0 > 0$. A straightforward, though tedious, estimate of the integral in (6.4) is carried out on p. 244 of [1]; it reveals the existence of a constant A_1, independent of z and of $y_0 \leqq 1$, such that

$$|\log P_+(z)| \leqq A_1 \, |z|^{-\sigma} (\log |z| + \log y_0^{-1}). \tag{6.5}$$

Next, in the region $0 < y < y_0$ we estimate the derivative

$$\frac{d}{dz} \log P_+(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log K(x + y\tau) - \log K(x)}{y(\tau - i)^2} \, d\tau, \tag{6.6}$$

in which we have first made the substitution $\xi = x + y\tau$ and then subtracted off a constant multiple of the integral

$$\int_{-\infty}^{\infty} \frac{d\tau}{(\tau - i)^2} = \frac{1}{i - \tau} \bigg|_{-\infty}^{\infty} = 0.$$

From (6.1) and the Hölder-continuity of K it follows that $\log K$ is also Hölder-continuous on the real axis. Hence from (6.6) we may immediately obtain an estimate

$$\left| \frac{d}{dz} \log P_+(z) \right| \leqq A_2/y^{1-q}, \tag{6.7}$$

where A_2 is a constant and $q < 1$ is a Holder exponent for $\log K$. But for $0 \leqq y \leqq y_0$

$$\log P_+(x + iy) = \log P_+(x + iy_0) + \int_{x+iy_0}^{x+iy} \frac{d}{dz} \log P_+(z) \, dz,$$

and (6.7) insures that this last integral is bounded above by $q^{-1}A_2y_0^q$. Combining with (6.5), we find that the following estimate is valid for all z in the closed upper half-plane:

$$|\log P_+(z)| \leqq A_1 \, |z|^{-\sigma} (\log |z| + \log y_0^{-1}) + q^{-1}A_2y_0^q. \tag{6.8}$$

But in this inequality y_0 is an arbitrary number in the range $0 < y_0 \leqq 1$, and we may therefore let it vary with z. The particular choice $y_0 = |z|^{-\sigma/\alpha}$ yields (6.2) and thus proves the lemma.

Lemma 2 does not apply directly to our $L(x)$, since $L(x)$ does not approach 1 at infinity but behaves instead like $c\epsilon\,|x|^\alpha$. Hence we must use at this point the standard reduction technique of Noble ([4], p. 41) to obtain factors which can be identified with the $Q_\pm(z)$ of Section 4 by means of our uniqueness theorem. Specifically, we set

$$\hat{K}(x) = (c\epsilon)^{-1}(x^2 + 1)^{-\alpha/2}L(x). \tag{6.9}$$

Because of (2.7) and (2.8), $\hat{K}(x)$ satisfies condition (6.1) with $\sigma = \beta$, and the Hölder-continuity of $\hat{K}(x)$ follows from that of $g(x)$. Thus Lemma 2 applies to $\hat{K}(x)$, which therefore has a factorization $\hat{K} = \hat{P}_+\hat{P}_-$ whose factors $\hat{P}_\pm$ satisfy (6.2). But then $L(x)$ has the factorization $L = \hat{Q}_+\hat{Q}_-$, where

$$\hat{Q}_\pm(z) = (c\epsilon)^{1/2}(z \pm i)^{\alpha/2}\hat{P}_\pm(z), \tag{6.10}$$

with the choice of branch $|\arg\,(z \pm i)^{\alpha/2}| \leqq \pi\alpha/2$. Since $L(x)$ can also be factored as $L = Q_+Q_-$, there must exist for each ϵ two complex constants $b_\pm = b_\pm(\epsilon)$ with product unity such that

$$Q_\pm(z) = b_\pm\hat{Q}_\pm(z). \tag{6.11}$$

Knowledge of $b_\pm$ would then give complete information about the behavior at infinity of $Q_\pm$.

To aid in determining $b_\pm$, we first deduce from (6.10) and (6.2) that

$$b_\pm = \lim_{z\to\infty} (c\epsilon)^{-1/2}(z \pm i)^{-\alpha/2}Q_\pm(z), \tag{6.12}$$

where the limit may be taken over any path in the closed upper (lower) half-plane. A particularly convenient path is the positive real axis, on which at least the modulus of $Q_\pm(z)$ is known in simple explicit form. Indeed, from (4.3) and (3.27) we may compute that for real x

$$\log|Q_\pm(x)| = \operatorname{Re}\log Q_\pm(x) = \tfrac{1}{2}\log L(x)$$

and so

$$|Q_\pm(x)| = [L(x)]^{1/2} \sim (c\epsilon)^{1/2}\,|x|^{\alpha/2}.$$

Thus we may take absolute values in (6.12) and obtain

$$|b_\pm| = \lim_{x\to+\infty} (c\epsilon)^{-1/2}\,|x|^{-\alpha/2}\,|Q_\pm(x)| = 1. \tag{6.13}$$

Hence $b_\pm = e^{\pm i\theta}$ with θ real, and we have proved the following theorem on the behavior of $Q_\pm(z)$ at infinity.

Theorem 4. Under the same assumptions as in Theorem 3, the factors Q_+ and Q_- possess for each fixed $\epsilon \geqq 0$ the asymptotic behavior

$$Q_\pm(z) = e^{\pm i\theta}(c\epsilon)^{1/2}z^{\alpha/2}[1 + O(|z|^{-\beta}\log|z|)] \tag{6.14}$$

as $z \to \infty$ along any path in the closed upper (lower) half plane. Here θ denotes a real constant which may depend on ϵ, while the branch of $z^{\alpha/2}$ is that for which $|\arg z| \leqq \pi$.

Corollary. *The modulus of $Q_\pm(z)$ satisfies for large $|z|$ the asymptotic relation*

$$|Q_\pm(z)| \sim [c\epsilon\, |z|^\alpha]^{1/2} \tag{6.15}$$

in the appropriate closed half-plane.

As an illustration of Theorem 4, we look again at Examples 1 and 2 of the preceding section. In both these examples, one has $\alpha = 2$ and $c = \beta = 1$. In Example 1, the factors are given by (5.3), and one sees that

$$Q_\pm(z) = \mp i\epsilon^{1/2}z + O(1),$$

which is (6.14) with $\theta = -\pi/2$ and a slightly improved error term. For Example 2, (5.5) yields

$$Q_\pm(z) = \mp i\left(\frac{1 \mp i\epsilon^{1/2}a}{1 \pm i\epsilon^{1/2}a}\right)^{1/2} \epsilon^{1/2}z + O(1),$$

which is again (6.14), but with

$$\theta = \theta(\epsilon) = -\pi/2 - \arctan(\epsilon^{1/2}a).$$

From these two examples, we may observe that cases do exist in which θ is a fixed constant, but that generally θ will vary with ϵ.

7. Remarks and extensions. In many important applications of Wiener-Hopf theory, the function to be factored is complex rather than real. It seems worth while, therefore, to verify that the asymptotic factorization procedure and results of this paper can be employed for complex $L(x)$, and even for complex (but non-negative) values of c. One has simply to separate $\log L(x)$ into its real and imaginary parts, apply the methods of Sections 3, 4, 6 to each part separately, and then add the results. The calculations required to do this are lengthy, but they contain no surprises. We therefore omit them, and merely report their outcome here. For complex $L(x)$ and complex non-negative c, Theorems 3 and 4 remain entirely valid with the single exception noted below, provided that one defines the branch of $c^{1/2}$ in (6.14) and of $\log(1 + ct^\alpha)$ in (4.7) by slitting the c-plane along its negative real axis. The exception concerns the constant θ in Theorem 4, which need no longer be real, so that the corollary (6.15) may fail to hold in the complex case.

A similarly straightforward extension may be made to situations in which c takes on different values at the two ends of the real axis. Here integrals such as (4.4) are broken up into separate sub-integrals over the two halves of the ξ-axis. For small ϵ, each sub-integral will have a leading asymptotic term which is just one-half of the term quoted in (4.6) for the value of c prevailing at its own end. Therefore one still has Theorem 4, but with coefficients in (4.6) containing an "averaged" value of c (or h). The behavior of the factors for large $|z|$ is harder to describe, since the standard reduction of $L(x)$ to a function approaching unity has in such situations a considerably more complicated appearance than (6.9). Standard reductions are, nevertheless, known (see [4], pp. 41–42, or [2], pp. 45–46), and using them one can proceed via Lemma 2 to the appropriate generalization of Theorem 4. When the two values of c are both (positive) real, one can even reproduce (6.13) and determine the exact behavior at infinity of the moduli $|Q_\pm(z)|$. The details are again omitted.

Finally, we observe that the problem of asymptotic factorization in a strip, which was treated in [1] and [2], may be included in the present treatment as a special case. Basically, this is because analyticity in a strip implies Hölder-continuity on lines interior to the strip. To be more precise, the assumptions of [1] and [2], stated in the notation of the present paper, are that (2.7), (2.8) and (2.9) hold not merely on the x-axis but in a whole strip $|\text{Im } x| \leqq \delta$ of a complex x-plane, and that $L(x)$ is an analytic function of x in this strip. But then $L(x)$ is certainly locally Hölder-continuous *on* the axis, and our whole procedure will apply provided the $g(x)$ of (2.7) is (strictly) Hölder-continuous near infinity for x real.

If $g(x)$ were itself analytic in the strip, this strict Hölder-continuity would follow immediately from (2.8) by writing down the Cauchy integral formula for the derivative $g'(x)$ on the real axis. Unfortunately, $g(x)$ is not analytic, but we may apply this argument instead to the analytic function $\mathfrak{g}(x)$ defined by

$$\frac{K_1(x)}{K_0(x)} = c(x^2 + 1)^{\alpha/2}[1 + \mathfrak{g}(x)]. \tag{7.1}$$

This function has near infinity the same decay rate and smoothness properties as $g(x)$, so that we may deduce from (2.8) the Hölder-continuity, first of $\mathfrak{g}$, then of g.

The results of Sections 4 and 6 therefore apply to the strip case; even the normalization (4.5) holds if the constants q^+ and q^- occurring in [1] and [2] are each fixed at $\frac{1}{2}$. It is not surprising, then, that our Theorems 3 and 4 look so much like the corresponding, but more special, results in [1] and [2].

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