EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 14, No. 1, 2021, 19-42
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Asymptotic laws for upper and strong record values in the extreme domain of attraction and beyond

Gane Samb Lo[1,2,3,*], Mohammad Ahsanullah[4], Moumouni Diallo[5], Modou Ngom[1]

[1] *LERSTAD, Gaston Berger University, Saint-Louis, Sénégal*
[2] *Department of Pure and Applied Mathematics, African University of Science and Technology, Abuja, Nigeria*
[3] *LSTA, Pierre and Marie Curie University, Paris VI, France*
[4] *Department of Management Sciences. Rider University. Lawrenceville, New Jersey, USA*
[5] *Université des Sciences Sociale et de Gestion de Bamako (USSGB), Faculté des Sciences Économiques et de Gestion (FSEG)*

Abstract. Asymptotic laws of record values have usually been investigated as limits in type. In this paper, we use functional representations of the tail of cumulative distribution functions in the extreme value domain of attraction to directly establish asymptotic laws of record values, not necessarily as limits in type and their rates of convergences. Results beyond the extreme value domain are provided. Explicit asymptotic laws concerning very usual laws and related rates of convergence are listed as well. Some of these laws are expected to be used in fitting distribution.
2020 Mathematics Subject Classifications: 62G30, 60G70, 60Fxx
Key Words and Phrases: Record values and record times, normal Asymptotic theory, extreme value theory, generalized extreme values distributions

---

1. Introduction

Let X, X_1, X_2, $\cdots$ be a sequence of independent real-valued randoms, defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$, with common cumulative distribution function F, which has the lower and upper endpoints, the first asymptotic moment function and the generalized inverse function defined by

$$lep(F) = \inf\{x \in \mathbb{R},\ F(x) > 0\},\ uep(F) = \sup\{x \in \mathbb{R},\ F(x) < 1\},$$

$$R(x, F) = \frac{1}{1 - F(y)} \int_x^{uep(F)} (1 - F(y))\ dy,\ x \in]lep(F), uep(F)[$$

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v14i1.3872

Email addresses: gane-samb.lo@ugb.edu.sn (G. S. Lo), ahsan@rider.edu (M. Ahsanullah), moudiallo1@gmail.com (M. Diallo), ngom.modou1@ugb.edu.sn (M. Ngom)

and

$$F^{-1}(u) = \inf\{x \in \mathbb{R},\ F(x) \geq u\}\ for\ u \in]0,1[\ and\ F^{-1}(0) = F^{-1}(0+).$$

respectively. Finally, let us consider the sequence of strong record values $X^{(1)} = X_1$, $X^{(n)}$, $\cdots$ (see [7]) and the sequence of record times $U(1) = 1$, $U(2)$, $\cdots$.

Before beginning an asymptotic theory, we should be sure that we have an infinite sequence $(X^{(n)})_{n\geq 1}$. For a bounded random variable with finite upper bound $uep(F)$ such that $\mathbb{P}(X = uep(F)) > 0$, we have $(X^{(n)} < uep(F))$ finitely often. This happens for classical integer-valued and bounded random variables as Binomial laws. In such cases, the asymptotic theory is meaningless. But, an interesting question would be the characterization the infinite random sequence $(n_k)_{k\geq 1}$ such that $X_{n_k} = uep(F)$ for all $k \geq 1$.

In all other cases, even if $uep(F)$ is bounded, the sequence $(X^{(n)})_{n\geq 1}$ is infinite. So, the results of this paper apply to *cdf*'s F such that $\mathbb{P}(X = uep(F)) = 0$. In that context, asymptotic laws have been proposed in the literature by many authors like [8], [10], [9], etc., in relation with Extreme Value Theory, as limits in type in the form

$$(\exists (A_n)_{n\geq 1} \subset \mathbb{R}_+ \setminus \{0\}),\ \exists\ (B_n)_{n\geq 1} \subset \mathbb{R},\ \frac{X^{(n)} - B_n}{A_n} \rightsquigarrow Z, \tag{1}$$

where $\rightsquigarrow$ stands for the convergence in distribution and Z is a non-degenerate random variable. The motive beneath this search is the following. If we denote by $M(n) = \max(X_1, \cdots, X_n)$ as the n-th maximum for $n \geq 1$, it is clear that we have

$$\forall\ n \geq 1,\ X^{(n)} = M(U(n)). \tag{2}$$

Since for any F in the extremal domain of attraction $\mathcal{D}$, we have that for some $\gamma \in \mathbb{R}$,

$$(\exists (a_n)_{n\geq 1} \subset \mathbb{R}_+ \setminus \{0\}),\ (\exists (b_n)_{n\geq 1} \subset \mathbb{R},\ \frac{M(n) - a_n}{b_n} \rightsquigarrow Z_\gamma, \tag{3}$$

where the *cdf* of Z_γ is the Generalized Extreme Value distribution defined by

$$G_\gamma(x) = \exp(-(1+\gamma x)^{1/\gamma}),\ with\, 1+\gamma x > 0,\ and\, G_0(x) = \exp(-\exp(-x))\ for\ x \in \mathbb{R}.$$

In Extreme value Theory, Formula (3) is rephrased as F *is attracted by* G_γ denoted by $F \in D(G_\gamma)$.

From Formulas (2) and (3) and from the fact that $U(n) \to +\infty$ as $n \to +\infty$, the investigation of the validity of (1) was justified enough. The results of the cited authors and others were positive with the stunning result that the *cdf* of Z should be on the form $\Phi(g(x))$, $x \in \mathbb{R}$, where Φ is the *cdf* of the standard normal law and g satisfies one of three definitions (in which c is a positive constant)

$$g(x) = x, \ \ x \in \mathbb{R}.$$
$$g(x) = -\infty 1_{(x<0)} + (c \log x) \ 1_{(x \geq 0)}, x \in \mathbb{R}.$$
$$g(x) = (-c \log -x) \ 1_{(x<0)} + \infty \ 1_{(x>0)}, \ \ x \in \mathbb{R}.$$

Instead of using this mathematically appealing approach based on functional equations, an other approach consisting in directly finding the asymptotic laws of $X^{(n)}$, not necessarily in the form of Formula (1) is possible and we proceed to it here. That approach is based on representations of $F \in \mathcal{D}$ due to Karamata and to de Haan for example.

Our achievement is the finding of the asymptotic laws of the records for all $F \in \mathcal{D}$ and the related rates of convergence : first, for $\gamma \neq 0$, outside the frame of Formula (1), that is as limits in type, and without any further condition and secondly, for $\gamma = 0$, within the frame of Formula (1), under a general regularity condition. That regularity condition generally holds for usual *cdf*'s.

We also give general conditions to ensure the asymptotic normality of the record values for F not necessarily in the extremal domain. As well general rates of convergence are given. These rates can be explicitly stated for usual *cdf*'s in $\mathcal{D}$. Finally, we give detailed asymptotic laws of the records of a list of remarkable *cdf*'s with specific coefficients.

In this paper that we want short, we use many results from Extreme Value Theory and Record Values Theory. So, for more details, we refer the reader to the books of [7] and [9], for an easy introduction to records and to those of [2], [1], [10] and [6], concerning Extreme Value Theory.

To end this introduction, we recall two important tools of extreme value theory that form the basis of our method. The first is the following proposition. Suppose that $X \geq 0$, that is $F(0) = 0$. In that case, we define $Y = \log X$ with *cdf* $G(x) = F(e^x)$, $x \in \mathbb{R}$ and we have the proposition below,

Proposition 1. *(see [5]) We have the following equivalences.*

(1) If $\gamma > 0$,

$$F \in D(G_\gamma) \ \Leftrightarrow \ (G \in D(G_0) \text{ and } R(x, G) \to \gamma \text{ as } x \to uep(G)).$$

(2) If $\gamma = 0$,

$$F \in D(G_0) \ \Leftrightarrow \ (G \in D(G_0) \text{ and } R(x,G) \to 0 \text{ as } x \to uep(G)).$$

(3) If $\gamma < 0$,

$$F \in D(G_\gamma) \ \Leftrightarrow \ (G \in D(G_\gamma)).$$

In the second place, we recall the following representations of *cdf*'s in the extreme value domain that repeatedly will be used in the sequel.

Proposition 2. *([3] and [1]) We have the following characterizations for the three extremal domains.*

(a) $F \in D(H_\gamma)$, $\gamma > 0$, *if and only if there exist a constant* c *and functions* $a(u)$ *and* $\ell(u)$ *of* $u \to u \in]0,1]$ *satisfying*

$$(a(u), \ell(u)) \to (0,0) \text{ as } u \to 0,$$

such that F^{-1} *admits the following representation of Karamata*

$$F^{-1}(1-u) = c(1+a(u))u^{-\gamma} \exp\left(\int_u^1 \frac{\ell(t)}{t} dt\right). \tag{4}$$

(b) $F \in D(H_\gamma)$, $\gamma < 0$, *if and only if* $uep(F) < +\infty$ *and there exist a constant* c *and functions* $a(u)$ *and* $\ell(u)$ *of* $u \in]0,1]$ *satisfying*

$$(a(u), \ell(u)) \to (0,0) \text{ as } u \to 0,$$

such that F^{-1} *admit the following representation of Karamata*

$$uep(F) - F^{-1}(1-u) = c(1+a(u))u^{-\gamma} \exp\left(\int_u^1 \frac{\ell(t)}{t} dt\right). \tag{5}$$

(c) $F \in D(H_0)$ *if and only if there exist a constant* d *and a slowly varying function* $s(u)$ *such that*

$$F^{-1}(1-u) = d + s(u) + \int_u^1 \frac{s(t)}{t} dt, 0 < u < 1, \tag{6}$$

and there exist a constant c *and functions* $a(u)$ *and* $\ell(u)$ *of* $u \in]0,1]$ *satisfying*

$$(a(u), \ell(u)) \to (0,0) \text{ as } u \to 0,$$

such that the function $s(u)$ of $u \in]0,1[$ admits the representation

$$s(u) = c(1 + a(u)) \exp\left(\int_u^1 \frac{\ell(t)}{t} dt\right). \tag{7}$$

Moreover, if $F^{-1}(1-u)$ is differentiable for small values of u such that

$$r(u) = -u(F^{-1}(1-u))' = u \ dF^{-1}(1-u)/du$$

is slowly varying at zero, then (6) may be replaced by

$$F^{-1}(1-u) = d + \int_u^{u_0} \frac{r(t)}{t} dt, 0 < u < u_0 < 1, \tag{8}$$

which will be called a reduced de Haan representation of F^{-1}.

The rest of the paper is organized as follows. The results are stated in Section 2. Examples and Applications are given in Section 3. The proofs are stated in Section 4. The computations related to examples in Section 3 are detailed in the Appendix Section 6 (Appendix I, page 37). The paper is closed by a conclusion in Section 5.

2. Results

Before stating our results, we recall that any $F \in \mathcal{D}$ is associated to a pair of functions $(a(u), b(u))$ of $u \in [0,1]$ as defined in the representations of Proposition 2 for $F \in D(G_\gamma)$, $\gamma \neq 0$. In the special case where $\gamma = 0$, the pair of functions $(a(\circ), b(\circ))$ is used in the representation of the function $s(u)$ of $u \in [0,1]$ in equation (6).

We will need the following condition. Let us define, for any $n \geq 1$, a finite sum of n standard exponential random variables

$$S_{(n)} = E_{1,n} + \cdots + E_{n,n},$$

and denote

$$V_n = \exp(-S_{(n)}) \ and \ v_n = \exp(-n), \ n \geq 1$$

and finally set the hypothesis

$$(Ha) \ : \ \sup\left\{\left|\frac{u}{v} - 1\right|, \ \min(v_n, V_n) \leq u, v \leq \max(v_n, V_n)\right\} \to_{\mathbb{P}} 0 \ as \ n \to +\infty,$$

$$(Hb) \ : \ (\exists \alpha > 0), \ \sqrt{n} \ s(v_n) \to \alpha \ as \ n \to +\infty,$$

where $\to_{\mathbb{P}}$ stands for the convergence in probability.

Here are our results that cover the whole extreme value domain of attraction. For $\gamma \neq 0$, we need any condition.

Let us begin by asymptotic laws for $F \in \mathcal{D}$.

Theorem 1. *Let $F \in D(G_\gamma)$, $\gamma \in \mathbb{R}$. We have :*

(a) If $\gamma > 0$, the asymptotic law of $X^{(n)}$ is lognormal, precisely

$$\left(\frac{X^{(n)}}{F^{-1}\left(1-e^{-n}\right)}\right)^{n^{-1/2}} \rightsquigarrow LN(0,\gamma^2),$$

where $LN(m,\sigma^2)$ is the lognormal law of parameters m and $\sigma > 0$.

(b) If $\gamma > 0$ and $X \geq 0$, $Y = \log X \in D(G_0))$ and $R(x,G) \to \gamma$ as $x \to uep(G)$ and we have

$$\frac{Y^{(n)} - G^{-1}\left(1-e^{-n}\right)}{\sqrt{n}} \rightsquigarrow \mathcal{N}(0,\gamma^2).$$

(c) If $\gamma < 0$, the asymptotic law of $X^{(n)}$ is lognormal, precisely

$$\left(\frac{uep(F) - X^{(n)}}{uep(F) - F^{-1}\left(1-e^{-n}\right)}\right)^{n^{-1/2}} \rightsquigarrow \exp(\mathcal{N}(0,\gamma^2)).$$

(d) Suppose that $\gamma = 0$ and $R(x,G) \to 0$ as $x \to uep(G)$. If (Ha) and (Hb) hold both, we have

$$X^{(n)} - F^{-1}\left(1-e^{-n}\right) \rightsquigarrow \mathcal{N}(0,\alpha^2).$$

More precisely, we have : Given $\gamma = 0$, $R(x,G) \to 0$ as $x \to uep(G)$ and (Ha), the above asymptotic normality is valid if and only if (Hb) holds.

Beyond distributions in $\mathcal{D}$, we may use the delta-method as follows. Drawing lessons from Theorem 1, we might be tempted to generalize point (a) by imposing that F^{-1} satisfies, for some coefficient γ,

$$\forall \lambda > 0,\ F^{-1}(1-\lambda u)/F^{-1}(1-u) = \lambda^{\gamma}(1+o(1)),\ u \in]0,1[.$$

But, by Extreme Value Theory, this would imply that $F \in D(G_\gamma)$ and nothing new would happen. But trying a generalization from Point (c) would be successful. Let us define the following hypothesis :

(Ga) F is differentiable in some left neighborhood of $uep(F)$.

(Gb) The function

$$s(x) = e^{-x}\left[F^{-1}(1-t)\right]'_{t=e^{-x}}, \ e^x < u_0 < 1, \ for\ some\ u_0 \in]0,1[$$

decreases to 0 as $x \to +\infty$ and is such that : for any sequence $(x_n, y_n)_{n\geq 1}$ such that

$$\limsup_{n\to+\infty} |x_n - y_n|/\sqrt{n} < +\infty,$$

we have, for some $\alpha > 0$,

$$\lim_{n\to+\infty} \sqrt{n}\ s\left(\exp(\min(x_n, y_n))\right) = \lim_{n\to+\infty} \sqrt{n}\ s\left(\exp(\max(x_n, y_n))\right) = \alpha.$$

We have the following generalization.

Theorem 2. *If F satisfies Assumptions (Ga) and (Gb), then we have*

$$X^{(n)} - F^{-1}\left(1 - e^{-n}\right) \rightsquigarrow \mathcal{N}(0, \alpha^2).$$

Comments I . A firm look at the results shows that for any $F \in \mathcal{D}$, we found the direct asymptotic law of $X^{(n)}$ or that of a function of $X^{(n)}$, mainly $\log X^{(n)}$. For example, Point (d) of Theorem 1 cannot be applied when X follows a lognormal law but can be applied to $\exp(X)$. This leads to the following rule for all $F \in \mathcal{D}$:

(e) If $F \in D(G_\gamma)$, $\gamma \neq 0$, we apply Points (a) or (c) without any further condition.

(f) If $F \in D(G_0)$ and $\exp(X) \in D(G_\gamma)$ for some $\gamma > 0$, we apply Point (b) without any further condition.

(g) If $F \in D(G_0)$ and $s(u) \to 0$ as $u \to 0$. If (Ha) and (Hb) hold, we conclude by applying Point (d). If not (as it is for a lognormal law), we search whether $X_1 = \exp(X) \in D(G_\gamma)$ for some $\gamma > 0$ or $X_1 = \exp(X)$ fulfills (Ha) and (Hb). If yes, we conclude by Point (b) or by Point (d). If not, we consider $X_2 = \exp(X_1)$, and we continue until we reach $X_p = \exp(X_{p-1}) \in D(G_\gamma)$ for some $\gamma > 0$ or $X_p = \exp(X_{p-1})$ for some $p \geq 1$.

Finally, we handle the rates of convergences in the theorems stated above. Let us introduce the following notations.

Theorem 3. *Let $F \in D(G_\gamma)$, $\gamma \in \mathbb{R}$. Then, there exists a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ holding a sequence of independent standard exponential random variables $(E_n)_{n\geq 1}$ and a Brownian Process $\{W(t),\ t \geq 0\}$ such that the record values $X^{(n)}$, $n \geq 1$, of the sequence $X_j = F^{-1}\left(1 - e^{E_j}\right)$, $j \geq 1$, satisfy the following representations below under the appropriate conditions. Here, $S_n = E_1 + ... + E_n$, $n \geq 1$, are the partial sums of the sequence $(E_n)_{n\geq 1}$, $S_n^* = n^{-1/2}(S_n - n)$, $v_n = e^{-n}$ and $V_n = e^{-S_n}$. Below, the function $a(u)$, $b(u)$ and $s(u)$ of $u \in]0,1[$ are those in the representations in Proposition 2.*

By denoting $W_n^* = n^{-1/2}W(n)$ and $c_n = n^{-1/2}\log n$, we have

$$W_n^* \sim \mathcal{N}(0,1) \quad \text{and} \quad |S_n^* - W_n^*| = O_{\mathbb{P}}(c_n).$$

Further, we have the following results.

(a) Let $\gamma > 0$. Suppose that

$$1 - \frac{1 + a(V_n)}{1 + a(v_n)} = O(a_n),\ \sup\{|b(t)|,\ 0 \leq t \leq v_n \vee V_n\} = O_{\mathbb{P}}(b_n)\ . \tag{9}$$

Then, we have

$$\begin{aligned} \left(\frac{X^{(n)}}{F^{-1}\left(1 - e^{-n}\right)}\right)^{n^{-1/2}} &= \exp(\gamma S_n^*) + O_{\mathbb{P}}(a_n \vee b_n) \\ &= \exp(\gamma W_n^*) + O_{\mathbb{P}}(a_n \vee b_n \vee c_n). \end{aligned}$$

(b) Let $\gamma > 0$ and $X \geq 0$, $Y = \log X \in D(G_0)$) and $R(x, G) \to \gamma$ as $x \to uep(G)$ and we have

$$\begin{aligned} \frac{Y^{(n)} - G^{-1}\left(1 - e^{-n}\right)}{\sqrt{n}} &= \gamma S_n^* + O_{\mathbb{P}}(a_n \vee b_n) \\ &= \gamma W_n^* + O_{\mathbb{P}}(a_n \vee b_n \vee c_n). \end{aligned}$$

(c) Let $\gamma < 0$. Then, by using the rates of convergence in Formula (9), we have

$$\begin{aligned} \left(\frac{uep(F) - X^{(n)}}{uep(F) - F^{-1}\left(1 - e^{-n}\right)}\right)^{n^{-1/2}} &= \exp(\gamma S_n^*) + O_{\mathbb{P}}(a_n \vee b_n) \\ &= \exp(\gamma W_n^*) + O_{\mathbb{P}}(a_n \vee b_n \vee c_n). \end{aligned}$$

(d) Suppose that $\gamma = 0$ and $R(x, G) \to 0$ as $x \to uep(G)$. Suppose that (Ha) and (Hb) hold both. If

$$\sup\left\{\left|\frac{s(u)}{s(v)}-1\right|,\ \min(v_n,V_n)\leq u,v\leq \max(v_n,V_n)\right\}=O_{\mathbb{P}}(d_n),\ and\ \sqrt{n}s(v_n)-\alpha=O(e_n),$$

we have

$$\begin{aligned} X^{(n)}-F^{-1}\left(1-e^{-n}\right) &= \alpha S_n^*+O_{\mathbb{P}}(\vee d_n\vee e_n)\\ &= \alpha W_n^*+O_{\mathbb{P}}(c_n\vee d_n\vee e_n). \end{aligned}$$

Comments II . In the domain of extremal attraction, most of the *cdf*'s which are used in applications are differentiable in a left-neighborhood of the upper endpoint. In such a case, we may take $a\equiv 0$ in Representation (4) and (5) in Proposition 2. By solving easy differential equations, we have the representation for

$$b(u)=-u(G^{-1}(1-u))'-\gamma,\ u\in(0,1)\ \ and\ \ a\equiv 0 \tag{10}$$

for $\gamma>0$ and

$$b(u)=\frac{u(uep(F)-F^{-1}(1-u))}{F'(F^{-1}(1-u))},\ u\in(0,1) \tag{11}$$

for $\gamma<0$, whenever we have $b(u)\to 0$ as $u\to 0$. Consequently, the rate of convergence is reduced to $O_{\mathbb{P}}(b_n\vee c_n)$.

For $\gamma=0$, Representation (8) in Proposition 2 holds for

$$s(u)=-u(F^{-1}(1-u))',\ 0<u<1,$$

whenever it is slowly varying at zero and the rate of convergence d_n becomes useless. In such cases, the rate of convergence is reduced to $O_{\mathbb{P}}(d_n\vee c_n)$.

Furthermore, based on the limit $S_n/n\to 1$ as $n\to+\infty$, we get that have for any $\eta\in]0,1[$,

$$\liminf_{n\to+\infty}\mathbb{P}\left(e^{-n/\eta}\leq e^{-S_n}\leq e^{-\eta n}\right)=1. \tag{12}$$

So we may replace the rates of convergence d_n *and* b_n by $d_n(\eta)$ and $b_n(\eta)$ defined as follows, for $\eta\in]0,1[$,

$$\sup\{|b(t)|,\ 0\leq t\leq e^{-\eta n}\}=O(b_n(\eta)) \tag{13}$$

and

$$\sup\left\{\left|\frac{s(u)}{s(v)}-1\right|,\ e^{-n/\eta}\leq u,v\leq e^{-\eta n}\right\}=O(d_n(\eta)).. \tag{14}$$

Specific rates of convergence will be given in the examples below as illustrations $\Diamond$

3. Examples and applications

Let us begin to explain how to apply the results for $\gamma = 0$. Generally, we may find the function $s(u)$, with $u \in]0,1[$, by the π-variation formula

$$\forall\ \lambda > 0,\ \frac{F^{-1}(1-\lambda u) - F^{-1}(1-u)}{s(u)} \to -\log\lambda\ as\ u \to 0.$$

Another method concerns the special case where F is differentiable on a left neighborhood of $uep(F)$. It is proved in [5] that if $u\left(F^{-1}(1-u)\right)'$ is slowly varying at zero, we have for some $u_0 \in]0,1[$,

$$s(u) = -u\left(F^{-1}(1-u)\right)'\ for\ u \in]0,u_0[.$$

Checking hypothesis, (Ha) and (Hb) can be done with the function $s(u)$ with $u \in]0,1[$, as explained above.

Here are some specific examples of asymptotic laws and related rates of convergence. The details for each case are given in the Appendix (Section 6).

Let us recall that $\{W(t),\ t \geq 1\}$ is a Brownian motion defined on the same probability space as the sequence of records. We begin for light tails :

I - $F \in D(G_0)$.

(1) X **follows an exponential law** $\mathcal{E}(\lambda)$**,** $\lambda > 0$. By Point (b) of Theorem 1,

$$\frac{X^{(n)} - n}{\sqrt{n}} \rightsquigarrow \mathcal{N}(0, \lambda^{-2}).$$

But for a fixed n, we have for a random variable V following a gamma law of parameters $n \geq 1$ and 1,

$$\frac{X^{(n)} - n}{\sqrt{n}} \sim \frac{V - \mathbb{E}(V)}{\mathbb{V}ar(V)^{1/2}}.$$

The rate of convergence is

$$\frac{X^{(n)}-n}{\sqrt{n}} = S_n^* = W_n^* + O_{\mathbb{P}}(n^{-1}\log n).$$

(2) X **follows a standard normal law** $\mathcal{N}(0,1)$. By Point (d) of Theorem 1,

$$X^{(n)} - (2n)^{1/2} \rightsquigarrow \mathcal{N}(0,1/2).$$

The rate of convergence is given by

$$\begin{aligned} X^{(n)} - (2n)^{1/2} &= S_n^* + O_{\mathbb{P}}\left(\frac{(\log n)^2}{n}\right) \\ &= W_n^* + O_{\mathbb{P}}\left(\frac{\log n}{n}\right). \end{aligned}$$

(3) X **follows a Rayleigh law of parameter** $\rho > 0$, with *cdf*

$$1 - F(x) = \exp(-\rho x^2),\ x \geq 0.$$

By Point (d) of Theorem 1, we have

$$X^{(n)} - \left(\frac{n}{\rho}\right)^{1/2} \rightsquigarrow \mathcal{N}(0, \rho^{-1}/4).$$

We also have

$$X^{(n)} - \left(\frac{n}{\rho}\right)^{1/2} = \rho^{-1/2} S_n^* + O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right).$$

(4) X **follows the logistic law**, with *cdf*

$$F(x) = \frac{1}{1+e^{-x}},\ x \in \mathbb{R}.$$

By Point (b) of Theorem 1, we have

$$\frac{X^{(n)}-n}{\sqrt{n}} \rightsquigarrow \mathcal{N}(0,1).$$

The rate of convergence is given, for any $\eta \in]0,1[$,

$$\frac{X^{(n)}-n}{\sqrt{n}} = S_n^* + O_{\mathbb{P}}\left(\frac{e^{-n\eta}}{1-e^{-n/\eta}}\right).$$

(5) $X > 0$ **follows a standard lognormal law, that is** $\log X$ **follows a standard normal law**. We have

$$\log X^{(n)} - (2n)^{1/2} \rightsquigarrow \mathcal{N}(0,1/2).$$

The rate of convergence is given by

$$\log X^{(n)} - (2n)^{1/2} = S_n^* + O_{\mathbb{P}}(n^{-1}(\log n)^2).$$

(6) $X > 0$ **follows a Gumbel law** with *cdf*

$$F(x) = \exp\left(-e^{-x}\right),\ x \in \mathbb{R}.$$

By Point (b) of Theorem 1, we have

$$\frac{X^{(n)}-n}{\sqrt{n}} \rightsquigarrow \mathcal{N}(0,1).$$

The rate of convergence is given, for any $\eta \in]0,1[$, by

$$\frac{X^{(n)}-n}{\sqrt{n}} = S_n^* + O\left(e^{-\eta n}\right).$$

II - $F \in D(G_\gamma)$**,** $\gamma > 0$.

(7) X **follows a log-logistic law of parameter** $p > 0$, with *cdf*

$$F(x) = \frac{x^p}{1+x^p},\ x \geq 0.$$

By Point (a) of Theorem 1,

$$\left(e^{-n/p}X^{(n)}\right)^{-1/2} \rightsquigarrow LN(0,p^2).$$

For the rate of convergence, we take $\eta \in]0,1[$ and

$$b_n(\eta) = \frac{e^{-\eta n}}{p\left(1-e^{-n/\eta}\right)}.$$

We have

$$\left(e^{-n/p}X^{(n)}\right)^{-1/2} - \exp\left(S_n^*\right) = O_{\mathbb{P}}(b_n).$$

(8) X **follows a sing-Maddala law of parameters** $a > 0$**,** $b > 0$ **and** $c > 0$, with *cdf*

$$1 - F(x) = \left(\frac{1}{1+ax^b}\right)^c, \ x \geq 0.$$

By Point (a) of theorem 1, we have

$$\left(a^{1/b}\exp(-n/(bc))X^{(n)}\right)^{1/\sqrt{n}} \rightsquigarrow LN(0,(bc)^{-2}).$$

The rate of convergence is given as follows. Let $\eta \in]0,1[$, and

$$b_n(\eta) = \frac{e^{-\eta n/c}}{b\left(1-e^{-n/(c\eta)}\right)}.$$

We have

$$\left(a^{1/b}\exp(-n/(bc))X^{(n)}\right)^{1/\sqrt{n}} = \exp(S_n^*) + O_{\mathbb{P}}(b_n(\eta)).$$

4. Proofs

(I) - Proof of Theorem 1.

We begin by describing the main tools which are based on following results of record theory. Suppose that $\{T, T_j > 0, \ 1 \leq j \leq k\}$ are $(k+1)$ non-negative real-valued, *iid* random variables and define

$$X_0 = 0, \ T_j = X_j - X_{j-1}, \ 1 \leq j \leq k.$$

It is clear that if $T \sim \mathcal{E}(\lambda)$, $\lambda > 0$, then the absolutely continuous *pdf* of $T = (T_1, \cdots, T_k)^t$ is given by

$$f_T(t_1, \cdots, t_k) = \lambda^k e^{-\lambda t_k} \, 1_{(0 \leq t_1 \leq \cdots \leq t_k)}. \tag{15}$$

Suppose if T_j's are independent and follow an exponential law $\mathcal{E}(\lambda)$, $\lambda > 0$, we have

$$r(x) = \frac{dF(x)/dx}{1 - F(x)} = \lambda \text{ and } f(x) = \lambda e^{-\lambda x}, \; x \geq 0.$$

As stated in page 3 of [7], the joint distribution of the k first record values $(T^{(1)}, \cdots, T^{(k)})$ of the sequence $(T_n)_{n \geq 1}$ is the one given in Formula (15). As a consequence, we have

Fact 1. *If the T_j's are independent and follow an exponential law $\mathcal{E}(\lambda)$, then the k-th record value, $k \geq 1$, has the same law as the sum of k independent $\mathcal{E}(\lambda)$-random variables $E_{1,k}$, $\cdots$, $E_{k,k}$, i.e.*

$$T^{(k)} =_d E_{1,k} + \cdots + E_{k,k},$$

where $=_d$ stands for the equality in distribution. By the Renyi's representation, we can represente the random variable X of *cdf* F by a standard exponential random variable E

$$X =_d F^{-1}\left(1 - e^{-E}\right).$$

It comes that, by considering *iid* sequence $(X_n)_{n \geq 1}$ and $(E_n)_{n \geq 1}$ from X and E, and by denoting the two n-th record values $X^{(n)}$ and $E^{(n)}$ from the two sequences respectively, we have the following representations

$$X^{(n)} =_d F^{-1}\left(1 - e^{-E^{(n)}}\right),$$

where

$$E^{(n)} = E_{1,n} + \cdots + E_{n,n} = S_{(n)}.$$

In the sequel, we can and do use the equality : $X^{(n)} = F^{-1}\left(1 - e^{-S_{(n)}}\right)$. Let us apply the representations by using the simple central limit theorem

$$\frac{S_{(n)} - n}{\sqrt{n}} \rightsquigarrow \mathcal{N}(0, 1) \text{ as } n \to +\infty.$$

In the sequel, any unspecified limit is meant as $n \to +\infty$.

Let us suppose that $X \in D(G_{1/\gamma})$. If $X \geq 0$, we will consider $Y = \log X$ of *cdf* G defined by $G(x) = F(e^x)$, $x \in \mathbb{R}$. Let us prove the theorem 1.

(a) - Asymptotic law of $X^{(n)}$ for $\gamma > 0$. We recall that $V_n = e^{-S_{(n)}}$ and $v_n = e^{-n}$, $n \geq 1$. By Representation (4), we have

$$F^{-1}\left(1 - e^{-S_{(n)}}\right) = (1 + a(V_n))V_n^{-\gamma} \exp\left(\int_{V_n}^{1} \frac{b(t)}{t}\, dt\right),\ n \geq 1$$

and

$$F^{-1}\left(1 - e^{-n}\right) = (1 + a(v_n))v_n^{-\gamma} \exp\left(\int_{v_n}^{1} \frac{b(t)}{t}\, dt\right),\ n \geq 1.$$

We get that $V_n \rightarrow_{\mathbb{P}} 0$, $(1 + a(V_n))/(1 + a(v_n)) \equiv 1 + p_n \rightarrow_{\mathbb{P}} 1$. We get

$$\log\left(\frac{X^{(n)}}{F^{-1}\left(1 - e^{-n}\right)}\right) = p_n(1 + o_{\mathbb{P}}(1)) - \gamma(S_{(n)} - n) + \int_{v_n}^{V_n} \frac{b(t)}{t}\, dt. \tag{16}$$

We have

$$\left|\int_{v_n}^{1} \frac{b(t)}{t}\, dt\right| \leq \left(\sup_{0 \leq t \leq (v_n \vee V_n)} |b(t)|\right)|S_{(n)} - n|. \tag{17}$$

By combining the two last formulae, we have

$$n^{-1/2} \log\left(\frac{X^{(n)}}{F^{-1}\left(1 - e^{-n}\right)}\right) \rightsquigarrow \mathcal{N}(0, \gamma^2).$$

(b) - Asymptotic law of $Y^{(n)}$ for $\gamma > 0$. From the previous theorem, it is immediate for the following result. It is clear that $G^{-1} = \log F^{-1}$. So, the previous theorem implies

$$n^{-1/2}\left(Y^{(n)} - G^{-1}\left(1 - e^{-n}\right)\right) \rightsquigarrow \mathcal{N}(0, \gamma^2).$$

Here, it is clear that $Y \in D(G_0)$ and $R(x, G) \rightarrow \gamma$ as $x \rightarrow uep(G)$. Hence this result says that

$$n^{-1/2}\left(X^{(n)} - F^{-1}\left(1 - e^{-n}\right)\right) \rightsquigarrow \mathcal{N}(0, \gamma^2),$$

if $F \in D(G_0)$ and $R(x, F) \rightarrow \gamma$ as $x \rightarrow uep(F)$.

(c) - Asymptotic law of $Y^{(n)}$ for $\gamma < 0$. We have $\mathbb{P}(X = uep(F)) = 0$. By using Representation (5), we may and do prove this point exactly as for Point (a).

(d) - Asymptotic law of $Y^{(n)}$ for $\gamma = 0$. We did not have yet the general law. Let us learn for a no-trivial example.

(A) - $X \sim \mathcal{N}(0,1)$. Let us recall the expansion of the tail of F as follows

$$\begin{aligned} F^{-1}(1-s) &= (2\log(1/s))^{1/2} - \frac{\log 4\pi + \log\log(1/s)}{2(2\log(1/s))^{1/2}} \\ &+ O((\log\log(1/s))^2(\log 1/s)^{-1/2}). \end{aligned} \tag{18}$$

We get

$$\begin{aligned} X^{(n)} &= (2S_{(n)})^{1/2} - \frac{\log 4\pi + \log S_{(n)}}{2(2S_{(n)}))^{1/2}} + O(S_{(n)}^{-1/2}\log S_{(n)}) \\ &= (2S_{(n)})^{1/2} - \frac{\log 4\pi + \log S_{(n)}}{2(2S_{(n))^{1/2}}} + O_{\mathbb{P}}(n^{-1}(\log n)^2) \qquad (19) \\ &= (2S_{(n)})^{1/2} + O_{\mathbb{P}}(n^{-1/2}\log n). \qquad (20) \end{aligned}$$

Furthermore

$$\begin{aligned} F^{-1}(1-e^{-n}) &= (2n)^{1/2} - \frac{\log 4\pi + \log n}{2(2n)^{1/2}} + O_{\mathbb{P}}(n^{-1}(\log n)^2) \qquad (21) \\ &= (2n)^{1/2} + O_{\mathbb{P}}(n^{-1/2}\log n). \qquad (22) \end{aligned}$$

Combining relations (20) and (22) leads to

$$X^{(n)} - F^{-1}(1-e^{-n}) = \frac{1}{\sqrt{2}}\frac{S_{(n)} - n}{\sqrt{n}}(n/\zeta_n)^{1/2} + O_{\mathbb{P}}(n^{-1}(\log n)^2), \tag{23}$$

with $n \wedge S_{(n)} < \zeta_n < n \vee S_{(n)}$ and next, by the weak law of large numbers, $(n/\zeta_n)^{1/2} \to_{\mathbb{P}} 1$ and thus

$$X^{(n)} - (2n)^{1/2} \rightsquigarrow \mathcal{N}(0, 1/2). \tag{24}$$

(B) - General proof. It is known that $s(u) \sim R(F^{-1}(1-u), F)$ and so, $s(u) \to 0$ as $u \to 0$. By Representation (6) of Proposition 2 and Hypothesis (Ha) together lead to

$$
\begin{aligned}
X^{(n)} - F^{-1}(1-e^{n}) &= s(V_n) - s(v_n) + \int_{v_n}^{V_n} \frac{s(u)}{u}\, du \qquad (25)\\
&= s(v_n)\left(\frac{s(V_n)}{s(v_n)} - 1\right) + (1 + o_{\mathbb{P}}(d_n))s(v_n)(S_{(n)} - n).\\
&= \{s(v_n)\sqrt{n}\}\left(\frac{1}{\sqrt{n}}\left(\frac{s(V_n)}{s(v_n)} - 1\right)\right) + \{s(v_n)\sqrt{n}\}(1 + O_{\mathbb{P}}(d_n))\ S_n^{*}\\
&= (\alpha + O_{\mathbb{P}}(e_n))\left(\frac{1}{\sqrt{n}}\left(\frac{s(V_n)}{s(v_n)} - 1\right)\right) + (\alpha + O_{\mathbb{P}}(e_n))(1 + O_{\mathbb{P}}(d_n))S_n^{*}\\
&= \alpha S_n^{*} + O_{\mathbb{P}}(e_n \vee d_n)\\
&= \alpha W_n^{*} + O_{\mathbb{P}}(c_n \vee e_n \vee d_n).
\end{aligned}
\qquad (26)
$$

From there, the conclusion is immediate by using Hypothesis (Hb) ■

(II) - Proof of Theorem 2. We have $g(x) = F^{-1}(1 - e^{-x})$, $g'(x) = S(x)$, $x \in]lep(F), uep(F)[$. The mean value theorem gives, for $\forall n > 0$,

$$
X^{(n)} - F^{-1}(1 - e^{-n}) = \frac{S_{(n)} - n}{\sqrt{n}}\left(\sqrt{n}S(\exp(-\zeta_n))\right), \qquad (27)
$$

where

$$
\zeta_n \in]\min(n, S_{(n)}),\ \max(n, S_{(n)})[.
$$

From there, the conclusion is direct ■

Proof of Theorem 3.

We will prove that theorem in a special space but it will be valid in any probability space. Following [4], we consider the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ holding a sequence $(E_n)_{n\geq 1}$ and a Wiener process W such that

$$
\frac{|(S_n - n) - W(n)|}{\sqrt{n}} = O\left(\frac{\log n}{\sqrt{n}}\right). \qquad (28)
$$

We set $U_n = 1 - \exp(-E_n)$, $n \geq 1$ and finally $X_n = F^{-1}(1 - \exp(-E_n))$, $n \geq 1$. From that point, all the notations above remain valid. The proofs of the different points of the theorems derive easily from the proof of the same points in Theorem 1. Here are the details.

Proof of Points (a) and (b). By combining Formulas (16) and (17) with Formula (28) approximation, we easily isolate a multiple of S_n^* or of W_n^* and find the rates of convergence as stated for Point (b). Point (a) is obtained by an exponential transformation.

Proof of Point (c). This is proved exactly as Point (a).

Proof of Point (d). From Formulas (25) and (28), we simply use the rates of convergence in Hypothesis (Ha) and (Hb) to conclude $\square$

5. Conclusion

After the statements of the asymptotic laws of the strong record values from *iid* random variables and their rates of convergences, and after some examples have been given, it should be interesting to have a review of such asymptotic laws for *cdf*'s as much as possible, $F \in \mathcal{D}$.

6. Appendix

Appendix. Let us give the details concerning the results listed in Section 3.

(1) X **follows an exponential law** $\mathcal{E}(\lambda)$**,** $\lambda > 0$. We have $exp(X) \in D(G_{-1})$ and $F^{-1}(1-e^{-n}) = n$. We apply Point (b) to conclude that

$$\frac{X^{(n)} - n}{\sqrt{n}} \rightsquigarrow \mathcal{N}(0, \lambda^{-2}).$$

The rate of convergence is the one in the approximation between S_n^* and W_n^* which is c_n.

(2) X **follows a standard normal law** $\mathcal{N}(0,1)$. The result of this point is justified by Formula 24, page 34.

To find the rate of convergence, we proceed to a direct proof based on Formulas (19) and (21). We get

$$\begin{aligned}
X^{(n)} - F^{-1}(1-e^{-n}) &= (2S_{(n)})^{1/2}(2n)^{1/2} && (L1)\\
&- \left(\frac{\log 4\pi + \log S_{(n)}}{2(2S_{(n)}))^{1/2}} - \frac{\log 4\pi + \log S_{(n)}}{2(2S_{(n))^{1/2}}}\right) && (L2)\\
&+ O_{\mathbb{P}}(n^{-}(\log n)^2). && (L3)
\end{aligned}$$

By using the result of the application of the mean value theorem in Formula (23) (page 34) in Line (L1), by applying the mean value theorem in Line (L2) above and by using the fact $(n/\zeta_n) \to 1$, we get

$$\begin{aligned}
X^{(n)} - F^{-1}(1-e^{-n}) &= \frac{1}{\sqrt{2}}\frac{S_{(n)} - n}{\sqrt{n}} + \frac{1}{\sqrt{2}}\frac{S_{(n)} - n}{\sqrt{n}}\left((n/\zeta_n)^{1/2} - 1\right)\\
&- O_{\mathbb{P}}(n^{-1}(\log n))\\
&+ O_{\mathbb{P}}(n^{-1/2}\log n).
\end{aligned}$$

Finally, by using $(n/\zeta_n) - 1 = O_{\mathbb{P}}(n^{-1}(\log n))$ and $(S_n - n)/\sqrt{n} = O_{\mathbb{P}}(1)$, we conclude that

$$X^{(n)} - F^{-1}(1-e^{-n}) = S_n^* + O_{\mathbb{P}}(n^{-1/2}\log n),$$

which was the target.

(3) X **follows a Rayleigh law of parameter** $\rho > 0$. We have

$$F^{-1}(1-u) = \left(-\frac{1}{\rho}\log u\right)^{1/2}, \ u \in]0,1[$$

and

$$s(u) = -u\left(F^{-1}(1-u)\right)' = \frac{1}{2\rho(-(1/\rho)\log u)^{1/2}} \to 0 \ as \ u \to 0.$$

Furthermore, $s(u)$ is decreasing in $u \in]0,1[$ and $s(V_n)/s(v_n) \to 0$ as $n \to +\infty$. Finally,

$$\sqrt{n}s(v_n) \to \rho^{-1/2}/2.$$

We conclude the case by applying Point (d) of Theorem 1.

For finding the rate of convergence, we have $e_n = 0$ since

$$\sqrt{n}\ s(v_n) = (2\sqrt{\rho}) = \alpha.$$

The rate of convergence corresponding to d_n is obtained by remarking that $s(\circ)$ is decreasing to zero and so

$$\sup\left\{\left|\frac{u}{v}-1\right|, \ \min(v_n, V_n) \leq u, v \leq \max(v_n, V_n)\right\} \leq \left|\frac{s(e^{-n} \vee e^{-S_n})}{s(e^{-n} \wedge e^{-S_n})} - 1\right|$$

$$\left|\frac{s(e^{-n} \vee e^{-S_n})}{s(e^{-n} \wedge e^{-S_n})} - 1\right| \leq \left|\left(\frac{S_n}{n}\right)^{1/2} - 1\right| + \left|\left(\frac{S_n}{n}\right)^{-1/2} - 1\right| = O_{\mathbb{R}}\left(\frac{1}{\sqrt{n}}\right).$$

This closes the discussions on the rate of convergence.

(4) X **follows the logistic law**. It is immediate that $\exp(X) \in D(G_{-1})$ and we have

$$F^{-1}(1-u) = \log(u/(1-u)), \ u \in]0,1[.$$

We conclude with Point (b) of Theorem 1.

The rate of convergence can be found from that $Z = exp(X)$ of *cdf*

$$H(t) = t(1+t)^{-1}, \ t > 0.$$

We have that $H \in G_1$ and, for $\gamma = 1$,

$$-u(\log H^{-1}(1-u))' - \gamma = u(1-u)^{-1}.$$

By applying Point (1) of Theorem 3, we take for any $\eta \in]0,1[$ and

$$b_n(\eta) = \frac{e^{-\eta n}}{p\left(1 - e^{-n/\eta}\right)}$$

and get

$$\left(\frac{Z^{(n)}}{n - \log(1-e^{-n})}\right)^{1/\sqrt{n}} = \exp(S_n^*) + O_{\mathbb{P}}(b_n).$$

(5) $X > 0$ **follows a standard lognormal law, that is** $\log X$ **follows a standard normal law**.

Since $\log X^{(n)}$ has the same law as the n-th record $Z^{(n)}$ from *iid* $\mathcal{N}(0,1)$ random variables. So we have

$$\log X^{(n)} - (2n)^{1/2} \to \mathcal{N}(0, 1/2).$$

The conclusion is done by taking the logarithm of both members.

We still can use the rate of convergence from normal records to have:

$$\log X^{(n)} - (2n)^{1/2} = S_n^* + O_{\mathbb{P}}(n^{-1}(\log n)^2).$$

(6) $X > 0$ **follows a Gumbel law**. We have

$$F^{-1}(1-u) = -\log\log(1/(1-u)),\ u \in]0,1[$$

and for any $\lambda > 0$,

$$F^{-1}(1-\lambda u) - F^{-1}(1-u) \to -\log\lambda \ as\ u \to 0.$$

So, $\exp(X) \in D(G_1)$. From there, an application of Point (b) of Theorem 1 closes the case.

The rate of convergence can be found from that of $Z = exp(X)$ of *cdf*

$$H(t) = \exp(-1/t),\ t > 0.$$

We have that $H \in G_1$ and, for $\gamma = 1$,

$$\begin{aligned} -u(\log H^{-1}(1-u))' &= -u\left[\log \frac{1}{-\log(1-u)}\right]' \\ &= \frac{-u}{(1-u)\log(1-u)} \\ &= \frac{u}{(1-u)(u-u^2/2+O(u^3))} \\ &= \frac{1}{(1-3u/2+O(u^2))}. \end{aligned}$$

Hence

$$-u(\log H^{-1}(1-u))' - \gamma = 3u/2(1+o(1)).$$

By applying Point (1) of Theorem 3, we take for any $\eta \in]0,1[$ and

$$b_n(\eta) = 1.5e^{-\eta n}$$

and get

$$\left(\frac{Z^{(n)}}{n - \log(1-e^{-n})}\right)^{1/\sqrt{n}} = \exp(S_n^*) + O_{\mathbb{P}}(b_n).$$

The conclusion is done by taking the logarithm of both members.

(7) X **follows a log-logistic law of parameter** $p > 0$, with *cdf*

$$F(x) = \frac{x^p}{1+x^p},\ x \geq 0.$$

We have $F \in D(G_{1/p})$ since

$$F^{-1}(1-u) = u^{-1/p}(1-u)^{1/p}, u \in]0,1[.$$

By Point (a) of Theorem 1,

$$\left(e^{-n/p}X^{(n)}\right)^{1/\sqrt{n}} \rightsquigarrow LN(0,p^2).$$

To find the rate of convergence, we apply the recommendations in *Comments II* (page 27). We have

$$b(u) = -u(G^{-1}(1-u))' - (1/p) = \frac{u}{p(1-u)}, u \in]0,1[.$$

For any $0 < \eta < 1$, we get the rate of convergence

$$b_n(\eta) = \frac{e^{-\eta n}}{p\left(1-e^{-n/\eta}\right)}.$$

(8) X **follows a sing-Maddala law of parameters** $a > 0$**,** $b > 0$ **and** $c > 0$. We have

$$1 - F(x) = x^{-bc}(x^{-b}+a)^{-c} \equiv x^{-bc}L(x),\ x \geq 0,$$

and L is a slowly varying function at $+\infty$. So $F \in G_{1/(bc)}$. Applying the Point (a) of Theorem 1, when combined with

$$F^{-1}(1-u) = a^{-1/b}u^{-1/(bc)}(1-u^{1/c})^{1/b},\ u \in]0,1[,$$

and with,

$$F^{-1}(1-e^{-n}) = a^{-1/b}e^{n/(bc)}(1-e^{-n/c})^{1/b},$$

for $n \geq 1$. To find the rate of convergence, we check that we have $\gamma = 1/(bc)$ and

$$b(u) = -u(G^{-1}(1-u))' - (1/(bc)) = \frac{u^{1/c}}{b(1-u^{1/c})}, u \in]0,1[.$$

For any $0 < \eta < 1$, we get the rate of convergence

$$b_n(\eta) = \frac{e^{-\eta n/c}}{b\left(1-e^{-n/(c\eta)}\right)}\square$$

References

[1] L. de Haan. *On regular variation and its application to the weak convergence of sample extremes.* Mathematical Center Tracts,Amsterdam. (MR0286156), 1970.

[2] J. Galambos. *The Asymptotic theory of Extreme Order Statistics.* Wiley, Nex-York. (MR0489334), 1985.

[3] Karamata J. *Some theorems concerning slowly varying.* Mathematics Research Center, Tech. Rep.,n^o 369. University of Winconsin, Madison., 1962.

[4] Komlós J., Major P., and Tusnády. An approximation of partial sums of independent rv's and the sample df. *I. Z. Wahrsch. Verw. Gebiete,32 111X96 131. MR0375412*, 1975.

[5] G.S. Lo. *On some estimators of the index of the Pareto law and limit theorems for extreme value sums.PhD Thesis.* Pierre and Marie Curie University, Paris VI. France., 1986.

[6] G.S. Lo, T. A. Kpanzou, M. Ngom, and M. Diallo. *Weak Convergence (IIA) - Functional and Random Aspects of the Univariate Extreme Value Theory.* Arxiv : 1810.01625, 2018.

[7] Ahsanullah M. *Record Statistics.* Nova Science Publishers Inc. New-York, USA, 1995.

[8] Tata M.N. On outstanding values in a sequence of random variables. *Zeitschrift. Wehrs. Verw. Gebiete*, 12,:09–20, 1969.

[9] V. B. Nevzorov. *Records : Mathematical Theory.Translation of Mathematical Monographs, Volume 194.* American Mathematical Society. Providence, RI, USA, 2001.

[10] S.I. Resnick. *Extreme Values, Regular Variation and Point Processes.* Springer-Verbag, New-York. (MR0900810), 1987.