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ASYMPTOTIC INDEPENDENCE AND LIMIT THEOREMS FOR POSITIVELY AND NEGATIVELY DEPENDENT RANDOM VARIABLES[1]

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For random variables which are associated or which exhibit certain related types of positive and negative dependence, the independence structure is largely determined by the covariance structure. We survey results of this sort with particular emphasis on limit theorems for partial sums of stationary sequences.

1. Introduction. The purpose of this paper is to survey a number of results concerning the degree to which the independence structure is determined by the covariance structure for families of random variables which exhibit certain types of positive or negative dependence. The original such result is due to Lehmann (1966). We first recall Lehmann's definition of positive and negative quadrant dependent (PQD and NQD) random variables. X_1 and X_2 are said to be PQD if

(1.1) $H_{1,2}(x_1,x_2) \equiv P[X_1>x_1, X_2>x_2] - P[X_1>x_1]\,P[X_2>x_2] \geq 0$ for all $x_1,x_2 \in \mathcal{R}$;

They are said to be NQD if X_1 and $(-X_2)$ are PQD.

Note that an equivalent condition to (1.1) is that $\mathrm{Cov}(f(X_1), g(X_2)) \geq 0$ for all real increasing (i.e. nondecreasing) f and g (such that $f(X_1)$ and $g(X_2)$ have finite variance). *In the following statement of Lehmann's result and throughout the rest of the paper we will assume, unless otherwise mentioned, that all random variables have finite variance.*

THEOREM 1 *(Lehmann (1966)). If X_1 and X_2 are PQD or NQD, then they are independent if and only if* $\mathrm{Cov}(X_1,X_2) = 0$.

Proof. This theorem is an immediate consequence of the identity (obtained from integration by parts),

(1.2) $$\mathrm{Cov}(X_1,X_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_{1,2}(x_1,x_2)\,dx_1\,dx_2$$

and the pointwise positivity (resp. negativity) of $H_{1,2}$ for PQD (resp. NQD) variables. □

The results which we discuss in this paper concern multivariate generalizations of Theorem 1 of two types. The first type is a direct generalization in which joint uncorrelatedness implies joint independence. The second type is an indirect generalization in which approximate uncorrelatedness implies approximate independence in a sufficiently quantitative sense to lead to useful limit theorems for sums of dependent variables. In Section 2, we review all the results of the first type along with an ergodicity result of the second type; with one exception, these are based on inequalities for *distribution* functions. In Section 3, we review a number of results of the second type, including a triangular array limit theorem and a central limit theorem; these are based on inequalities for *characteristic* functions. In Section 4, we present some recent results which extend the inequalities and limit

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theorems of Section 3 to, e.g., nonmonotonic functions of associated variables; some of the results of Section 4 are new. Finally, in Section 5, we review briefly some related results and open problems; these concern nonstationary sequences, Berry-Esseen asymptotics, invariance principles, and demimartingales.

We conclude this section by noting that many of the results given below, which are stated for both positively and negatively dependent variables, were originally derived, in the referenced papers, only for the positively dependent case; the derivation for the negative case is usually essentially unchanged. There is however a simple, but striking, distinction between the two cases, which can be seen in Theorems 7, 12 and 17. Namely, as a consequence of the elementary Lemma 8, it follows that for a stationary sequence $Y_1, Y_2, \ldots$, the decrease in $|\mathrm{Cov}(Y_1, Y_j)|$ as $j \to \infty$ which must be specifically assumed in the positive case in order to have ergodicity or a central limit theorem is automatically valid in the negative case: *stationary negatively dependent sequences are automatically asymptotically independent.*

2. Distribution Function Inequalities and Applications. A finite family $\{X_1, \ldots, X_n\}$ of random variables is said to be *associated* if $\mathrm{Cov}(f(X_1, \ldots, X_n), g(X_1, \ldots, X_n)) \geq 0$ for any real (coordinatewise) increasing functions f and g on $\mathcal{R}^n$; it is said to be *negatively associated* if for any disjoint $A, B \subset \{1, \ldots, n\}$ and any increasing functions f on $\mathcal{R}^A$ and g on $\mathcal{R}^B$, $\mathrm{Cov}(f(X_k; k \in A), g(X_l; l \in B)) \leq 0$. The first definition is due to Esary, Proschan, and Walkup (1967) and the second to Joag-Dev and Proschan (1983). Infinite families are associated (resp. negatively associated) if every finite subfamily is associated (resp. negatively associated). These definitions are two of the many possible multivariate generalizations of Lehmann's PQD and NQD; for further discussion of these and related concepts, see Karlin and Rinott (1980a; 1980b), Shaked (1982a), Block, Savits and Shaked (1982), and the references therein. All of the results discussed in this paper apply to associated and most apply to negatively associated families; many results apply under weaker hypotheses as will be discussed below.

There are two almost independent bodies of literature on the subject of associated random variables. One developed from the work of Esary, Proschan and Walkup (1967) and Sarkar (1969) and is oriented towards reliability theory and statistics; the other developed from the work of Harris (1960) and of Fortuin, Kastelyn and Ginibre (1971) and is oriented towards percolation theory and statistical mechanics. It should be noted that in the latter literature, the term "associated" is usually not used but rather variables are said to satisfy the FKG inequalities. Some people use the term FKG inequalities only when the joint distribution satisfies some version of the lattice-theoretic sufficient condition for being associated which was analyzed by Fortuin, Kastelyn and Ginibre (1971). When the joint distribution has a smooth density $p(x_1, \ldots, x_n)$, which is strictly positive on all of $\mathcal{R}^n$, this condition is equivalent to

$$(2.1) \qquad (\partial^2/\partial x_i \partial x_j) \ln p \geq 0 \quad \text{for all } i \neq j \text{ and all } x_1, \ldots, x_n,$$

which is further equivalent to the "TP_2 in pairs" condition obtained independently (and previously) in Sarkar (1969).

That condition (2.1) is not necessary for association can be seen by considering a trivariate normal vector whose covariance matrix, although (entrywise) positive, is not the inverse of a matrix with nonpositive off-diagonal entries. Such normal variables can easily be constructed; they are associated by the results of Pitt (1982), but do not satisfy (2.1).

A paper which has a nice proof of the sufficiency of the FKG condition along with references to many of the papers in both bodies of literature is Karlin and Rinott (1980a).

The recognition that association is useful in the study of approximate independence seems to have first occurred in Lebowitz (1972). The central limit theorem reviewed in Section 3 below was largely motivated by Lebowitz' results. Also in Lebowitz' paper is an inequality on distribution functions, which we state as Theorem 2, which gives a very simple proof that uncorrelated implies independent for associated variables. This latter result was apparently first stated as a theorem in Wells (1977), but the proof given there was a complicated one based on generalizations of other theorems of Lebowitz (1972) (see Simon (1973)); it was not noticed until recently that this result is an immediate consequence of Lebowitz' basic distribution function inequality. We define for A and B subsets of $\{1, \ldots, n\}$, and real x_j's,

(2.2) $$H_{A,B} = P(X_j>x_j; j\epsilon A \cup B) - P(X_k>x_k; k\epsilon A)\, P(X_l>x_l; l\epsilon B);$$

note that according to (1.1) $H_{i,j} = H_{\{i\},\{j\}}$.

THEOREM 2 *(Lebowitz (1972)). If the X_j's are associated, then*

(2.3a) $$0 \leq H_{A,B} \leq \Sigma_{k\epsilon A}\Sigma_{l\epsilon B} H_{k,l};$$

if the X_j's are negatively associated, then for disjoint A,B

(2.3b) $$0 \geq H_{A,B} \geq \Sigma_{k\epsilon A}\Sigma_{l\epsilon B} H_{k,l}.$$

Proof. Let ρ_j denote the indicator function of the event $\{X_j>x_j\}$ and define

(2.4) $$\rho_A = \Pi_{j\epsilon A}\rho_j \quad , \quad S_A = \Sigma_{j\epsilon A}\rho_j.$$

It is easy to see that ρ_A, ρ_B, $S_A - \rho_A$, S_A, and $S_B - \rho_B$ are all increasing functions of the X_j's; it follows that for associated X_j's

$$0 \leq \mathrm{Cov}(\rho_A, \rho_B) \leq \mathrm{Cov}(S_A, \rho_B) \leq \mathrm{Cov}(S_A, S_B).$$

This yields (2.3a) since $H_{A,B} = \mathrm{Cov}(\rho_A, \rho_B)$ while the right hand side of (2.3a) equals $\mathrm{Cov}(S_A, S_B)$; the case of negative association is similar. □

COROLLARY 3. *Suppose the X_j's are either associated or negatively associated. It follows that $\{X_k; k\epsilon A\}$ is independent of $\{X_l; l\epsilon B\}$ if and only if $\mathrm{Cov}(X_k, X_l) = 0$, for all $k\epsilon A$ and $l\epsilon B$; similarly the X_j's are jointly independent if and only if $\mathrm{Cov}(X_k, X_l) = 0$ for all $k \neq l$.*

Proof. This is an immediate consequence of Theorems 1 and 2. □

It is clear that (2.3a) remains valid for disjoint A,B, if the hypothesis of association is weakened to make it analogous to a positive version of negative association; there seems to be no standard term for this weakened version of association. Both parts of Corollary 3 are valid for this weakened association (as well as for negative association); the second part of Corollary 3 can also be shown to be valid under even weaker hypotheses on the dependence of X_j's as we now discuss.

We let $\bar{\rho}_j = 1-\rho_j =$ the indicator function of the event $\{X_j \leq x_j\}$, $\bar{\rho}_A = \Pi_{j\epsilon A}\bar{\rho}_j$, and then following Joag-Dev (1983) we define $\{X_1, \ldots, X_n\}$ to be *strongly positive orthant dependent* (SPOD) if for any disjoint $A,B \subset \{1, \ldots, n\}$ and any real x_j's,

(2.5a) $$\mathrm{Cov}(\rho_A, \rho_B) \geq 0,\ \mathrm{Cov}(\bar{\rho}_A, \bar{\rho}_B) \geq 0,\ \mathrm{Cov}(\rho_A, \bar{\rho}_B) \leq 0,$$

and *strongly negative orthant dependent* (SNOD) if analogously

(2.5b) $$\mathrm{Cov}(\rho_A,\rho_B)\leq 0,\ \mathrm{Cov}(\bar\rho_A,\bar\rho_B)\leq 0,\ \mathrm{Cov}(\rho_A,\bar\rho_B)\geq 0.$$

It is immediate that association (resp. negative association) implies SPOD (resp. SNOD) which in turn implies pairwise PQD (resp. NQD).

The following theorem is due to Joag-Dev, but the proof given here is somewhat different than the original one.

THEOREM 4. *[Joag-Dev (1983)]. Suppose the X_j's are either SPOD or SNOD. It follows that they are jointly independent if and only if* $\mathrm{Cov}(X_k,X_l)=0$ *for all* $k\neq l$.

Proof. The theorem is an immediate consequence of Theorem 1 together with the following distribution function inequality. □

LEMMA 5. *Suppose* $X_1,\ldots,X_m$ *are SPOD; then*

(2.6a) $$0\leq P[X_j>x_j, j=1,\ldots,m]-\Pi_{j=1}^m P(X_j>x_j)\leq K_m\sum_{\substack{k,l=1\\k<l}}^m H_{k,l}$$

where K_m is a constant depending only on m. If the X_j's are SNOD, then

(2.6b) $$0\geq P[X_j>x_j, j=1,\ldots,m]-\Pi_{j=1}^m P(X_j>x_j)\geq K_m\sum_{\substack{k,l=1\\k<l}}^m H_{k,l}.$$

Proof. We consider the SPOD case; the SNOD case is treated similarly. The quantity of interest in the center of (2.6a) may be rewritten as

$$G_m\equiv \mathrm{E}(\Pi_{j=1}^m\rho_j)-\Pi_{j=1}^m E(\rho_j).$$

Its positivity follows easily from repeated application of (2.5a); we wish to obtain the upper bound of (2.6a). Denoting $\Pi_{j=1}^m\rho_j$ by ρ^m, we have

(2.7) $$G_{m+1}=E(\rho_{m+1})G_m+\mathrm{Cov}(\rho^m,\rho_{m+1})\leq G_m+\mathrm{Cov}(\rho^m,\rho_{m+1}),$$

while for $j>n+1$,

$$\mathrm{Cov}(\rho^{n+1},\rho_j)=\mathrm{Cov}(\rho^n,\rho_{n+1}\rho_j)-E(\rho_j)E(\rho^n\rho_{n+1})+E(\rho_{n+1}\rho_j)E(\rho^n)$$

(2.8) $$=\mathrm{Cov}(\rho^n,\rho_{n+1})+\mathrm{Cov}(\rho^n,\rho_j)+\mathrm{Cov}(\rho^n,\bar\rho_{n+1}\bar\rho_j)+\mathrm{Cov}(\rho_{n+1}\rho_j)\cdot E(\rho^n)$$
$$-E(\rho_j)\mathrm{Cov}(\rho^n,\rho_{n+1})\leq \mathrm{Cov}(\rho^n,\rho_{n+1})+\mathrm{Cov}(\rho^n,\rho_j)+\mathrm{Cov}(\rho_{n+1},\rho_j).$$

The last inequality follows from the fact that SPOD implies

$$\mathrm{Cov}(\rho^n,\bar\rho_{n+1}\bar\rho_j)\leq 0,\ \mathrm{Cov}(\rho_{n+1},\rho_j)\geq 0,\ \mathrm{Cov}(\rho^n,\rho_{n+1})\geq 0,$$

while $E(\rho^n)\leq 1$ and $E(\rho_j)\geq 0$. The right hand inequality of (2.6a) follows from (2.7) and (2.8) by induction. □

Remark. If the X_j's are associated or negatively associated, then it is easily seen that (2.3) implies (2.6) with $K_m=1$. It is not known to the author whether this value of K_m is valid under the weaker hypothesis of SPOD or SNOD. It is also not known to the author whether inequality (2.3) (possibly modified by a factor analogous to K_m) and the *first* result of Corollary 3 are valid when only assuming SPOD or SNOD.

An alternative improvement to the second result of Corollary 3 can be obtained from the characteristic function inequalities discussed in Section 3 below. We define X_j's to be *linearly positive quadrant dependent* (LPQD) if for any disjoint A,B and positive λ_j's, $\Sigma_{k\in A}\lambda_k X_k$ and $\Sigma_{l\in B}\lambda_l X_l$ are PQD; *linearly negative quadrant dependent* (LNQD) is defined in the obvious analogous manner. The next theorem is an immediate corollary of Theorem 10 of the next section. We include it here for comparison with Theorem 4.

THEOREM 6. *Suppose the X_j's are either LPQD or LNQD. It follows that they are jointly independent if and only if* $\mathrm{Cov}(X_k,X_l)=0$ *for all* $k\neq l$.

Remark. As in the previous remark, it is not known to the author whether the first result of Corollary 3 is valid under the weaker hypothesis of LPQD or LNQD.

In order to compare Theorems 4 and 6, we present two examples of M. Shaked (1982b) which show that neither SPOD nor LPQD implies the other. Consider three discrete random variables with joint density $p(x_1,x_2,x_3) \equiv P[X_1 = x_1, X_2 = x_2, X_3 = x_3]$. In the first example, $p(0,1,0) = p(0,2,0) = p(1,0,1) = p(1,1,0) = 1/14$, $p(0,2,1) = p(1,0,0) = 2/14$ and $p(0,0,0) = p(1,2,1) = 3/14$; here $\{X_1,X_2,X_3\}$ is not LPQD since $3/14 = P[X_1>0, X_2+X_3 > 1] < P[X_1>0] \cdot P[X_2 + X_3>1] = (7/14)^2$ while a lengthy verification shows that it is in fact SPOD. In the second example, $p(2,2,1) = p(3,2,1) = p(2,3,1) = p(3,3,1) = p(1,1,2) = p(2,1,2) = p(3,1,2) = p(1,2,2) = p(1,3,2) = 1/17$ and $p(1,1,1) = p(3,3,2) = 4/17$; here $\{X_1,X_2,X_3\}$ is not SPOD since $P[X_1>1, X_2>1, X_3>1] = 4/17 < P[X_1>1, X_2>1] \cdot P[X_3>1] = (8/17) \cdot (9/17)$ while a lengthy verification shows that it is in fact LPQD. For another example showing that LPQD does not imply SPOD with more details, see Joag-Dev (1983).

The next theorem on ergodicity is a consequence of Theorem 2. It is implicitly contained in the work of Lebowitz (1972) and is explicitly mentioned in a remark of Newman (1980) in the more general context of sequences indexed by Z^d. A somewhat simpler proof than the following one can be based on Theorem 16 below.

THEOREM 7 *[Lebowitz (1972)]. Let $X_1, X_2, \ldots$ be a strictly stationary sequence which is either associated or negatively associated and let T denote the usual shift transformation, defined so that $T(f(X_{j_1}, \ldots, X_{j_m})) = f(X_{j_1+1}, \ldots, X_{j_m+1})$. Then T is ergodic (i.e., every T-invariant event in the σ-field generated by the X_j's has probability 0 or 1) if and only if*

$$\text{(2.9)} \qquad \lim_{n\to\infty} n^{-1}\Sigma_{j=1}^{n}\mathrm{Cov}(X_1,X_j) = 0.$$

In particular, if (2.9) is valid, then for any f such that $f(X_1)$ is L_1,

$$\lim_{n\to\infty} n^{-1}\Sigma_{j=1}^{n} f(X_j) = E(f(X_1)) \quad \text{a.s.}$$

In the negatively associated case, (2.9) is automatically valid.

Proof. The necessity of (2.9) follows from the L_2 ergodic theorem which implies $n^{-1}\Sigma_{j=1}^{n} X_j \to E(X_1)$ in L_2. To prove the sufficiency of (2.9) we note that by standard ergodic theory/Hilbert space arguments, it suffices to find two sets S_1 and S_2 of random variables (measurable with respect to the X_j's) each of whose linear combinations are dense in L_2 and a subsequence n_i, such that for any $W_1 \in S_1$, $W_2 \in S_2$,

$$\text{(2.10)} \qquad \lim_{i\to\infty} \mathrm{Cov}(W_1, n^{-1}\Sigma_{j=1}^{n_i} T^j W_2) = 0,$$

since that would imply that the eigenvalue 1 of T is simple. For $\ell = 1,2$, we take $S_\ell = \{\Pi_{j=1}^{m} \rho_j(x_j) : m = 1,2, \ldots ; \text{each } x_j \in D_\ell\}$ where $\rho_j(x_j)$ is the indicator function of $\{X_j > x_j\}$ and D_ℓ is a dense subset of $\mathcal{R}$ to be chosen. To see that linear combinations of S_ℓ are dense in L_2, note that for $x_j \leq x_j'$, $\Pi_{j=1}^{m}[\rho_j(x_j) - \rho_j(x_j')]$ is the indicator function of the rectangle, $\{x_j < X_j \leq x_j', \text{ for all } j\}$. Defining $H_j(x_1,x_2) = \mathrm{Cov}(\rho_1(x_1), \rho_j(x_2))$ and $\tilde{H}_n = n^{-1}\,\Sigma_{j=1}^{n} H_j$, we see from Theorem 2 that to obtain (2.10) it suffices to show that $\tilde{H}_{n_i}(x_1,x_2) \to 0$ for $x_1 \in D_1, x_2 \in D_2$. But by (2.9) and identity (1.2) we know that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{H}_n(x_1,x_2)dx_1dx_2 \to 0$ as $n \to \infty$ and moreover $|\tilde{H}_n| \leq 1$; it follows there is a subsequence so that $\tilde{H}_{n_i} \to 0$ pointwise (except on a set of zero Lebesgue measure) in $\mathcal{R}^2$ and thus that D_1 and D_2 exist. The final statement of the theorem is a consequence of the following lemma. □

LEMMA 8. *If $X_1, X_2, \ldots$ is a (wide sense) stationary sequence with $\mathrm{Cov}(X_i,X_j) \leq 0$ for $i \neq j$, then $\Sigma_{j=2}^{\infty}\mathrm{Cov}(X_1,X_j)$ is absolutely convergent and*

(2.11) $$\sigma^2 \equiv \text{var}(X_1)+2\Sigma_{j=2}^{\infty}\text{Cov}(X_1,X_j)\in[0,\text{Var}(X_1)].$$

Proof. This is a consequence of the negativity of $\text{Cov}(X_1,X_j)$ for $j\geq 2$, which implies that $\sigma_n^2 \equiv \text{Var}(n^{-1/2}\Sigma_{j=1}^{\infty}X_j)$ satisfies

(2.12) $$0 \leqslant \lim_{n\to\infty}(\text{Var}(X_1)-\sigma_n^2) = -2\,\Sigma_{j=2}^{\infty}\text{Cov}(X_1,X_j)$$

Since $\sigma_n^2 \geqslant 0$, we must have the right hand side of (2.12) finite and bounded by $\text{Var}(X_1)$ which completes the proof. □

3. Characteristic Function Inequalities and Applications. We begin with a simple extension of Theorem 1 which gives a quantitative estimate of the approximate independence between a pair of variables in terms of the covariance.

PROPOSITION 9 *(Newman (1980)). If X and Y are PQD or NQD, then*

(3.1) $$|E(e^{irX+isY})-E(e^{irX})E(e^{isY})|\leqslant|rs\,\text{Cov}(X,Y)|,\ \textit{for all real } r,s.$$

Proof. Integration by parts yields, analogously to (1.2), the identity,

(3.2) $$\text{Cov}(e^{irX},e^{isY}) = \textstyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ire^{irx}ise^{isy}H(x,y)dxdy.$$

where H is defined as in (1.1). The triangle inequality, the pointwise positivity (resp. negativity) of H for PQD (resp. NQD) variables, and equation (1.2) then yield (3.1). □

The next theorem is the main ingredient used to obtain the limit theorems of this section.

THEOREM 10 *(Newman (1980)). Suppose $X_1, \ldots, X_m$ are LPQD or NPQD; then*

(3.3) $$|\phi(r_1,\ldots,r_m) - \Pi_{j=1}^{m}\phi_j(r_j)| \leqslant \Sigma_{\substack{k,l=1\\k<l}}^{m}|r_kr_l\text{Cov}(X_k,X_l)|$$

where ϕ and ϕ_j are given by

$$\phi = E(\exp[i\Sigma_{j=1}^{m}r_jX_j]),\ \phi_j = E(\exp[i\,r_jX_j]).$$

Proof. (3.3) follows from (3.1) by induction on m. The first step of the induction argument is to choose a nontrivial subset A of $\{1,\ldots,m\}$ so that the r_j's have a common sign in A and a common sign in $\bar{A}$, the complement of A. Defining $\phi_B=E(\exp[i\,\Sigma_{j\in B}r_jX_j])$, we then have the left hand side of (3.3) bounded by

(3.4) $$|\phi-\phi_A\phi_{\bar{A}}| + |\phi_A|\,|\phi_{\bar{A}}-\Pi_{l\in\bar{A}}\phi_l| + |\Pi_{\ell\in\bar{A}}\phi_\ell||\phi_A-\Pi_{k\in A}\phi_k|.$$

The first term of (3.4) is bounded by (3.1) while the other two terms are bounded by the induction hypothesis (and the fact that $|\phi_A|,|\phi_l|\leqslant 1$) to yield the right hand side of (3.3). □

The next theorem is an immediate corollary of Theorem 10. I appears in Newman, Rinot and Tversky (1982) and independently in Wood (1982). It was used in the latter reference for a general analysis of limit theorems for sums of associated variables and in the former reference for a specific application to a model arising in mathematical psychology. In that model there is a collection of "distances," $\{D_{ij};\ 0\leqslant i<j\leqslant n\}$, between objects i and j, which are exchangeable random variables and one is interested in the asymptotic behavior of $S_n=$ number of objects in $\{1,\ldots,n\}$ which have object 0 as their nearest neighbor. S_n can be represented as $\Sigma_{j=1}^{n}Y_{n,j}$ where $Y_{n,j}$ is the indicator function of the event that j has 0 as its nearest neighbor. Each $Y_{n,j}$ is Bernoulli ($p=1/n$) and although they are not independent (for fixed n) they can be shown to be associated. The following theorem can then be used to give a particularly simple proof that S_n converges in distribution to Poisson ($\lambda=1$).

THEOREM 11 *(Newman, Rinott, Tversky (1982); Wood (1982)). Suppose $Y_{n,j}$ and $W_{n,j}$*

($n=1,2, \ldots ; j=1,2, \ldots , M_n$) are triangular arrays such that for each n and j, $Y_{n,j}$ is equidistributed with $W_{n,j}$ and such that for each n, the $Y_{n,j}$'s are LPQD or NPQD while the $W_{n,j}$'s are independent. If in addition,

$$\lim_{n\to\infty} \Sigma_{k,l=1 \atop k<l}^{M_n} \mathrm{Cov}(Y_{n,k}, Y_{n,l}) = 0, \tag{3.5}$$

then $\Sigma_{j=1}^{M_n} Y_{n,j}$ converges in distribution to (some) X if and only if $\Sigma_{j=1}^{M_n} W_{n,j}$ converges in distribution to (the same) X.

Proof. This is an immediate consequence of Theorem 10 and standard arguments. □

The next theorem was the original application of (and motivation for) the characteristic function inequality (3.3). It was first given in Newman (1980) in the more general context of sequences indexed by Z^d. In this paper we sketch a proof based on Theorem 11; for a more detailed proof (using Theorem 10 directly rather than Theorem 11) see Newman (1980) or Newman and Wright (1981). The theorem itself (or more accurately its Z^d indexed generalization) was applied to Ising model magnetization fluctuations (or the equivalent lattice gas model density fluctuations) in Newman (1980) and to the density fluctuations of infinite clusters in percolation models in Newman and Schulman (1981). In the statement of the theorem, note that for LPQD (resp. NPQD) Y_j's

$$\mathrm{Cov}(Y_1,Y_j) \geq 0 \quad (\text{resp. } \mathrm{Cov}(Y_1,Y_j) \leq 0) \quad \text{for all } j \geq 2.$$

THEOREM 12 *(Newman (1980)). Let $Y_1,Y_2, \ldots$ be a strictly stationary sequence which is LPQD or LNQD. Then*

$$\sigma^2 \equiv \mathrm{Var}(Y_1) + 2\,\Sigma_{j=2}^{\infty}\mathrm{Cov}(Y_1,Y_j)$$

always exists and $\sigma^2 \epsilon [\mathrm{Var}(Y_1), \infty]$ (resp. $\sigma^2 \epsilon [0, \mathrm{Var}(Y_1)]$) in the LPQD (resp. LNQD) case. If $\sigma^2 \neq \infty$; i.e. if in the LPQD case we additionally assume that

$$\Sigma_{j=2}^{\infty}\mathrm{Cov}(Y_1,Y_j) < \infty, \tag{3.6}$$

then

$$\lim_{n\to\infty} n^{-1/2}\Sigma_{j=1}^{n}(Y_j - EY_j) = \sigma Z \tag{3.7}$$

where Z is standard normal and (3.7) refers to convergence in distribution.

Sketch of Proof. The first part of the theorem follows from the positivity or negativity of (Y_1,Y_j) for $j \geq 2$ and Lemma 8 above. In particular, this, together with the nonnegativity of the variance of $\Sigma_{j=1}^{n} Y_j$, *yields in the LNQD case the bound,* $2\Sigma_{j=2}^{n}\mathrm{Cov}(Y_1,Y_j) \geq -\mathrm{Var}(Y_1)$. For the rest of the theorem, we define "block variables,"

$$Y_k^m = m^{-1/2}\Sigma_{j=(k-1)m+1}^{km}(Y_j - EY_j) \quad ; \quad m = 1,2, \ldots ; j = 1,2, \ldots \tag{3.8}$$

By straightforward variance estimates, it can be seen that it suffices to show $\lim_{\ell\to\infty} S_\ell^{m_\ell} = \sigma Z$, where

$$S_\ell^m = (m\ell)^{-1/2}\Sigma_{j=1}^{m\ell}(Y_j - EY_j) = \ell^{-1/2}\Sigma_{j=1}^{\ell} Y_j^m, \tag{3.9}$$

and m_ℓ is some nondecreasing sequence of positive integers such that $(m_{\ell+1} - m)/m_\ell \to 0$. If for each m, we define $W_j^m (j=1,2, \ldots)$ to be i.i.d. and equidistributed with Y_1^m, then defining

$$\tilde{S}_\ell^m = \ell^{-1/2}\Sigma_{j=1}^{\ell} W_j^m,$$

we have (by the standard central limit theorem) that for fixed m, $\lim_{\ell\to\infty}\tilde{S}_\ell^m = \sigma_m Z$ *where* $\sigma_m^2 = \mathrm{Var}(Y_1^m)$. More variance estimates show that $\sigma_m \to \sigma$ and thus that for any sequence m_ℓ growing to ∞ sufficiently slowly, $\lim_{\ell\to\infty}\tilde{S}^{m_\ell}_\ell = \sigma Z$. The desired result follows from Theorem 11 by taking $Y_{\ell,j} = \ell^{-1/2} Y_j^{m_\ell}$ and $W_{\ell,j} = \ell^{-1/2} W_j^m$, providing we show that (3.5) is valid.

But (3.5) is a simple consequence of σ_m converging to σ. □

Remark. Note that $\sigma^2>0$ in the LPQD case (except when the Y_j*'s are constant) but* σ^2 can in fact vanish in the LNQD case. A trivial example of the latter phenomenon is obtained by taking $Y_j = Z_j - Z_{j-1}$ where $Z_0, Z_1, \ldots$ are i.i.d. standard normal; these Y_j's are not only LNQD but are negatively associated by a result of Joag-Dev and Proschan (1983).

We present the following theorem of Herrndorf (1983) without proof. It disproves a conjecture of Newman (1980) and Newman and Wright (1981) concerning the weakening of condition (3.6).

THEOREM 13 *(Herrndorf (1983)). There exists, $Y_1, Y_2, \ldots$, a strictly stationary non-constant associated sequence with $K(R) \equiv \mathrm{Var}(Y_1) + 2\,\Sigma_{j\leq R}\mathrm{Cov}(Y_1, Y_j)$ slowly varying as $R\to\infty$ (i.e. $K(\lambda R)/K(R)\to 1$ as $R\to\infty$ for any $\lambda>0$) such that*

$$[nK(n)]^{-1/2}\Sigma_{j=1}^{n}(Y_j - EY_j)$$

does not converge in distribution to a standard normal Z.

4. More Characteristic Function Inequalities and Applications. In this section, we present a number of recent results, one of whose motivations is the desire to extend Theorem 12 to a central limit theorem for sums of $f(Y_j)$'s; some of the results are presented here for the first time. If the Y_j's are associated or negatively associated and f is either increasing or decreasing, then Theorem 12 can be directly applied to the $f(Y_j)$'s. We begin with a number of inequalities which are applicable to more general f's.

For f and f_1 complex functions on $\mathcal{R}^m$, we write $f << f_1$ if $f_1 - Re(e^{i\alpha}f)$ is (coordinatewise) nondecreasing for all real α. Note first that $f_1 = [(f_1 - Re(f)) + (f_1 - Re(-f))]/2$ and hence is automatically nondecreasing and second that $f << f_1$ for real f if and only if $f_1 + f$ and $f_1 - f$ are both nondecreasing. We write $f <<_A f_1$ if $f << f_1$ and both f_1 and f depend only on x_j's with $j\epsilon A$. The next two propositions will be used to obtain useful characteristic function inequalities.

PROPOSITION 14. *If h is real, $h << h_1$, and φ is a complex function on $\mathcal{R}$ such that $|\varphi(t) - \varphi(s)| \leq |t-s|$ for all t,s, then $\varphi(h) << h_1$. This applies in particular to $\varphi(h) = \exp(ih)$.*

Proof. We denote by Δg the increment in the function g when one or more of the x_j's is increased. We wish to show that for any real α, $\Delta[h_1 - Re(e^{i\alpha}\varphi(h))] \geq 0$. But $|\Delta Re(e^{i\alpha}\varphi(h))| \leq |\Delta(e^{i\alpha}\varphi(h))| = |\Delta\varphi(h)| \leq |\Delta h|$ because of the properties of φ, while $|\Delta h| \leq \Delta h_1$ because $h << h_1$. □

PROPOSITION 15 *(Newman (1983)). Suppose $f <<_A f_1$ and $g <<_B g_1$. Define $<f,g> = \mathrm{Cov}(f(X_1, X_2, \ldots), g(X_1, X_2, \ldots))$ where the X_j's are either associated or negatively associated. In the negatively associated case, assume in addition that A and B are disjoint. Then $|<f,g>| \leq |<f_1,g_1>|$ iff and/or g is real; otherwise $|<f,g>| \leq 2|<f_1,g_1>|$.*

Proof. First suppose f is real. Since $|<f,g>| = \sup(Re(e^{i\alpha}<f,g>) : \alpha\epsilon\mathcal{R})$, it suffices to show that $Re(e^{i\alpha}<f,g>) \leq |<f_1,g_1>|$. This follows from the assumption that $h \equiv Re(e^{i\alpha}g) << g_1$ and $f << f_1$ and the identities,

$$|<f_1,g_1>| - <f,h> = <f_1,g_1> - <f,h> = \tfrac{1}{2}[<f_1+f, g_1-h> + <f_1-f, g_1+h>] \geq 0$$

for the associated case, and

$$|<f_1,g_1>| - <f,h> = -<f_1,g_1> - <f,h> = \tfrac{1}{2}[<f_1+f, g_1+h> + <f_1-f, g_1-h>] \geq 0$$

for the negatively associated case. If g is real the argument is the same and if neither are real, one has

$$|\langle f,g\rangle| = |\langle Re\,f,h\rangle + i\langle Im\,f,g\rangle| \leq |\langle Re\,f,g\rangle| + |\langle Im\,f,g\rangle|$$

so that the desired inequality follows from the real f inequality. □

Remark. Proposition 15 is a generalization of Theorem 2. It is possible that the factor 2 appearing when both f and g are complex could be eliminated by a better proof; that would also eliminate the corresponding factors of 2 in the next theorem.

THEOREM 16. *(Newman (1983)). Suppose that for each j,* $X_j = f_j(Y_1,Y_2,\ldots)$, $\tilde{X}_j = \tilde{f}_j(Y_1,Y_2,\ldots)$ *where the* Y_i*'s are associated or negatively associated. Suppose further that* $f_j \ll_{A_j} \tilde{f}_j$ *for each j and, in the negatively associated case, additionally that the* A_j*'s are disjoint. Then the characteristic functions of the* X_j*'s,* ϕ, ϕ_j, ϕ_C, *defined as in Theorem 10 and its proof, satisfy (for disjoint A,B)*

$$|\phi_{A\cup B} - \phi_A\phi_B| \leq 2\,\Sigma_{k\in A}\Sigma_{l\in B}|r_k r_l \mathrm{Cov}(\tilde{X}_k,\tilde{X}_l)| \tag{4.1}$$

and

$$|\phi - \Pi_{j=1}^{m}\phi_j| \leq 2\,\Sigma_{\substack{k,l=1\\k<l}}^{m}|r_k r_l \mathrm{Cov}(\tilde{X}_k,\tilde{X}_l)|. \tag{4.2}$$

Proof. (4.1) follows from Propositions 14 and 15 since the left hand side of (4.1) is $|\langle f,g\rangle|$ with $f = \exp(i\,\Sigma_{j\in A}r_jX_j)$, $g = \exp(i\,\Sigma_{j\in B}r_jX_j)$ and since $\Sigma r_jf_j \ll \Sigma|r_j|\tilde{f}_j$. (4.2) follows from (4.1) essentially as in the proof of Theorem 10 from Proposition 9. □

There is a natural extension of Theorem 11 which follows from Theorem 16 in the same way as Theorem 11 follows from Theorem 10. To save space, we do not state that extension explicitly but rather go on to an extension of Theorem 12. This latter extension was applied in Newman (1983) to the fluctuations in Ising model energy densities and to the fluctuations of infinite cluster surfaces in percolation models.

THEOREM 17 *(Newman (1983)). Let* $Y_1,Y_2,\ldots$ *be a strictly stationary sequence which is associated (resp. negatively associated). Let* $X_j = f(Y_j,Y_{j+1},\ldots)$ *and* $\tilde{X}_j = \tilde{f}(Y_j,Y_{j+1},\ldots)$ *(resp.* $X_j = f(Y_j)$ *and* $\tilde{X}_j = \tilde{f}(Y_j)$*) with* $f \ll \tilde{f}$; *in the associated case, assume in addition that*

$$\Sigma_{j=2}^{\infty}\mathrm{Cov}(\tilde{X}_1,\tilde{X}_j) < \infty. \tag{4.3}$$

Then

$$\lim_{n\to\infty} n^{-1/2}\Sigma_{j=1}^{n}(X_j - EX_j) = \sigma Z, \tag{4.4}$$

where Z is standard normal and

$$\sigma^2 = \mathrm{Var}(X_1) + 2\,\Sigma_{j=2}^{\infty}\mathrm{Cov}(X_1,X_j) \in [0,\infty). \tag{4.5}$$

Proof. This theorem follows from Theorem 16 in the same way as Theorem 12 follows from Theorem 10. □

To investigate Theorem 17 in more detail, we restrict attention to $X_j = f(Y_j)$ even in the associated case. In this context, we define for $y,y' \in \mathcal{R}$

$$H(y,y') = H_1(y,y') + \Sigma_{j=2}^{\infty}[H_j(y,y') + H_j(y',y)], \tag{4.6}$$

where

$$H_j(y,y') = P(Y_1 > y,\ Y_j > y') - P(Y_1 > y)\,P(Y_j > y'). \tag{4.7}$$

PROPOSITION 18. *Let* $Y_1,Y_2,\ldots$ *be a strictly stationary (not necessarily* L_2*) sequence* *which is either associated or negatively associated. Then $H(y,y')$ exists for all y,y' with $0 \leq H_1 \leq H \leq \infty$ in the associated case and $-1 \leq H \leq H_1 \leq 1$ in the negatively associated case. H and H_1 are positive semidefinite in the sense that for any real g such that $g(y)H(y,y')g(y')$ is in $L_1(\mathscr{R}^2)$,*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(y)H(y,y')g(y')\,dy\,dy' \geq 0, \tag{4.8}$$

and similarly for H_1.

Proof. Denoting by $\rho_j(y)$ the indicator function of $\{Y_j > y\}$, and defining

$$H_{(n)}(y,y') = n^{-1}\mathrm{Cov}(\Sigma_{j=1}^{n}\rho_j(y), \Sigma_{k=1}^{n}\rho_k(y')),$$

we see that $H_{(n)}$ is positive semidefinite. In the associated case, the positivity of the H_j's implies that H exists, that $0 \leq H_1 \leq H \leq \infty$ and that $H = \lim H_{(n)}$ is positive semidefinite. In the negatively associated case, the negativity of the H_j's implies that H exists, that $H \leq H_1$, and that

$$\lim_{n\to\infty}[H_{(1)}(y,y') - H_{(n)}(y,y')] = -2\,\Sigma_{j=2}^{\infty}[H_j(y,y') + H_j(y',y)] \equiv G(y,y').$$

Since $H_{(n)}(y,y) \geq 0$ we have that $G(y,y) \leq H_{(1)}(y,y) = H_1(y,y)$ and thus that $H(y,y) = H_1(y,y) - G(y,y) \geq 0$. By the positive semidefiniteness of $H_{(n)}$, we have that

$$[H(y,y)\cdot H(y',y')] - [H(y,y')]^2 = \lim_{n\to\infty}([H_{(n)}(y,y)\cdot H_{(n)}(y',y')] - [H_{(n)}(y,y')]^2) \geq 0,$$

which implies that for any y,y',

$$|H(y,y')| \leq [H(y,y)\cdot H(y',y')]^{1/2} \leq [H_1(y,y)\cdot H_1(y',y')]^{1/2} \leq 1.$$

The positive semidefiniteness of $H = \lim H_{(n)}$ follows from that of $H_{(n)}$. □

Remark. In the associated case, $H(y,y')$ may equal $+\infty$ for some or all values of y,y'. For example, straightforward estimates show that when the Y_i's are jointly normal, then for any y,y', $\Sigma\, H_j(y,y')$ is absolutely convergent if and only if $\Sigma\, \mathrm{Cov}(Y_1,Y_j)$ is absolutely convergent.

We define D_H to be the set of real functions such that $g(y)H(y,y')g(y')$ is in $L_1(\mathscr{R}^2)$ and similarly for H_1, and we say that a real function f on $\mathscr{R}$ is absolutely continuous if it is the indefinite integral of a locally L_1 function f'. Note that $f' \in D_{H_1}$ if and only if the random variable, $\int_0^{Y_1}|f'(t)|dt$, has finite variance.

THEOREM 19. *Let $Y_1, Y_2, \ldots$ be a strictly stationary (not necessarily L_2) sequence which is either associated or negatively associated and let $X_j = f(Y_j)$ where f is an absolutely continuous function. Define $\bar{X}_j = \bar{f}(Y_j)$ where $\bar{f}(y) = \int_0^y|f'(t)|dt$. In the associated (resp. negatively associated) case, assume in addition that $f' \in D_H$ (resp. $\overline{X}_1$ is L_2); it follows that $\bar{X}_1$ is L_2 (resp. $f' \in D_H$) and that (4.4) is valid with σ^2 given by (4.5) or equivalently by*

$$\sigma^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f'(y)\,H(y,y')f'(y')\,dy\,dy'. \tag{4.9}$$

Proof. A straightforward generalization of (1.2) and (3.2) yields

$$\mathrm{Cov}(g_1(Y_1),g_j(Y_j)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g'_1(y)g'_j(y')H_j(y,y')\,dy\,dy', \tag{4.10}$$

providing g_1, g_j are absoluted continuous and the random variables on the left hand side are L_2. If we take $g_1 = g_j = \bar{f}$ and $j = 1$, then the identity (4.10) shows that $\bar{X}_1$ is L_2 if and only if $f' \in D_{H_1}$. In the associated case the inequalities $H \geq H_1 \geq 0$ show that $f' \in D_H$ implies $f' \in D_{H_1}$. In the negatively associated case the positive semidefiniteness of H, the negativity of $H - H_1$, and the positivity of H_1 imply

$$\int |f' H f'|\, dy\, dy' \leq \int |f' H_1 f'|\, dy\, dy' + \int |f'(H - H_1) f'|\, dy\, dy'$$
$$= \int |f'| H_1 |f'|\, dy\, dy' + \int |f'| (H_1 - H) |f'|\, dy\, dy' \leq 2 \int |f'| H_1 |f'|\, dy\, dy',$$

which shows that $f' \epsilon D_{H_1}$ imples $f' \epsilon D_H$. In either case, we now obtain the desired results by applying Theorem 17 and using (4.10) first with $g_1 = g_j = \tilde{f}$ to verify (4.3) and then with $g_1 = g_j = f$ to obtain (4.9). □

Remark. In the associated case, the automatic convergence of $\Sigma \mathrm{Cov}(Y_1, Y_j)$ implies by (2) that $H \in L_1(\mathcal{R}^2)$ and then the hypothesis, $f' \epsilon D_H$, of Theorem 19 will be satisfied for any f with $f' \epsilon L_\infty(\mathcal{R}^1)$. In the associated case, if one assumes the convergence of $\Sigma_j \sup(H(y,y') : y, y' \epsilon \mathcal{R})$, then one would have H bounded on $\mathcal{R}^2$ while in the negatively associated case this is automatically the case; in either of these situations, if $X_j = f(Y_j)$ where f has bounded total variation then an $\tilde{f}$ can be chosen to satisfy the hypothesis of Theorem 17. The central limit result of Theorem 17 can then be interpreted in terms of the scaled, centered, empirical distribution function,

$$I_n(y) = n^{-1/2}[(\text{number of } i \epsilon \{1, \ldots, n\} \text{ with } Y_i \leq y) - n P(Y_i \leq y)]$$
$$= n^{-1/2}[(\text{number of } i \epsilon \{1, \ldots, n\} \text{ with } Y_i > y) - n P(Y_i > y)];$$

the result is that I_n converges to a Gaussian process with mean zero and covariance $H(y,y')$ (at least) in the sense of convergence of finite dimensional distributions.

5. Related Results and Open Problems. We present the following triangular array central limit theorem without proof; for more details, see Cox and Grimmett (1982) where this theorem is proved as a consequence of Theorem 10.

THEOREM 20 *(Cox and Grimmett (1982)). Let* $S_n = \Sigma_{j=1}^{M_n}(Y_{n,j} - E\, Y_{n,j})$ *where for each n, the* $Y_{n,j}$*'s are LPQD. Suppose there exist* $c_1, c_2, c_3 \epsilon (0, \infty)$ *and a sequence* $u_l \to 0$ *so that for all* n, j, l, the following hold:

(5.1) $$Var(Y_{n,j}) \geq c_1, E(|Y_{n,j} - E\, Y_{n,j}|^3) \leq c_2.$$

(5.2) $$\Sigma_{k=1}^{M_n} \mathrm{Cov}(Y_{n,j}, Y_{n,k}) \leq c_3$$

(5.3) $$\sum_{\substack{k=1 \\ |k-j| \geq l}}^{M_n} \mathrm{Cov}(Y_{n,j}, Y_{n,k}) \leq u_l \quad ;$$

then

(5.4) $$\lim_{n \to \infty} (\mathrm{Var}(S_n))^{-1/2} S_n = Z.$$

Remark. There should be extensions of this theorem to the LNQD case and to sums of nonmonotonic functions of associated or negatively associated $Y_{n,j}$'s (analogous to the extension of Theorem 12 to Theorems 17 and 19); the details of such extensions have not been worked out.

The next theorem is due to Wood (1982; 1983). As in Theorem 20 the existence of absolute third moments is assumed; as a consequence a Berry-Esseen type result is obtained. In order to use this theorem to obtain uniform rates of convergence in the limit (3.7) of Theorem 12, one must control (usually in an ad hoc manner) the asymptotics of the parameters,

for more details and for the proof of the theorem, see Wood (1982; 1983).

THEOREM 21 *(Wood (1982)). Let* $S_n = Y_1 + \ldots + Y_n - nE(Y_1)$, *where* $Y_1, Y_2, \ldots$ *is* *a strictly stationary LPQD sequence with* $E(|Y_1|^3) < \infty$ *and such that*

(5.6) $$0 < \sigma^2 \equiv \mathrm{Var}(Y_1) + 2\,\Sigma_{j=2}^{\infty}\mathrm{Cov}(Y_1,Y_j) < \infty.$$

Then for $n = m \cdot k$,

(5.7) $$|P[n^{-1/2}S_n \le z] - P[\sigma Z \le z]| \le 16\sigma_k^4 m(\sigma^2-\sigma_k^2)/9\pi\nu_k^2 + 3\nu_k/\sigma_k^3 m^{1/2}.$$

The next result extends Theorem 12 in the associated case to an invariance principle (functional central limit theorem); such extensions in the negatively associated case and/or in the more general limits of Theorems 17 and 19 have not been investigated.

THEOREM 22 *(Newman and Wright (1981)). Let* $S_n = Y_1 + \ldots + Y_n - nEY_1$, *where* $Y_1, Y_2, \ldots$ *is a strictly stationary associated sequence with (5.6) valid. Define the stochastic processes,* $W_n(t)$ *for* $0 \leqslant t \leqslant T$ *by*

(5.8) $$W_n(t) = (\sigma^2 n)^{-1/2}[S_m + (nt-m)(Y_{m+1} - E\,Y_{m+1})] \text{ for } m/n \leqslant t \leqslant (m+1)/n;$$

then W_n *converges in distribution (on* $C[0,T]$ *to the standard Wiener process.*

Sketch of proof. A slight extension of Theorem 12 shows that the finite dimensional distributions of W_n converge to those of W. It remains to show that the distributions of W_n are tight. This is done in Newman and Wright (1981) as a consequence of the inequality (for $\alpha_2 - \alpha_1 > 1$),

(5.9) $$P[\mathrm{Max}(|S_l|, \ldots, |S_n|) \ge \alpha_2\sqrt{n}\,\sigma_n] \le [(\alpha_2-\alpha_1)^2/((\alpha_2-\alpha_1)^2-1)] \cdot P[|S_n| \ge \alpha_1 n^{1/2}\sigma_n],$$

which is derived by using the association of the Y_j's. □

A version of Theorem 12 for sequences indexed by d-dimensional parameters was already given in Newman (1980). The problem of obtaining a d-dimensional invariance principle for $d>1$ by deriving appropriate d-dimensional maximal inequalities was solved in Newman and Wright (1982) for $d=2$ by somewhat ad hoc methods; the problem is still open for $d>2$. In the process of obtaining results for $d=2$, the status of maximal and other inequalities for $d=1$ was clarified by realizing that there is a close connection between martingales and sums of associated variables. The following definition is due to Newman and Wright (1982).

Definition. An L_1 *sequence,* $S_0=0, S_1, S_2, \ldots$, *is a demimartingale (resp. demisubmartingale) if for* $j=1,2,\ldots$,

(5.10) $$E((S_{j+1}-S_j)f(S_1, \ldots, S_j)) \geqslant 0,$$

for all non decreasing (resp. nonnegative and nondecreasing) f such that the expectation is defined.

Note that with the natural choice of σ-fields, $S_0, S_1, \ldots$ would be a martingale (resp. submartingale) if the nondecreasing hypothesis were dropped. Note also that the assumption that the $(S_{j+1}-S_j)$'s are mean zero and associated implies that the sequence S_n is a demimartingale. It was shown in Newman and Wright (1982) that many standard martingale (resp. submartingale) inequalities, including Doob's maximal inequality and upcrossing inequality, remain valid for demimartingales (resp. demisubmartingales). In particular, we note that the inequalities of Corollary 6 of Newman and Wright (1982) are sufficient (without recourse to (5.9)) to yield tightness once convergence of finite dimensional distributions to those of a Wiener process is known. This fact, among others, suggests that both an ordinary and functional central limit theorem should be valid in the demimartingale context as it is in the martingale context (see, e.g. Billingsley (1968)).

CONJECTURE 23. *Let $S_0 = 0$, $S_1, S_2, \ldots$ be an L_2 demimartingale whose difference sequence $Y_1 = S_1 - S_0$, $Y_2 = S_2 - S_1$, $\ldots$ is strictly stationary and ergodic with (5.6) valid. Then W_n defined by (5.8) converges in distribution to the standard Wiener process; in particular,* $\lim n^{-1/2} S_n = \sigma Z$.

Remark. The status of maximal inequalities for sequences which satisfy (5.10) for all nonincreasing f has not yet been investigated. We do not consider that case further.

As a first step toward proving the above conjecture, we have the following result, presented here for the first time.

THEOREM 24. *Let $T_0 = 0$, $T_1, T_2, \ldots$ be an L_2 demimartingale and let $\mathcal{F}_n$ be the σ field generated by $T_0, T_1, \ldots, T_n$. If the T_j's have uncorrelated increments (i.e. if $\mathrm{Cov}((T_{j+1}-T_j), (T_{k+1}-T_k)) = 0$ for all $0 \leq k < j$), then the sequence $(T_n, \mathcal{F}_n)$ is a martingale.*

Proof. It follows immediately from the uncorrelated increment hypothesis that for each j,

$$E((T_{j+1}-T_j)T_k) = \mathrm{Cov}(T_{j+1}-T_j, T_k) = 0, \text{ for } k = 1, \ldots, j. \tag{5.11}$$

We have used the fact that $T_{j+1}-T_j$ has zero mean, as can be seen by taking f in (5.10) to be alternately $+1$ and -1. It suffices to show that

$$E((T_{j+1}-T_j)\exp[i\textstyle\sum_{k=1}^{j} r_k T_k]) = 0, \text{ for } r_1, \ldots, r_j \in \mathcal{R}, \tag{5.12}$$

in order to conclude that $E(Y_{j+1}|\mathcal{F}_j) = 0$ and thus that $(T_j, \mathcal{F}_j)$ is a martingale. The next proposition shows that (5.12) is a consequence of (5.11). □

PROPOSITION 25. *Suppose f and f_1 are complex functions on $\mathcal{R}^j$ such that $f << f_1$; then for a demimartingale T_n*

$$|E((T_{j+1}-T_j)f(T_1, \ldots, T_j))| \leq E((T_{j+1}-T_j)f_1(T_1, \ldots, T_j))$$

In particular this is the case for $f(t_1, \ldots, t_j) = \exp[i \sum_{k=1}^{j} r_k t_k]$ *and* $f_1(t_1, \ldots, t_j) = \sum_{k=1}^{j} |r_k| t_k$.

Proof. The proposition follows easily from Proposition 14 and a portion of the proof of Proposition 15. □

Remark. To clarify somewhat the distinction between martingales, sums of associated variables, and demimartingales we let $S_0 = 0$, $S_n = Z_1 + \ldots + Z_n (n = 1, 2, \ldots)$ where the Z_j's are jointly normal. The sequence S_n is a martingale (resp. submartingale) if and only if $\mathrm{Cov}(Z_k, Z_l) = 0$ for all $k > l$ and $E\,Z_j = 0$ (resp. $E\,Z_j \geq 0$) for all j. By the results of Pitt (1982), the Z_j's are associated if and only if $\mathrm{Cov}(Z_k, Z_l) \geq 0$ for all $k > l$ while the sequence S_n is a demimartingale (resp. subdemimartingale) if and only if $\sum_{j=1}^{\ell} \mathrm{Cov}(Z_k, Z_j) \geq 0$ for all $k > \ell$ and $EZ_j \geq 0$) for all j.

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