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Asymptotic for a second-order evolution equation with convex potential and vanishing damping term

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Abstract: In this short note, we recover by a different method the new result due to Attouch, Chbani, Peyrouqet, and Redont concerning the weak convergence as $t \to +\infty$ of solutions $x(t)$ to the second-order differential equation

$$x''(t) + \frac{K}{t}x'(t) + \nabla\Phi(x(t)) = 0,$$

where $K > 3$ and Φ is a smooth convex function defined on a Hilbert space $\mathcal{H}$. Moreover, we improve their result on the rate of convergence of $\Phi(x(t)) - \min\Phi$.

Key words: Dynamical systems, asymptotically small dissipation, asymptotic behavior, energy function, convex function, convex optimization

1. Introduction and statement of the result

Let $\mathcal{H}$ be a real Hilbert space with inner product and norm respectively denoted by $\langle .,. \rangle$ and $\|.\|$. In a very recent work [1], Attouch et al. considered the following second-order differential equation:

$$x''(t) + \gamma(t)x'(t) + \nabla\Phi(x(t)) = 0, \tag{1.1}$$

where $\gamma(t) = \frac{K}{t}$ with K as a nonnegative constant and $\Phi : \mathcal{H} \to \mathbb{R}$ is a convex continuously differentiable function. By developing a method due to Su et al. [5], they proved the following result:

Theorem 1.1 (Attouch, Chbani, Peypouquet, and Redont) *Assume that* $K > 3$ *and the set* $\arg\min\Phi \equiv \{x \in \mathcal{H} : \ \Phi(x) \leq \Phi(y) \ \forall y \in \mathcal{H}\}$ *is nonempty. Let* $x : [t_0, +\infty[\to \mathcal{H}$ *be a solution to (1.1). Then* $x(t)$ *converges weakly in* $\mathcal{H}$ *as* $t \to +\infty$ *to some element of* $\arg\min\Phi$*. Moreover, the energy function*

$$W(t) \equiv \frac{1}{2}\|x'(t)\|^2 + \Phi(x(t)) - \min\Phi \tag{1.2}$$

satisfies $W(t) = O(t^{-2})$ *as* $t \to +\infty$.

In this note, we establish, by using a different method, a slightly improved version of the previous theorem. Precisely, we prove the following result.

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Theorem 1.2 *Assume that $K > 3$ and $\arg\min\Phi \neq \emptyset$. Let $x : [t_0, +\infty[\to \mathcal{H}$ be a solution to (1.1). Then $x(t)$ converges weakly in $\mathcal{H}$ as $t \to +\infty$ to some element of $\arg\min\Phi$. Moreover, $W(t) = \circ(t^{-2})$ as $t \to +\infty$.*

Remark 1.1 *In [3], we studied the asymptotic behavior as $t \to +\infty$ of the solution to Equation (1.1) when the damping term $\gamma(t)$ behaves, for t large enough, like $\frac{K}{t^\alpha}$ with $K > 0$ and $\alpha \in [0, 1[$. We proved that if $\arg\min\Phi \neq \emptyset$ then every solution to (1.1) converges weakly in $\mathcal{H}$ to some element of $\arg\min\Phi$. Hence, Theorem 1.1 and Theorem 1.2 extend this result to the limit case corresponding to $\alpha = 1$.*

2. Proof of Theorem 1.2

We will prove Theorem 1.2 in a more general setting. Indeed, we will assume that the damping term γ in Equation (1.1) is a real function defined on $[t_0, +\infty[$ that belongs to the class $W^{1,1}_{loc}([t_0, +\infty[, \mathbb{R})$ and satisfies:

$$\text{There exists } K > 3 \text{ such that } \gamma(t) \geq \frac{K}{t} \ \forall t \geq t_0, \tag{2.1}$$

and

$$\int_{t_0}^{+\infty} [(t\gamma(t))']_+ \, dt < +\infty, \tag{2.2}$$

where $[(t\gamma(t))']_+ \equiv \max\{(t\gamma(t))', 0\}$ is the positive part of $(t\gamma(t))'$.

Typical examples of functions γ satisfying (2.1) and (2.2) are $\gamma(t) = \frac{K}{a+t}$ with $a \in \mathbb{R}$ and $K > 3$.

Proof [Proof of Theorem 1.2]We will use a modified version of a method introduced by Cabot and Frankel in [2] and recently developed in [3].

Let $x^* \in \arg\min\Phi$ and define the function $h : [t_0, +\infty[\to \mathbb{R}^+$ by $h(t) = \frac{1}{2}\|x(t) - x^*\|^2$. By differentiating, we have

$$h'(t) = \langle x'(t), x(t) - x^*\rangle,$$
$$h''(t) = \|x'(t)\|^2 + \langle x''(t), x(t) - x^*\rangle.$$

Combining these last equalities and using Equation (1.1), we get

$$h''(t) + \gamma(t)h'(t) = \|x'(t)\|^2 + \langle \nabla\Phi(x(t)), x^* - x(t)\rangle. \tag{2.3}$$

Using now the convexity inequality

$$\Phi(x^*) \geq \Phi(x) + \langle \nabla\Phi(x), x^* - x\rangle, \tag{2.4}$$

and the definition (1.2) of the energy function W, we obtain

$$W(t) \leq \frac{3}{2}\|x'(t)\|^2 - h''(t) - \gamma(t)h'(t). \tag{2.5}$$

On the other hand, in view of (1.1),

$$\begin{aligned} W'(t) &= \langle x'(t), x(t)\rangle + \langle \nabla\Phi(x(t)), x'(t)\rangle \\ &= -\gamma(t)\|x'(t)\|^2. \end{aligned}$$

Hence

$$(t^2W(t))' = 2tW(t) - t^2\gamma(t)\left\|x'(t)\right\|^2. \tag{2.6}$$

Using now assumption (2.1), we get

$$\begin{aligned}\frac{3}{2}t\left\|x'\right\|^2 &\leq \frac{3}{2K}t^2\gamma(t)\left\|x'(t)\right\|^2\\ &= \frac{3}{K}tW(t) - \frac{3}{2K}\left(t^2W(t)\right)'. \end{aligned}\tag{2.7}$$

Multiplying (2.5) by t and using inequality (2.7), we obtain

$$(1-\frac{3}{K})tW(t) + \frac{3}{2K}\left(t^2W(t)\right)' \leq -th''(t) - t\gamma(t)h'(t).$$

Integrating this last inequality on $[t_0,t]$, we get after simplification

$$\begin{aligned}(1-\frac{3}{K})\int_{t_0}^t sW(s)ds + \frac{3}{2K}\left(t^2W(t)\right) &\leq C_0 - th'(t) + (1-t\gamma(t))h(t)\\ &+ \int_{t_0}^t (s\gamma(s))'h(s)ds, \end{aligned}\tag{2.8}$$

where $C_0 = \frac{3}{2K}\left(t_0^2W(t_0)\right) + t_0h'(t_0) - h(t_0)$.

Let $\varepsilon > 0$ such that $K > 3+3\varepsilon$. By using (2.1), we obtain from the inequality (2.8)

$$\begin{aligned}(1-\frac{3}{K})\int_{t_0}^t sW(s)ds + \frac{3}{2K}\left(t^2W(t)\right) + \varepsilon h(t) &\leq C_0 - th'(t) - (K-1-\varepsilon)h(t)\\ &+ \int_{t_0}^t \left[(s\gamma(s))'\right]_+ h(s)ds.\end{aligned}$$

Using now the fact that

$$\begin{aligned}t\left|h'(t)\right| &\leq t\left\|x'(t)\right\|\left\|x(t)-x^*\right\|\\ &\leq 2\sqrt{t^2W(t)}\sqrt{h(t)},\end{aligned}$$

and applying the elementary inequality

$$\forall a>0 \forall b,x\in\mathbb{R}, \quad -ax^2+bx \leq \frac{b^2}{4a}$$

with $x=\sqrt{h(t)}$, we get

$$A\int_{t_0}^t sW(s)ds + Bt^2W(t) + \varepsilon h(t) \leq C_0 + \int_{t_0}^t \left[(s\gamma(s))'\right]_+ h(s)ds, \tag{2.9}$$

where $A = 1-\frac{3}{K}$ and $B = \frac{3}{2K} - \frac{1}{K-1-\varepsilon}$.

Since $K > 3 + 3\varepsilon$, the constants A and B are positive; then

$$\varepsilon h(t) \leq C_0 + \int_{t_0}^{t} \left[(s\gamma(s))'\right]_+ h(s)ds.$$

Hence, by using Gronwall's inequality and the assumption (2.2), we deduce that the function h is bounded, more precisely, we get

$$\sup_{t \geq t_0} h(t) \leq \frac{C_0}{\varepsilon} \exp\left(\frac{1}{\varepsilon} \int_{t_0}^{+\infty} \left[(s\gamma(s))'\right]_+ ds\right).$$

Therefore, we infer from (2.9) that

$$\sup_{t \geq t_0} t^2 W(t) < +\infty, \tag{2.10}$$

$$\int_{t_0}^{+\infty} sW(s)ds < +\infty. \tag{2.11}$$

Combining (2.6) and (2.11) yields that the positive part $\left[(t^2W(t))'\right]_+$ of $(t^2W(t)'$ belongs to $L^1([t_0, +\infty[, \mathbb{R})$; hence $m := \lim_{t\to+\infty} t^2W(t)$ exists. This limit m must be equal to 0, since otherwise $tW(t) \simeq \frac{m}{t}$ as $t \to +\infty$, which contradicts (2.11). It remains to prove the weak convergence of $x(t)$ as $t \to +\infty$. Let us note that (2.10) implies that $\Phi(x(t)) \to \min \Phi$ as $t \to +\infty$. Hence by using the weak lower semicontinuity of the function Φ, we deduce that if $x(t_n) \rightharpoonup \bar{x}$ weakly in $\mathcal{H}$ with $t_n \to +\infty$ then $\Phi(\bar{x}) \leq \min \Phi$, which is equivalent to $\bar{x} \in \arg\min \Phi$. On the other hand, from the convex inequality (2.4) we deduce that $\langle \nabla\Phi(x), x^* - x\rangle \leq 0$ for every $x \in \mathcal{H}$. Then Equation (2.3) implies

$$h''(t) + \gamma(t)h'(t) \leq \left\|x'(t)\right\|^2.$$

Multiply this last equation by $e^{\Gamma(t,t_0)}$, where $\Gamma(t,s) = \int_s^t \gamma(\tau)dt$, and integrate between t_0 and t, and we obtain

$$h'(t) \leq e^{-\Gamma(t,t_0)}h'(t_0) + \int_{t_0}^{t} e^{-\Gamma(t,\tau)} \left\|x'(\tau)\right\|^2 d\tau. \tag{2.12}$$

In view of the assumption (2.1), a simple calculation gives

$$\forall s \geq t_0, \quad \int_s^{+\infty} e^{-\Gamma(t,s)}dt \leq \frac{s}{K-1}.$$

Hence by using (2.12) and Fubini Theorem, we get

$$\int_{t_0}^{+\infty} [h'(t)]_+ dt \leq \frac{t_0\left|h'(t_0)\right|}{K-1} + \frac{1}{K-1}\int_{t_0}^{+\infty} \tau \left\|x'(\tau)\right\|^2 d\tau.$$

Thanks to (2.11), the right-hand side of the last inequality is finite; thus $\int_{t_0}^{+\infty}[h'(t)]_+dt < +\infty$, which implies that $\lim_{t\to+\infty} h(t)$ exists. Hence, for every $x^* \in \arg\min \Phi$, the limit of $\|x(t) - x^*\|$ as $t \to +\infty$ exists. Therefore, Opial's lemma [4], which we recall below, guarantees the required weak convergence of $x(t)$ in $\mathcal{H}$ to some element of $\arg\min \Phi$. $\square$

Lemma 2.1 (Opial's lemma) *Let* $x : [t_0, +\infty[\to \mathcal{H}$*. Assume that there exists a nonempty subset* S *of* $\mathcal{H}$ *such that:*

i) If $t_n \to +\infty$ *and* $x(t_n) \rightharpoonup x$ *weakly in* $\mathcal{H}$*, then* $x \in S$*.*

ii) For every $z \in S$*,* $\lim_{t\to+\infty} \|x(t) - z\|$ *exists.*

Then there exists $z_\infty \in S$ *such that* $x(t) \rightharpoonup z_\infty$ *weakly in* $\mathcal{H}$ *as* $t \to +\infty$*.*

3. Conclusion

In this paper, we have proved that if the damping term $\gamma(t)$ behaves at infinity like $\frac{K}{t}$ with $K > 3$, then every solution $x(t)$ of the equation (1.1) converges weakly as $t \to +\infty$ to a minimizer of Φ and the energy function $W(t)$ is $\circ(t^{-2})$. However, two important questions remain open. The first one is on the behavior of the solution $x(t)$ in the limit case $K = 3$ and the second one is about the effect of the constant K on the convergence rate of the associated energy function $W(t)$.

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