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ASYMPTOTIC INVARIANTS OF BASE LOCI

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Abstract. — The purpose of this paper is to define and study systematically some asymptotic invariants associated to base loci of line bundles on smooth projective varieties. The functional behavior of these invariants is related to the set-theoretic behavior of base loci.

Résumé. — Le but de cet article est de définir et d'étudier systématiquement quelques invariants asymptotiques associés aux lieux de base des fibrés en droites sur les variétés projectives lisses. Le comportement fonctionnel de ces invariants est lié au comportement ensembliste des lieux de base.

Introduction

Let X be a normal complex projective variety, and D a big divisor on X. Recall that the *stable base locus* of D is the Zariski-closed set

$$\mathbf{B}(D) = \bigcap_{m>0} \mathrm{Bs}(mD), \tag{0.1}$$

where $\mathrm{Bs}(mD)$ denotes the base locus of the linear system $|mD|$. This is an interesting and basic invariant, but well-known pathologies associated to linear series have discouraged its study. Recently, however, a couple of results have appeared suggesting that the picture might be more structured than expected. To begin with, Nakayama [20] attached an asymptotically-defined multiplicity $\sigma_\Gamma(D)$ to any divisorial component Γ of $\mathbf{B}(D)$, and proved that $\sigma_\Gamma(D)$ varies continuously as D varies over the cone $\mathrm{Big}(X)_{\mathbb{R}} \subseteq N^1(X)_{\mathbb{R}}$ of numerical equivalence classes of big divisors on X. More recently, the fourth author showed in [18] that many pathologies disappear if

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one perturbs D slightly by subtracting a small ample divisor. Inspired by this work, the purpose of this paper is to define and explore systematically some asymptotic invariants that one can attach to base loci of linear series, and to study their variation with D.

We start by specifying the invariants in question. Let v be a discrete valuation of the function field $K(X)$ of X and let R be the corresponding discrete valuation ring. Every effective Cartier divisor D on X determines an ideal in R, and we denote by $v(D)$ the order via v of this ideal.

Let D be a big divisor on X with $|D| \neq \emptyset$. We wish to quantify how nasty are the singularities of a general divisor $D' \in |D|$ along v: we define the *order of vanishing* of $|D|$ along v by

$$v\big(|D|\big) = v(D')$$

where D' is a general divisor in $|D|$. If v is the valuation associated to a prime divisor E on X, then $v(|D|)$ is the coefficient of E in a general element D' in $|D|$.

Our focus will be on asymptotic analogues of these invariants. Let D be a big divisor on X, and v a valuation as above. The *asymptotic order of vanishing* of D along v is defined as

$$v(\|D\|) := \lim_{p\to\infty} \frac{v(|pD|)}{p}. \tag{0.2}$$

It is easy to see that this limit exists. By taking p to be sufficiently divisible this definition extends naturally to $\mathbb{Q}$-divisors such that $v(\|D\|)$ is homogeneous of degree one in D. The most important example is obtained when X is smooth, considering the valuation given by the order of vanishing at the generic point of a subvariety Z – we denote the corresponding invariant by $\mathrm{ord}_Z(\|D\|)$. When Z is a prime divisor E on X, this is the invariant $\sigma_E(D)$ introduced and studied by Nakayama. In general these invariants may be irrational (Example 3.6).

Our first result shows that these quantities vary nicely as functions of D:

THEOREM A. — *Let X be a normal projective variety, v a fixed discrete valuation of the function field of X, and D a big $\mathbb{Q}$-divisor on X.*

(i) *The asymptotic order of vanishing $v(\|D\|)$ depends only on the numerical equivalence class of D, so it induces a function on the set $\mathrm{Big}(X)_{\mathbb{Q}}$ of numerical equivalence classes of big $\mathbb{Q}$-divisors.*

(ii) *This function extends uniquely to a continuous function on the set $\mathrm{Big}(X)_{\mathbb{R}}$ of numerical equivalence classes of big $\mathbb{R}$-divisors.*

When v is the valuation corresponding to a prime divisor E on X, this result is due to Nakayama; some of Nakayama's results were rediscovered and extended to an analytic setting by Boucksom [2]. The theorem was also suggested to us by results of the second author [16, Chap. 2.2.C] concerning continuity of the volume of a big divisor.

It can happen that the stable base locus $\mathbf{B}(D)$ does not depend only on the numerical class of D (Example 1.1). Motivated by the work [18] of the fourth author, we consider instead the following approximations of $\mathbf{B}(D)$.

Let D be a big $\mathbb{Q}$-divisor on X. The stable base locus $\mathbf{B}(D)$ is defined in the natural way, e.g. by taking m to be sufficiently divisible in (0.1). The *augmented base locus* of D is the closed set

$$\mathbf{B}_+(D) := \bigcap_A \mathbf{B}(D - A),$$

where the intersection is over all ample $\mathbb{Q}$-divisors A. Similarly, the *restricted base locus* of D is given by

$$\mathbf{B}_-(D) := \bigcup_A \mathbf{B}(D + A),$$

where the union is over all ample $\mathbb{Q}$-divisors A. This is a potentially countable union of irreducible subvarieties of X (it is not known whether $\mathbf{B}_-(D)$ is itself Zariski-closed). It follows easily from the definition that both $\mathbf{B}_-(D)$ and $\mathbf{B}_+(D)$ depend only on the numerical class of D. Moreover, since the definitions involve perturbations there is a natural way to define the augmented and the restricted base loci of an arbitrary real class $\xi \in \mathrm{Big}(X)_{\mathbb{R}}$.

The restricted base locus of a big $\mathbb{Q}$-divisor D is the part of $\mathbf{B}(D)$ which is accounted for by numerical properties of D. For example, $\mathbf{B}_-(D)$ is empty if and only if D is nef.[(1)] At least when X is smooth, we have the following

THEOREM B. — *Let X be a smooth projective variety, v a discrete valuation of the function field of X, and Z the center of v on X. If ξ is in $\mathrm{Big}(X)_{\mathbb{R}}$, then $v(\|\xi\|) > 0$ if and only if Z is contained in $\mathbf{B}_-(\xi)$.*

It is natural to distinguish the big divisor classes for which the restricted and the augmented base loci coincide. We call such a divisor class *stable*. Equivalently, ξ is stable if there is a neighborhood of ξ in $\mathrm{Big}(X)_{\mathbb{R}}$ such that $\mathbf{B}(D)$ is constant on the $\mathbb{Q}$-divisors D with class in that neighborhood. For example, if ξ is nef, then it is stable if and only if it is ample.

The set of stable classes is open and dense in $\mathrm{Big}(X)_{\mathbb{R}}$. Given an irreducible closed subset $Z \subseteq X$, we denote by $\mathrm{Stab}^Z(X)_{\mathbb{R}}$ the set of stable

(1) Note that $\mathbf{B}_-(D)$ appears also in [3], where it is called the *non-nef* locus of D.

classes ξ such that Z is an irreducible component of $\mathbf{B}_+(D)$. Suppose that v is a discrete valuation with center Z. We see that Theorems A and B show that if X is smooth, then $v(\|.\|)$ is positive on $\mathrm{Stab}^Z(X)_{\mathbb{R}}$ and $v(\|\xi\|)$ goes to zero only when the argument ξ approaches the boundary of a connected component of $\mathrm{Stab}^Z(X)_{\mathbb{R}}$ and when in addition Z "disappears" from the stable base locus as ξ crosses that boundary. We explain the structure of the union of these boundaries, i.e. the set of unstable classes, in Section 3.

Similar asymptotic functions can be defined starting with other invariants of singularities instead of valuations. For example, if X is smooth we may use the reciprocal of the log canonical threshold or the Hilbert-Samuel multiplicity (cf. Section 2). The resulting asymptotic invariants enjoy properties analogous to those of $v(\|.\|)$.

In general, these asymptotic invariants need not be locally polynomial (Example 3.6). However on varieties whose linear series satisfy sufficiently strong finiteness hypotheses, the picture is very simple. We start with a definition.

Definition C (Finitely generated linear series). — A normal projective variety X has *finitely generated linear series* if there exist integral Cartier divisors $D_1, \ldots, D_r$ on X with the properties:

(a) the classes of the D_i are a basis for $N^1(X)_{\mathbb{R}}$;

(b) the $\mathbb{Z}^r$-graded ring

$$\mathrm{Cox}(D_1, \ldots, D_r) := \bigoplus_{m=(m_i)\in\mathbb{Z}^r} H^0\big(X, \mathcal{O}_X(m_1D_1 + \cdots + m_rD_r)\big)$$

is a finitely generated $\mathbb{C}$-algebra.

The definition was inspired by the notion of a "Mori dream space" introduced by Hu and Keel [13]: these authors require in addition that the natural map $\mathrm{Pic}(X)_{\mathbb{Q}} \to N^1(X)_{\mathbb{Q}}$ be an isomorphism, but this is irrelevant for our purposes. It follows from a theorem of Cox that any projective toric variety has finitely generated linear series, and it is conjectured (and verified in dimension 3) [13] that the same is true for any smooth Fano variety. (For more examples see §4.)

Theorem D. — *If X has finitely generated linear series, then the closed cone*

$$\overline{\mathrm{Eff}}(X)_{\mathbb{R}} = \overline{\mathrm{Big}(X)}_{\mathbb{R}}$$

of pseudoeffective divisors[(2)] *on X is rational polyhedral. For every discrete*

[(2)] Recall that by definition a divisor is pseudoeffective if its class lies in the closure of the cone of effective divisors, or equivalently in the closure of the cone of big divisors.

valuation v of the function field of X, the function $v(\|.\|)$ can be extended by continuity to $\overline{\mathrm{Eff}}(X)_{\mathbb{R}}$. Moreover, there is a fan Δ whose support is the above cone, such that for every v, the function $v(\|.\|)$ is linear on each of the cones in Δ.

A similar statement holds for the usual volume function on varieties with finitely generated linear series (cf. Proposition 4.13).

The paper is organized as follows. We start in §1 with a discussion of various base loci and stable divisor classes. The asymptotic invariants are defined in §2 and the continuity is established in §3, where we also discuss the structure of the set of unstable classes and we give a number of examples. Finally, we prove Theorem D in §4.

The present paper is part of a larger project to explore the asymptotic properties of linear series on X. See [10] for an invitation to this circle of ideas.

1. Augmented and restricted base loci

We consider in this section the augmented and restricted base loci of a linear system, and the notion of stable divisor classes. The picture is that the stable base locus of a divisor changes only as the divisor passes through certain "unstable" classes.

We start with some notation. Throughout this section X is a normal complex projective variety. An integral divisor D on X is an element of the group $\mathrm{Div}(X)$ of Cartier divisors, and as usual we can speak about $\mathbb{Q}$- or $\mathbb{R}$-divisors. A $\mathbb{Q}$- or $\mathbb{R}$-divisor D is *effective* if it is a non-negative linear combination of effective integral divisors with $\mathbb{Q}$- or $\mathbb{R}$-coefficients. If D is effective, we denote by $\mathrm{Supp}(D)$ the union of the irreducible components which appear in the associated Weil divisor. Numerical equivalence between $\mathbb{Q}$- or $\mathbb{R}$-divisors will be denoted by $\equiv$. We denote by $N^1(X)_{\mathbb{Q}}$ and $N^1(X)_{\mathbb{R}}$ the finite dimensional $\mathbb{Q}$- and $\mathbb{R}$-vector spaces of numerical equivalence classes. One has $N^1(X)_{\mathbb{R}} = N^1(X)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$, and we fix compatible norms $\|.\|$ on these two spaces.

Recall next that the *stable base locus* of an integral divisor D is defined to be

$$\mathbf{B}(D) := \bigcap_{m \geqslant 1} \mathrm{Bs}(mD)_{\mathrm{red}},$$

considered as a reduced subset of X. It is elementary that there exists $p \geqslant 1$ such that $\mathbf{B}(D) = \mathrm{Bs}(pD)_{\mathrm{red}}$, and that

$$(*) \qquad \mathbf{B}(D) = \mathbf{B}(mD) \quad \text{for all } m \geqslant 1.$$

This allows us to define the stable base locus for any $\mathbb{Q}$-divisor D: take a positive integer k such that kD is integral and put $\mathbf{B}(D) := \mathbf{B}(kD)$. It follows from (*) that the definition does not depend on k.

Example 1.1 (Non-numerical nature of stable base locus). — Let C be an elliptic curve, A a divisor of degree 1 on C and let

$$\pi\colon X = \mathbf{P}\big(\mathcal{O}_C \oplus \mathcal{O}_C(A)\big) \longrightarrow C.$$

For $i = 1, 2$, let $L_i = \mathcal{O}_X(1) \otimes \pi^*\mathcal{O}(P_i)$, where P_1 and P_2 are divisors of degree zero on C, with P_1 torsion and P_2 non-torsion. It is shown in [16, Example 10.3.3] that L_1 and L_2 are numerically equivalent big and nef line bundles such that $\mathbf{B}(L_1) = \emptyset$ and $\mathbf{B}(L_2)$ is a curve. □

The augmented base locus. — The previous example points to the fact that the stable base locus of a divisor is not in general very well behaved. We introduce here an upper approximation of this asymptotic locus which has better formal properties. The importance of this "augmented" locus, and the fact that it eliminates pathologies, was systematically put in evidence by the fourth author in [18], [19].

Definition 1.2 (Augmented base locus). — *The* augmented base locus *of an $\mathbb{R}$-divisor D on X is the Zariski-closed set*

$$\mathbf{B}_+(D) := \bigcap_{D=A+E} \mathrm{Supp}(E),$$

where the intersection is taken over all decompositions $D = A + E$, where A and E are $\mathbb{R}$-divisors such that A is ample and E is effective.

To relate this definition with the definition given in the introduction, we note the following:

Remark 1.3 (Alternative construction of augmented base locus). — We remark that in the definition of $\mathbf{B}_+(D)$ one may take the intersection over all decompositions such that, in addition, E is a $\mathbb{Q}$-divisor. Furthermore,

$$\mathbf{B}_+(D) = \bigcap_A \mathbf{B}(D - A),$$

the intersection being taken over all ample $\mathbb{R}$-divisors A such that $D - A$ is a $\mathbb{Q}$-divisor. In fact, for the first assertion, note that if $D = A + E$, with A ample and E effective, then we can find effective $\mathbb{Q}$-divisors E_m

for $m \in \mathbb{N}$, such that $E_m \to E$ when m goes to infinity and such that $\mathrm{Supp}(E_m) = \mathrm{Supp}(E)$ for all m. Since $D - E$ is ample, so is $D - E_m$ for $m \gg 0$, hence we are done. The second assertion follows immediately from the first one and the definition of the stable base locus.

We observe first that – unlike the stable base locus itself – the augmented base locus depends only on the numerical equivalence class of a divisor.

PROPOSITION 1.4. — *If D_1 and D_2 are numerically equivalent $\mathbb{R}$-divisors, then*

$$\mathbf{B}_+(D_1) = \mathbf{B}_+(D_2).$$

Proof. — This follows from the observation that if we have a decomposition $D_1 = A + E$ as in the definition of $\mathbf{B}_+(D_1)$, then we get a corresponding decomposition $D_2 = (A + (D_2 - D_1)) + E$. □

The next statement shows that $\mathbf{B}_+(D)$ coincides with the stable base locus $\mathbf{B}(D - A)$ for any sufficiently small ample divisor A such that $D - A$ is a $\mathbb{Q}$-divisor.

PROPOSITION 1.5. — *For every $\mathbb{R}$-divisor D, there is $\varepsilon > 0$ such that*

$$\mathbf{B}_+(D) = \mathbf{B}(D - A)$$

for any ample A with $\|A\| < \varepsilon$ and such that $D - A$ is a $\mathbb{Q}$-divisor. More generally, if D' is any $\mathbb{R}$-divisor with $\|D'\| < \varepsilon$ and such that $D - D'$ is a $\mathbb{Q}$-divisor, then

$$\mathbf{B}(D - D') \subseteq \mathbf{B}_+(D).$$

Proof. — There exist ample $\mathbb{R}$-divisors $A_1, \ldots, A_r$ such that each $D - A_i$ is a $\mathbb{Q}$-divisor and so that moreover

$$\mathbf{B}_+(D) = \bigcap_i \mathbf{B}(D - A_i).$$

Choose $\varepsilon > 0$ so that $A_i - D'$ is ample for every i whenever $\|D'\| < \varepsilon$. Writing

$$D - D' = (D - A_i) + (A_i - D'),$$

we see that if $D - D'$ is a $\mathbb{Q}$-divisor, then $\mathbf{B}(D - D') \subseteq \mathbf{B}(D - A_i)$ for all i. This proves the second assertion. The first statement follows at once, since for any ample divisor A such that $D - A$ is a $\mathbb{Q}$-divisor, we have $\mathbf{B}_+(D) \subseteq \mathbf{B}(D - A)$ by Remark 1.3. □

COROLLARY 1.6. — *If D and $\varepsilon > 0$ are as in Proposition 1.5 and if D' is an $\mathbb{R}$-divisor such that $\|D'\| < \varepsilon$, then $\mathbf{B}_+(D - D') \subseteq \mathbf{B}_+(D)$. If D' is ample, then equality holds.*

Proof. — For every D' as above, we apply Proposition 1.5 to $D - D'$ to conclude that if A' is ample, with $\|A'\|$ small enough, and such that $D - D' - A'$ is a $\mathbb{Q}$-divisor, then we have

$$\mathbf{B}_+(D - D') = \mathbf{B}(D - D' - A').$$

Since $\|A'\|$ is small, we may assume that $\|D' + A'\| < \varepsilon$, hence

$$\mathbf{B}(D - D' - A') \subseteq \mathbf{B}_+(D).$$

Moreover, this is an equality if D' (hence also $D' + A'$) is ample. □

Example 1.7. — The augmented base locus is a proper subset of X if and only if D is big. Similarly, it follows from Proposition 1.5 that $\mathbf{B}_+(D) = \emptyset$ if and only if D is ample. □

Example 1.8. — For any $\mathbb{R}$-divisor D, $\mathbf{B}_+(D) = \mathbf{B}_+(cD)$ for any real number $c > 0$. □

Example 1.9. — For any $\mathbb{R}$-divisors D_1 and D_2, it can be easily shown that

$$\mathbf{B}_+(D_1 + D_2) \ \subseteq \ \mathbf{B}_+(D_1) \cup \mathbf{B}_+(D_2).$$

Example 1.10 (Augmented base locus of nef and big divisors). — Assume for the moment that X is non-singular, and let D be a nef and big divisor on X. Define the *null locus* $\mathrm{Null}(D)$ of D to be the union of all irreducible subvarieties $V \subseteq X$ of positive dimension with the property that $\left(D^{\dim V} \cdot V\right) = 0$, i.e. with the property that the restriction of D to V is not big. Then $\mathbf{B}_+(D) = \mathrm{Null}(D)$. This is proved for $\mathbb{Q}$-divisors in [18], and in general in [11]. □

Example 1.11 (Augmented base loci on surfaces). — Assume here that X is a smooth surface, and let D be a big divisor on X. Then D has a Zariski decomposition $D = P + N$ (see [5]) into a nef part P and "negative" part N. Then $\mathbf{B}_+(D)$ is the null locus $\mathrm{Null}(P)$ of P. To see this, note that if $D = A + E$, where A is ample and E is effective, then $E - N$ is effective. Therefore

$$\mathbf{B}_+(D) = \mathbf{B}_+(P) \cup \mathrm{Supp}(N).$$

Since $\mathrm{Supp}(E) \subseteq \mathrm{Null}(P)$, we get

$$\mathbf{B}_+(D) = \mathbf{B}_+(P) = \mathrm{Null}(P)$$

by the previous example. □

The restricted base locus. — Proposition 1.5 shows that the augmented base locus of a divisor is the stable base locus of a small negative perturbation of the divisor. When it comes time to discuss the behavior of the numerical asymptotic invariants of base loci, it will be helpful to have an analogous notion involving small positive perturbations:

DEFINITION 1.12. — *If D is an $\mathbb{R}$-divisor on X, then the* restricted base locus *of D is*

$$\mathbf{B}_{-}(D) = \bigcup_{A} \mathbf{B}(D + A),$$

where the union is taken over all ample divisors A, such that $D + A$ is a $\mathbb{Q}$-divisor.

Remark 1.13 (Warning on restricted base loci). — It is not known whether the restricted base locus of a divisor is Zariski closed in general. *A priori* $\mathbf{B}_{-}(D)$ could consist of a countable union of subvarieties whose Zariski closure is contained in $\mathbf{B}_{+}(D)$.

LEMMA 1.14. — *For every $\mathbb{R}$-divisor D, one has $\mathbf{B}_{-}(D) = \bigcup_A \mathbf{B}_{+}(D + A)$, the union being taken over all ample $\mathbb{R}$-divisors A.*

Proof. — It is enough to show that if A is ample, then $\mathbf{B}_{+}(D+A) \subseteq \mathbf{B}_{-}(D)$. If A_0 is ample with $0 < \|A_0\| \ll 1$, such that $D + A - A_0$ is a $\mathbb{Q}$-divisor, then $A - A_0$ is ample, and

$$\mathbf{B}_{+}(D + A) = \mathbf{B}(D + A - A_0) \subseteq \mathbf{B}_{-}(D). \qquad \square$$

PROPOSITION 1.15. — (i) *For every $\mathbb{R}$-divisor D and every real number $c > 0$, we have*

$$\mathbf{B}_{-}(D) = \mathbf{B}_{-}(cD).$$

(ii) *If D_1 and D_2 are numerically equivalent $\mathbb{R}$-divisors, then*

$$\mathbf{B}_{-}(D_1) = \mathbf{B}_{-}(D_2).$$

Proof. — Both assertions follow from Lemma 1.14, since we already know the corresponding assertions for the augmented base locus. $\square$

Example 1.16. — For every $\mathbb{R}$-divisor D, we have $\mathbf{B}_{-}(D) \subseteq \mathbf{B}_{+}(D)$. If D is a $\mathbb{Q}$-divisor, then $\mathbf{B}_{-}(D) \subseteq \mathbf{B}(D) \subseteq \mathbf{B}_{+}(D)$. $\square$

Example 1.17. — Let D be a big divisor on a smooth projective surface, with Zariski decomposition $D = P + N$. Then $\mathbf{B}_{-}(D) = \mathrm{Supp}(N)$. See Example 3.4 below for a proof. $\square$

Example 1.18. — Given an $\mathbb{R}$-divisor D, $\mathbf{B}_-(D) = \emptyset$ if and only if D is nef. Similarly, $\mathbf{B}_-(D) = X$ if and only if the class of D in $N^1(X)_{\mathbb{R}}$ does not lie in the closure of the cone of big classes. □

As we have indicated, it isn't known whether $\mathbf{B}_-(D)$ is Zariski-closed in general. However it is at worst a countable union of closed subvarieties:

PROPOSITION 1.19. — *If $\{A_m\}_{m\in\mathbb{N}}$ are ample divisors with*

$$\lim_{m\to\infty} \|A_m\| = 0,$$

and such that $D + A_m$ are $\mathbb{Q}$-divisors, then

$$\mathbf{B}_-(D) = \bigcup_m \mathbf{B}(D + A_m).$$

In particular, $\mathbf{B}_-(D)$ is a countable union of Zariski closed subsets of X.

Proof. — The statement follows since for every ample A, such that $D+A$ is a $\mathbb{Q}$-divisor, $A - A_m$ is ample, for $m \gg 0$. Since we can write

$$D + A = (D + A_m) + (A - A_m),$$

we get $\mathbf{B}(D + A) \subseteq \mathbf{B}(D + A_m)$. □

Remark 1.20. — Suppose that A_m is a sequence of ample $\mathbb{R}$-divisors, where $A_m \to 0$. As in Proposition 1.19, we can show that

$$\mathbf{B}_-(D) = \bigcup_m \mathbf{B}_+(D + A_m).$$

We note that if $A_m - A_{m+1}$ is ample then

$$\mathbf{B}_+(D + A_m) \subseteq \mathbf{B}_+(D + A_{m+1}).$$

In particular, if $\mathbf{B}_-(D)$ is closed, then it is equal to $\mathbf{B}_+(D + A)$ for all sufficiently small ample $\mathbb{R}$-divisors A.

PROPOSITION 1.21. — *For every $\mathbb{R}$-divisor D, there is an $\varepsilon > 0$ such that $\mathbf{B}_-(D - A) = \mathbf{B}_+(D - A) = \mathbf{B}_+(D)$, for every ample A with $\|A\| < \varepsilon$.*

Proof. — Apply Corollary 1.6 to D to find $\varepsilon > 0$ such that for every ample A, with $\|A\| < \varepsilon$, we have $\mathbf{B}_+(D - A) = \mathbf{B}_+(D)$. For every such A, we have

$$\mathbf{B}_+(D) = \mathbf{B}_+(D - \tfrac{1}{2}A) \ \subseteq\ \mathbf{B}_-(D - A) \ \subseteq\ \mathbf{B}_+(D),$$

as required. □

Stable divisors. — We now single out those divisors for which the various base loci we have considered all coincide.

DEFINITION 1.22. — *An $\mathbb{R}$-divisor D on X is called* stable *if*

$$\mathbf{B}_+(D) = \mathbf{B}_-(D),$$

Remark 1.23. — Note that as both $\mathbf{B}_-(D)$ and $\mathbf{B}_+(D)$ depend only on the numerical class of D, so does the stability condition.

The next statement gives various characterizations of stability for an $\mathbb{R}$-divisor.

PROPOSITION 1.24. — *For an $\mathbb{R}$-divisor D on X, the following are equivalent:*

(i) *D is stable;*

(ii) *there is an ample $\mathbb{R}$-divisor A such that*

$$\mathbf{B}_+(D) = \mathbf{B}_+(D + A);$$

(iii) *there is an $\varepsilon > 0$ such that for every $\mathbb{R}$-divisor D' with $\|D'\| < \varepsilon$, we have*

$$\mathbf{B}_+(D) = \mathbf{B}_+(D + D');$$

(iv) *there is an $\varepsilon > 0$ such that for every $\mathbb{R}$-divisor D' with $\|D'\| < \varepsilon$, we have*

$$\mathbf{B}_-(D) = \mathbf{B}_-(D + D');$$

(v) *there exists a positive number $\varepsilon > 0$ such that all the closed sets $\mathbf{B}(D + D')$ coincide whenever D' is an $\mathbb{R}$-divisor with $\|D'\| < \varepsilon$ and $D + D'$ rational.*

Proof. — If D is stable, then in particular $\mathbf{B}_-(D)$ is closed. Remark 1.20 implies that there is a sufficiently small ample $\mathbb{R}$-divisor A such that

$$\mathbf{B}_+(D) = \mathbf{B}_-(D) = \mathbf{B}_+(D + A).$$

This shows that (i) $\Rightarrow$ (ii).

We assume (ii). By Corollary 1.6, there is an $\varepsilon > 0$ such that if $|D'| < \varepsilon$ then $\mathbf{B}_+(D + D') \subseteq \mathbf{B}_+(D)$. We may also assume that $A - D'$ is ample, so

$$\mathbf{B}_+(D + A) \subseteq \mathbf{B}_+(D + D').$$

We see that (ii) $\Rightarrow$ (iii).

We assume (iii). Suppose that $|D'| < \varepsilon$. Note that

$$\mathbf{B}_-(D + D') = \bigcup_A \mathbf{B}_+(D + D' + A),$$

where A is ample and $\|A\| < \varepsilon - \|D'\|$. By hypothesis,

$$\mathbf{B}_+(D + D' + A) = \mathbf{B}_+(D),$$

and therefore

$$\mathbf{B}_-(D+D') = \mathbf{B}_+(D) = \mathbf{B}_-(D),$$

hence (iv) holds.

Assume (iv). We choose ε as in (iv) and such that it satisfies Proposition 1.21. Assume that $\|D'\| < \varepsilon$ and that $D + D'$ is rational. It follows that

$$\mathbf{B}_-(D) = \mathbf{B}_-(D+D') \subseteq \mathbf{B}(D+D') \subseteq \mathbf{B}_+(D+D') \subseteq \mathbf{B}_+(D).$$

Note that if we take D' such that $-D'$ is ample, then the hypothesis gives $\mathbf{B}_-(D) = \mathbf{B}_+(D)$. We deduce that $\mathbf{B}(D+D') = \mathbf{B}_+(D)$ for all D' such that $|D'| < \varepsilon$ and $D + D'$ is rational. Hence (iv) impies (v).

Assume (v). For any sufficiently small ample divisor A such that $D - A$ is rational, Proposition 1.5 implies that $\mathbf{B}(D-A) = \mathbf{B}_+(D)$. Now for a sufficiently small ample divisor A' such that $D + A'$ is rational, we have that

$$\mathbf{B}(D-A) = \mathbf{B}(D+A') \subseteq \mathbf{B}_-(D).$$

We conclude that $\mathbf{B}_+(D) = \mathbf{B}_-(D)$, hence D is stable. □

Remark 1.25. — If the class of D is not in the closure of the big cone, then D is trivially stable, as $\mathbf{B}_-(D) = \mathbf{B}_+(D) = X$. On the other hand, if the class of D is in the boundary of this cone (so that D is not big), then D is not stable, because $\mathbf{B}_+(D) = X$, but $\mathbf{B}_-(D) \neq X$.

Proposition 1.26. — *The set of stable divisor classes is open and dense in $N^1(X)_{\mathbb{R}}$. In fact, for every D there is $\varepsilon > 0$ such that if A is ample, and $\|A\| < \varepsilon$, then $D - A$ is stable.*

Proof. — The set of stable classes is open, as the condition in Proposition 1.24, (v) is an open condition. To show that it is dense, it is enough to prove the last assertion. This follows from Proposition 1.21. □

Example 1.27. — Let $X = \mathrm{Bl}_P(\mathbf{P}^n)$ be the blowing-up of $\mathbf{P}^n$ at a point P. Write H and E respectively for the pullback of a hyperplane and the exceptional divisor. For $x, y \in \mathbb{R}$ consider the $\mathbb{R}$-divisor $D_{x,y} = xH - yE$. Identifying $N^1(X)_{\mathbb{R}}$ with the xy-plane in the evident way, the set of unstable classes consists of three rays: the negative y-axis, the positive x-axis, and the ray of slope $= 1$ in the first quadrant. The corresponding augmented base loci are indicated in Figure 1.1. □

Example 1.28. — Suppose X is a smooth surface and D a big divisor with Zariski decomposition $D = P + N$. Then D is stable if and only if $\mathrm{Null}(P) = \mathrm{Supp}(N)$. □

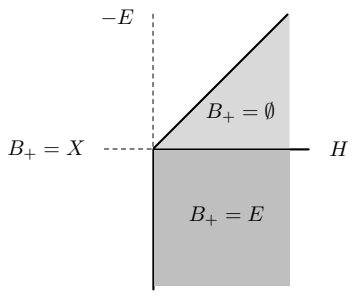

Figure 1.1. Stable classes on blow-up of $\mathbf{P}^n$

Example 1.29. — For a big $\mathbb{Q}$-divisor D, one introduces in [19] an asymptotic version of the Seshadri constant at a point $x \in X$, denoted by $\varepsilon_m(x, D)$. This invariant describes the augmented base locus of D, namely $x \in \mathbf{B}_+(D)$ if and only if $\varepsilon_m(x, D) = 0$. The main result of [19] is a continuity statement with respect to D for this invariant. □

2. Asymptotic numerical invariants

In this section we define the asymptotic numerical invariants with which we shall be concerned. Let X be a normal projective variety. We fix a discrete valuation v of the function field of X, let R be the corresponding DVR and Z the center of v on X. Given a big integral divisor D, we denote by $\mathfrak{a}_p$ the image in R of the ideal $\mathfrak{b}(|pD|)$ defining the base locus of $|pD|$. These ideals form a graded system of ideals in the sense of [12], i.e. $\mathfrak{a}_p \cdot \mathfrak{a}_q \subseteq \mathfrak{a}_{p+q}$ for every p and q. Note that since D is big, we have $|pD| \neq \emptyset$, and therefore $\mathfrak{a}_p \neq (0)$ for $p \gg 0$.

For every p such that $|pD| \neq \emptyset$, we put $v(|pD|)$ for the order $v(\mathfrak{a}_p)$ of the ideal $\mathfrak{a}_p$. Equivalently, $\nu(|pD|)$ is equal to $v(g)$, where g is an equation of a general element in $|pD|$ at the generic point of Z.

The convexity property of our invariants will be crucial in what follows. A first indication is given by the lemma below. It is an immediate consequence of the fact that the ideals $\mathfrak{a}_p$ form a graded system of ideals.

LEMMA 2.1. — *With the above notation, if p and q are such that $|pD|$ and $|qD|$ are nonempty, then*

$$v\big(|(p+q)D|\big) \leqslant v\big(|pD|\big) + v\big(|qD|\big).$$

We are now in a position to define our asymptotic invariants. The existence of the limit in the following definition follows from Lemma 2.1. In fact, the limit is equal to the infimum of the corresponding quantities (see, for example, Lemma 1.4 in [17]).

Definition 2.2. — Given a big integral divisor D, set

$$v(\|D\|) = \lim_{p\to\infty} \frac{v(|pD|)}{p}.$$

This is called the *asymptotic order of vanishing* along v.

Remark 2.3 (Rescaling and extension to $\mathbb{Q}$-divisors). — It follows from the definition as a limit that for any $m \in \mathbb{N}$:

$$v\big(\|mD\|\big) = m \cdot v\big(\|D\|\big).$$

In particular, by clearing denominators we see that our invariants are defined in the natural way for any big $\mathbb{Q}$-divisor D.

Proposition 2.4 (Convexity). — *If D and E are big $\mathbb{Q}$-divisors on X, then*

$$v(\|D+E\|) \leqslant v(\|D\|) + v(\|E\|).$$

Proof. — The assertion follows from the fact that if $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ are the ideals defining the base loci of the linear systems $|pD|$, $|pE|$ and respectively $|p(D+E)|$, then $\mathfrak{a}\cdot\mathfrak{b} \subseteq \mathfrak{c}$. □

Computation via multiplier ideals. — We show now that these asymptotic invariants can be computed using multiplier ideals. For the theory of multiplier ideals we refer to [16, Part Three].

Note that if $f\colon X' \to X$ is a proper, birational morphism, with X' normal, then we have an asymptotic order of vanishing along v defined for big $\mathbb{Q}$-divisors on X'. It is clear that for a big $\mathbb{Q}$-divisor D on X, we have $v(\|D\|) = v(\|f^*D\|)$. In particular, by taking f such that X' is smooth, we reduce the computation of the asymptotic order of vanishing along v to the case of a smooth variety. In this case, we can make use of multiplier ideals.

Recall that R is the DVR corresponding to the valuation v. If D is a big integral divisor, we denote by $\mathfrak{j}_p$ the image in R of the asymptotic multiplier ideal $\mathcal{J}(X, \|pD\|)$. We show that $v(\|D\|)$ can be computed using the orders $v(\mathfrak{j}_p)$ of the ideals $\mathfrak{j}_p$.

The set of ideals $\{\mathfrak{j}_p\}_p$ is not a graded sequence anymore. However, the Subadditivity Theorem of [9] gives $\mathfrak{j}_{p+q} \subseteq \mathfrak{j}_p\cdot\mathfrak{j}_q$ for every p and q, and hence

$$v(\mathfrak{j}_{p+q}) \geqslant v(\mathfrak{j}_p) + v(\mathfrak{j}_q).$$

Moreover, if $p < q$ then $\mathfrak{j}_q \subseteq \mathfrak{j}_p$, so $v(\mathfrak{j}_q) \geqslant v(\mathfrak{j}_p)$. It is easy to deduce from these facts that

$$\lim_{p\to\infty} \frac{v(\mathfrak{j}_p)}{p} = \sup_p \frac{v(\mathfrak{j}_p)}{p} \tag{2.1}$$

(see, for example, Lemma 2.2 in [17]). The above limit is finite: for every p we have $\mathfrak{a}_p \subseteq \mathfrak{j}_p$ so $v(\mathfrak{j}_p) \leqslant v(\mathfrak{a}_p)$, and therefore the above limit is bounded above by $v(\|D\|)$. The next proposition shows that in fact we have equality.

PROPOSITION 2.5. — *With the above notation, for every big integral divisor D we have*

$$v\big(\|D\|\big) = \lim_{p\to\infty} \frac{v(\mathfrak{j}_p)}{p}.$$

Proof. — It follows from [16, Theorem 11.2.21] that there is an effective divisor E on X such that for every $p \gg 0$, we have

$$\mathcal{J}\big(X, \|pD\|\big) \otimes \mathcal{O}_X(-E) \subseteq \mathfrak{b}\big(|pD|\big).$$

In particular, there is a nonzero element u in R such that $u \cdot \mathfrak{j}_p \subseteq \mathfrak{a}_p$ for all $p \gg 0$. Therefore $v(\mathfrak{a}_p) \leqslant v(\mathfrak{b}_p) + v(u)$, so dividing by p and passing to limit gives

$$v\big(\|D\|\big) \leqslant \lim_{p\to\infty} \frac{v(\mathfrak{j}_p)}{p}.$$

As we have already seen the opposite inequality, this completes the proof. □

Remark 2.6. — It follows from the above proposition that $v(\|D\|) = 0$ if and only if $\mathfrak{j}_p = R$ for every p. By definition, this is the case if and only if the center Z of v is not contained in $\mathcal{Z}(\mathcal{J}(X, \|pD\|))$ for any p.

COROLLARY 2.7. — *If D and E are numerically equivalent big $\mathbb{Q}$-divisors on the normal projective variety X, then $v(\|D\|) = v(\|E\|)$ for every discrete valuation v of the function field of X.*

Proof. — By taking a resolution of singularities, we may assume that X is smooth. Moreover, we may assume that D and E are integral divisors. Since D and E are big and numerically equivalent, it follows from [16, Example 11.3.12] that

$$\mathcal{J}\big(X, \|pD\|\big) = \mathcal{J}\big(X, \|pE\|\big)$$

for every p. The assertion now follows from Proposition 2.5. □

We can give now a proof of Theorem B from the Introduction for $\mathbb{Q}$-divisors. In fact, in this case we prove the following more precise

PROPOSITION 2.8. — *Let X be a smooth projective variety and D a big $\mathbb{Q}$-divisor on X. If v is a discrete valuation of the function field of X having center Z on X, then the following are equivalent:*

(i) *there is a constant $C > 0$ such that $v(|pD|) \leqslant C$ whenever $|pD|$ is nonempty;*
(ii) $v(\|D\|) = 0$;
(iii) $Z \not\subseteq \mathbf{B}_-(D)$.

Proof. — We may assume that D is an integral divisor. Note that (i) clearly implies (ii).

Suppose now that $v(\|D\|) = 0$ and let us show that Z is not contained in $\mathbf{B}_-(D)$. By Remark 2.6, we see that $\mathcal{J}(X, \|pD\|) = \mathcal{O}_X$ at the generic point of Z. Let A be a very ample divisor on X, and $G = K_X + (n+1)A$, where n is the dimension of X. It follows from [16, Corollary 11.2.13] that $\mathcal{J}(X, \|pD\|) \otimes \mathcal{O}_X(G + pD)$ is globally generated for every p in $\mathbb{N}$. This shows that Z is not contained in the base locus of $|G + pD|$ for every p. Since G is ample, we deduce from Proposition 1.19 that Z is not contained in $\mathbf{B}_-(D)$.

We show now (ii) $\Rightarrow$ (i). With the above notation, we see that Z is not contained in the base locus of $|G + pD|$ for every p. On the other hand, since D is big, by Kodaira's Lemma we can find a positive integer p_0 and an integral effective divisor B such that $p_0 D$ is linearly equivalent to $G + B$. For $p \geqslant p_0$, pD is linearly equivalent with $(p - p_0)D + G + B$, so $v(|pD|) \leqslant v(|B|)$. The assertion in (i) follows easily.

In order to prove (iii)$\Rightarrow$(ii) we proceed similarly. By Kodaira's Lemma we can find a positive integer p_0 and integral divisors H and B, with H ample and B effective such that $p_0 D$ is linearly equivalent to $H + B$. For $p \geqslant p_0$, we have pD linearly equivalent to $(p - p_0)D + H + B$. Since Z is not contained in $\mathbf{B}_-(D)$, it follows that Z is not contained in $\mathbf{B}_+((p - p_0)D + H)$, hence

$$v\big(\|pD\|\big) \leqslant v\big(\|(p - p_0)D + H\|\big) + v\big(\|B\|\big) = v\big(\|B\|\big).$$

Hence $v(\|D\|) \leqslant v(\|B\|)/p$ for every p, and therefore $v(\|D\|) = 0$. □

DEFINITION 2.9. — *For every irreducible subvariety Z of a normal variety X, there is a discrete valuation v whose center is Z. For example, take the normalized blow-up of X along Z, and v the valuation corresponding to a component of the exceptional divisor that dominates Z. If X is smooth, we get this way the valuation given by the order of vanishing at the generic point of Z. For a $\mathbb{Q}$-divisor D, this gives the* asymptotic order of vanishing

of D along Z:

$$\mathrm{ord}_Z\big(\|D\|\big) := \lim_{p\to\infty} \frac{\mathrm{ord}_Z(|pD|)}{p},$$

where $\mathrm{ord}_Z(|pD|)$ *is the order of vanishing along* Z *of a generic divisor in* $|pD|$.

COROLLARY 2.10. — *If* X *is smooth, we have the equality of sets*

$$\mathbf{B}_-(D) = \bigcup_{m\in\mathbb{N}} \mathcal{Z}\big(\mathcal{J}(X, \|mD\|)\big).$$

Proof. — The assertion follows from Propositions 2.5 and 2.8, using the fact that every irreducible subvariety Z of X is the center of some discrete valuation. □

Other invariants. — Similar asymptotic invariants can be defined starting from different invariants of singularities, instead of valuations. For example, if X is a smooth variety and Z is an irreducible subvariety, we can consider either the Arnold multiplicity or the Samuel multiplicity at the generic point of Z. If $R = \mathcal{O}_{X,Z}$ and $\mathfrak{a}$ is an ideal in R, then the Arnold multiplicity of $\mathfrak{a}$ at the generic point of Z is $\mathrm{Arn}_Z(\mathfrak{a}) := 1/\mathrm{lct}(\mathfrak{a})$, where $\mathrm{lct}(\mathfrak{a})$ is the log canonical threshold of the pair $(\mathrm{Spec}(R), V(\mathfrak{a}))$. If $\mathfrak{a}$ is of finite colength in R, then as usual

$$\mathrm{e}_Z(\mathfrak{a}) := \lim_{\ell\to\infty} \frac{\mathrm{colength}_R(\mathfrak{a}^\ell)}{\ell^d/d\,!}$$

denotes the Samuel multiplicity of $\mathfrak{a}$, where d is the codimension of Z.

Given a big divisor D, if $\mathfrak{a}_p$ denotes the image of $b(|pD|)$ in R, then

$$\mathrm{Arn}_Z\big(|pD|\big) := \mathrm{Arn}_Z(\mathfrak{a}_p)$$

whenever $|pD|$ is nonempty. The corresponding asymptotic invariant is

$$\mathrm{Arn}_Z(\|D\|) := \lim_{p\to\infty} \frac{\mathrm{Arn}_Z(|pD|)}{p}.$$

If Z is not properly contained in any irreducible component of $\mathbf{B}_+(D)$, then for $p \gg 0$ we put $\mathrm{e}_Z(|pD|)^{1/d} := \mathrm{e}_Z(\mathfrak{a}_p)^{1/d}$ and

$$\mathrm{e}_Z\big(\|D\|\big)^{1/d} := \lim_{p\to\infty} \frac{\mathrm{e}_Z(|pD|)^{1/d}}{p}.$$

These invariants satisfy analogous properties with $v(\|.\|)$. In particular, they can be extended in the obvious way to big $\mathbb{Q}$-divisors and depend only on the numerical class of the divisor. Moreover, they are positive at D if

and only if Z is contained in $\mathbf{B}_{-}(D)$. The proofs are analogous using the fact that for two ideals $\mathfrak{a}$ and $\mathfrak{b}$ in R we have

$$\mathrm{Arn}_Z(\mathfrak{a}\mathfrak{b}) \leqslant \mathrm{Arn}_Z(\mathfrak{a}) + \mathrm{Arn}_Z(\mathfrak{b}) \quad \text{and} \quad \mathrm{e}_Z(\mathfrak{a}\mathfrak{b})^{1/d} \leqslant \mathrm{e}_Z(\mathfrak{a})^{1/d} + \mathrm{e}_Z(\mathfrak{b})^{1/d}$$

(where in the second inequality we assume that $\mathfrak{a}$ and $\mathfrak{b}$ have finite colength). The inequality for Arnold multiplicities follows directly from the definition of log canonical thresholds, while the inequality for Samuel multiplicities is proved in [21].

3. Asymptotic invariants as functions on the big cone

In this section we study the variation of the asymptotic invariants of base loci. In particular, we prove Theorems A and B from the Introduction. We also present some examples.

A general uniform continuity lemma. — The continuity statement (Theorem A) follows formally from an elementary general statement about convex functions on cones, which we formulate here.

Consider an open convex cone $C \subset \mathbb{R}^n$ and suppose we have a function

$$f : C \cap \mathbb{Q}^n \longrightarrow \mathbb{R}_+.$$

We assume that this function satisfies the following properties:

(i) (*homogeneity*) $f(q \cdot x) = q \cdot f(x)$, for all $q \in \mathbb{Q}_+^*$ and $x \in C \cap \mathbb{Q}^n$;
(ii) (*convexity*) $f(x+y) \leqslant f(x) + f(y)$, for all $x, y \in C \cap \mathbb{Q}^n$;
(iii) (*"ample" basis*) there exists a basis $a_1, \ldots, a_n$ for $\mathbb{Q}^n$, contained in C, such that $f(a_i) = 0$ for all i.

Proposition 3.1. — *Under the assumptions above, the function f satisfies the following (locally uniformly Lipschitz-type) property on $C \cap \mathbb{Q}^n$: for every $x \in C$, there exists a compact neighborhood K of x contained in C, and a constant $M_K > 0$, such that for all rational points $x_1, x_2 \in K$*

$$(\star) \qquad \big|f(x_1) - f(x_2)\big| \ \leqslant \ M_K \|x_1 - x_2\|.$$

In particular, f extends uniquely by continuity to a function on all of C satisfying $(\star)$.

Proof. — Consider a cube $K \subset C$ with rational endpoints. With respect to the chosen basis, we can write

$$K = [c_1, d_1] \times [c_2, d_2] \times \cdots \times [c_n, d_n].$$

We work with the norm given by $\|\Sigma u_i a_i\| = \max_i\{|u_i|\}$. We have to show that there is $M_K \geqslant 0$ such that

$$|f(x_1) - f(x_2)| \leqslant M_K \cdot \|x_1 - x_2\|,$$

for every x_1, $x_2 \in K \cap \mathbb{Q}^n$.

Since K is compact, there exists $\delta \in \mathbb{Q}_+^*$ such that $x - \sum \delta a_i \in C$ for all $x \in K$. We can also assume (by subdividing K if necessary), that all the sides of K have length $< \delta$. Take now any rational points x_1, $x_2 \in K$, write $x_2 - x_1 = \sum_i \lambda_i a_i$, and set $\lambda = \|x_2 - x_1\|$. Note that we must have $\lambda < \delta$. We will estimate the difference $|f(x_1) - f(x_2)|$.

By repeatedly using properties (i)–(iii) we get

$$\begin{aligned}
f(x_1) &= f\big(x_2 - (x_2 - x_1)\big) = f\Big(x_2 - \sum_i \lambda a_i\Big) \\
&= f\Big((1-\lambda/\delta)x_2 + \lambda/\delta\Big(x_2 - \sum_i \delta a_i\Big)\Big) \\
&\leqslant f\big((1-\lambda/\delta)x_2\big) + f\Big(\lambda/\delta\Big(x_2 - \sum_i \delta a_i\Big)\Big) \\
&= (1-\lambda/\delta) f(x_2) + (\lambda/\delta) f\Big(x_2 - \sum_i \delta a_i\Big) \\
&\leqslant f(x_2) + \frac{f(x_2 - \sum_i \delta a_i)}{\delta} \cdot \lambda = f(x_2) + \frac{f(x_2 - \sum_i \delta a_i)}{\delta} \cdot \|x_2 - x_1\|.
\end{aligned}$$

On the other hand, $f(x_2 - \sum_i \delta a_i)$ can be bounded uniformly. Indeed, since $x_2 \in K$, we have that $x_2 - \sum_i c_i a_i$ is a positive combination of the a_i's, so it belongs to C and $f(x_2 - \sum_i c_i a_i) = 0$. Thus we get that

$$f\Big(x_2 - \sum_i \delta a_i\Big) \leqslant f\big(\sum_i (c_i - \delta) a_i\big).$$

If we take $M_K = f\big(\sum_i (c_i - \delta) a_i\big)/\delta$, it follows that $|f(x_1) - f(x_2)| \leqslant M_K \|x_1 - x_2\|$ for every x_1, $x_2 \in K \cap \mathbb{Q}^n$, as required. □

Proof of Theorem A and Theorem B. — We now explain how Proposition 3.1 applies to complete the proofs of these two results from the Introduction. Let v be a discrete valuation of the function field of X. The dependence on the numerical equivalence class in part (i) of Theorem A, as well as Theorem B, have already been proved for $\mathbb{Q}$-classes in Corollary 2.7 and Proposition 2.8 respectively.

The cone C will be the cone of big divisors $\mathrm{Big}(X)_{\mathbb{R}} \subset N^1(X)_{\mathbb{R}}$. The fact that the three properties required in the proposition are satisfied for $v(\|.\|)$ has already been checked in previous sections:

(i) On rational classes, $v(\|.\|)$ is homogeneous of degree one thanks to Remark 2.3.

(ii) The convexity property was noted in Proposition 2.4.

(iii) This follows from the general fact that one can choose an ample basis for the Néron-Severi space and the obvious fact that $v(\|A\|) = 0$ if A is an ample $\mathbb{Q}$-divisor.

Note that the proof implies the slightly stronger statement that the three invariants extend to locally uniformly continuous functions on the real big cone.

It remains only to show that Theorem B holds for arbitrary big $\mathbb{R}$-classes ξ. Suppose first that $Z \not\subseteq \mathbf{B}_-(\xi)$, so that for every ample class α with $\xi + \alpha$ rational, we have $Z \not\subseteq \mathbf{B}_-(\xi + \alpha)$. Corollary 2.8 gives $v(\|\xi + \alpha\|) = 0$. Letting α go to 0, and using continuity, we get $v(\|\xi\|) = 0$. On the other hand, suppose that $Z \subseteq \mathbf{B}_-(\xi)$. It follows from Proposition 1.19 that there is an ample class α such that $\xi + \alpha$ is rational and $Z \subseteq \mathbf{B}_-(\xi + \alpha)$. Therefore Corollary 2.8 gives $v(\|\xi\|) \geqslant v(\|\xi + \alpha\|) > 0$. This completes the proof of Theorems A and B.

Remark 3.2. — If X is a smooth variety and Z is an irreducible subvariety of codimension d, Proposition 3.1 applies for $f = \mathrm{Arn}_Z$, so we get analogues of Theorems A and B for $\mathrm{Arn}_Z(\|.\|)$. The same applies for $\mathrm{e}_Z(\|.\|)$ with one change: the domain on which it is defined is $\mathrm{Big}^Z(X)_{\mathbb{R}}$, consisting of classes of big $\mathbb{R}$-divisors D such that Z is not a proper subset of an irreducible component of $\mathbf{B}_+(D)$.

Examples and complements. — We next give some examples and further information about our invariants. We start with an alternative computation of the order along a valuation for a real class. If v is a discrete valuation of the function field of X and D is an effective divisor on X, then we define $v(D)$ as the order of an equation of D in the local ring R of v. This extends by linearity to the case of an $\mathbb{R}$-divisor D.

LEMMA 3.3. — *If $\alpha \in N^1(X)_{\mathbb{R}}$ is big, then*

$$v\big(\|\alpha\|\big) = \inf_D v(D), \tag{3.1}$$

where the minimum is over all effective $\mathbb{R}$-divisors D with numerical class α.

Proof. — Let us temporarily denote by $v'(\|\alpha\|)$ the infimum in (3.1). It is easy to check from the definition that v' satisfies properties (i), (ii) and (iii) in Proposition 3.1. Hence v' is continuous, and it is enough to show that

$$v'\big(\|\alpha\|\big) = v\big(\|\alpha\|\big)$$

when α is the class of a big integral divisor E. Inequality $v'(\|\alpha\|) \leqslant v(\|\alpha\|)$ follows from the definition of the two functions.

For the reverse inequality, suppose that D is an effective $\mathbb{R}$-divisor, numerically equivalent to E. We have to check that $v(\|D\|) \leqslant v(D)$. This is clearly true if D is a $\mathbb{Q}$-divisor. In the general case, it is enough to vary the coefficients of the components of D to get a sequence of effective $\mathbb{Q}$-divisors with limit D. Taking the limit, we get the desired inequality. □

Recall that if X is smooth and $Z \subset X$ is an irreducible subvariety, we denote by ord_Z the valuation given by the order of vanishing at the generic point of Z.

Example 3.4. — We check the assertion in Example 1.17. Let X be a smooth projective surface, and D a big $\mathbb{R}$-divisor with Zariski decomposition $D = P + N$. We prove that $\mathbf{B}_-(D) = \mathrm{Supp}(N)$. If A is ample, then $P + A$ is ample, hence $\mathbf{B}_+(D + A) \subseteq \mathrm{Supp}(N)$. This shows that $\mathbf{B}_-(D) \subseteq \mathrm{Supp}(N)$.

For the reverse inclusion, we use the previous lemma. If E is an effective $\mathbb{R}$-divisor numerically equivalent with D, then $E - N$ is effective, so $\mathrm{ord}_Z(\|D\|) \geqslant \mathrm{ord}_Z(N)$ for every Z. If Z is a component of N, we deduce from Theorem B that Z is contained in $\mathbf{B}_-(D)$.

A similar argument, based on Lemma 3.3, shows that if C is a curve in X, then $\mathrm{ord}_C(\|D\|)$ is equal to the coefficient of C in N. □

Example 3.5. — Let $X = \mathrm{Bl}_{\{P,Q\}}(\mathbf{P}^n)$ be the blowing-up of $\mathbf{P}^n$ at two points P and Q. We assume $n \geqslant 2$. The Néron-Severi group of X is generated by the classes of the exceptional divisors E_1 and E_2 and by the pull-back H of a hyperplane in $\mathbf{P}^n$. A line bundle $L = \alpha H - \beta_1 E_1 - \beta_2 E_2$ is big if and only if

$$\alpha > \max\{\beta_1, \beta_2, 0\}.$$

We describe now the decomposition of the set of stable classes into five chambers and the behavior of our asymptotic invariants on each of these chambers.

The first region is described by $\beta_1 < 0$ and $\alpha > \beta_2 > 0$. If L is inside this region, then L is stable and $\mathbf{B}(L) = E_1$. Moreover, we have $\mathrm{ord}_{E_1}(\|L\|) = -\beta_1$. A similar behavior holds inside the second region, described by $\beta_2 < 0$ and $\alpha > \beta_1 > 0$. The third chamber is given by β_1, $\beta_2 < 0$ and $\alpha > 0$. If L belongs to this chamber, we have $\mathbf{B}(L) = E_1 \cup E_2$, and $\mathrm{ord}_{E_1}(\|L\|) = -\beta_1$ and $\mathrm{ord}_{\|E_2\|}(L) = -\beta_2$.

From now on we assume that $\beta_1, \beta_2 > 0$. The fourth chamber is given by adding the condition $\alpha > \beta_1 + \beta_2$. This chamber gives precisely the ample

cone. The last region is given by the opposite inequality $\alpha < \beta_1 + \beta_2$. Every L in this chamber is stable, and $\mathbf{B}(L) = \ell$, the proper transform of the line $\overline{PQ}$. In order to compute the invariants associated to L along ℓ, we may assume that $P = (1 : 0 : \ldots : 0)$ and $Q = (0 : 1 : 0 : \ldots : 0)$. We see that $H^0(X, L)$ is spanned by

$$\Big\{ \prod_{i=0}^{n} X_i^{p_i} \,;\; p_0 \leqslant \alpha - \beta_1,\; p_1 \leqslant \alpha - \beta_2,\; \sum_i p_i = \alpha \Big\}.$$

Therefore we get coordinates $x_2, \ldots, x_n$ at the generic point of ℓ such that the base locus of L is defined at this point by $(\prod_{i=2}^{n} x_i^{q_i} \,;\; \sum_i q_i \geqslant \beta_1 + \beta_2 - \alpha)$ and $\mathrm{ord}_\ell(\|L\|) = \beta_1 + \beta_2 - \alpha$. Note that we also have

$$\mathrm{Arn}_\ell\big(\|L\|\big) = (\beta_1 + \beta_2 - \alpha)/(n-1) \quad \text{and} \quad \mathrm{e}_\ell\big(\|L\|\big)^{1/(n-1)} = (\beta_1 + \beta_2 - \alpha).$$

We will see in the next section that for every toric variety (or more generally, for every variety with finitely generated linear series) there is a fan refining the big cone as above, such that on each of the subcones our asymptotic invariants are polynomial. □

Example 3.6. — We give now an example when the asymptotic invariants can take irrational values for $\mathbb{Q}$-divisors. Moreover, we will see that in this case the invariants are not locally polynomial. The idea of this example is due to Cutkosky [8]. We follow the approach in Küronya [15] where this is used to give an example when the volume function is not locally polynomial.

We start by recalling the notation and the definitions from [15]. Let $S = E \times E$, where E is a general elliptic curve. If F_1 and F_2 are fibers of the respective projections, and if Δ is the diagonal, then the classes of F_1, F_2 and Δ span $N^1(X)_{\mathbb{R}}$. If h is an ample class on S and if $\alpha \in N^1(X)_{\mathbb{R}}$, then α is ample (equivalently, it is big) if and only if $(\alpha^2) > 0$ and $(\alpha \cdot h) > 0$. We consider the following ample divisors on S: $D = F_1 + F_2$ and $H = 3(F_2 + \Delta)$.

Let $\pi : X = \mathbf{P}(\mathcal{O}_S(D) \oplus \mathcal{O}_S(-H)) \to S$ be the canonical projection. If $0 \leqslant t \ll 1$, with $t \in \mathbb{Q}$, we take $L_t = \mathcal{O}(1) + t \cdot \pi^* F_1$, which is big. We consider the section of π induced by the projection $\mathcal{O}_S(D) \oplus \mathcal{O}_S(-H) \to \mathcal{O}_S(-H)$, and denote by E its image. We will compute $\mathrm{ord}_E(\|D_t\|)$. If k is a positive integer such that $kt \in \mathbb{N}$, then

$$H^0\big(X, \mathcal{O}_X(kD_t)\big) \simeq \bigoplus_{i+j=k} H^0\big(S, \mathcal{O}_S(iD - jH + ktF_1)\big).$$

An easy computation shows that if

$$\sigma(t) = \frac{9 + 5t - \sqrt{49t^2 + 78t + 45}}{18 - 12t},$$

then $H^0(S, \mathcal{O}_S(iD - jH + (i+j)tF_1))$ is zero if $j/i > \sigma(t)$ and it is non-zero if $j/i < \sigma(t)$. Note also that $\mathcal{O}_X(kD_t - pE)|_E \simeq \mathcal{O}_S(pD - (k-p)H + ktF_1)$. We deduce

$$\big|\mathrm{ord}_E\big(|kD_t|\big) - \big\lceil k/(1+\sigma(t))\big\rceil\big| \leqslant 1.$$

This implies $\mathrm{ord}_E(\|D_t\|) = 1/(1+\sigma(t))$. We get $\mathrm{ord}_E(D_0) \notin \mathbb{Q}$ by taking $t = 0$. Moreover, it is clear that ord_E is not a locally polynomial function in any neighbourhood of D_0. □

Example 3.7 (Surfaces). — The case of surfaces has been studied recently both from the point of view of the volume function and of asymptotic base loci in [1]. We interpret now their results in our framework.

Let X be a smooth projective surface, and let $B \subset X$ be a collection of curves having negative definite intersection form. Consider the (possibly empty) set $\mathcal{S}_B$ consisting of stable classes $\alpha \in N^1(X)_{\mathbb{R}}$ with $\mathbf{B}_+(\alpha) = B$. It is clear that if it is non-empty then $\mathcal{S}_B$ is an open cone. Moreover, it is also convex, since if $\alpha_1 = P_1 + N_1$ and $\alpha_2 = P_2 + N_2$ are two Zariski decompositions with $\mathrm{Supp}(N_1) = \mathrm{Supp}(N_2)$, then $\alpha_1 + \alpha_2 = (P_1 + P_2) + (N_1 + N_2)$ is the Zariski decomposition of $\alpha_1 + \alpha_2$.

If $E_1, \ldots, E_r$ are the irreducible components of B, and if $\alpha \in \mathcal{S}_B$ has Zariski decomposition $\alpha = P + \sum_{j=1}^r a_j E_j$, then $(\alpha \cdot E_i) = \sum_{j=1}^r (E_i \cdot E_j) a_j$. Therefore the coefficients a_j depend linearly on α, hence for every curve C on X, the function $\mathrm{ord}_C(\|.\|)$ is linear on $\mathcal{S}_B$, with rational coefficients.

The closed cones $\overline{\mathcal{S}_B}$ give a cover of $\overline{\mathrm{Big}(X)}_{\mathbb{R}}$ that is locally finite inside the big cone. Indeed, suppose that $\alpha \in \mathrm{Big}(X)_{\mathbb{R}}$. It follows from Corollary 1.6 that if β is in a suitable open neighbourhood U of α and if $\beta \in \mathcal{S}_B$, then $B \subseteq \mathbf{B}_+(\alpha)$. In particular, there are only finitely many possibilities for B.

We show now that each cone $\overline{\mathcal{S}_B}$ is rational polyhedral inside the big cone. We keep the above notation, and without any loss of generality, we may assume that the open subset U is a convex cone. We have seen that U is covered by finitely many $\overline{\mathcal{S}_{B_i}}$, and for every curve C in X, we have linear functions L_i such that $\mathrm{ord}_C(\|.\|) = L_i$ on $\overline{\mathcal{S}_{B_i}}$. Since the asymptotic order function along C is convex, it follows from general considerations that $\mathrm{ord}_C(\|.\|) = \max_i L_i$ on U, i.e. $\mathrm{ord}_C(\|.\|)$ is piecewise linear on U. On the other hand, $\mathcal{S}_B \cap U$ is the set of those $\beta \in U$ such that $\mathrm{ord}_C(\|\xi\|) = 0$ for ξ in a neighborhood of β, for every C in $\mathbf{B}_+(\alpha)$ but not in B, and

$\mathrm{ord}_C(\|\beta\|) \neq 0$ for C in B. Therefore $\overline{\mathcal{S}}_B \cap U$ is the intersection of U with finitely many half-spaces, which proves our assertion. □

The structure of the unstable locus. — We discuss the structure of the locus of unstable classes inside the big cone. The picture is similar to that given by a theorem of Campana and Peternell (cf. [16, Chapter 1.5]) for the structure of the boundary of the nef cone. We assume that X is smooth.

We first fix a closed subset $Z \subseteq X$. Let v_Z be a fixed discrete valuation of the function field of X such that Z is the center of v_Z on X. We use the asymptotic order function $v_Z(\|.\|)$ to obtain information on the locus of Z-unstable points. The zero locus

$$\mathcal{N}_Z := \big\{\xi \in \mathrm{Big}(X)_{\mathbb{R}}\,;\; v_Z(\|\xi\|) = 0\big\}$$

is a convex cone which is closed in $\mathrm{Big}(X)_{\mathbb{R}}$. By Theorem B this is the set of big classes ξ such that Z is not contained in $\mathbf{B}_-(\xi)$. We call it the null cone determined by Z. A class $\xi \in \mathrm{Big}(X)_{\mathbb{R}}$ is called *Z-unstable* if $Z \subseteq \mathbf{B}_+(\xi)$, but $Z \not\subseteq \mathbf{B}_-(\xi)$. It is easy to see that the Z-unstable classes are precisely the big classes that lie on the boundary of $\mathcal{N}_Z$.

By definition, a class $\xi \in \mathrm{Big}(X)_{\mathbb{R}}$ is unstable if and only if it is Z-unstable for some irreducible component $Z \subseteq \mathbf{B}_+(\xi)$. Thus ξ is unstable if and only if it is Z-unstable for *some* subvariety Z. Thus the picture is that we have convex null-cones $\mathcal{N}_Z$ in $\mathrm{Big}(X)_{\mathbb{R}}$ indexed by all subvarieties $Z \subseteq X$, and

$$(*) \qquad \mathrm{Unstab}(X) = \bigcup_Z \partial\mathcal{N}_Z.$$

It follows for example that the set of unstable classes does not contain isolated rays. (This is just a general statement about boundaries of convex cones. Visually, this says that in any section of the big cone the unstable locus does not have isolated points.)

Note that the union in (*) can be taken over countably many Z. Indeed, it is enough to consider those Z which are irreducible components of augmented base loci (and we may restrict to $\mathbb{Q}$-divisors by Proposition 1.5). Since $\mathbf{B}_+(D)$ depends only on the numerical equivalence class of the $\mathbb{Q}$-divisor D, we have to consider only countably many subvarieties.

Since $\partial\mathcal{N}_Z$ is the boundary of a convex cone in the Néron-Severi space, it has measure zero and therefore so does $\mathrm{Unstab}(X)$.

Remark 3.8. — In fact, one can show that there is an open dense subset $V \subset \mathrm{Unstab}(X)$ which looks locally like the boundary of a unique $\mathcal{N}_Z$: for

every $\xi \in V$, there is an open neighborhood $U(\xi)$ of ξ, and an irreducible closed subset $Z \subseteq X$, such that $\mathrm{Unstab}(X) \cap U(\xi) = \partial \mathcal{N}_Z \cap U(\xi)$.

4. Asymptotic invariants on varieties with finitely generated linear series

Our goal in this section is to prove Theorem D from the Introduction. In fact, we will prove a somewhat stronger local statement.

Let X be a normal projective variety, and fix r integral divisors $D_1, \ldots, D_r$ on X such that some linear combination of the D_i (with rational coefficients) is big. Setting $N = \mathbb{Z}^r$ and $N_{\mathbb{R}} = N \otimes \mathbb{R} = \mathbb{R}^r$, the choice of the D_i gives linear maps

$$\phi : N \longrightarrow N^1(X), \quad \phi_{\mathbb{R}} : N_{\mathbb{R}} \longrightarrow N^1(X)_{\mathbb{R}}.$$

We denote by $B \subseteq N_{\mathbb{R}}$ the pull-back $\phi_{\mathbb{R}}^{-1}\big(\mathrm{Big}(X)_{\mathbb{R}}\big)$, so that $\overline{B}$ is the pull-back of the closure of $\mathrm{Big}(X)_{\mathbb{R}}$. The main result of this section is:

THEOREM 4.1. — *Assume that the graded $\mathbb{C}$-algebra*

$$\mathrm{Cox}(D_1, \ldots, D_r) := \bigoplus_{m=(m_i)\in\mathbb{Z}^r} H^0\big(X, \mathcal{O}_X(m_1 D_1 + \cdots + m_r D_r)\big) \tag{4.1}$$

is finitely generated. Then $\overline{B}$ is a rational polyhedral cone and for every discrete valuation v of the function field of X, the pull-back to B of the function $v(\|.\|)$ can be extended by continuity to $\overline{B}$. Moreover, there is a fan Δ with support $\overline{B}$ such that every $v(\|.\|)$ is linear on the cones in Δ.

Before proving Theorem 4.1 we give a few examples of finitely generated Cox rings.

Example 4.2. — Suppose $N^1(X)_{\mathbb{R}}$ has dimension 1. Let D be any ample divisor on X. Then the $\mathbb{Z}$-graded ring

$$\mathrm{Cox}(D) = \bigoplus_{m\in\mathbf{Z}} H^0(X, mD)$$

is finitely generated since it is isomorphic to the projective coordinate ring of X. Hence X has finitely generated linear series. □

Example 4.3. — If X is the projective plane blown up at an arbitrary number of *collinear* points, then it is shown in [14] that X has finitely generated linear series.

Example 4.4. — If $X = \mathrm{Bl}_{p_1,\ldots,p_r}(\mathbf{P}^n)$, where $n \geqslant 2$, $r \geqslant n+3$, and $p_1, \ldots, p_r$ are distinct points lying on a rational normal curve in $\mathbf{P}^n$, then it is shown in [6] that X has finitely generated linear series. (Cf. *loc. cit.* for a few other examples.)

To prove Theorem 4.1, it is convenient to pass to a local statement involving families of ideals.

Definition 4.5. — *Let V be any variety, and let $S \subseteq \mathbb{Z}^r$ be a subsemigroup (in most cases we will take $S = \mathbb{N}^r$ or $S = \mathbb{Z}^r$). An S-graded system of ideals on V is a collection $\mathfrak{a}_\bullet = \{\mathfrak{a}_m\}_{m\in S}$ of ideal sheaves on V, with $\mathfrak{a}_0 = \mathcal{O}_V$, which satisfies*

$$\mathfrak{a}_m \cdot \mathfrak{a}_{m'} \subseteq \mathfrak{a}_{m+m'}$$

for all $m, m' \in S$. The Rees algebra of $\mathfrak{a}_\bullet$ is the S-graded $\mathcal{O}_V$-algebra

$$R(\mathfrak{a}_\bullet) = \bigoplus_{m\in S} \mathfrak{a}_m,$$

and $\mathfrak{a}_\bullet$ is finitely generated if $R(\mathfrak{a}_\bullet)$ is a finitely generated $\mathcal{O}_V$-algebra. □

For example, starting with divisors $D_1, \ldots, D_r$ on a projective variety X, we have an $\mathbb{N}^r$-graded sequence of ideals $\mathfrak{b}_\bullet$ such that $\mathfrak{b}_m$ is the ideal defining the base locus of $|m_1D_1 + \cdots + m_rD_r|$. If the Cox ring $\mathrm{Cox}(D_1, \ldots, D_r)$ in (4.1) is finitely generated, then the corresponding system $\mathfrak{b}_\bullet$ of base ideals is likewise finitely generated.

Remark 4.6 (Invariants for graded systems). — It would be very interesting to know what sort of regularity properties the functions defined by the invariants introduced in §2 satisfy. For example, are they piecewise analytic on a dense open set in their domains? As the reader has probably noticed, these invariants can also be defined for an arbitrary graded sequence of ideals. As a consequence, most of what we have done in the previous sections can be transposed into the abstract setting of S-graded systems (in this case the Néron-Severi space is replaced by the group generated by S). One can define analogues of the effective and of the nef cones in this setting and under mild hypotheses (for example that the system in question contain a non-empty "ample" cone), one can prove the continuity of the asymptotic invariants in this setting. See [22] and [10] for more on this point of view. Work of Wolfe [22] suggests that in this abstract setting one can't generally expect any good behavior other than that implied by convexity. One might hope however that this sort of pathology does not occur in the global geometric setting.

For the proof of Theorem 4.1, the main point is to show that a finitely generated graded system essentially is given by products of powers of finitely many ideals. This is the content of the following Proposition. We fix a lattice $N \simeq \mathbb{Z}^r \subset N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ and a finitely generated, saturated subsemigroup $S \subseteq N$. This means that if C is the convex cone generated by S, then C is a rational, polyhedral cone, and $S = C \cap N$.

We denote by $\overline{\mathfrak{a}}$ the integral closure of an ideal $\mathfrak{a}$.

PROPOSITION 4.7. — *With the above notation, let $\mathfrak{a}_\bullet$ be a finitely generated S-graded system of ideals on the variety V (more generally, V can be an arbitrary Noetherian scheme). Then there is a smooth fan Δ with support C, such that for every smooth refinement Δ' of Δ there is a positive integer d with the following property. For every cone $\sigma \in \Delta'$, if we denote by $e_1, \ldots, e_s$ the generators of $S_\sigma := \sigma \cap N$, then*

$$\overline{\mathfrak{a}_{d\sum_i p_i e_i}} = \overline{\prod_i \mathfrak{a}_{de_i}^{p_i}}, \tag{4.2}$$

for every $p = (p_i) \in \mathbb{N}^s$.

It is clear that it is enough to prove Proposition 4.7 when $X = \mathrm{Spec}(R_0)$ is affine. Before giving the proof we need a few lemmas. The following one is well known, but we include a proof for the benefit of the reader.

LEMMA 4.8. — *With S as above, suppose $R = \bigoplus_{m \in S} R_m$ is an S-graded ring that is finitely generated as an R_0-algebra. If $S' \subseteq S$ is a (finitely generated, saturated) subsemigroup, and if $R' = \bigoplus_{m \in S'} R_m$, then R' is a finitely generated R_0-algebra.*

Proof. — Choose homogeneous generators $x_1, \ldots, x_q$ of R as an R_0-algebra, and let $m_i = \deg(x_i)$. We get a surjective morphism of R_0-algebras

$$\Phi : R_0[X_1, \ldots, X_q] \longrightarrow R$$

given by $\Phi(X_i) = x_i$. This is homogeneous with respect to the semigroup homomorphism $\phi : \mathbb{N}^q \longrightarrow S$ that takes the i-th coordinate vector to m_i.

If $T := \phi^{-1}(S')$, then T is cut out in $\mathbb{N}^q$ by finitely many linear inequalities, hence T is finitely generated by Gordan's Lemma. If $w = (w_1, \ldots, w_q)$ belongs to $\mathbb{N}^q$, we put X^w for the monomial $\prod_i X_i^{w_i}$. If we choose generators $v^{(1)}, \ldots, v^{(p)}$ for T, and let $y_i = \Phi(X^{v^{(i)}})$, then R' is generated over R_0 by $y_1, \ldots, y_p$. For this, it is enough to note that by the surjectivity of Φ, every homogeneous element in R' is a linear combination (with coefficients in R_0) of images of monomials with degrees in T. □

We will prove Proposition 4.7 by induction on $\dim(S)$. The following lemma which covers the case $S = \mathbb{N}$ is standard (see [4], Chap. III, Section 1, Prop. 2). Note that in this case we get a stronger statement than in Proposition 4.7.

LEMMA 4.9. — *If R_0 is a Noetherian ring, and if $R = \bigoplus_{m\in\mathbb{N}} R_m$ is an $\mathbb{N}$-graded, finitely generated R_0-algebra, then there is a positive integer d, such that $R_{dm} = R_d^m$, for every $m \in \mathbb{N} \setminus \{0\}$.*

We need one more easy result about cones and semigroups.

LEMMA 4.10. — *Let $N \subset N_{\mathbb{R}}$ be a lattice, and let $C \subseteq N_{\mathbb{R}}$ be a rational, polyhedral strongly convex cone. If $S = C \cap N$, and if $m_1, \ldots, m_p$ are the first non-zero integral vectors on the rays of C, then there is a positive integer d such that for every $m \in S$, dm lies in the semigroup T generated by $m_1, \ldots, m_p$.*

Proof. — Consider a smooth fan Δ that refines the cone C. By taking the first non-zero integral vectors on the rays in Δ, we get t extra vectors $m_1', \ldots, m_t'$. Since each of the cones in Δ is smooth, it follows that S is equal to the semigroup generated by the m_i and the m_j'. On the other hand, it is clear that the m_i span the cone $C \cap N_{\mathbb{Q}}$ over $\mathbb{Q}$. Therefore for every $j \leqslant t$, we can find $d_j \in \mathbb{N} \setminus \{0\}$ such that $d_j m_j'$ is in T. Take d to be the least common multiple of the d_j. □

Proof of Proposition 4.7. — We have already noticed that it is enough to prove the statement when $V = \mathrm{Spec}(R_0)$ is affine. Moreover, after taking a refinement of S, we may assume that the cone C spanned by S is strongly convex. We use induction on $\dim(S)$. If $\dim(S) = 1$, then we are done by Lemma 4.9.

Suppose now that $\dim(S) > 1$ and that we know the assertion in smaller dimensions. We use the construction in the proof of Lemma 4.8. Let $R = R(\mathfrak{a}_\bullet)$ be the Rees algebra of $\mathfrak{a}_\bullet$, and let $x_1, \ldots, x_q$ be homogeneous generators of R as an R_0-algebra. We put $m_i = \deg(x_i)$. Consider the surjective homomorphism of R_0-algebras $\Phi : R_0[X_1, \ldots, X_q] \to R$, given by $\Phi(X_i) = x_i$, and the corresponding semigroup homomorphism $\phi : \mathbb{N}^q \to S$ which takes the i^{th} coordinate vector to m_i. Let $\phi_{\mathbb{R}}$ be the extension of ϕ as a map $\mathbb{R}^q \to N_{\mathbb{R}}$.

Consider a smooth fan Δ refining C such that every m_i is on a ray of Δ. We apply now the induction hypothesis for each cone in Δ of dimension $\dim(S) - 1$ (note that Lemma 4.8 ensures the finite generation of the corresponding R_0-subalgebras). By refining Δ, we may assume that each face of dimension $\dim(S) - 1$ (as well as its refinements) satisfies (4.2) for a given

positive integer d. For example, we take d to be the least common multiple of the positive integers we get for each face. Note that every refinement of such Δ still satisfies these conditions (for a possibly different d). In order to complete the induction step, it is enough to show that every maximal cone $\sigma \in \Delta$ satisfies (4.2) for this d.

Let $e_1, \ldots, e_s$ be the generators of $S_\sigma := \sigma \cap N$ (hence $s = \dim(S)$). We put $\widetilde{S}_\sigma := \phi^{-1}(S_\sigma)$, and $\widetilde{\sigma} := \phi_{\mathbb{R}}^{-1}(\sigma)$ so that $\widetilde{S}_\sigma = \widetilde{\sigma} \cap \mathbb{N}^q$.

It is clear that $\widetilde{\sigma}$ is a rational, polyhedral, strongly convex cone. Now we claim that every element on ray of $\widetilde{\sigma}$ is mapped by $\phi_{\mathbb{R}}$ to the boundary of σ. Indeed, suppose that w is nonzero and lies on a ray of $\widetilde{\sigma}$. If w is also on a ray of $\mathbb{R}_+^q$, then $\phi_{\mathbb{R}}(w)$ is also on a ray of σ by our construction. Otherwise, w is in the interior of an r-dimensional face F of $\mathbb{R}_+^q$, where $2 \leqslant r \leqslant q$. If $\phi_{\mathbb{R}}(w)$ is in the interior of σ, then since $\phi_{\mathbb{R}}$ is continuous and σ is of maximal dimension, we can find an open convex neighborhood V of w in F, such that $\phi_{\mathbb{R}}(V)$ is contained in the interior of σ. But this contradicts the fact that w lies on a ray of $\widetilde{\sigma}$. We conclude that $\phi_{\mathbb{R}}(w)$ is in the boundary of σ.

We apply now Lemma 4.10 to find d' such that every element in $(d' \cdot \mathbb{N})^q \cap \widetilde{S}_\sigma$ is in the semigroup generated by the first integral points on the rays of $\widetilde{\sigma}$.

Suppose that $f \in \mathfrak{a}_{\sum_i p_i e_i}$, with $p_i \in d \cdot \mathbb{N}$. Since Φ is surjective, we can write $f = \sum_\alpha c_\alpha f_\alpha$, where $c_\alpha \in R_0$ and each f_α is of the form $\Phi(X^u)$, with $u \in \phi^{-1}(\deg(f)) \subseteq \widetilde{S}_\sigma$. Since $d'u$ lies in the semigroup generated by the first integral points on the rays of $\widetilde{S}_\sigma$, it follows that we can write $f_\alpha^{d'} = \prod_i g_i$, where each g_i is homogeneous, and $\deg(g_i) = \sum_j \theta_{ij} e_j$ lies in the boundary of σ. It follows from the induction hypothesis that

$$g_i^d \in \overline{\prod_j \mathfrak{a}_{de_j}^{\theta_{ij}}}, \quad \text{so that} \quad f_\alpha^{dd'} \in \overline{\prod_j \mathfrak{a}_{de_j}^{d'p_j}}.$$

Since $d \mid p_j$ for every j, we deduce

$$f_\alpha \in \overline{\prod_j \mathfrak{a}_{de_j}^{p_j/d}}.$$

Since $f = \sum_\alpha c_\alpha f_\alpha$, this implies that $\mathfrak{a}_{\sum_i p_i e_i} \subseteq \overline{\prod_j \mathfrak{a}_{de_j}^{p_j/d}}$. As we clearly have the inclusion $\prod_j \mathfrak{a}_{de_j}^{p_j/d} \subseteq \mathfrak{a}_{\sum_i p_i e_i}$, this completes the proof. □

We apply now Proposition 4.7 to prove Theorem 4.1.

Proof of Theorem 4.1. — Consider the set C consisting of those $m = (m_i) \in \mathbb{Q}^r$ such that

$$h^0\big(X, \mathcal{O}_X(pm_1 D_1 + \cdots + pm_r D_r)\big) \neq 0$$

for some positive integer p with $pm_i \in \mathbb{Z}$ for all i. It is clear that C is the set of points in $\mathbb{Q}^r$ of a rational convex cone. If we take a finite set of homogeneous generators of $\mathrm{Cox}(D_1, \ldots, D_r)$ as a $\mathbb{C}$-algebra, then their degrees span C, so C is polyhedral. Denote by $\overline{C}$ the closure of C in $\mathbb{R}^r$.

We have the following inclusions

$$\overline{\phi_{\mathbb{Q}}^{-1}\big(\mathrm{Big}(X)_{\mathbb{Q}}\big)} \subseteq \overline{C} \subseteq \phi_{\mathbb{R}}^{-1}\big(\overline{\mathrm{Big}(X)_{\mathbb{R}}}\big).$$

Since we have assumed that some linear combination of the D_i is big, we deduce that the above inclusions are equalities, so $\overline{B} = \overline{C}$, and therefore it is polyhedral.

We consider now the $\mathbb{Z}^r$-graded system $\mathfrak{b}_\bullet = (\mathfrak{b}_m)_{m \in \mathbb{Z}^r}$, where $\mathfrak{b}_m$ defines the base locus of $|m_1 D_1 + \cdots + m_r D_r|$. Our hypothesis implies that this is a finitely generated system, so we can find a fan Δ refining $\mathbb{Z}^r$ as in Proposition 4.7. If v is a discrete valuation of the function field of X, then we define as in §2

$$\widetilde{v} : \overline{B} \cap \mathbb{Q}^r \longrightarrow \mathbb{R}_+, \quad \widetilde{v}(m) = \lim_{p \to \infty} \frac{v(|pm_1 D_1 + \cdots + pm_r D_r|)}{p},$$

where the limit is over those p which are divisible enough. Since the valuation of an ideal is equal to that of its integral closure, it follows from (4.2) that this function is linear on each cone in Δ. It follows that $\widetilde{v}$ can be uniquely extended by continuity to $\overline{B}$ (and the extension is again piecewise linear). Moreover, it is clear from definition that $\widetilde{v}$ agrees with the pull-back of $v(\|.\|)$ on B. □

Remark 4.11. — If X is smooth, similar considerations apply to the functions Arn_Z and e_Z introduced at the end of §2. Note however that the function

$$m = (m_i) \in \mathbb{Z}^r \longmapsto \mathrm{Arn}_Z(\mathfrak{a}_1^{m_1} \cdots \mathfrak{a}_r^{m_r})$$

is not necessarily linear. It is however piecewise linear (it is linear on a fan refinement which does not depend on Z, but only on the log resolution of the ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$). Therefore we get our conclusion after passing to a suitable refinement of Δ.

In the case of e_Z, it follows from (4.2) that the set of those $m \in \mathbb{Q}^r$ such that Z is not properly contained in an irreducible component of $\mathbf{B}(m_1 D_1 + \cdots + m_r D_r)$ is the set of rational points in a union of cones in Δ. For such m we define $\widetilde{\mathrm{e}_Z}(m)$ in the obvious way, and (4.2) implies that $\widetilde{\mathrm{e}_Z}$ is polynomial of degree d on each of these cones.

The case of varieties with finitely generated linear series, which was stated in the Introduction, follows easily from Proposition 4.7.

Proof of Theorem D. — Take divisors $D_1, \ldots, D_r$ as in Definition C. If we consider the corresponding map $\phi : \mathbb{Z}^r \to N^1(X)$, then $\phi_{\mathbb{R}}$ is an isomorphism. All the assertions now follow from Theorem 4.1. □

Remark 4.12. — In the context of Theorem D, note that if L is a line bundle whose class α is on the boundary of $\overline{\mathrm{Eff}}(X)_{\mathbb{R}}$, then it is not clear that $v(\|\alpha\|)$ (or the other functions) can be defined in terms of linear series of multiples of L. On the other hand, it follows from the proof of Theorem 4.1 that there does exist *some* line bundle M numerically equivalent to L such that $v(\|\alpha\|)$ can be defined using linear series of multiples of M.

We conclude with another application of Proposition 4.7 to the study of the volume function. We fix a smooth n-dimensional variety X. Recall that if $L \in \mathrm{Pic}(X)$, then the volume of L is given by

$$\mathrm{vol}(L) := \limsup_{m \to \infty} \frac{n\,!\, h^0(X, L^m)}{m^n}.$$

This induces a continuous function on $N^1(X)_{\mathbb{R}}$ such that

$$\mathrm{vol}(mL) = m^n \cdot \mathrm{vol}(L)$$

and such that $\mathrm{vol}(L) > 0$ if and only if L is big. For a detailed study of the volume function we refer to [16, Chapter 2].

We will need the following formula for the volume of a line bundle which is a consequence of Fujita's Approximation Theorem (see [9] or [16, Chap. 11]). If L is a line bundle with $Bs(L)$ defined by $\mathfrak{b} \neq \mathcal{O}_X$, and if $\pi : X' \to X$ is a projective, birational morphism, with X' smooth and such that $\pi^{-1}(\mathfrak{b}) = \mathcal{O}_{X'}(-F)$ is an invertible ideal, then we put $(L^{[n]}) := (M^n)$, where $M = \pi^* L - F$. If $\mathfrak{b} = \mathcal{O}_X$, then we put $(L^{[n]}) = 0$. With this notation, we have

$$\mathrm{vol}(L) = \sup_{m \in \mathbb{N}} \frac{((mL)^{[n]})}{m^n}. \tag{4.3}$$

Note that in the above definition of $(L^{[n]})$ we may replace the ideal $\mathfrak{b}$ by its integral closure.

PROPOSITION 4.13. — *If X has finitely generated linear series then the closed cone $\overline{\mathrm{Big}(X)}_{\mathbb{R}}$ has a fan refinement Δ such that the volume function is piecewise polynomial with respect to this fan.*

Proof. — The proof is analogous to the proof of Theorem D. In fact, we use the same fan refinement. By Proposition 4.7, it is enough to prove the following assertion: suppose that $L_1, \ldots, L_r$ are line bundles on X whose

classes are linearly independent in $N^1(X)_{\mathbb{R}}$, and let us denote by $\mathfrak{a}_p$ the base ideal of $\sum_i p_i L_i$ for $p \in \mathbb{N}^r$; if there is $d \geqslant 1$ such that

$$\overline{\mathfrak{a}_{dp}} = \overline{\prod_i \mathfrak{a}_{pe_i}^{p_i}} \tag{†}$$

for all $p \in \mathbb{N}^n$, then the volume function is polynomial on the cone spanned by the classes of $L_1, \ldots, L_r$.

It is clear that it is enough to show that the map

$$p \longmapsto \mathrm{vol}\Big(\sum_{i=1}^r dp_i L_i\Big)$$

is a polynomial function of degree n for $p \in \mathbb{N}^r$. Let $\pi : X' \to X$ be a projective birational morphism, with X' smooth and such that $\pi^{-1}(\mathfrak{a}_{de_i}) = \mathcal{O}(-F_i)$ are invertible for all i. If $M_i = \pi^*(dL_i) - F_i$, then it follows from (†) that for every $p \in \mathbb{N}^r$ we have

$$\Big(\Big(\sum_i dp_i L_i\Big)^{[n]}\Big) = \Big(\Big(\sum_i p_i M_i\Big)^n\Big).$$

Together with (4.3), this implies

$$\mathrm{vol}\Big(\sum_i dp_i L_i\Big) = \Big(\Big(\sum_i p_i M_i\Big)^n\Big),$$

which completes the proof. □

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