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ASYMPTOTIC INDEPENDENCE OF THE NUMBERS OF HIGH AND LOW LEVEL CROSSINGS OF STATIONARY GAUSSIAN PROCESSES[1]

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1. Introduction. Let $X(t)$, $t \geqq 0$, be a real valued separable stationary Gaussian process with mean 0, variance 1 and continuous covariance function $r(t)$. Suppose there exists $\gamma > 0$ such that

$$1 - r(t) \sim \gamma^2 t^2/2, \qquad t \to 0. \tag{1.1}$$

Such a process will be called a "standard process." Well-known theorems imply that the sample functions of the process are continuous on every finite interval [1], and that the expected number of upcrossings (and downcrossings) of a fixed level is finite [12]. For $\sigma_1 > 0$ and $\sigma_2 > 0$ put

$$u = (2 \log (t\gamma/2\pi\sigma_1))^{\frac{1}{2}} \tag{1.2}$$

$$v = (2 \log (t\gamma/2\pi\sigma_2))^{\frac{1}{2}};$$

and

$$M(t) = \text{number of upcrossings of } u \text{ by } X(s), \qquad 0 \leqq s \leqq t,$$

$$N(t) = \text{number of downcrossings of } -v \text{ by } X(s), \qquad 0 \leqq s \leqq t.$$

Our main result is: *If either*

$$\lim_{t\to\infty} r(t) \log t = 0 \qquad \textit{or} \tag{1.3}$$

$$\textstyle\int_0^\infty r^2(s)\, ds < \infty, \tag{1.4}$$

then the joint limiting distribution of $M(t)$ *and* $N(t)$ *exists and is a product of Poisson distributions with means* σ_1 *and* σ_2, *respectively.*

The limit theorem for the number of upcrossings of u has been obtained by several writers under increasingly more general conditions (Volkonskii and Rozanov [11], Cramér [4], Beljaev [2], and Qualls [8]). Results for "ε-upcrossings" for processes more general than those satisfying (1.1) have been obtained by Pickands [6]. All of these results are about the *marginal* limiting distribution of $M(t)$. The joint distribution of crossings of high multiple levels is considered in a recent paper by Qualls [9]. The main novelty of the present work is the consideration of the *joint* distribution of the numbers of high and low extreme crossings; furthermore, our conditions (1.1), (1.3) and (1.4) are more general than any previously used in

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the upcrossing theorem. While they are identical with those of Pickands, his work is concerned only with ε-upcrossings, not the usual upcrossings.

A direct consequence of our theorem is that the maximum and minimum of the process are asymptotically independent; this is analogous to the well-known property of the maximum and minimum in a sequence of independent random variables with a common distribution.

2. Approximation by a random trigonometric function. We shall show that the distribution of the maximum (or minimum) of a standard process on a small interval is very nearly that of the same functional for a specific elementary process.

LEMMA 2.1. *Let Y_1 and Y_2 be independent random variables with a common standard Gaussian distribution. Put*

$$Y(t) = Y_1 \cos t + Y_2 \sin t, \qquad t \geqq 0,$$

$$\Phi(u) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{u} \exp(-y^2/2)\, dy;$$

then for every $u > 0$ and $0 < t < \pi/2$:

$$P\{\max(Y(s): 0 \leqq s \leqq t) \leqq u\} = \Phi(u) - (t/2\pi) \exp(-u^2/2). \tag{2.1}$$

PROOF. The inequalities $Y(s) \leqq u, \forall\, s: 0 \leqq s \leqq t$ are satisfied if and only if the pair (Y_1, Y_2) falls in the intersection of the half-spaces of the xy-plane:

$$\{(x, y): x \cos s + y \sin s \leqq u\}, \qquad 0 \leqq s \leqq t.$$

This is a union of disjoint regions $A_1, \cdots, A_5$ in the plane for which

$$\int\int_{A_i} d\Phi(x)\, d\Phi(y), \qquad i = 1, \cdots, 5$$

is computed. The double integral over

$$A_1 = \{0 \leqq x \leqq u, y \leqq 0\}$$

is equal to $\frac{1}{2}(\Phi(u) - \frac{1}{2})$. Let A_2 be the region bounded by: the circle centered at the origin, of radius u; the x-axis; and the line $y = x \tan t$. The double integral is equal to $(t/2\pi)[1 - \exp(-u^2/2)]$. Let A_3 be the infinite subregion of the half-space $y \geqq 0$ bounded on three sides by the lines

$$y = x \tan t, \qquad y = -x/\tan t, \qquad y = \frac{-x}{\tan t} + \frac{u}{\sin t}.$$

By a rotation of the axes of t radians, this region is transformed into A_1, and so the double integral is the same. Let A_4 be the region in the second quadrant bounded by the x-axis and the line $y = -x/\tan t$; then the double integral over it is equal to $(\frac{1}{4}) - (t/2\pi)$. Finally, let A_5 be the third quadrant; then the integral over it is equal to $\frac{1}{4}$. The right-hand side of (2.1) is equal to the sum of these five integrals.

LEMMA 2.2. *Let $X(t)$, $t \geqq 0$, be a standard process. For every ε, $0 < \varepsilon < 1$, there exists $T > 0$ such that*

$$(2.2) \quad \Phi(u)-[t\gamma(1+\varepsilon)/2\pi]\exp(-u^2/2) \leqq P\{\max(X(s)\colon 0 \leqq s \leqq t) \leqq u\}$$
$$\leqq \Phi(u)-[t\gamma(1-\varepsilon)/2\pi]\exp(-u^2/2), \qquad u \geqq 0, \quad 0 < t \leqq T.$$

PROOF. It follows from (1.1) and the expansion of $\cos t$ for small t that for every $\varepsilon > 0$ there exists $T > 0$ such that

$$(2.3) \qquad \cos[(1+\varepsilon)\gamma t] \leqq r(t) \leqq \cos[(1-\varepsilon)\gamma t], \qquad 0 < t \leqq T.$$

Since T may be taken arbitrarily small, we may suppose also that $(1+\varepsilon)\gamma T < \pi/2$. Let $Y(t)$ be the process defined in Lemma 2.1. The maximum of $Y(s)$ on the interval $[0, (1\pm\varepsilon)\gamma t]$ is equivalent to the maximum of the process $Y((1\pm\varepsilon)\gamma s)$ on the interval [0, t]. The latter process is a standard one and its covariance function is $\cos[(1\pm\varepsilon)\gamma s]$. A well-known result of Slepian [10] implies that if $X(t)$ and $Y(t)$ are standard processes, and the covariance function of X is at most equal to that of Y, then the maximum of X is stochastically larger than the maximum of Y. Applying this to the covariance functions appearing in (2.3), and noting the form of the distribution of the maximum in Lemma 2.1, we infer the double inequality (2.2).

LEMMA 2.3. *Let J be a subset of* $[0, t]$. *For every $\varepsilon > 0$, let T be the number whose existence is asserted by Lemma 2.2. Suppose that J can be covered by m intervals each of length $h > T$; then*

$$P\{\sup(X(s)\colon s \in J) > u\} \leqq m\left[1-\Phi(u)+\frac{h\gamma(1+\varepsilon)}{2\pi}\exp(-u^2/2)\right], \qquad \textit{for} \quad u > 0.$$

PROOF. By Boole's inequality we have:

$$P\{\sup(X(s)\colon s \in J) > u\} \leqq mP\{\max(X(s)\colon 0 \leqq s \leqq h) > u\}.$$

The lemma follows from this and the first inequality in (2.2).

3. Separation of the crossings. Let $(jt/n, j = 0, 1, \cdots, n)$ be $n+1$ equally spaced points in $[0, t]$. Let $M = M(t)$ be the number of upcrossings defined in Section 1, and M' the number of upcrossings of u be the finite set of values $(X(jt/n), j = 0, 1, \cdots, n)$, that is, the number of events

$$(3.1) \qquad X((j-1)t/n) \leqq u < X(jt/n), \qquad j = 1, \cdots, n,$$

that occur. The continuity of X implies that, as $n \to \infty$, M' converges to M. We are concerned with the limiting distribution of M for t and $u \to \infty$; therefore, if n is allowed to increase with t—and sufficiently rapidly—we expect M' to be asymptotically near M. The following lemma indicates a sufficient rate of increase of n with t:

LEMMA 3.1. *Let $g(t)$ be a positive function of $t > 0$ such that $g(t) \to \infty$ for $t \to \infty$; let $u = u(t)$ be given by* (1.2); *and put*

$$n = \text{integral part of } tu\, g(t). \tag{3.2}$$

With this choice of n, we have:

$$\lim_{t\to\infty} E\,|M-M'| = 0.$$

A similar statement holds for downcrossings of $-v$.

PROOF. Here and in what follows we shall refer to the well-known formula for the expected number of upcrossings of the level u:

$$E(M) = (t\gamma/2\pi)\exp(-u^2/2) = \sigma_1 \tag{3.3}$$

(cf. [5], p. 197). By stationarity, and (3.1):

$$E(M') = nP\{X(0) \leqq u < X(t/n)\}.$$

Writing

$$P\{X(0) \leqq u < X(t/n)\} = P\{X(t/n) > u\} - P\{X(0) > u, \quad X(t/n) > u\},$$

and adapting the formula in [5], page 27, we find:

$$P\{X(0) < u \leqq X(t/n)\} = (2\pi)^{-1} \int_{r(t/n)}^{1} \exp[-u^2/1+y](1-y^2)^{-\frac{1}{2}}\, dy.$$

The latter may be expressed as

$$(2\pi)^{-1}\exp(-u^2/2)\int_{r(t/n)}^{1} \exp[-u^2(1-y)/2(1+y)](1-y^2)^{-\frac{1}{2}}\, dy.$$

This is asymptotic to σ_1/n for $t \to \infty$; it can be verified by changing the variable of integration from y to $u^2(1-y)$, applying the relation $1-y^2 \sim 2(1-y)$ for $y \to 1$, and then using (1.1) and the explicit forms (1.2) and (3.2) of u and n. From this we get:

$$E(M') \to \sigma_1,$$

and so, from (3.3), $E(M-M') \to 0$. Since $M' \leqq M$, we have $|M-M'| = M-M'$, and the assertion of the lemma follows.

4. An inequality for the distribution of the extreme values on a set of intervals. Let [0, t] be a fixed interval, and for an integer $n \geqq 1$, let G_t be the set of numbers of the form jt/n, for $j = 0, 1, \cdots, n$. For $\varepsilon > 0$ and a positive integer m, let $I_1, \cdots, I_m$ be closed disjoint subintervals of $[0, t]$ which are indexed in natural order:

$$\text{If } j < k \text{ and } x \in I_j,\ y \in I_k, \text{ then } x < y;$$

and are separated from each other by intervals of length at least ε:

$$\text{If } j \neq k \text{ and } x \in I_j,\ y \in I_k, \text{ then } |x-y| \geqq \varepsilon.$$

Put

$$U_j = \max(X(s)\colon s \in I_j \cap G_t),\ V_j = \min(X(s)\colon s \in I_j \cap G_t), \qquad j = 1, \cdots, m;$$

these are extremes of finitely many random variables. One of the main points in the subsequent derivation of the limiting distribution of the numbers of crossings is to show that, for t and $n \to \infty$, the pairs (U_j, V_j) are, in a sense, asymptotically independent. This is based on the following:

LEMMA 4.1. *Let* $\phi(x, y; \rho)$ *be the standard bivariate Gaussian density with correlation coefficient* ρ:

$$\phi(x, y; \rho) = \frac{1}{2\pi(1-\rho^2)^{\frac{1}{2}}} \exp\left\{\frac{x^2 - 2\rho xy - y^2}{2(1-\rho^2)}\right\}. \tag{4.1}$$

If $u_1, \cdots, u_m$ *are variables each assuming only the values* u *and* $+\infty$, *and* $v_1, \cdots, v_m$ *similarly assuming just* v *and* ∞, *then*

$$\left|P\{U_j \leqq u_j, V_j \geqq -v_j, j = 1, \cdots, m\} - \textstyle\prod_{j=1}^m P\{U_j \leqq u_j, V_j \geqq -v_j\}\right| \tag{4.2}$$
$$\leqq n \textstyle\sum_{\{j : (n\varepsilon/t) \leqq j \leqq n\}} \int_{-|r(jt/n)|}^{|r(jt/n)|} [\phi(u, u; y) + \phi(v, v; y) + 2\phi(u, v; y)]\, dy.$$

PROOF. We generalize the method in [3], which was further developed in [5], page 268. Let (r_{ij}) and (s_{ij}) be $k \times k$ nonsingular covariance matrices with 1's on their diagonals; then, for each λ, $0 \leqq \lambda \leqq 1$, the matrix $(\lambda r_{ij} + (1-\lambda)s_{ij})$ is of the same kind. Let $\hat{\phi}(x_1, \cdots, x_k; \lambda)$ be the k-variate Gaussian density with all means 0 and covariance matrix $(\lambda r_{ij} + (1-\lambda)s_{ij})$. From the well-known partial differential equation for $\hat{\phi}$ ([5] page 26), we get

$$\frac{\partial \hat{\phi}}{\partial \lambda}(x_1, \cdots, x_k; \lambda) = \sum_{i,j} \frac{\partial^2 \hat{\phi}}{\partial x_i \partial x_j}(r_{ij} - s_{ij});$$

thus:

$$\hat{\phi}(x_1, \cdots, x_k; 1) - \hat{\phi}(x_1, \cdots, x_k; 0) = \sum_{i,j} (r_{ij} - s_{ij}) \int_0^1 \frac{\partial^2 \hat{\phi}}{\partial x_i \partial x_j}\, d\lambda. \tag{4.3}$$

On each side of the equation integrate with respect to x_i over $-\omega_i \leqq x_i \leqq y_i$, $i = 1, \cdots, k$. We shall find a bound on the integral of the right-hand side. Integration of the term

$$\int_0^1 \frac{\partial^2 \hat{\phi}}{\partial x_i \partial x_j}\, d\lambda$$

is carried out first with respect to the variables x_i and x_j. When the order of integration is interchanged with that over λ, the integral becomes

$$\textstyle\int_0^1 [\hat{\phi}(\cdots y_i \cdots y_j \cdots; \lambda) - \hat{\phi}(\cdots y_i \cdots -\omega_j \cdots; \lambda)$$
$$- \hat{\phi}(\cdots -\omega_i \cdots y_j \cdots; \lambda) + \hat{\phi}(\cdots -\omega_i \cdots -\omega_j \cdots; \lambda)]\, d\lambda;$$

this is not more than

$$\textstyle\int_0^1 [\hat{\phi}(\cdots y_i \cdots y_j \cdots; \lambda) + \hat{\phi}(\cdots y_i \cdots -\omega_j \cdots; \lambda) + \hat{\phi}(\cdots -\omega_i \cdots y_j \cdots; \lambda) \tag{4.4}$$
$$+ \hat{\phi}(\cdots -\omega_i \cdots -\omega_j \cdots; \lambda)]\, d\lambda.$$

Now integrate over the remaining $k-2$ variables distinct from x_i and x_j. Since the integrand in (4.4) is nonnegative, the integral cannot decrease when the limits $-\omega_k$ and y_k are replaced by $-\infty$ and $+\infty$, respectively. When this integration is performed under the integral sign in (4.4) the integrand is changed to the sum of the bivariate densities,

$$\phi(y_i, y_j; \lambda r_{ij}+(1-\lambda)s_{ij})+\phi(y_i, -\omega_j; \lambda r_{ij}+(1-\lambda)s_{ij}) \\ +\phi(-\omega_i, y_j; \lambda r_{ij}+(1-\lambda)s_{ij})+\phi(-\omega_i, -\omega_j; \lambda r_{ij}+(1-\lambda)s_{ij}),$$

where ϕ is given by (4.1). From this and (4.3) we get:

$$(4.5) \qquad \int_{-\omega_k}^{y_k}\cdots\int_{-\omega_1}^{y_1} \hat{\phi}(x_1, \cdots, x_k; 1)-\hat{\phi}(x_1, \cdots, x_k; 0)\, dx_1 \cdots dx_k \\ \leqq \sum_{i,j} \Big| \int_{s_{ij}}^{r_{ij}}[\phi(y_i, y_j; \lambda)+\phi(y_i, -\omega_j; \lambda)+\phi(-\omega_i, y_j; \lambda) \\ +\phi(-\omega_i, -\omega_j; \lambda)]\, d\lambda. \Big|$$

We apply this inequality to the joint distribution of the random variables $X(s)$, where $s\in I_j \cap G_t$, $j = 1, \cdots, m$, and where $u_1, \cdots, u_m$ and $v_1, \cdots, v_m$ are in places of $\omega_1, \cdots, \omega_m$ and $y_1, \cdots, y_m$, respectively. Let (r_{ij}) be the covariance matrix of these random variables, and let (s_{ij}) be the matrix obtained from it by the alteration $EX(s)X(s') = 0$ if $s\in I_i$, $s'\in I_j$, $i\neq j$; in other words, the variables belonging to different intervals are independent under (s_{ij}). Now u_i assumes only the values u and $+\infty$, and $\phi(\infty, v; \rho) = 0$; furthermore, an analogous relation holds for v_i and v; thus

$$(4.6) \qquad \phi(u_i, u_j; \rho) \leqq \phi(u, u; \rho), \qquad \phi(u_i,-v_j; \rho) \leqq \phi(u, -v; \rho), \\ \phi(-v_i, -v_j; \rho) \leqq \phi(-v, -v; \rho).$$

The left-hand side of (4.2) is of the form of the integral on the corresponding side of (4.5). By the inequalities (4.6), the integrands on the right-hand side of (4.5) are bounded by

$$\phi(u, u; \lambda)+\phi(u, -v; \lambda)+\phi(-v, u; \lambda)+\phi(-v, -v; \lambda).$$

In this application of (4.5) we note that $r_{ij} = s_{ij}$ or 0 depending on whether or not the corresponding pair of random variables belongs to the same interval I; further we observe that $r_{ij}\neq s_{ij}$ only if the indices of the corresponding pair of variables are separated by a distance at least equal to ε on the t-axis; thus, by (4.5), the left-hand side of (4.2) is at most equal to

$$(4.7) \qquad \sum_{\{i,j:\, i,j=1,\ldots,n,\, |i-j|\geqq n\varepsilon/t\}} \Big| \int_0^{r((i-j)t/n)} [\phi(u, u; \lambda) \\ +\phi(-v, -v; \lambda)+2\phi(u, -v; \lambda)]\, d\lambda \Big|$$

which, by the nonnegativity of ϕ, and the form (4.1) of ϕ, is not more than

$$n\sum_{\{j:\, n\varepsilon/t\leqq j\leqq n\}} \int_{-|r(jt/n)|}^{|r(jt/n)|} [\phi(u, u; \lambda)+\phi(v, v; \lambda)+2\phi(u, v; \lambda)]\, d\lambda.$$

This completes the proof.

5. The convergence of the right-hand side of the inequality to 0. One interesting point about the inequality (4.2) is that the right-hand side, unlike the left, does not depend on m or on the relative sizes of the subintervals. The right-hand side will be shown to converge to 0 as $t \to \infty$ and as n increases with t at a prescribed rate.

LEMMA 5.1. *If* $r(t)$ *satisfies* (1.1) *and* (1.3), *and if* $g(t)$ *is a positive function tending to* $+\infty$ *in such a way that*

$$\lim_{t\to\infty} g(t)t^{-p} = 0, \qquad \textit{for every } p > 0, \tag{5.1}$$

$$\lim_{t\to\infty} (\log g(t)/\log t) = 0, \tag{5.2}$$

$$\lim_{t\to\infty} g^2(t)\cdot \sup_{s \geq t^{\beta/2}} r(s)\cdot \log s = 0 \quad \textit{for all} \quad \beta > 0, \tag{5.3}$$

then the right-hand side of (4.2) *tends to* 0 *if* n *is defined as in* (3.2). (It is clear that (5.3) is consistent with the condition (1.3).)

PROOF. Under (1.3) $r(t)$ is bounded away from 1 outside every neighborhood of 0; indeed, the zeros of $1-r(t)$ form an additive subgroup of the reals, which, under (1.3), consists only of $\{0\}$. This implies that for any $\varepsilon > 0$—in particular, the ε appearing on the right-hand side of (4.2)—there exists δ, $0 < \delta < 1$, such that

$$\sup\,(|r(s)| : s > \varepsilon) < \delta. \tag{5.4}$$

First we prove the convergence to 0 of

$$n \sum_{\{j:\, n\varepsilon/t \leq j \leq n\}} \int_{-|r(jt/n)|}^{|r(jt/n)|} \phi(u, u; \lambda)\, d\lambda, \tag{5.5}$$

which, by (4.1), is dominated by

$$(n/\pi) \sum_{\{j:\, n\varepsilon/t \leq j \leq n\}} |r(jt/n)|(1-r^2(jt/n))^{-\frac{1}{2}} \exp\{-u^2/1+|r(jt/n)|\},$$

which, by (5.4), is of the order

$$n \sum_{\{j:\, n\varepsilon/t \leq j \leq n\}} |r(jt/n)| \exp\{-u^2/1+|r(jt/n)|\}. \tag{5.6}$$

Let β be an arbitrary number satisfying

$$0 < \beta < (1-\delta)/(1+\delta). \tag{5.7}$$

Split (5.6) into two subsums, the first over indices $j < n^\beta$ and the second over $j \geq n^\beta$. By (5.4), the part corresponding to the first subsum is of the order

$$n^{\beta+1} \exp(-u^2/1+\delta),$$

which, by (1.2), (5.1) and (5.7), converges to 0 as $t \to \infty$. Next we evaluate the part of (5.6) with indices greater than n^β. From the inequality $1-r \leq (1+r)^{-1}$, $|r| < 1$, this portion of (5.6) is not more than

$$n \exp(-u^2) \sum_{\{j:\, n^\beta \leq j \leq n\}} |r(jt/n)| \exp[u^2|r(jt/n)|]. \tag{5.8}$$

Assumption (5.1) implies that

$$tn^{\beta-1} > t^{\beta/2} \tag{5.9}$$

for all sufficiently large t; therefore, by (1.2) and (1.3):

$$\sup_{j \geq n^\beta} u^2 |r(jt/n)| \leqq \text{constant}(\log t^{\beta/2} \cdot \sup_{s \geq t^{\beta/2}} |r(s)|)$$

for all sufficiently large t. This and condition (1.3) imply that the second exponential in (5.8) converges to 1 uniformly in j as $t \to \infty$; therefore, it suffices to estimate

$$n \exp(-u^2) \textstyle\sum_{\{j:\, n^\beta \leqq j \leqq n\}} |r(jt/n)|,$$

which, by (1.2), is of the order

(5.10) $$nt^{-2} \textstyle\sum_{\{n^\beta \leqq j \leqq n\}} |r(jt/n)|.$$

This is not more than

(5.11) $$nt^{-2} \textstyle\sum_{\{n^\beta \leqq j \leqq n\}} 1/\log(jt/n) \cdot \sup_{s \geq tn^{\beta-1}} |r(s)| \log s.$$

By (5.9):

$$\sup_{s \geq tn^{\beta-1}} |r(s)| \cdot \log(s) \leqq \sup_{s \geq t^{\beta/2}} |r(s)| \log s, \qquad \textit{for all sufficiently large } t;$$

furthermore:

$$\begin{aligned} nt^{-2} \textstyle\sum_{\{n^\beta \leqq j \leqq n\}} 1/\log(jt/n) &\sim nt^{-2} \textstyle\sum_{\{n^\beta \leqq j \leqq n\}} 1/\log j \qquad \text{(by (5.2))} \\ &\leqq nt^{-2} \textstyle\int_2^{n+1} (\log x)^{-1}\, dx \sim (n/t)^2/\log n \qquad \text{(L'Hospital's rule)} \\ &\sim 2(nt/u)^2 = 2g^2(t) \qquad \text{(by (1.2), (3.2), (5.2))}; \end{aligned}$$

thus, by (5.3), the expression (5.11) converges to 0 as $t \to \infty$. The proof of the convergence of (5.5) is complete.

When v is substituted for u in (5.5), the same argument shows that this expression also converges to 0.

In order to complete the proof of the lemma we show that (5.5) converges to 0 even when $\phi(u, v; \lambda)$ is substituted for $\phi(u, u; \lambda)$. By (4.1):

$$\phi(u, v; \lambda) = \phi(u, u; \lambda) \exp\left[\frac{u(u-v)}{1+\lambda} - \frac{(u-v)^2}{2(1-\lambda^2)}\right].$$

From (1.2):

$$u = (2 \log t)^{\frac{1}{2}} + \frac{\log(\gamma/2\pi\sigma_1)}{(2 \log t)^{\frac{1}{2}}} + O\left(\frac{1}{\log t}\right), \qquad t \to \infty,$$

and v has a similar expansion; therefore:

$$u(u-v) \to \log(\sigma_2/\sigma_1), \qquad (u-v)^2 \to 0.$$

With these estimates the convergence of the expression (5.5) with $\phi(u, v; \lambda)$ can be demonstrated in the same way as for $\phi(u, u; \lambda)$. The proof of the lemma is now complete.

Now we state and prove a version of Lemma 5.1 when the hypothesis (1.3) is replaced by (1.4). The conditions (5.2) and (5.3) are not needed. ((5.3) is not necessarily consistent with (1.4).)

LEMMA 5.2. *If $r(t)$ satisfies* (1.1) *and* (1.4), *and if $g(t) \to \infty$ in such a way that* (5.1) *holds, then the right-hand side of* (4.2) *tends to* 0 *if n is defined as in* (3.2).

PROOF. Under (1.4), $r(t)$ is the Fourier transform of a square-integrable function $f(\lambda)$, which is necessarily even:

$$r(t) = 2\int_0^\infty \cos \lambda t\, f(\lambda)\, d\lambda. \tag{5.12}$$

f is the spectral density. By the well-known property of the convolution, $r^2(t)$ is the Fourier transform of the L_1 function

$$f^*(\lambda) = \int_{-\infty}^{\infty} f(\lambda-w)f(w)\, dw; \tag{5.13}$$

thus, by the Riemann-Lebesgue Lemma, $r^2(t) \to 0$ for $t \to \infty$ and so, of course,

$$r(t) \to 0 \quad \text{for} \quad t \to \infty. \tag{5.14}$$

As in the proof of Lemma 4.1, this condition implies (5.4); thus, it suffices to show that (5.6) converges to 0. Applying condition (5.1) as in the previous proof, we find it sufficient to prove that (5.8) converges to 0.

By (1.2) and (5.14) the second exponential in (5.8) is, for every $q > 0$, of the order $t^q (t \to \infty)$ uniformly in $j \geqq n^\beta$. Equations (1.2) and (3.2) imply that $n \exp(-u^2)$ is of the order $ug(t)/t$; thus, (5.8) is of the order

$$t^{q-1}ug(t)\cdot \sum_{\{n^\beta \leqq j \leqq n\}} |r(jt/n)|. \tag{5.15}$$

We estimate the sum in (5.15):

$$\begin{aligned}\sum_{\{n^\beta \leqq j \leqq n\}} |r(jt/n)| &\leqq [n\textstyle\sum_{j=1}^n r^2(jt/n)]^{\frac12} \qquad \text{(Cauchy-Schwarz)}\\ &= [2n\textstyle\int_0^\infty \sum_{j=1}^n \cos(\lambda tj/n) f^*(\lambda)\, d\lambda]^{\frac12} \qquad \text{(by (5.12), (5.13))}\\ &\leqq \left\{ n\int_0^\infty \frac{\sin[\lambda t(1+1/2n)]}{\sin(\lambda t/2n)} f^*(\lambda)\, d\lambda\right\}^{\frac12}\end{aligned}$$

(by a well-known summation formula).

By a standard decomposition of the domain of integration into bounded and unbounded parts, and by the relation $t/n \to 0$, it can be shown that the last expression displayed above is asymptotic to

$$\left\{ n\int_0^\infty \frac{\sin[\lambda t(1+1/2n)]}{\lambda t/2n} f^*(\lambda)\, d\lambda\right\}^{\frac12},$$

which, for arbitrary β, $0 < \beta < 1$, is at most

$$\left\{ n\int_0^\infty \frac{\lambda^\beta t^\beta (1+1/2n)^\beta}{\lambda t/2n} f^*(\lambda)\, d\lambda\right\}^{\frac12} \qquad (\text{by } |\sin\theta| \leqq |\theta|^\beta)$$

$$\sim \left\{ 2n^2 t^{\beta-1} \int_0^\infty \lambda^{\beta-1} f^*(\lambda)\, d\lambda\right\}^{\frac12}.$$

The last integral is finite because f^*, as the convolution of square-integrable functions, is continuous. It follows from the calculations just completed that (5.15) is of the order

$$t^{q-1} u g(t) \cdot n t^{(\beta-1)/2}, \qquad t \to \infty.$$

Since q and β are arbitrary, and $g(t)$ satisfies (5.1), the expression above tends to 0 as $t \to \infty$. The proof of the lemma is now completed as that of the previous one.

6. The magnitude of the probability of both an upcrossing of u and a downcrossing of $-v$ in an interval of fixed length.

LEMMA 6.1. *Let M and N be the numbers of upcrossings of u and downcrossings of $-v$, respectively, by $X(s)$, $0 \leqq s \leqq T$, where $T > 0$ is fixed; then, under* (1.3) *or* (1.4):

$$\lim_{t\to\infty} tP\{M \geqq 1, N \geqq 1\} = 0. \tag{6.1}$$

PROOF. For simplicity of notation take T equal to 1. Let M' and N' be the numbers of upcrossings of u and downcrossings of $-v$, respectively, by the finite sequence $X(jt/n)$, $j = 1, \cdots, [n/t]$, where n is given by (3.2) (cf. (3.1)). Since $\{M' \geqq 1\}$ and $\{N' \geqq 1\}$ imply $\{M \geqq 1\}$ and $\{N \geqq 1\}$, respectively, it follows that

$$\begin{aligned} t\big|P\{M \geqq 1, N \geqq 1\} - P\{M' \geqq 1, N' \geqq 1\}\big| &\leqq t[P\{M - M' \geqq 1\} \\ &\quad + P\{N - N' \geqq 1\}] \leqq tE(M - M') + tE(N - N'). \end{aligned} \tag{6.2}$$

From the calculation in the proof of Lemma 3.1 it follows that the last member of (6.2) converges to 0 as $t \to \infty$; thus, the relation

$$\lim_{t\to\infty} tP\{M' \geqq 1, N' \geqq 1\} = 0 \tag{6.3}$$

is sufficient for (6.1).

If $X(jt/n)$, $j = 0, 1, \cdots, [n/t]$ has at least one upcrossing of u and at least one downcrossing of $-v$, then for some pair i, j either

$$X(it/n) > u \quad \text{and} \quad X(jt/n) < -v, \qquad \text{or}$$

$$X(it/n) < -v \quad \text{and} \quad X(jt/n) > u;$$

thus, by the stationarity of the process and the symmetry of the standard bivariate Gaussian distribution it follows that

$$tP\{M' \geqq 1, N' \geqq 1\} \leqq 2n \textstyle\sum_{j=1}^{[n/t]} P\{X(0) < -v, X(jt/n) > u\}.$$

If, in the definitions of u and v in (1.2), σ_1 is assumed to be less than or equal to σ_2, then $u > v$ for all t; therefore, it is sufficient for the proof of (6.3) that

$$n \textstyle\sum_{j=1}^{[n/t]} P\{X(0) < -u, X(jt/n) > u\} \tag{6.4}$$

tend to 0 for $t \to \infty$. Of $\sigma_2 \leqq \sigma_1$, then u is replaced by v in the expression (6.4) and the reasoning is the same.

Since $r(0) = 1$, there exists $\varepsilon > 0$ sufficiently small so that

$$r(s) \geqq \tfrac{1}{2} \quad \text{for} \quad |s| \leqq \varepsilon. \tag{6.5}$$

The event $\{X(0) < -u,\ X(jt/n) > u\}$ implies that $X(jt/n) - X(0) > 2u$. Since $X(jt/n) - X(0)$ has a Gaussian distribution with mean 0 and variance $2(1 - r(jt/n))$, it follows from the well-known estimate

$$1 - \Phi(x) \leqq ((2\pi)^{\frac{1}{2}} x)^{-1} \exp(-x^2/2), \qquad x > 0, \tag{6.6}$$

that

$$P\{X(jt/n) - X(0) > 2u\} \leqq \exp[-u^2/1 - r(jt/n)]/2u\pi^{\frac{1}{2}}. \tag{6.7}$$

By the definition (1.2) and the inequalities (6.5) and (6.6), the portion of (6.4) corresponding to terms of index $j \leqq n\varepsilon/t$,

$$n \textstyle\sum_{j=1}^{[n\varepsilon/t]} P\{X(0) < -u,\ X(jt/n) > u\},$$

is of the order $n^2 u t^{-5}$, which converges to 0 if g increases sufficiently slowly, e.g., subject to (5.1).

Now we estimate the other portion of (6.4). Put

$$\delta = \sup_{|s| \geqq \varepsilon} |r(s)|;$$

then, by the argument in the proofs of Lemmas 5.1 and 5.2, δ is less than 1. By a variation of the identity in [5], page 27 (cf. proof of Lemma 3.1), we get

$$P\{X(0) < -u,\ X(jt/n) > u\} = [1 - \Phi(u)]^2 + \textstyle\int_0^{-r(jt/n)} \phi(u, u; y)\, dy. \tag{6.8}$$

By the definition (1.2) of u and from the inequality (6.6), we obtain

$$1 - \Phi(u) = O(1/tu), \qquad t \to \infty. \tag{6.9}$$

The definition of δ implies

$$\left| \int_0^{-r(jt/n)} \phi(u, u; y)\, dy \right| \leqq \int_{-\delta}^{\delta} \phi(u, u; y)\, dy \leqq \frac{\delta}{\pi(1-\delta^2)^{\frac{1}{2}}} \exp(-u^2/1+\delta),$$
$$\text{for} \quad j > n\varepsilon/t;$$

by (1.2), the last member is of the order $t^{-2/1+\delta}$. From this, (6.8), and (6.9), it follows that

$$n \textstyle\sum_{\{j:\, n\varepsilon/t \leqq j \leqq n/t\}} P\{X(0) < -u,\ X(jt/n) > u\} = O(n^2 u^{-2} t^{-3})$$
$$+ O(n^2 t^{-(3+\delta)/(1+\delta)}).$$

The last expressions converge to 0 if g increases slowly enough, e.g., as in (5.1). This completes the proof that (6.4) converges to 0 as $t \to 0$.

7. Limiting distribution of the maximum and absolute maximum. We shall show that $\max(X(s)\colon 0 \leqq s \leqq t)$ and $\max(|X(s)|\colon 0 \leqq s \leqq t)$ have the familiar limiting extreme value distribution for $t \to \infty$. The result about $\max X(s)$ is a special case of Pickands' [7] limit theorem; however, his method does not directly extend to $\max |X(s)|$, so that we shall give another proof for $\max X(s)$ and indicate why it is also valid for $\max |X(s)|$.

THEOREM 7.1. *If X is a standard process satisfying either* (1.3) *or* (1.4), *then*

$$\lim_{t\to\infty} P\{\max(X(s)\colon 0 \leqq s \leqq t) \leqq u\} = e^{-\sigma_1}, \tag{7.1}$$

$$\lim_{t\to\infty} P\{\min(X(s)\colon 0 \leqq s \leqq t) \geqq -u\} = e^{-\sigma_1}, \tag{7.2}$$

$$\lim_{t\to\infty} P\{\max(|X(s)|\colon 0 \leqq s \leqq t) \leqq u = e^{-2\sigma_1}. \tag{7.3}$$

PROOF. First we note that the equivalence of (7.1) with the result of Pickands can be verified by putting $x = -\log \sigma_1$ and using the asymptotic expansion of u at the end of the proof of Lemma 5.1.

Since $-X(s)$ has the same finite-dimensional distributions as $X(s)$, (7.2) follows from (7.1). We shall now prove (7.1) and then (7.3).

The positive t-axis is decomposed into a series of alternating segments of lengths of two magnitudes. Let β and τ be arbitrary numbers satisfying $0 < \beta < 1$, $0 < \tau < 1$, and I_k and J_k the intervals

$$I_k = [k\tau, (k+1-\beta)\tau], \qquad J_k = [(k+1-\beta)\tau, (k+1)\tau],$$

of lengths $\tau(1-\beta)$ and $\tau\beta$, respectively, for $k = 0, 1, \cdots$; finally, put

$$I = \bigcup_{k\geqq 0} I_k, \qquad J = \bigcup_{k\geqq 0} J_k.$$

We shall prove that the limiting distribution of $\max X(s)$ on $[0, t]$ is arbitrarily close to that of the maximum over the subset $I\cap[0, t]$ if β is chosen sufficiently small: *for $\varepsilon > 0$, the inequality*

$$\limsup_{t\to\infty} |P\{\max(X(s)\colon s \in I\cap[0, t]) \leqq u\} - P\{\max(X(s)\colon 0 \leqq s \leqq t) \leqq u\}| \leqq \beta(1+\varepsilon)\sigma_1 \tag{7.4}$$

holds for all sufficiently small $\beta > 0$. Since $[0, t]$ is the union of $I\cap[0, t]$ and $J\cap[0, t]$, the event $\{\max(X(s)\colon 0 \leqq s \leqq t) \leqq u\}$ implies $\{\max(X(s)\colon s \in I\cap[0, t]) \leqq u\}$, and the difference between their probabilities is at most

$$P\{\max(X(s)\colon s \in J\cap[0, t]) > u\}.$$

The set $J\cap[0, t]$ consists of at most t intervals of length β or less. If β is sufficiently small, then, by Lemma 2.3, the last probability displayed above is not more than

$$t[1-\Phi(u)+\frac{\beta\gamma(1+\varepsilon)}{2\pi}\exp(-u^2/2)],$$

which, by (6.9) and (1.2), is asymptotic to the right-hand side of (7.4).

Now we replace the maximum on $I \cap [0, t]$ by that over $I \cap G_t$, where G_t is the finite subset defined in Section 4. We prove:

(7.5) $\quad \lim_{t\to\infty} \left|P\{\max (X(s): s \in I \cap G_t) \leqq u\} - P\{\max (X(s): I \cap [0, t]) \leqq u\}\right| = 0.$

Since $G_t \subset [0, t]$, the event $\{\max (X(s): s \in I \cap [0, t]) \leqq u\}$ implies $\{\max (X(s): s \in I \cap G_t) \leqq u\}$, and the difference between their probabilities is less than or equal to the probability of the event:

$X(s)$, $0 \leqq s \leqq t$, has at least one more upcrossing of u than the sequence

$$X(jt/n), \qquad j = 1, \cdots, n.$$

By Lemma 3.1 the probability of this event, which is at most $E(M - M')$, converges to 0 as $t \to \infty$.

The maximum on $I \cap G_t$ is representable as the maximum of approximately $[t/\tau]$ sub-maxima

$$U_j = \max (X(s): s \in I_j \cap G_t).$$

(The last interval may be incomplete because t may fall in it, but this does not affect the limiting values in the following calculations.) By Lemma 4.1 and with the appropriate function g in Lemmas 5.1 and 5.2, we may, in the derivation of the limiting distribution, suppose that the variables U_j are mutually independent. By the "separation of crossings" argument following (7.5), the limiting distribution of the maximum of the (independent) sub-maxima U_j is the same as that of the maximum of independent random variables each distributed as

$$\max (X(s): 0 \leqq s \leqq \tau(1-\beta)).$$

The maximum of $[t/\tau)$ such independent random variables has the distribution function

(7.6) $\qquad P^{[t/\tau]} \{\max (X(s): 0 \leqq s \leqq \tau(1-\beta)) \leqq u\}.$

For $\varepsilon > 0$ chosen as in (7.4), let τ be smaller than the number T in Lemma 2.2: by that lemma, the distribution (7.6) is, for $u > 0$, bounded above and below (corresponding to $-$ and $+$, respectively) by

$$\left\{\Phi(u) - \frac{\tau\gamma(1-\beta)(1+\varepsilon)}{2\pi} \exp (-u^2/2)\right\}^{[t/\tau]}.$$

By (1.2) and (6.9), this converges to $\exp [-\sigma_1(1-\beta)(1\pm\varepsilon)]$ for $t \to \infty$; thus, by (7.4), (7.5) and (7.6):

$$\begin{aligned} \exp [-\sigma_1(1-\beta)(1+\varepsilon)] - \beta\sigma_1(1+\varepsilon) &\leqq \liminf_{t\to\infty} P\{\max (X(s): 0 \leqq s \leqq t) \leqq u\} \\ &\leqq \limsup_{t\to\infty} P\{\max (X(s): 0 \leqq s \leqq t) \leqq u\} \\ &\leqq \exp [-\sigma_1(1-\beta)(1-\varepsilon)] + \beta\sigma_1(1+\varepsilon). \end{aligned}$$

Since ε and β are arbitrary, (7.1) follows.

In proving (7.3), we first show that if max $|X(s)|$ exceeds u, then it is very *unlikely that both* max $X(s)$ *and* $|\min X(s)|$ *exceed* u. It is elementary that

$$(7.7)\qquad P\{\max |X(s)| > u\} = P\{\max X(s) > u\}+P\{\min X(s) < -u\} - P\{\max X(s) > u, \min X(s) < -u\}.$$

Since $-X(s)$ is stochastically equivalent to $X(s)$ we have

$$(7.8)\qquad P\{\max X(s) > u\} = P\{\min X(s) < -u\}.$$

Write the last probability in (7.7) as

$$(7.9)\qquad P\{\max X(s) > u, \min X(s) < -u, |X(0)| < u\} + P\{\max X(s) > u, \min X(s) < -u, |X(0)| \geqq u\}.$$

All maxima are now over $0 \leqq s \leqq T$, for fixed $T > 0$. The sum (7.9) is less than

$$P\{\text{at least one upcrossing of } u \text{ and one downcrossing of } -u \text{ by } X(s), 0 \leqq s \leqq T\}+P\{|X(0)| \geqq u\},$$

which, by Lemma 6.1 and by (6.9), is of smaller order than $1/t$. It now follows from (7.7) and (7.8) that

$$(7.10)\qquad tP\{\max (|X(s)|: 0 \leqq s \leqq T) > u\} \sim 2tP\{\max (X(s): 0 \leqq s \leqq T) > u\}, \quad \text{for} \quad t \to \infty.$$

From the relation $-u \sim \log (1-u)$, for $u \to 0$, we get:

$$tP\{\max (X(s): 0 \leqq s \leqq \tau(1-\beta)) > u\} \sim -\log P^t \{\max (X(s): 0 \leqq s \leqq \tau(1-\beta)) \leqq u\}.$$

The latter, by the estimates following (7.6), is asymptotically equal to $-\sigma_1\tau(1-\beta)$ except for a multiple of $(1 \pm \varepsilon)$. Take $T = \tau(1-\beta)$ in (7.10); then

$$(7.11)\qquad tP\{\max (|X(s)|: 0 \leqq s \leqq \tau(1-\beta)) > u\} \to -2\sigma_1\tau(1-\beta), \quad \textit{up to a multiple of } (1 \pm \varepsilon).$$

The logarithmic expansion above now implies that (7.11) is equivalent to

$$(7.12)\qquad P^{[t/\tau]} \{\max (|X(s)|: 0 \leqq s \leqq \tau(1-\beta)) \leqq u\} \to \exp [-2\sigma_1(1-\beta)], \quad \textit{up to a multiple of } \exp (\pm 2\sigma_1 \varepsilon).$$

The proof of (7.3) is now very similar to that of (7.1). The maximum of $|X(s)|$ is broken up into maxima over I and J. From (7.7) and (7.8) we get

$$P\{\max |X(s)| > u\} \leqq 2P\{\max X(s) > u\}.$$

This implies, as in (7.4), that the maximum of $|X(s)|$ over $[0, t]$ is like the maximum over $[0, t]\cap I$. By the separation principle (Lemma 3.1), applied to upcrossings of u and downcrossings of $-u$, the maximum over $[0, t]\cap I$ is replaced by that over

$[0, t] \cap G_t$. By the argument preceding (7.6) the maximum over $[0, t] \cap G_t$ has a limiting distribution equal to that of the *left*-hand side of (7.12); thus, by (7.12) and the arbitrariness of ε and β, the limiting distribution is given by (7.3). The proof of the theorem is complete.

COROLLARY 7.1. *For* $\theta > 0$:

$$\lim_{t\to\infty} P\{\max (X(s): 0 \leqq s \leqq \theta t) \leqq u\} = \exp(-\theta\sigma_1),$$

$$\lim_{t\to\infty} P\{\max (|X(s)|: 0 \leqq s \leqq \theta t) \leqq u\} = \exp(-2\theta\sigma_1).$$

PROOF. This is a consequence of the fact that the limiting distributions are the limits of (7.6) and (7.12), respectively: a scale change in the time parameter corresponds to an exponential change in the limiting distribution.

8. Limiting independence and Poisson distribution of the numbers of upcrossings and downcrossings. In applying the asymptotic independence property of the extreme values (Lemmas 4.1, 5.1 and 5.2) to the limiting distribution (Theorem 7.1), we used intervals I_j and J_j of *fixed* length. As remarked in the beginning of Section 5, the asymptotic independence does not depend on the number of intervals or their sizes. In deriving the limiting distribution of the numbers of crossings, we shall apply Lemma 4.1 with a fixed number of subintervals whose lengths are proportional to t.

For an arbitrary positive integer m, cut $[0, t]$ into m subintervals of equal length t/m; then, for an arbitrary β, $0 < \beta < 1$, clip the segment of length $\beta t/m$ from the right end of each one, leaving m intervals $I_1, \cdots, I_m$, which are taken as closed:

$$I_j = [(j-1)t/m, \qquad j(1-\beta)t/m], \qquad j = 1, \cdots, m.$$

The set G_t is defined as before (Section 4). Define the random variables:

$$\begin{aligned} \xi_j &= 1 \quad \text{if } \max (X(s): s \in I_j \cap G_t) > u, \\ &= 0 \quad \text{otherwise}; \\ \eta_j &= 1 \quad \text{if } \min (X(s): s \in I_j \cap G_t) \leqq -v, \\ &= 0 \quad \text{otherwise.} \end{aligned} \qquad j = 1, \cdots, m. \tag{8.1}$$

LEMMA 8.1. *If* x_j *and* y_j, $j = 1, \cdots, m$ *are variables assuming only the values* 0 *and* 1, *then, under* (1.3) *or* (1.4),

$$\lim_{t\to\infty} \left|P\{\xi_j = x_j, \eta_j = y_j, j = 1, \cdots, m\} - \prod_{j=1}^{m} P\{\xi_j = x_j, \eta_j = y_j\}\right| = 0 \tag{8.2}$$

or all 2^m *values of* (x_j, y_j).

PROOF. It suffices to prove a statement formally weaker than (8.2), involving only x_j and y_j values equal to 0; precisely, we prove: *For any decomposition of the set* 1, $\cdots$, m *into disjoint subsets* A, B *and* C:

$$\lim_{t\to\infty} \big|P\{\xi_i = 0, \eta_i = 0, i \in A; \xi_j = 0, j \in B; \eta_k = 0, k \in C\} - \prod_{i\in A} P\{\xi_i = 0, \eta_i = 0\}\cdot\prod_{j\in B} P\{\xi_j = 0\}\cdot\prod_{k\in C} P\{\eta_k = 0\}\big| = 0. \tag{8.3}$$

A typical example demonstrates that (8.3) implies (8.2): If

$$\big|P\{\xi_1 = 0, \eta_1 = 0, \xi_2 = 0, \eta_2 = 0\} - P\{\xi_1 = 0, \eta_1 = 0\}\cdot P\{\xi_2 = 0, \eta_2 = 0\}\big| \to 0,$$

and

$$\big|P\{\xi_1 = 0, \eta_1 = 0, \xi_2 = 0\} - P\{\xi_1 = 0, \eta_1 = 0\}\cdot P\{\xi_2 = 0\}\big| \to 0,$$

then

$$\big|P\{\xi_1 = 0, \eta_1 = 0, \xi_2 = 0, \eta_1 = 1\} - P\{\xi_1 = 0, \eta_1 = 0\}P\{\xi_2 = 0, \eta_2 = 1\}\big| \to 0$$

because

$$P\{\xi_1 = 0, \eta_1 = 0, \xi_2 = 0, \eta_2 = 1\} = P\{\xi_1 = 0, \eta_1 = 0, \xi_2 = 0\} - P\{\xi_1 = 0, \eta_1 = 0, \xi_1 = 0, \xi_2 = 0\}, \quad \text{and} \quad P\{\xi_1 = 0, \eta_1 = 0\} \times P\{\xi_2 = 0, \eta_2 = 1\} = P\{\xi_1 = 0, \eta_1 = 0\}\cdot P\{\xi_2 = 0\} - P\{\xi_1 = 0, \eta_1 = 0\}P\{\xi_2 = 0, \eta_2 = 0\}.$$

To complete the proof, we note that (8.3) is a consequence of Lemmas 4.1 (with appropriate choices of u_j and v_j), 5.1 and 5.2.

Define:

$$\xi_j' = 1 \quad \text{if there is at least one upcrossing of } u \text{ by } X(s), \quad s \in I_j$$
$$= 0 \quad \text{otherwise;}$$

and define η_j' similarly in terms of downcrossings of $-v$.

LEMMA 8.2.

$$\lim_{t\to\infty} P\{\xi_j' = \xi_j, \eta_j' = \eta_j, j = 1, \cdots, m\} = 1.$$

PROOF. By Lemma 3.1, we may suppose that ξ_j''s are defined as upcrossings by $X(s)$, $s \in I_j \cap G_t$; thus $\xi_j' \leqq \xi_j, j = 1, \cdots, m$. If $\xi_j' < \xi_j$ for some j, then max $(X(s)$: $s \in I_j \cap G_t) > u$ but there are no upcrossings; therefore, $X(\cdot)$ must "start" $I_j \cap G_t$ above u, i.e., $X(t_0) > u$ for $t_0 = \min(s: s \in I_j \cap G_t)$. The probability of this event is at most $1 - \Phi(u)$; therefore,

$$\sum_{j=1}^{m} P\{\xi_j' < \xi_j\} \leqq m(1-\Phi(u)) \to 0.$$

By symmetry the same is true for η_j' and η_j.

Define:

$$M_j = \text{number of upcrossings of } u \text{ by } X(s), \qquad s \in I_j$$
$$N_j = \text{number of downcrossings of } -v \text{ by } X(s), \qquad s \in I_j.$$

LEMMA 8.3. *Under* (1.3) *or* (1.4):

$$\limsup_{t\to\infty} P\{M_j \geqq 2, \textit{for some } j = 1, \cdots, m\} \leqq (1-\beta)\sigma_1 + m(1-\exp(-\sigma_1(1-\beta)/m),$$
$$\limsup_{t\to\infty} P\{N_j \geqq 2, \textit{for some } j = 1, \cdots, m\} \leqq (1-\beta)\sigma_2 + m(1-\exp(-\sigma_2(1-\beta)/m).$$

PROOF. It is elementary that for a nonnegative integer-valued random variable M: $P\{M \geqq 2\} \leqq E(M) - P\{M \geqq 1\}$; therefore, since $\xi_j' \leqq M_j$:

$$P\{M_j \geqq 2 \quad \text{for some} \quad j = 1, \cdots, m\} \leqq \textstyle\sum_{j=1}^m P\{M_j \geqq 2\} \leqq \textstyle\sum_{j=1}^m E(M_j) - \textstyle\sum_{j=1}^m P\{\xi_j' = 1\}. \tag{8.4}$$

Formula (3.3) implies:

$$\textstyle\sum_{j=1}^m E(M_j) = (1-\beta)\sigma_1.$$

Lemma 8.2 and stationarity imply:

$$\textstyle\sum_{j=1}^m P\{\xi_j' = 1\} \sim \textstyle\sum_{j=1}^m P\{\xi_j = 1\} = mP\{\max(X(s): s \in [0, (1-\beta)t/m] \cap G_t) > u\}.$$

The last expression is, in accordance with the proof of Theorem 7.1, asymptotically the same as

$$mP\{\max(X(s): 0 \leqq s \leqq (1-\beta)t/m) > u\}.$$

By Corollary 7.1, this converges to $m(1-\exp(-\sigma_1(1-\beta)/m))$; thus, the right-hand member of (8.4) converges to the right-hand member of the first equation of the lemma. The proof of the second equation is similar.

LEMMA 8.4. *Under* (1.3) *or* (1.4):

$$\limsup_{t\to\infty} P\{\xi_1 = \eta_1 = 1\} \leqq (\sigma(1-\beta)/m)^2 \quad \textit{where} \quad \sigma = \max(\sigma_1, \sigma_2).$$

PROOF. Suppose that $\sigma_1 \geqq \sigma_2$ so that $u \leqq v$; then

$$P\{\xi_1 = \eta_1 = 1\} \leqq P\{\max X(s) > u \quad \text{and} \quad \min X(s) \leqq -v \quad \text{for} \quad 0 \leqq s \leqq (1-\beta)t/m\} \leqq P\{\max X(s) > u \quad \text{and} \quad \min X(s) \leqq -u \quad \text{for} \quad 0 \leqq s \leqq (1-\beta)t/m\},$$

which, by the stochastic equivalence of X and $-X$, and an elementary identity, is equal to

$$1 - 2P\{\max(X(s): 0 \leqq s \leqq t(1-\beta)/m) \leqq u\} + P\{\max(|X(s)|: 0 \leqq s \leqq t(1-\beta)/m) \leqq u\}.$$

By Corollary 7.1 this converges to the limit

$$(1-\exp(-\sigma_1(1-\beta)/m))^2 \leqq (\sigma(1-\beta)/m)^2.$$

Since $\sigma = \sigma_1$ the proof is complete.

LEMMA 8.5. *Under* (1.3) *or* (1.4) *the joint generating function of* ξ_1 *and* η_1,

$$E(w^{\xi_1}z^{\eta_1}), \qquad 0 < w < 1, \qquad 0 < z < 1,$$

is asymptotically bounded above and below (*corresponding to* + *and* −, *respectively by*

$$(8.5) \qquad 1-(1-w)(1-\exp(-\sigma_1(1-\beta)/m))+(1-z)(1-\exp(-\sigma_2(1-\beta)/m)), \\ \pm(1-w)(1-z)\sigma^2(1-\beta)^2/m^2,$$

where $\sigma = \max(\sigma_1, \sigma_2)$.

PROOF. By elementary identities:

$$E(w^{\xi_1}z^{\eta_1}) = 1-(1-w)P\{\xi_1 = 1\}-(1-z)P\{\eta_1 = 1\} \\ +(1-w)(1-z)P\{\xi_1 = \eta_1 = 1\}.$$

As in the proof of Lemma 8.3, $P\{\xi_1 = 1\}$ and $P\{\eta_1 = 1\}$ converge to the indicated limits; finally, an application of Lemma 8.4 to $P\{\xi_1 = \eta_1 = 1\}$ completes the proof.

THEOREM 8.1. *If* $X(t)$ *is a standard process satisfying either* (1.3) *or* (1.4), *then* $M(t)$ *and* $N(t)$ *have a joint limiting distribution which is a product of Poisson distributions with means* σ_1 *and* σ_2, *respectively.*

PROOF. By (3.3) and the inequality $M(t) \geqq M_1 + \cdots + M_m$:

$$E|M(t)-\textstyle\sum_{j=1}^m M_j| = EM(t)-\sum_{j=1}^m EM_j = \beta\sigma_1;$$

similarly:

$$E|N(t)-\textstyle\sum_{j=1}^m N_j| = \beta\sigma_2;$$

therefore:

$$(8.6) \qquad P\{|M(t)-\Sigma M_j| > 0 \quad \text{or} \quad |N(t)-\Sigma N_j| > 0\} \leqq \beta(\sigma_1+\sigma_2).$$

By definition:

$$\xi_j' = \min(M_j, 1), \qquad \eta_j' = \min(N_j, 1);$$

thus, by Lemma 8.3:

$$\limsup_{t\to\infty} P\{\xi_j' \neq M_j \quad \text{or} \quad \eta_j' \neq N_j \text{ for some } j = 1, \cdots, m\} \leqq (1-\beta)(\sigma_1+\sigma_2) \\ -m[2-\exp(-\sigma_1(1-\beta)/m)-\exp(-\sigma_2(1-\beta)/m)].$$

By Lemma 8.2 the same inequality holds when ξ_j' and η_j' are replaced by ξ_j and η_j, respectively; therefore:

$$(8.7) \qquad \limsup_{t\to\infty} P\{|\Sigma M_j-\Sigma\xi_j| > 0 \quad \text{or} \quad |\Sigma N_j-\Sigma\eta_j| > 0\} \leqq (1-\beta) \\ \times(\sigma_1+\sigma_2)-m[2-\exp(-\sigma_1(1-\beta)/m)-\exp(-\sigma_2(1-\beta)/m)].$$

By asymptotic independence (Lemma 8.1), stationarity, and Lemma 8.5, the joint generating function of $\Sigma\xi_j$ and $\Sigma\eta_j$ is asymptotically bounded above and below (+ and −, respectively) by

$$\{1-(1-w)(1-\exp(-\sigma_1(1-\beta)/m))+(1-z)(1-\exp(-\sigma_2(1-\beta)/m)) \pm(1-w)(1-z)\sigma^2(1-\beta)^2/m^2\}^m. \tag{8.8}$$

It is elementary that

$$E\left|w^{\Sigma\xi_j}z^{\Sigma\eta_j}-w^{\Sigma M_j}z^{\Sigma N_j}\right| \leqq P\{\left|\Sigma M_j-\Sigma\xi_j\right|>0 \quad \text{or} \quad \left|\Sigma N_j-\Sigma\eta_j\right|>0\}. \tag{8.9}$$

It is a consequence of (8.6), (8.7), (8.8), and (8.9) that $E(w^{M(t)}z^{N(t)})$ is asymptotically bounded above and below (corresponding to + and −, respectively) by

$$\begin{aligned}\{1-(1-w)(1-\exp(-\sigma_1(1-\beta)/m))+(1-z)(1-\exp(-\sigma_2(1-\beta)/m))\\ \pm(1-w)(1-z)\sigma^2(1-\beta)^2/m^2\}^m\\ \pm\{\sigma_1+\sigma_2-m[2-\exp(-\sigma_1(1-\beta)/m)-\exp(-\sigma_2(1-\beta)/m)]\}.\end{aligned} \tag{8.10}$$

Since $\beta>0$ and $m\geqq 1$ are arbitrary, we let $\beta\to 0$ and then $m\to\infty$; then the expression (8.10) converges to $\exp[-\sigma_1(1-w)-\sigma_2(1-z)]$, which is the product of Poisson generating functions.

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