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**ASYMPTOTIC LOWER BOUNDS FOR THE FUNDAMENTAL  
FREQUENCY OF CONVEX MEMBRANE**

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**1. Introduction.** Let the bounded, simply connected, open region  $R$  of the  $(x, y)$ -plane have the boundary curve  $C$ . If a uniform ideal elastic membrane of unit density is uniformly stretched upon  $C$  with unit tension across each unit length, then  $\lambda$ , the square of the fundamental frequency, satisfies the conditions (subscripts denote differentiation)

$$(1a) \quad \begin{cases} \Delta u = u_{xx} + u_{yy} = -\lambda u & \text{in } R, \\ \lambda = \text{minimum}, \end{cases}$$

with the boundary condition

$$(1b) \quad u(x, y) = 0 \quad \text{on } C.$$

Variational methods of the Rayleigh-Ritz type are frequently used to approximate  $\lambda$ . They always yield upper bounds for  $\lambda$ , and the upper bounds can be made arbitrarily close.

Another common practical method of approximating  $\lambda$  is to calculate the least eigenvalue  $\lambda_h$  of a suitably chosen finite-difference operator  $\Delta_h$  over a network with small mesh width  $h$ . For one choice of  $\Delta_h$  it was shown by Courant, Friedrichs, and Lewy [3, p. 57] without details that  $\lambda_h \rightarrow \lambda$  as  $h \rightarrow 0$ . For convex regions  $R$  of a special polygonal form the author has shown [4] that a special case of (11) below is valid for a common choice of  $\Delta_h$ , and hence that  $\lambda_h$  is asymptotically a lower bound for  $\lambda$  as  $h \rightarrow 0$ . For an unusual finite-difference approximation to problem (1) when  $R$  is the union of squares of the network, Polya [12] has found that  $\lambda_h > \lambda$  for all  $h$ , and also for the higher eigenvalues. The author knows of no other study of the sign or order of decrease of  $\lambda - \lambda_h$  to 0.

In the present paper the investigation of [4] is extended to a much wider class of regions: those with piecewise analytic boundary curves and convex corners. The new theorems are stated and proved in §§ 3 and 4. Theorem 2 contains the theorem of [4] as a special case. Lemmas used in the proof of Theorem 1 are given in § 5. Identity (31) of Lemma 7 is interesting in itself.

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When  $C$  is no longer made up of line segments of the network, it is necessary when using finite-difference methods either to move  $C$  or to alter  $\mathcal{A}_h$  near the boundary. The latter procedure is potentially more accurate, and has been adopted in deriving the rather delicate results proved below. The definition of  $\mathcal{A}_h$  given in § 2 is a self-adjoint modification of Mikeladze's approximation [10; 11], and is believed to be new. The cruder approximations to  $\mathcal{A}$  near  $C$  proposed by Collatz in 1933 and expounded in [2, p. 357], while easier to compute in practice, appear to introduce an unmanageable term  $O(h^2)$  into (19). It is therefore doubted that Theorem 2 would remain valid for these cruder operators.

The technique of the present paper could be applied to study the asymptotic behavior of  $\lambda_h$  also for other difference approximations to  $\mathcal{A}$  in the interior of  $R$ —for example, for those associated with a triangular net [2, p. 367].

It is not clear that one could revise the argument of the paper to prove an inequality of the type

$$\frac{\lambda}{\lambda_h} \leq 1 + bh^2 + o(h^2) .$$

**2. Definitions.** Assume the bounded, simply connected, open region  $R$  to have a closed boundary curve  $C$ :  $x(s) + iy(s)$  ( $0 \leq s \leq s_m$ ) which is *piecewise analytic*. That is,  $x(s)$  and  $y(s)$  are real analytic functions of the arc length  $s$  of  $C$  in each of a finite number  $m$  of closed intervals

$$0 = s_0 \leq s \leq s_1, \quad s_1 \leq s \leq s_2, \quad \dots, \quad s_{m-1} \leq s \leq s_m .$$

Moreover, we demand that the corners of  $C$  be convex; that is, at any point  $x(s_j) + iy(s_j)$  ( $0 \leq j < m$ ) where distinct analytic curves meet, the interior angle of  $C$  must be less than  $\pi$ .

For  $h > 0$ , let a *net* consist of the lines  $x = \mu h$ ,  $y = \nu h$  ( $\mu, \nu = 0, \pm 1, \pm 2, \dots$ ). The points  $(\mu h, \nu h)$  in  $R$  are the *interior nodes*  $R_h$  of the net. The *boundary nodes*  $C_h$  of the net consist of (i) all points  $(\mu h, \nu h)$  on  $C$ , and (ii) all *isolated* points of intersection of the net with  $C$ . Thus each node  $(\mu h, \nu h)$  of  $R_h$  has two *neighboring nodes* in  $R_h \cup C_h$  on the line  $x = \mu h$ , and two in  $R_h \cup C_h$  on the line  $y = \nu h$ . Moreover, each node in  $C_h$  has at least one neighbor in  $R_h \cup C_h$ .

We now move toward a definition of the difference operator  $\mathcal{A}_h$ . Let us denote the neighboring nodes of the node

$$(2) \quad (x, y) \text{ of } R_h \text{ by } (x - h_1, y), (x + h_2, y), (x, y - h_3), \text{ and } (x, y + h_4),$$

where  $0 < h_i \leq h$  for  $i = 1, 2, 3, 4$ . For nodes remote from  $C_h$ , all  $h_i = h$ . Let  $v$  be any net function defined on the nodes of  $R_h \cup C_h$ , vanishing

on  $C_h$ . Define  $D_x^{(h)}v$  as the (constant) second derivative of the quadratic polynomial function of  $x$  assuming the three values  $v(x-h_1, y)$ ,  $v(x, y)$ , and  $v(x+h_3, y)$ . That is,

$$(3) \quad D_x^{(h)}v(x, y) = \frac{2}{h_1 + h_2} \left[ \frac{v(x+h_2, y) - v(x, y)}{h_2} - \frac{v(x, y) - v(x-h_1, y)}{h_1} \right].$$

Also,  $D_y^{(h)}v(x, y)$  is defined analogously. We next define

$$(4) \quad \begin{aligned} \mathcal{A}^{(h)}v(x, y) &= D_x^{(h)}v(x, y) + D_y^{(h)}v(x, y) \\ &= -\left(\frac{2}{h_1 h_2} + \frac{2}{h_3 h_4}\right)v(x, y) \\ &\quad + \frac{2}{h_1(h_1+h_2)}v(x-h_1, y) + \frac{2}{h_2(h_1+h_2)}v(x+h_2, y) \\ &\quad + \frac{2}{h_3(h_3+h_4)}v(x, y-h_3) + \frac{2}{h_4(h_3+h_4)}v(x, y+h_4). \end{aligned}$$

The operator  $\mathcal{A}^{(h)}$  is the approximation to  $\mathcal{A}$  recommended in [10]. It linearly transforms the net function  $v$  defined over  $R_h$  into the net function  $\mathcal{A}^{(h)}v$ , also defined over  $R_h$ . But  $\mathcal{A}^{(h)}$  is not a self-adjoint linear operator; that is, the matrix  $A^{(h)}$  of the linear transformation of  $v$  into  $\mathcal{A}^{(h)}v$  is not symmetric.

We define the matrix  $A_h$  as the symmetric part of the matrix  $A^{(h)}$ :

$$(5) \quad A_h = \frac{1}{2}[A^{(h)} + A^{(h)T}],$$

where  $T$  means transpose. Finally, we define  $\mathcal{A}_h$  to be the self-adjoint linear operator corresponding to  $A_h$ .

The explicit expressions for  $\mathcal{A}_h$  assume 16 different forms, depending on the location of  $(x, y)$  with respect to  $C_h$ . Although we shall not need these expressions for the present paper, we describe them briefly. If, in any of the four directions from  $(x, y)$ , the neighboring node—say  $(x-h_1, y)$ , for definiteness—is in  $R_h$ , then  $h_1=h$ , and there is another node  $(x-h-h_1', y)$  in  $R_h \cup C_h$ . Then the term  $2v(x-h_1, y)/h_1(h_1+h_2)$  of (4) is to be replaced by

$$(6) \quad \frac{h_1' + 2h + h_2}{(h_1' + h)h(h+h_2)}v(x-h, y).$$

For any  $(x, y)$ , the expression for  $\mathcal{A}_h$  is obtained from (4) by making replacements like (6) corresponding to all neighbors of  $(x, y)$  in  $R_h$ .

When  $(x, y)$  is more than two nodes away from  $C_h$ , so that all  $h_i=h_i'=h$ , the values of both  $\mathcal{A}^{(h)}$  and  $\mathcal{A}_h$  reduce to the familiar form used in [4]:

$$(7) \quad \Delta_h v(x, y) = \Delta^{(h)} v(x, y)$$

$$= \frac{1}{h^2} [v(x-h, y) + v(x+h, y) + v(x, y-h) + v(x, y+h) - 4v(x, y)] .$$

Let  $\lambda_h$  satisfy the following difference equation for a net function  $v$  defined in  $R_h \cup C_h$ :

$$(8a) \quad \begin{cases} \Delta_h v = -\lambda_h v & \text{in } R_h , \\ \lambda_h = \text{minimum} , \end{cases}$$

where  $v$  is extended to satisfy the boundary condition

$$(8b) \quad v = 0 \quad \text{on } C_h .$$

It is readily shown that  $\lambda_h$  is the minimum over all net functions  $v$  satisfying (8b) of the quotient

$$\rho_h(v) = \frac{-h^2 \sum_{R_h} v \Delta_h v}{h^2 \sum_{R_h} v^2} .$$

(This is simply the minimum principle for a definite quadratic form.) By (5), we can write  $\rho_h(v)$  in the following equivalent form, simpler to use:

$$(9) \quad \rho_h(v) = \frac{-h^2 \sum_{R_h} v \Delta^{(h)} v}{h^2 \sum_{R_h} v^2} .$$

The reason for not using the least eigenvalue  $\mu_h$  of  $\Delta^{(h)}$  in this investigation is that  $\mu_h$  does not have the foregoing minimum property and, in fact, might turn out to be complex. On the other hand, it is known [9, p. 27] that  $\lambda_h \leq \mathcal{R}(\mu_h)$ , so that when  $\mu_h$  is real it could conceivably be a better approximation to  $\lambda$  than  $\lambda_h$  is. The relative magnitude of  $|\lambda_h - \lambda|$  to  $|\mu_h - \lambda|$  is not known.

**3. The results.** The following new result will be proved in § 4:

**THEOREM 1.** *Let  $R$  be a bounded, open, simply connected region bounded by a piecewise analytic curve  $C$  whose corners are convex in the sense of § 2. Let  $\tau$  be the angle between the tangent to  $C$  and the  $x$  axis. Let  $u$  solve problem (1) for  $R$ , and let  $u_n$  be the normal derivative of  $u$  on  $C$ . Define  $\lambda_h$  as in § 2. Let*

$$(10) \quad a = a(R) = \frac{\iint_R (u_{xx}^2 + u_{yy}^2) dx dy + \int_C u_n^2 \sin^2 2\tau d\tau}{12 \iint_R (u_x^2 + u_y^2) dx dy} .$$

Then  $-\infty < a < \infty$  and, as  $h \rightarrow 0$ , one has

$$(11) \quad \frac{\lambda_h}{\lambda} \leq 1 - ah^2 + o(h^2) \quad (h \rightarrow 0).$$

In Theorem 1 the quantity  $a$  can probably be negative for certain nonconvex  $R$ , because  $d\tau$  in (10) will be negative at some points of  $C$ . But if  $R$  is convex we get a stronger result, as an immediate consequence of Theorem 1.

**THEOREM 2.** *Under the hypotheses of Theorem 1, if  $R$  is also convex, then  $0 < a < \infty$ , and there exists  $h_0 > 0$  such that  $\lambda_h < \lambda$  for all  $h < h_0$ .*

For the operator  $A_h$  of § 2 the methods of [3] can undoubtedly be followed to show that  $\lambda_h \rightarrow \lambda$  as  $h \rightarrow 0$ ; the author has not attempted to carry through the details. When  $\lambda_h \rightarrow \lambda$  as  $h \rightarrow 0$ , the lower bounds  $\lambda_{h_0}$  can be made arbitrarily close by choice of  $h_0$  sufficiently small. Thus for these  $R$  the Rayleigh-Ritz methods and the finite-difference methods (8) are theoretically complementary, and together could confine  $\lambda$  to an arbitrarily short interval if one knew an upper bound for  $h_0$ .

The author has not developed an upper bound for  $h_0$  in Theorem 2, although it would be desirable to do so by estimating the term  $o(h^2)$ . One could always make an intelligent guess based on the behavior of  $\lambda_h$  for certain  $h$ .

The constant  $a$  of (10) is the best possible for certain rectangular regions; see [4]. That the corners of  $C$  be convex seems essential to the validity of Theorem 1. Indeed, for one nonconvex polygon some heuristics and an experiment mentioned in [4] make it appear that  $\lambda_h = \lambda + Ah^{4/3} + o(h^{4/3})$ , where  $A > 0$ . It would be interesting to know the sign of  $a$  for the general case of Theorem 1, or in particular when  $C$  is a nonconvex analytic curve.

Corners of angle  $\pi$  are frequent in engineering practice, and it would be desirable to know how  $\lambda_h$  behaves when  $R$  has such corners. For such corners Lemma 2 is no longer valid. Lewy [7] provides new tools for an attack on corners of angle  $\pi$ .

**4. Proof of Theorem 1.** Let  $u$  henceforth be the solution of problem (1) for the fundamental eigenvalue  $\lambda$ . It is known that

$$(12) \quad \lambda \iint_R u^2 dx dy = \iint_R (u_x^2 + u_y^2) dx dy.$$

The proof of Theorem 1, following [4], consists in setting the values of the function  $u$  at the nodes of  $R_h \cup C_h$  into the Rayleigh quotient (9) of problem (8). It will be shown that

$$(13) \quad \frac{\rho_h(u)}{\lambda} = 1 - ah^2 + o(h^2) \quad (h \rightarrow 0).$$

Since  $\lambda_h \leq \rho_h(u)$ , the theorem follows from (13).

The denominator  $h^2 \sum u^2$  of  $\rho_h(u)$  differs from a Riemann sum for  $\iint_R u^2 dx dy$  at most by the terms corresponding to squares or part-squares at the boundary  $C$ . The total contribution of these terms does not exceed the order of magnitude  $Lh \max_R u^2$ , where  $L$  is the length of  $C$ . Hence a fortiori

$$(14) \quad h^2 \sum_{R_h} u^2 = \iint_R u^2 dx dy + o(1) \quad (h \rightarrow 0).$$

Let the nodes of  $R_h$  be divided into three classes:

$$(15) \quad \begin{cases} R_h^1: & \text{those within a distance } h \text{ of some corner of } C; \\ R_h^2: & \text{those not in } R_h^1 \text{ but within a distance } h \text{ of } C; \\ R_h^3: & \text{the other nodes of } R_h. \end{cases}$$

Split the numerator of  $\rho_h(u)$  accordingly:

$$-h^2 \sum_{R_h} u \Delta^{(h)} u = \sum_{i=1}^3 \left( -h^2 \sum_{R_h^i} u \Delta^{(h)} u \right) = \sum_{i=1}^3 S_h^i(u).$$

There are a fixed number of corners, not exceeding  $m$ , and at most two nodes of  $R_h^1$  per corner. Moreover  $|\nabla u(x, y)|^2 \rightarrow 0$  as  $(x, y) \rightarrow$  a corner of  $C$ , by Lemma 1 in § 5. At any node  $(x, y)$  of  $R_h^1$  with neighbors denoted as in (2), we find from (3) that

$$h^2 |u \Delta^{(h)} u| \leq \frac{h^2(u-0)}{\min h_i} \sum_{i=1}^4 \left| \frac{u - u_i}{h_i} \right| \leq 4h^2 \max |\nabla u|^2,$$

where the  $u_i$  are the values of  $u$  at the four neighbors of  $(x, y)$ , and where the maximum of  $|\nabla u|^2$  is taken over all points within a distance  $2h$  of some vertex. Hence

$$(16) \quad |S_h^1(u)| \leq 8mh^2 \max |\nabla u|^2 = o(h^2) \quad (h \rightarrow 0).$$

Using the notation and assertion of Lemma 3, we have

$$(17) \quad S_h^2(u) = -h^2 \sum_{R_h^2} u \Delta u - \frac{2h^3}{3} \sum_{R_h^2} u (\theta_x u'_{xxx} + \theta_y u'_{yyy}).$$

Since  $u$  satisfies (1a),

$$(18) \quad -h^2 \sum_{R_h^2} u \Delta u = \lambda h^2 \sum_{R_h^2} u^2.$$

By (17), (18), and Lemma 4,

$$|S_h^2(u) - \lambda h^2 \sum_{R_h^2} u^2| \leq \frac{2}{3} h^3 \sum_{R_h^2} u(|u'_{xxx}| + |u''_{yyy}|) = o(h^2) \quad (h \rightarrow 0).$$

Thus

$$(19) \quad S_h^2(u) = \lambda h^2 \sum_{R_h^2} u^2 + o(h^2) \quad (h \rightarrow 0).$$

Similarly, using the notation and assertion of Lemma 5, and by (1a), we have

$$(20) \quad S_h^3(u) = \lambda h^2 \sum_{R_h^3} u^2 - \frac{h^4}{12} \sum_{R_h^3} u(u'_{xxxx} + u''_{yyyy}).$$

Now

$$(21) \quad h^2 \sum_{R_h^2 \cup R_h^3} u^2 = h^2 \sum_{R_h} u^2 - h^2 \sum_{R_h^1} u^2 = h^2 \sum_{R_h} u^2 + o(h^2),$$

since  $u(x, y) \rightarrow 0$  as  $(x, y) \rightarrow C$ , and since there are at most  $2m$  vertices in  $R_h^1$ . Adding (19) and (20), and using (21), we find that

$$\begin{aligned} S_h^2(u) + S_h^3(u) &= \lambda h^2 \sum_{R_h} u^2 - \frac{h^4}{12} \sum_{R_h^3} u(u'_{xxxx} + u''_{yyyy}) + o(h^2) \\ &= \lambda h^2 \sum_{R_h} u^2 - \frac{h^4}{12} \iint_R u(u'_{xxxx} + u''_{yyyy}) dx dy + o(h^2), \end{aligned}$$

by Lemma 6. Adding  $S_h^1(u)$  to the above, and dividing by (14), we find that

$$\begin{aligned} (22) \quad \rho_h(u) &= \frac{\sum_{i=1}^3 S_h^i(u)}{h^2 \sum_{R_h} u^2} \\ &= \lambda - \frac{h^2}{12} \frac{\iint_R u(u'_{xxxx} + u''_{yyyy}) dx dy}{\iint_R u^2 dx dy} + o(h^2). \end{aligned}$$

Finally, dividing (22) by  $\lambda$ , and applying Lemma 7 and (12), one proves (13) and hence Theorem 1.

**5. Some lemmas.** The following lemmas are basic to the proof of Theorem 1. In all of them  $R$  satisfies the conditions stated at the start of § 2, while  $u=u(x, y)$  solves problem (1).

**LEMMA 1.** *The function  $u$  is an analytic function of  $x$  and  $y$  in  $R \setminus C$ , except possibly at the corners of  $C$ . Let  $r$  be the distance of  $(x, y)$  from a corner  $P$  with interior angle  $\pi/\alpha$ ,  $1 < \alpha < \infty$ . Then for  $m=0, 1, 2, \dots$ , any partial derivative of  $u$  of order  $m$  has the local representation*

$$(23) \quad \frac{\partial^m u}{\partial x^\mu \partial y^\nu} = r^{\alpha-m} f_m(x, y) \quad (\mu + \nu = m),$$

where  $f_m$  is continuous at  $P$ .

*Proof.* By [1, p. 179],  $u$  is analytic in  $R$ . The representation (27') below shows that the interior normal derivative  $u_n$  is integrable on  $C$ . Then the analyticity of  $u$  on  $C$  (corners excluded) was shown by Hadamard [5, p. 25].<sup>1</sup>

Let  $t = \xi + i\eta$  and  $z = x + iy$ . For each  $t \in R$  let  $w = \phi(z, t)$  map  $R$  conformally onto the circle  $|w| < 1$ , with  $\phi(t, t) = 0$ . We may assume without loss of generality that  $P$  is at  $z = 0$ , and that  $\phi(0, t) = 1$ . Lichtenstein [8, pp. 255–256 and footnote 273] showed<sup>2</sup> that for  $m = 0, 1, 2, \dots$ , and  $z \in R$ ,

$$(24) \quad \frac{\partial^m \phi(z, t)}{\partial z^m} = z^{\alpha-m} \varphi_m(z, t),$$

where  $\varphi_m$  is continuous at  $z = 0$ . It follows from (24) that

$$(25) \quad \frac{\partial^m \log \phi(z, t)}{\partial z^m} = z^{\alpha-m} \psi_m(z, t),$$

where  $\psi_m$  is continuous at  $z = 0$ . Let  $G(z, t) = G(\xi, \eta; x, y)$  be Green's function for  $\Delta u$  in  $R$ . Since

$$G(z, t) = -(2\pi)^{-1} \log |f(z, t)|,$$

it follows from (25) that for  $m = 0, 1, 2, \dots$  and  $z \in R$ ,

$$(26) \quad \frac{\partial^m G(z, t)}{\partial x^\mu \partial y^\nu} = r^{\alpha-m} \Psi_m(z, t) \quad (\mu + \nu = m),$$

where  $\Psi_m$  is continuous at  $z = 0$ .

Now the function  $u$  has the integral representation [1, pp. 182–183]

$$u(x, y) = \lambda \iint_R G(x, y; \xi, \eta) u(\xi, \eta) d\xi d\eta.$$

Hence

$$(27) \quad \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x}$$

<sup>1</sup> The author wishes to thank Professor Lewy for this reference.

<sup>2</sup> Lichtenstein actually asserts that (24) is without question true for all  $\alpha$ , but that his proof is valid only for irrational  $\alpha$ . Warschawski [13] has found a simple proof of (24), valid for all  $\alpha$  in the range  $\frac{1}{2} \leq \alpha < \infty$ .

Added in April 1954: For asymptotic expansions of  $\phi$  at a corner, see R. Sherman Lehmann, "Development of the mapping function at an analytic corner," Technical Report No. 21, Applied Mathematics and Statistics Laboratory, Stanford University, California, March 31, 1954, 17 pp.

$$\begin{aligned}
&= \lambda \iint_R \frac{G(x+\Delta x, y; \xi, \eta) - G(x, y; \xi, \eta)}{\Delta x} u(\xi, \eta) d\xi d\eta \\
&= \lambda \iint_R \frac{\partial G}{\partial x}(x + \theta \Delta x, y; \xi, \eta) u(\xi, \eta) d\xi d\eta ,
\end{aligned}$$

where  $0 < \theta = \theta(x, y, \Delta x) < 1$ . Since  $G(z, t) = G(t, z)$ , it is clear that  $\partial G / \partial x = \partial G / \partial \xi$  and, as a function of  $t$ ,  $\partial G / \partial x$  behaves like  $|t - t_0|^{\alpha-1}$  at any corner  $t_0$  of  $R$ , uniformly in  $z$  for  $z$  bounded away from  $C$ . Hence  $(\partial G / \partial x)u(\xi, \eta)$  in (27) is dominated by an integrable function of  $\xi, \eta$ , uniformly with respect to  $\Delta x$ . By Lebesgue's convergence theorem, letting  $\Delta x \rightarrow 0$  in (27) proves that

$$(27') \quad \frac{\partial u}{\partial x} = \lambda \iint_R \frac{\partial G}{\partial x}(x, y; \xi, \eta) u(\xi, \eta) d\xi d\eta .$$

Setting the expression (26) for  $m=\mu=1$  into the last equation proves the case  $m=\mu=1$  of (23).

In a similar way one can prove all the cases  $m=0, 1, 2, 3, 4$  of (23), and the lemma is established.

**LEMMA 2.** *The functions  $u_{xx}^2, u_x u_{xxx}, uu_{xxxx}, u_{yy}^2, u_y u_{yyy}$ , and  $uu_{yyyy}$  are Lebesgue integrable in  $R$ . The Lebesgue integrals  $\int_C u_x u_{xx} dy$  and  $\int_C u_y u_{yy} dx$  exist.*

*Proof.* By Lemma 1 the functions  $u_{xx}^2, \dots, uu_{yyyy}$  are continuous in  $R \setminus C$  except possibly at the corners, where they are  $O(r^{2\alpha-4})$ . Since  $0 < \alpha$ , the first sentence follows. The second sentence is proved analogously.

**REMARK.** The proof of Lemma 2 breaks down for corners of angle  $\pi(\alpha-1)$ , as  $r^{-2}$  is not integrable.

**LEMMA 3.** *At any node  $(x, y)$  of  $R_h$  whose neighbors are denoted as in (2), one has*

$$\Delta^{(h)} u = \Delta u + \frac{2}{3} h [\theta_x u'_{xxx} + \theta_y u''_{yyy}] ,$$

where  $-1 < \theta_x < 1$ ,  $-1 < \theta_y < 1$ , and where

$$(28) \quad \begin{cases} u'_{xxx} = u_{xxx}(x', y), & x - h_1 < x' < x + h_2, \\ u''_{yyy} = u_{yyy}(x, y'), & y - h_3 < y' < y + h_4. \end{cases}$$

*Proof.* By Lemma 1,  $u_{xxx}$  is continuous in the open line segment from  $(x-h_1, y)$  to  $(x+h_2, y)$ , but may become infinite if the endpoint is a corner of  $C$ . Since  $u$  is continuous in  $R \setminus C$ , it nevertheless follows

from Taylor's formula as stated in [6, p. 357] that, if we fix  $y$  and set  $\phi(x)=u(x, y)$ ,

$$\frac{\phi(x+h_2)-\phi(x)}{h_2} = \phi'(x) + \frac{h_2}{2}\phi''(x) + \frac{h_2^2}{6}\phi'''(x+\theta_2 h_2),$$

where  $0 < \theta_2 < 1$ .

Writing a similar formula for  $h_1$  and subtracting, we find in the notation of (3) that

$$D_x^{(h)}\phi(x) = \phi''(x) + \left[ \frac{h_2^2}{3}\phi'''(x+\theta_2 h_2) - \frac{h_1^2}{3}\phi'''(x-\theta_1 h_1) \right] (h_1+h_2)^{-1}.$$

If one writes  $k = \max(h_1, h_2) \leq h$ , the last term can be bounded in absolute value by

$$\frac{2k^2}{3k} \max[|\phi'''(x+\theta_2 h_2)|, |\phi'''(x-\theta_1 h_1)|],$$

and hence can be written in the form  $(2h/3)\theta_x u'_{xxx}$ . Addition of a similar expression for  $D_y^{(h)}u(x, y)$  proves the lemma.

**LEMMA 4.** *For each node  $(x, y)$  of  $R_h^2$  defined in (15) use the notation of (28). Then, as  $h \rightarrow 0$ , one has*

$$(29) \quad h \sum_{R_h^2} u(|u'_{xxx}| + |u''_{yyy}|) = o(1) \quad (h \rightarrow 0).$$

*Proof.* The lemma is proved much like Lemma 6 of [4]. The functions  $u|u'_{xxx}|$  and  $u|u''_{yyy}|$  are continuous in  $R \setminus C$ , except at a corner of interior angle  $\pi\alpha$ , where Lemma 1 states that they behave like  $r^{2\alpha-3}$  with  $2\alpha-3 > -1$ . The sum (29) can be majorized by the Lebesgue integral of a step function over a polygonal arc in  $R$  which converges in length to  $C$  as  $h \rightarrow 0$ . The integrability of  $r^{2\alpha-3}$  in  $(0, 1)$  permits the application of Lebesgue's convergence theorem as  $h \rightarrow 0$ . Since  $u=0$  on  $C$ , (29) follows. Details are omitted.

**LEMMA 5.** *At each node in  $R_h^3$ , defined in (15), one has*

$$\Delta^{(h)}u = \Delta u + \frac{1}{12}h^2(u'_{xxxx} + u''_{yyyy}),$$

where

$$(30) \quad \begin{cases} u'_{xxxx} = u_{xxxx}(x+\theta'h, y), & -1 < \theta' < 1, \\ u''_{yyyy} = u_{yyyy}(x, y+\theta''h), & -1 < \theta'' < 1. \end{cases}$$

*Proof.* In [4]; the points of  $R_h^3$  all have four neighbors in  $R_h^3$ ,

each at a distance  $h$ .

LEMMA 6. *At each node of  $R_h^3$ , defined in (15), use the notation of (30). Then, as  $h \rightarrow 0$ , one has*

$$h^2 \sum_{R_h^3} u(u'_{xxxx} + u''_{yyyy}) = \iint_R u(u_{xxxx} + u_{yyyy}) dx dy + o(1) \quad (h \rightarrow 0).$$

*Proof.* In [4].

LEMMA 7. *Define  $u_n$  and  $\tau$  as in Theorem 1. One then has*

$$\iint_R u(u_{xxxx} + u_{yyyy}) dx dy = \iint_R (u_{xx}^2 + u_{yy}^2) dx dy + \int_C u_n^2 \sin^2 2\tau d\tau,$$

where the latter is a Riemann-Stieltjes integral.

*Proof.* The proof repeats that of Lemma 7 in [4] down to (29) of that paper. It then remains only to prove for smooth convex curves  $C$  that

$$(31) \quad \int_C u_{yy}(u_y dx + u_x dy) = \int_C u_n^2 \sin^2 2\tau d\tau.$$

Let  $s$  denote arclength on  $C$ , and let primes denote  $d/ds$ . Differentiating the relations  $u_x = -u_n \sin \tau$ ,  $u_y = u_n \cos \tau$ , we find that, on  $C$ ,

$$(32) \quad \begin{cases} u_x' = -u_n' \sin \tau - u_n \tau' \cos \tau = u_{xy} \sin \tau + u_{xx} \cos \tau, \\ u_y' = u_n' \cos \tau - u_n \tau' \sin \tau = u_{xy} \cos \tau + u_{yy} \sin \tau. \end{cases}$$

Changing  $u_{xx}$  to  $-u_{yy}$  by (1), we can solve (32) for  $u_{yy}$  on  $C$ :

$$u_{yy} = u_n' \sin 2\tau + u_n \tau' \cos 2\tau.$$

Since  $dx = ds \cos \tau$  and  $dy = ds \sin \tau$ , we obtain

$$(33) \quad \begin{aligned} \int_C u_{yy}(u_y dx + u_x dy) &= \int_C (u_n' \sin 2\tau + u_n \tau' \cos 2\tau)(u_n \cos 2\tau) ds \\ &= \int_C u_n^2 \tau' \cos^2 2\tau ds + \int_C u_n u_n' \cos 2\tau \sin 2\tau ds. \end{aligned}$$

By partial integration, we have

$$(34) \quad \begin{aligned} \int_C u_n u_n' \cos 2\tau \sin 2\tau ds &= \frac{1}{4} \int_C (u_n^2)' \sin 4\tau ds \\ &= \frac{1}{4} [u_n^2 \sin 4\tau]_C - \int_C u_n^2 \tau' \cos 4\tau ds. \end{aligned}$$

Since  $\cos^2 2\tau - \cos 4\tau \equiv \sin^2 2\tau$ , substitution of (34) into (33) shows that

$$\int_C u_{yy} (u_y dx + u_x dy) = \int_C u_n^2 \tau' \sin^2 2\tau ds .$$

Since  $\tau' ds = d\tau$ , the identity (31) is proved, and with it, the lemma.

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